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Research Article

Existence and Algorithm for the Systems of Hierarchical Variational Inclusion Problems

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We study the existence and approximation of a solution for a system of hierarchical variational inclusion problems in Hilbert spaces. In this study, we use Maingé's approach for finding the solutions of the system of hierarchical variational inclusion problems. Our result in this paper improves and generalizes some known corresponding results in the literature.

1. Introduction

Let H be a real Hilbert space with inner product and norm being $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively, and let C be a nonempty closed convex subset of H. A mapping $T : H \to H$ is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in H. \tag{1}$$

We use $F(T)$ to denote the set of *fixed points* of T; that is, $F(T) = \{x \in H : Tx = x\}$. It is well known that $F(T)$ is a closed convex set, if T is nonexpansive mappings.

A variational inclusion problem [1–3] is the problem of finding a point $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{2}$$

where $A : H \to H$ is a single-valued nonlinear mapping and $M : H \to 2^H$ is a multivalued mapping. We use Ω to denote the set of solutions of the variational inclusion (2).

On the other hand, *a hierarchical fixed point problem* [4–11] is the problem of finding a point $x^* \in F(T)$ such that

$$\langle Ax^*, x - x^*\rangle \ge 0, \quad \forall x \in F(T). \tag{3}$$

If the set $F(T)$ is replaced by the solution set of the variational inequality, then the hierarchical fixed point problems are called *hierarchical variational inequality problems* or *hierarchical optimization problems*. Many problems in mathematics, for example, the signal recovery [12], the power control problem [13], and the beamforming problem [14], can be considered in the framework of this kind of the hierarchical variational inequality problems.

Recently, Chang et al. [15] introduced *bilevel hierarchical variational inclusion problems*; that is, find $(x^*, y^*) \in \Omega_1 \times \Omega_2$ such that, for given positive real numbers ρ and η, the following inequalities hold:

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^*\rangle \ge 0, \quad \forall x \in \Omega_1, \\ &\langle \eta F(x^*) + y^* - x^*, y - y^*\rangle \ge 0, \quad \forall y \in \Omega_2, \end{aligned} \tag{4}$$

where $F, A_1, A_2 : H \to H$ are mappings, $M_1, M_2 : H \to 2^H$ are multivalued mappings, and Ω_i is the set of solutions to variational inclusion problem (2) with $A = A_i$, $M = M_i$ for $i = 1, 2$. They solved the convex programming problems and quadratic minimization problems by using Maingés scheme.

In this paper, we consider the following *system of hierarchical variational inclusion problem*: find $(x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3$, such that, for given positive real numbers ρ, η, and ξ, the following inequalities hold:

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1, \\ &\langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in \Omega_2, \\ &\langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in \Omega_3. \end{aligned} \tag{5}$$

Some special cases of the system of hierarchical variational inclusion problem (5) are as follows.

(I) If $M_i = 0$, $A_i = I - T_i$, where $T_i : H \to H$ is a nonlinear mapping for each $i = 1, 2, 3$, in (5), then $\Omega_i = F(T_i)$ and the system of hierarchical variational inclusion problem (5) reduces to the following *system of hierarchical optimization problem*: find $(x^*, y^*, z^*) \in F(T_1) \times F(T_2) \times F(T_3)$, such that

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T_1), \\ &\langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in F(T_2), \\ &\langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in F(T_3), \end{aligned} \tag{6}$$

which was studied by Li [16].

(II) If $T_i = P_{K_i}$ for each $i = 1, 2, 3$, where P_{K_i} is the metric projection from H onto a nonempty closed convex subset K_i in (6), then it is clear that the $\Omega_i = F(T_i) = K_i$ and the system of hierarchical optimization problem (6) reduces to the following *system of optimization problem*: find $(x^*, y^*, z^*) \in K_1 \times K_2 \times K_3$ such that

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K_1, \\ &\langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in K_2, \\ &\langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in K_3. \end{aligned} \tag{7}$$

(III) If $K_1 = K_2 = K_3$, then the system of optimization problem (7) reduces to the following *system of variational inequality problem*: find $(x^*, y^*, z^*) \in K_1 \times K_1 \times K_1$ such that

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K_1, \\ &\langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in K_1, \\ &\langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in K_1. \end{aligned} \tag{8}$$

(IV) If $\xi = 0, \rho, \eta > 0, \Omega_1 = \Omega_3$, and $x^* = z^*$ in (5) then the system of hierarchical variational inclusion problem (5) reduces to the following *bilevel hierarchical variational inclusion problem*: find $(x^*, y^*) \in \Omega_1 \times \Omega_2$ such that

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1, \\ &\langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in \Omega_2, \end{aligned} \tag{9}$$

which was studied by Chang et al. [15].

(V) In (9), if $M_i = 0$, $A_i = I - T_i$, for each $i = 1, 2$, then bilevel hierarchical variational inclusion problem (9) reduces to the following *bilevel hierarchical optimization problem*: find $(x^*, y^*) \in F(T_1) \times F(T_2)$ such that

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T_1), \\ &\langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in F(T_2), \end{aligned} \tag{10}$$

which was studied by Maingé [17] and Kraikaew and Saejung [18].

(VI) In (10), if $T_i = P_{K_i}$ for each $i = 1, 2$, then bilevel hierarchical optimization problem (10) reduces to the following problem [19–21]: find $(x^*, y^*) \in K_1 \times K_2$ such that

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K_1, \\ &\langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in K_2. \end{aligned} \tag{11}$$

(VII) In (11), if $K_1 = K_2$ then the problem (11) reduces to the following problem: find $(x^*, y^*) \in K_1 \times K_1$ such that

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K_1, \\ &\langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in K_1. \end{aligned} \tag{12}$$

(VIII) In (5), if $\xi = \eta = 0$, $\rho > 0$, $\Omega_1 = \Omega_2 = \Omega_3$, and $x^* = y^* = z^*$ then the system of hierarchical variational inclusion problem (5) reduces to the following *hierarchical variational inclusion problem*: find $x^* \in \Omega_1$ such that

$$\langle F(y^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1. \tag{13}$$

(IX) In (13), if $M_1 = 0$, $A_i = I - T_1$ then the hierarchical variational inclusion problem (13) reduces to the following *hierarchical fixed point problem*: find $x^* \in F(T_1)$ such that

$$\langle F(y^*), x - x^* \rangle \geq 0, \quad \forall x \in F(T_1). \tag{14}$$

(X) In (15), if $T_1 = P_{K_1}$ then the hierarchical fixed point problem (15) reduces to the following *classic variational inequality problem*: find $x^* \in K_1$ such that

$$\langle F(y^*), x - x^* \rangle \geq 0, \quad \forall x \in K_1. \tag{15}$$

Motivated and inspired by Chang et al. [15], we introduce the system of a hierarchical variational inclusion problem (5) and investigate a more general variant of the scheme proposed by Chang et al. [15] to solve the system of a hierarchical variational inclusion problem. Our analysis and method allow us to prove the existence and approximation of solutions to the system of a hierarchical variational inclusion problem (5). The results presented in this paper extend and improve the results of Chang et al. [15], Maingé [17], Kraikaew and Saejung [18], and some authors.

2. Preliminaries

This section collects some definitions and lemmas which can be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive. We use $\rightarrow$ for strong convergence and $\rightharpoonup$ for weak convergence.

Definition 1. Let $A, T, F : H \rightarrow H$ be a mapping and let $M : H \rightarrow 2^H$ be a multivalued mapping.

(1) A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H. \tag{16}$$

(2) A mapping T is called *quasinonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in H, \; p \in F(T). \tag{17}$$

It should be noted that T is quasinonexpansive if and only if for all $x \in H$, $p \in F(T)$

$$\langle x - Tx, x - p \rangle \geq \frac{1}{2} \|x - Tx\|^2. \tag{18}$$

(3) A mapping T is called *strongly quasinonexpansive* if T is quasinonexpansive and $x_n - Tx_n \rightarrow 0$, whenever $\{x_n\}$ is a bounded sequence in H and $\|x_n - p\| - \|Tx_n - p\| \rightarrow 0$ for some $p \in F(T)$.

(4) A mapping F is called *μ-Lipschitzian* if there exists $\alpha > 0$ such that

$$\|Fx - Fy\| \leq \mu \|x - y\|, \quad \forall x, y \in H. \tag{19}$$

(5) A mapping F is called *r-strongly monotone* if there exists $r > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq r \|x - y\|^2, \quad \forall x, y \in H. \tag{20}$$

It is easy to prove that if $F : H \rightarrow H$ is a μ-Lipschitzian and r-strongly monotone mapping and if $\rho \in (0, 2r/\mu^2)$, then the mapping $I - \rho F$ is a contraction.

(6) A mapping A is called *α-inverse-strongly monotone* if there exists $\mu > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in H. \tag{21}$$

(7) A multivalued mapping M is called *monotone* if for all $x, y \in H$, $u \in Mx$ and $v \in My$ imply that

$$\langle u - v, x - y \rangle \geq 0. \tag{22}$$

(8) A multivalued mapping M is called *maximal monotone* if it is monotone and for any $(x, u) \in H \times H$,

$$\langle u - v, x - y \rangle \geq 0 \tag{23}$$

for every $(y, v) \in \mathrm{Graph}(M)$ (the graph of mapping M) implies that $u \in Mx$.

Lemma 2 (see [22]). *Let $A : H \rightarrow H$ be an α-inverse-strongly monotone mapping. Then*

(1) *A is an $1/\alpha$-Lipschitz continuous and monotone mapping;*

(2) *for any constant $\lambda > 0$, one has*

$$\begin{aligned} &\|(I - \lambda A)x - (I - \lambda A)y\|^2 \\ &\quad \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2; \end{aligned} \tag{24}$$

(3) *if $\lambda \in (0, 2\alpha]$, then $I - \lambda A$ is a nonexpansive mapping, where I is the identity mapping on H.*

Lemma 3. *Let $x \in H$ and $z \in C$ be any points. Then one has the following.*

(1) *That $z = P_C[x]$ if and only if there holds the relation:*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C. \tag{25}$$

(2) *That $z = P_C[x]$ if and only if there holds the relation:*

$$\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in C. \tag{26}$$

(3) *There holds the relation:*

$$\begin{aligned} \langle P_C[x] - P_C[y], x - y \rangle &\geq \|P_C[x] - P_C[y]\|^2, \\ &\forall x, y \in H. \end{aligned} \tag{27}$$

Consequently, P_C is nonexpansive and monotone.

Definition 4. Let $M : H \rightarrow 2^H$ be a multivalued maximal monotone mapping. Then the mapping $J_{M,\lambda} : H \rightarrow H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad u \in H \tag{28}$$

is called *the resolvent operator associated with M*, where λ is any positive number and I is the identity mapping.

Proposition 5 (see [22]). *Let $M : H \rightarrow 2^H$ be a multivalued maximal monotone mapping, and let $A : H \rightarrow H$ be an α-inverse-strongly monotone mapping. Then the following conclusions hold.*

(1) *The resolvent operator $J_{M,\lambda}$ associated with M is single-valued and nonexpansive for all $\lambda > 0$.*

(2) *The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone; that is,*

$$\begin{aligned} \|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 &\leq \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y) \rangle, \\ &\forall x, y \in H. \end{aligned} \tag{29}$$

(3) *$u \in H$ is a solution of the variational inclusion (2) if and only if $u = J_{M,\lambda}(u - \lambda A u)$, for all $\lambda > 0$; that is, u is a fixed point of the mapping $J_{M,\lambda}(I - \lambda A)$. Therefore one has*

$$\Omega = F(J_{M,\lambda}(I - \lambda A)), \quad \forall \lambda > 0, \tag{30}$$

where Ω is the set of solutions of variational inclusion problem (2).

(4) *If $\lambda \in (0, 2\alpha]$, then Ω is a closed convex subset in H.*

Lemma 6 (see [23]). *For $x, y \in H$ and $\omega \in (0,1)$, the following statements hold:*

(1) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$;

(2) $\|(1-\omega)x+\omega y\|^2 = (1-\omega)\|x\|^2+\omega\|y\|^2-\omega(1-\omega)\|x-y\|^2$.

Lemma 7 (see [24]). *Let $\{a_n\}$ be a sequence of real numbers, and there exists a subsequence $\{a_{m_j}\}$ of $\{a_n\}$ such that $a_{m_j} < a_{m_j+1}$ for all $j \in N$, where N is the set of all positive integers. Then there exists a nondecreasing sequence $\{n_k\}$ of N such that $\lim_{k\to\infty} n_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in N$:*

$$a_{n_k} \le a_{n_k+1}, \qquad a_k \le a_{n_k+1}. \tag{31}$$

In fact, n_k is the largest number n in the set $\{1, 2, \ldots, k\}$ such that $a_n < a_{n+1}$ holds.

Lemma 8 (see [18]). *Let $\{a_n\} \subset [0,\infty)$, $\{\alpha_n\} \subset [0,1)$, $\{b_n\} \subset (-\infty, +\infty)$, and $h \in [0,1)$ be such that*

(1) *$\{a_n\}$ is a bounded sequence;*

(2) $a_{n+1} \le (1-\alpha_n)^2 a_n + 2\alpha_n h\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n$, *for all $n \ge 1$;*

(3) *whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying*

$$\liminf_{k\to\infty}\left(a_{n_k+1} - a_{n_k}\right) \ge 0, \tag{32}$$

it follows that $\limsup_{k\to\infty} b_{n_k} \le 0$;

(4) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 9 (see [15]). *Let $M : H \to 2^H$ be a multivalued maximal monotone mapping, let $A : H \to H$ be an α-inverse-strongly monotone mapping, and let Ω be the set of solutions of variational inclusion problem (2) and $\Omega \ne \emptyset$. Then the following statements hold.*

(1) *If $\lambda \in (0, 2\alpha]$, then the mapping $K : H \to H$ defined by*

$$K := J_{M,\lambda}(I - \lambda A) \tag{33}$$

is quasinonexpansive, where I is the identity mapping and $J_{M,\lambda}$ is the resolvent operator associated with M.

(2) *The mapping $I - K : H \to H$ is demiclosed at zero; that is, for any sequence $\{x_n\} \subset H$, if $x_n \rightharpoonup x$ and $(I-K)x_n \to 0$, then $x = Kx$.*

(3) *For any $\beta \in (0,1)$, the mapping K_β defined by*

$$K_\beta = (1-\beta)I + \beta K \tag{34}$$

is a strongly quasinonexpansive mapping and $F(K_\beta) = F(K)$.

(4) *$I - K_\beta$, $\beta \in (0,1)$ is demiclosed at zero.*

3. Main Results

Throughout this section, we always assume that the following conditions are satisfied:

(C1) $M_i : H \to 2^H$ is a multivalued maximal monotone mapping, $A_i : H \to H$ is an α_i-inverse-strongly monotone mapping, and Ω_i is the set of solutions to variational inclusion problem (2) with $A = A_i$, $M = M_i$, and $\Omega_i \ne \emptyset$, for all $i = 1, 2, 3$;

(C2) K_i and $K_{i,\beta}, \beta \in (0,1)$, $i = 1,2,3$, are the mappings defined by

$$\begin{aligned} K_i &:= J_{M_i,\lambda}(I - \lambda A_i), \quad \lambda \in (0, 2\alpha_i], \\ K_{i,\beta} &:= (1-\beta) I + \beta K_i, \quad \beta \in (0,1), \end{aligned} \tag{35}$$

respectively.

Next, there are our main results.

3.1. An Existence Theorem

Theorem 10. *Let A_i, M_i, Ω_i, K_i, and $K_{i,\beta}$ satisfy conditions (C1) and (C2), and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0,1)$, for all $i = 1,2,3$. Then there exists a unique element $(x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied:*

$$\begin{aligned} &\langle x^* - f_1(y^*), x - x^*\rangle \ge 0, \quad \forall x \in \Omega_1, \\ &\langle y^* - f_2(z^*), y - y^*\rangle \ge 0, \quad \forall y \in \Omega_2, \\ &\langle z^* - f_3(x^*), z - z^*\rangle \ge 0, \quad \forall z \in \Omega_3. \end{aligned} \tag{36}$$

Proof. The proof is a consequence of Banach's contraction principle but it is given here for the sake of completeness. By Proposition 5 and Lemma 9, Ω_1, Ω_2, and Ω_3 are nonempty closed and convex. Therefore the metric projection P_{Ω_i} is well defined for each $i = 1, 2, 3$.

Since f_i is a contraction mapping for each $i = 1,2,3$, then we have $P_{\Omega_i} f_i$ which is a contraction and also have

$$P_{\Omega_1} f_1 \circ P_{\Omega_2} f_2 \circ P_{\Omega_3} f_3 \tag{37}$$

which is a contraction. Hence there exists a unique element $x^* \in H$ such that

$$x^* = \left(P_{\Omega_1} f_1 \circ P_{\Omega_2} f_2 \circ P_{\Omega_3} f_3\right) x^*. \tag{38}$$

Putting $z^* = P_{\Omega_3} f_3(x^*)$ and $y^* = P_{\Omega_2} f_2(z^*)$, then $z^* \in \Omega_3$, $y^* \in \Omega_2$, and $x^* = P_{\Omega_1} f_1(y^*)$.

Suppose that there is an element $(\hat{x}, \hat{y}, \hat{z}) \in \Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied:

$$\begin{aligned} &\langle \hat{x} - f_1(\hat{y}), x - \hat{x}\rangle \ge 0, \quad \forall x \in \Omega_1, \\ &\langle \hat{y} - f_2(\hat{z}), y - \hat{y}\rangle \ge 0, \quad \forall y \in \Omega_2, \\ &\langle \hat{z} - f_3(\hat{x}), z - \hat{z}\rangle \ge 0, \quad \forall z \in \Omega_3. \end{aligned} \tag{39}$$

Then

$$\begin{aligned}\hat{x} &= P_{\Omega_1} f_1(\hat{y}),\\ \hat{y} &= P_{\Omega_2} f_2(\hat{z}),\\ \hat{z} &= P_{\Omega_3} f_3(\hat{x}).\end{aligned} \tag{40}$$

Therefore

$$\hat{x} = \left(P_{\Omega_1} f_1 \circ P_{\Omega_2} f_2 \circ P_{\Omega_3} f_3\right)\hat{x}. \tag{41}$$

This implies that $\hat{x} = x^*$, $\hat{y} = y^*$, and $\hat{z} = z^*$. This completes the proof. □

3.2. A Convergence Theorem

Theorem 11. *Let A_i, M_i, Ω_i, K_i, and $K_{i,\beta}$ satisfy conditions (C1) and (C2), and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0,1)$, for all $i = 1,2,3$. Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be three sequences defined by*

$$\begin{aligned}&x_0, y_0, z_0 \in H,\\ &x_{n+1} = (1-\alpha_n) K_{1,\beta} x_n + \alpha_n f_1\left(K_{2,\beta} y_n\right),\\ &y_{n+1} = (1-\alpha_n) K_{2,\beta} y_n + \alpha_n f_2\left(K_{3,\beta} z_n\right),\\ &z_{n+1} = (1-\alpha_n) K_{3,\beta} z_n + \alpha_n f_3\left(K_{1,\beta} x_n\right),\\ &\qquad n = 0,1,2,\ldots,\end{aligned} \tag{42}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated to be (42) converge to x^*, y^*, and z^*, respectively, where (x^*, y^*, z^*) is the unique element in $\Omega_1 \times \Omega_2 \times \Omega_3$ verifying (36).

Proof. (i) First we prove that sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded.

From Lemma 9, it follow that $K_{i,\beta}$ is strongly quasinonexpansive and $F(K_{i,\beta}) = F(K_i) = \Omega_i$ for each $i = 1,2,3$. Since f_i is contraction with the coefficient h_i for each $i = 1,2,3$ and $x^* \in F(K_{1,\beta})$, $y^* \in F(K_{2,\beta})$, and $z^* \in F(K_{3,\beta})$, it follows that

$$\begin{aligned}\|x_{n+1} - x^*\| &\le (1-\alpha_n)\left\|K_{1,\beta} x_n - x^*\right\|\\ &\quad + \alpha_n \left\|f_1\left(K_{2,\beta} y_n\right) - x^*\right\|\\ &\le (1-\alpha_n)\|x_n - x^*\|\\ &\quad + \alpha_n \left\|f_1\left(K_{2,\beta} y_n\right) - f_1(y^*)\right\|\\ &\quad + \alpha_n \|f_1(y^*) - x^*\|\\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n h_1 \left\|K_{2,\beta} y_n - y^*\right\|\\ &\quad + \alpha_n \|f_1(y^*) - x^*\|\\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n h_1 \|y_n - y^*\|\\ &\quad + \alpha_n \|f_1(y^*) - x^*\|\\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n h \|y_n - y^*\|\\ &\quad + \alpha_n \|f_1(y^*) - x^*\|,\end{aligned} \tag{43}$$

where $h = \max\{h_1, h_2, h_3\}$. Similarly, we can also compute that

$$\begin{aligned}\|y_{n+1} - y^*\| &\le (1-\alpha_n)\|y_n - y^*\| + \alpha_n h \|z_n - z^*\|\\ &\quad + \alpha_n \|f_2(z^*) - y^*\|,\\ \|z_{n+1} - z^*\| &\le (1-\alpha_n)\|z_n - z^*\| + \alpha_n h \|x_n - x^*\|\\ &\quad + \alpha_n \|f_3(x^*) - z^*\|.\end{aligned} \tag{44}$$

This implies that

$$\begin{aligned}&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| + \|z_{n+1} - z^*\|\\ &\quad \le (1 - \alpha_n(1-h))\left[\|x_n - x^*\| + \|y_n - y^*\| + \|z_n - z^*\|\right]\\ &\qquad + \alpha_n(1-h)\\ &\qquad \times \frac{\|f_1(y^*) - x^*\| + \|f_2(z^*) - y^*\| + \|f_3(x^*) - z^*\|}{1-h}\\ &\quad \le \max\Big\{\|x_n - x^*\| + \|y_n - y^*\| + \|z_n - z^*\|,\\ &\qquad (\|f_1(y^*) - x^*\| + \|f_2(z^*) - y^*\|\\ &\qquad + \|f_3(x^*) - z^*\|) \times (1-h)^{-1}\Big\}.\end{aligned} \tag{45}$$

By induction, we have

$$\begin{aligned}&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| + \|z_{n+1} - z^*\|\\ &\quad \le \max\Big\{\|x_0 - x^*\| + \|y_0 - y^*\| + \|z_0 - z^*\|,\\ &\qquad (\|f_1(y^*) - x^*\| + \|f_2(z^*) - y^*\|\\ &\qquad + \|f_3(x^*) - z^*\|) \times (1-h)^{-1}\Big\},\end{aligned} \tag{46}$$

for all $n \ge 1$.

Hence $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded. Consequently, $\{K_{1,\beta} x_n\}$, $\{K_{2,\beta} y_n\}$, and $\{K_{3,\beta} z_n\}$ are bounded.

(ii) Next we prove that for each $n \ge 1$ the following inequality holds:

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 + \|z_{n+1} - z^*\|^2\\ &\quad \le (1-\alpha_n)^2\left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2\right)\\ &\qquad + 2\alpha_n h\,(\|x_{n+1} - x^*\|\,\|y_n - y^*\| + \|y_{n+1} - y^*\|\\ &\qquad \times \|z_n - z^*\| + \|z_{n+1} - z^*\|\,\|x_n - x^*\|)\\ &\qquad + 2\alpha_n\,(\langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle\\ &\qquad + \langle f_2(z^*) - y^*, y_{n+1} - y^*\rangle\\ &\qquad + \langle f_3(x^*) - z^*, z_{n+1} - z^*\rangle).\end{aligned} \tag{47}$$

From (42) and Lemma 6, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&= \|(1-\alpha_n)(K_{1,\beta}x_n - x^*) + \alpha_n(f_1(K_{2,\beta}y_n) - x^*)\|^2\\
&\le \|(1-\alpha_n)(K_{1,\beta}x_n - x^*)\|^2\\
&\quad + 2\alpha_n \langle f_1(K_{2,\beta}y_n) - x^*, x_{n+1} - x^*\rangle\\
&= (1-\alpha_n)^2 \|K_{1,\beta}x_n - x^*\|^2\\
&\quad + 2\alpha_n \langle f_1(K_{2,\beta}y_n) - f_1(y^*), x_{n+1} - x^*\rangle\\
&\quad + 2\alpha_n \langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle\\
&\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \|f_1(K_{2,\beta}y_n) - f_1(y^*)\|\\
&\quad \times \|x_{n+1} - x^*\| + 2\alpha_n \langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle\\
&\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n h_1 \|K_{2,\beta}y_n - y^*\|\\
&\quad \times \|x_{n+1} - x^*\| + 2\alpha_n \langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle\\
&\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n h \|y_n - y^*\| \|x_{n+1} - x^*\|\\
&\quad + 2\alpha_n \langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle .
\end{aligned} \tag{48}$$

Similarly, we can also prove that

$$\begin{aligned}
\|y_{n+1}-y^*\|^2 &\le (1-\alpha_n)^2 \|y_n - y^*\|^2\\
&\quad + 2\alpha_n h \|z_n - z^*\| \|y_{n+1} - y^*\|\\
&\quad + 2\alpha_n \langle f_2(z^*) - y^*, y_{n+1} - y^*\rangle ,\\
\|z_{n+1}-z^*\|^2 &\le (1-\alpha_n)^2 \|z_n - z^*\|^2\\
&\quad + 2\alpha_n h \|x_n - x^*\| \|z_{n+1} - z^*\|\\
&\quad + 2\alpha_n \langle f_3(x^*) - z^*, z_{n+1} - z^*\rangle .
\end{aligned} \tag{49}$$

Adding up inequalities (48) and (49), inequality (47) is proved.

(iii) Next, we prove that if there exists a subsequence $\{n_k\} \subset \{n\}$ such that

$$\begin{aligned}
\liminf_{k\to\infty} \Big\{ &\left(\|x_{n_k+1} - x^*\|^2 + \|y_{n_k+1} - y^*\|^2 + \|z_{n_k+1} - z^*\|^2 \right)\\
&- \left(\|x_{n_k} - x^*\|^2 + \|y_{n_k} - y^*\|^2 + \|z_{n_k} - z^*\|^2 \right) \Big\} \ge 0,
\end{aligned} \tag{50}$$

then

$$\begin{aligned}
\limsup_{k\to\infty} \Big\{ &\langle f_1(y^*) - x^*, x_{n_k+1} - x^*\rangle\\
&+ \langle f_2(z^*) - y^*, y_{n_k+1} - y^*\rangle\\
&+ \langle f_3(x^*) - z^*, z_{n_k+1} - z^*\rangle \Big\} \le 0.
\end{aligned} \tag{51}$$

Since the norm $\|\cdot\|^2$ is convex and $\lim_{n\to\infty}\alpha_n = 0$, by (42), we have

$$\begin{aligned}
0 &\le \liminf_{k\to\infty} \Big\{ \Big(\|x_{n_k+1} - x^*\|^2 + \|y_{n_k+1} - y^*\|^2\\
&\qquad + \|z_{n_k+1} - z^*\|^2 \Big)\\
&\qquad - \Big(\|x_{n_k} - x^*\|^2 + \|y_{n_k} - y^*\|^2\\
&\qquad + \|z_{n_k} - z^*\|^2 \Big) \Big\}\\
&\le \liminf_{k\to\infty} \Big\{ (1-\alpha_{n_k}) \|K_{1,\beta}x_{n_k} - x^*\|^2\\
&\qquad + \alpha_{n_k} \|f_1(K_{2,\beta}y_{n_k}) - x^*\|^2\\
&\qquad + (1-\alpha_{n_k}) \|K_{2,\beta}y_{n_k} - y^*\|^2\\
&\qquad + \alpha_{n_k} \|f_2(K_{3,\beta}z_{n_k}) - y^*\|^2\\
&\qquad + (1-\alpha_{n_k}) \|K_{3,\beta}z_{n_k} - z^*\|^2\\
&\qquad + \alpha_{n_k} \|f_3(K_{1,\beta}x_{n_k}) - z^*\|^2\\
&\qquad - \Big(\|x_{n_k} - x^*\|^2 + \|y_{n_k} - y^*\|^2\\
&\qquad + \|z_{n_k} - z^*\|^2 \Big) \Big\}\\
&= \liminf_{k\to\infty} \Big\{ \left(\|K_{1,\beta}x_{n_k} - x^*\|^2 - \|x_{n_k} - x^*\|^2 \right)\\
&\qquad + \left(\|K_{2,\beta}y_{n_k} - y^*\|^2 - \|y_{n_k} - y^*\|^2 \right)\\
&\qquad + \left(\|K_{3,\beta}z_{n_k} - z^*\|^2 - \|z_{n_k} - z^*\|^2 \right) \Big\}\\
&\le \limsup_{k\to\infty} \Big\{ \left(\|K_{1,\beta}x_{n_k} - x^*\|^2 - \|x_{n_k} - x^*\|^2 \right)\\
&\qquad + \left(\|K_{2,\beta}y_{n_k} - y^*\|^2 - \|y_{n_k} - y^*\|^2 \right)\\
&\qquad + \left(\|K_{3,\beta}z_{n_k} - z^*\|^2 - \|z_{n_k} - z^*\|^2 \right) \Big\}\\
&\le 0.
\end{aligned} \tag{52}$$

This implies that

$$\begin{aligned}
&\lim_{k\to\infty} \left(\|K_{1,\beta}x_{n_k} - x^*\|^2 - \|x_{n_k} - x^*\|^2 \right)\\
&\quad = \lim_{k\to\infty} \left(\|K_{2,\beta}y_{n_k} - y^*\|^2 - \|y_{n_k} - y^*\|^2 \right)\\
&\quad = \lim_{k\to\infty} \left(\|K_{3,\beta}z_{n_k} - z^*\|^2 - \|z_{n_k} - z^*\|^2 \right) = 0.
\end{aligned} \tag{53}$$

Since the sequences $\{\|K_{1,\beta}x_{n_k}-x^*\|+\|x_{n_k}-x^*\|\}$, $\{\|K_{2,\beta}y_{n_k}-y^*\|+\|y_{n_k}-y^*\|\}$, and $\{\|K_{3,\beta}z_{n_k}-z^*\|+\|z_{n_k}-z^*\|\}$ are bounded, we have

$$\lim_{k\to\infty}\left(\left\|K_{1,\beta}x_{n_k}-x^*\right\|-\left\|x_{n_k}-x^*\right\|\right)=\lim_{k\to\infty}\left(\left\|K_{2,\beta}y_{n_k}-y^*\right\|-\left\|y_{n_k}-y^*\right\|\right)=\lim_{k\to\infty}\left(\left\|K_{3,\beta}z_{n_k}-z^*\right\|-\left\|z_{n_k}-z^*\right\|\right)=0. \quad (54)$$

By Lemma 9, $K_{1,\beta}$, $K_{2,\beta}$, and $K_{3,\beta}$ are strongly quasinonexpansive. We have

$$K_{1,\beta}x_{n_k}-x_{n_k}\longrightarrow 0,\qquad K_{2,\beta}y_{n_k}-y_{n_k}\longrightarrow 0,\qquad K_{3,\beta}z_{n_k}-z_{n_k}\longrightarrow 0. \quad (55)$$

Consequently, we obtain that

$$x_{n_k}-x_{n_k+1}\longrightarrow 0,\qquad y_{n_k}-y_{n_k+1}\longrightarrow 0,\qquad z_{n_k}-z_{n_k+1}\longrightarrow 0. \quad (56)$$

It follows from the boundedness of $\{x_{n_k}\}$ and H which is reflexive that there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}}\rightharpoonup p$ and

$$\lim_{l\to\infty}\left\langle f_1(y^*)-x^*,x_{n_{k_l}}-x^*\right\rangle=\limsup_{k\to\infty}\left\langle f_1(y^*)-x^*,x_{n_k}-x^*\right\rangle=\limsup_{k\to\infty}\left\langle f_1(y^*)-x^*,x_{n_k+1}-x^*\right\rangle. \quad (57)$$

By Lemma 9, $I-K_{1,\beta}$ is demiclosed at zero, and so $p\in F(K_{1,\beta})=\Omega_1$. Hence from (36) we have

$$\lim_{l\to\infty}\left\langle f_1(y^*)-x^*,x_{n_{k_l}}-x^*\right\rangle=\langle f_1(y^*)-x^*,p-x^*\rangle\le 0. \quad (58)$$

Therefore

$$\limsup_{k\to\infty}\left\langle f_1(y^*)-x^*,x_{n_k+1}-x^*\right\rangle=\lim_{l\to\infty}\left\langle f_1(y^*)-x^*,x_{n_{k_l}}-x^*\right\rangle\le 0. \quad (59)$$

Similarly, we can also prove that

$$\limsup_{k\to\infty}\left\langle f_2(z^*)-y^*,y_{n_k+1}-y^*\right\rangle\le 0,\qquad \limsup_{k\to\infty}\left\langle f_3(x^*)-z^*,z_{n_k+1}-z^*\right\rangle\le 0. \quad (60)$$

Hence, we have the desired inequality.

(iv) Finally, we prove that the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated to be (42) converge to x^*, y^*, and z^*, respectively.

It is clear that

$$\begin{aligned}&\|x_{n+1}-x^*\|\,\|y_n-y^*\|+\|y_{n+1}-y^*\|\,\|z_n-z^*\|+\|z_{n+1}-z^*\|\,\|x_n-x^*\|\\&\le\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2+\|z_n-z^*\|^2\right)^{1/2}\times\left(\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2+\|z_{n+1}-z^*\|^2\right)^{1/2}.\end{aligned} \quad (61)$$

Substituting (61) into (47), we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2+\|z_{n+1}-z^*\|^2\\&\le(1-\alpha_n)^2\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2+\|z_n-z^*\|^2\right)\\&\quad+2\alpha_n h\left\{\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2+\|z_n-z^*\|^2\right)^{1/2}\times\left(\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2+\|z_{n+1}-z^*\|^2\right)^{1/2}\right\}\\&\quad+2\alpha_n\left(\langle f_1(y^*)-x^*,x_{n+1}-x^*\rangle+\langle f_2(z^*)-y^*,y_{n+1}-y^*\rangle+\langle f_3(x^*)-z^*,z_{n+1}-z^*\rangle\right).\end{aligned} \quad (62)$$

Set

$$\begin{aligned}a_n&:=\|x_n-x^*\|^2+\|y_n-y^*\|^2+\|z_n-z^*\|^2,\\ b_n&:=2\left(\langle f_1(y^*)-x^*,x_{n+1}-x^*\rangle+\langle f_2(z^*)-y^*,y_{n+1}-y^*\rangle+\langle f_3(x^*)-z^*,z_{n+1}-z^*\rangle\right).\end{aligned} \quad (63)$$

Then, we have the following statements.

(i) From (i), $\{a_n\}$ is bounded sequence.

(ii) From (62), $a_{n+1}\le(1-\alpha_n)^2a_n+2\alpha_n h\sqrt{a_n}\sqrt{a_{n+1}}+\alpha_n b_n$, for all $n\ge 1$.

(iii) From (iii), whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying

$$\liminf_{k\to\infty}\left(a_{n_k+1}-a_{n_k}\right)\ge 0, \quad (64)$$

it follows that $\limsup_{k\to\infty}b_{n_k}\le 0$.

By Lemma 8, we have

$$\lim_{n\to\infty}\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2+\|z_n-z^*\|^2\right)=0. \quad (65)$$

Hence, we obtain that

$$\lim_{n\to\infty}\|x_n - x^*\| = \lim_{n\to\infty}\|y_n - y^*\| = \lim_{n\to\infty}\|z_n - z^*\| = 0. \tag{66}$$

This completes the proof. □

3.3. Consequence Results. Using Theorem 11, we can prove the following results.

Theorem 12. *Let A_i, M_i, Ω_i, K_i, and $K_{i,\beta}$ satisfy conditions (C1) and (C2), and let $F : H \to H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be three sequences defined by*

$$\begin{aligned} &x_0, y_0, z_0 \in H,\\ &x_{n+1} = (1-\alpha_n) K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n),\\ &y_{n+1} = (1-\alpha_n) K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n),\\ &z_{n+1} = (1-\alpha_n) K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n),\\ &\qquad n = 0,1,2,\ldots, \end{aligned} \tag{67}$$

where $f_1 := I - \rho F$, $f_2 := I - \eta F$, $f_3 := I - \xi F$ with $\rho, \eta, \xi \in (0, 2r/\mu^2)$, and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge to x^*, y^*, and z^*, respectively, where (x^*, y^*, z^*) is the unique element in $\Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied:

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in \Omega_1,\\ &\langle \eta F(z^*) + y^* - z^*, y - y^*\rangle \geq 0, \quad \forall y \in \Omega_2,\\ &\langle \xi F(x^*) + z^* - x^*, z - z^*\rangle \geq 0, \quad \forall z \in \Omega_3. \end{aligned} \tag{68}$$

Proof. It is easy to see that f_1, f_2, and f_3 are contraction mappings and all the conditions in Theorem 11 are satisfied. By Theorem 11, we have the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ which converge to $(x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied:

$$\begin{aligned} &\langle x^* - f_1(y^*), x - x^*\rangle \geq 0, \quad \forall x \in \Omega_1,\\ &\langle y^* - f_2(z^*), y - y^*\rangle \geq 0, \quad \forall y \in \Omega_2,\\ &\langle z^* - f_3(x^*), z - z^*\rangle \geq 0, \quad \forall z \in \Omega_3. \end{aligned} \tag{69}$$

Substituting $f_1 := I - \rho F$, $f_2 := I - \eta F$, and $f_3 := I - \xi F$ into (69), we obtain that the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge to $(x^*, y^*, z^*) \in \Omega_1\times\Omega_2\times\Omega_3$ such that the following three inequalities are satisfied:

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in \Omega_1,\\ &\langle \eta F(z^*) + y^* - z^*, y - y^*\rangle \geq 0, \quad \forall y \in \Omega_2,\\ &\langle \xi F(x^*) + z^* - x^*, z - z^*\rangle \geq 0, \quad \forall z \in \Omega_3. \end{aligned} \tag{70}$$

This completes the proof □

Setting $A_1 = A_2 = A_3$ in Theorem 11, we obtain the following corollary.

Corollary 13. *Let A_1, M_1, Ω_1, K_1, and $K_{1,\beta}$ satisfy conditions (C1) and (C2), and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0,1)$, for all $i = 1,2,3$. Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be three sequences defined by*

$$\begin{aligned} &x_0, y_0, z_0 \in H,\\ &x_{n+1} = (1-\alpha_n) K_{1,\beta}x_n + \alpha_n f_1(K_{1,\beta}y_n),\\ &y_{n+1} = (1-\alpha_n) K_{1,\beta}y_n + \alpha_n f_2(K_{1,\beta}z_n),\\ &z_{n+1} = (1-\alpha_n) K_{1,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n),\\ &\qquad n = 0,1,2,\ldots, \end{aligned} \tag{71}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated to be (42) converge to x^*, y^*, and z^*, respectively, where (x^*, y^*, z^*) is the unique element in $\Omega_1 \times \Omega_1 \times \Omega_1$ such that the following three inequalities are satisfied:

$$\begin{aligned} &\langle x^* - f_1(y^*), x - x^*\rangle \geq 0, \quad \forall x \in \Omega_1,\\ &\langle y^* - f_2(z^*), x - y^*\rangle \geq 0, \quad \forall x \in \Omega_1,\\ &\langle z^* - f_3(x^*), x - z^*\rangle \geq 0, \quad \forall x \in \Omega_1. \end{aligned} \tag{72}$$

Corollary 14. *Let A_1, M_1, Ω, K_1, and $K_{1,\beta}$ satisfy conditions (C1) and (C2), and let $F : H \to H$ be μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be three sequences defined by*

$$\begin{aligned} &x_0, y_0, z_0 \in H,\\ &x_{n+1} = (1-\alpha_n) K_{1,\beta}x_n + \alpha_n f_1(K_{1,\beta}y_n),\\ &y_{n+1} = (1-\alpha_n) K_{1,\beta}y_n + \alpha_n f_2(K_{1,\beta}z_n),\\ &z_{n+1} = (1-\alpha_n) K_{1,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n),\\ &\qquad n = 0,1,2,\ldots, \end{aligned} \tag{73}$$

where $f_1 := I - \rho F$, $f_2 := I - \eta F$, $f_3 := I - \xi F$ with $\rho, \eta, \xi \in (0, 2r/\mu^2)$, and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge to x^*, y^*, and z^*, respectively, where (x^*, y^*, z^*) is the unique element in $\Omega_1 \times \Omega_1 \times \Omega_1$ such that the following three inequalities are satisfied:

$$\begin{aligned} &\langle \rho F(y^*) + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in \Omega_1,\\ &\langle \eta F(z^*) + y^* - z^*, x - y^*\rangle \geq 0, \quad \forall x \in \Omega_1,\\ &\langle \xi F(x^*) + z^* - x^*, x - z^*\rangle \geq 0, \quad \forall x \in \Omega_1. \end{aligned} \tag{74}$$

Setting $A_1 = A_2 = A_3$, $f_1 = f_2 = f_3$, and $x_0 = y_0 = z_0$ in Theorem 11, we obtain the following corollary.

Corollary 15. *Let* A_1, M_1, Ω_1, K_1, *and* $K_{1,\beta}$ *satisfy conditions (C1) and (C2), and let* $f : H \to H$ *be contractions with a contractive constant* $h \in (0,1)$. *Let* $\{x_n\}$ *be the sequences defined by*

$$x_0 \in H,$$
$$x_{n+1} = (1-\alpha_n) K_{1,\beta} x_n + \alpha_n f(K_{1,\beta} x_n), \quad n = 0,1,2,\ldots, \tag{75}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Then the sequences* $\{x_n\}$ *converge to* $x^* \in \Omega_1$ *such that the following three inequalities are satisfied:*

$$\langle x^* - f_1(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1. \tag{76}$$

Corollary 16. *Let* A_1, M_1, Ω_1, K_1, *and* $K_{1,\beta}$ *satisfy conditions (C1) and (C2), and let* $F : H \to H$ *be* μ*-Lipschitzian and* r*-strongly monotone mapping. Let* $\{x_n\}$ *be the sequences defined by*

$$x_0 \in H,$$
$$x_{n+1} = (1-\alpha_n) K_{1,\beta} x_n + \alpha_n (I - \rho F)(K_{1,\beta} x_n), \quad n = 0,1,2,\ldots, \tag{77}$$

where $\rho \in (0, 2r/\mu^2)$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Then the sequences* $\{x_n\}$ *converge to* $x^* \in \Omega_1$ *such that the following three inequalities are satisfied:*

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1. \tag{78}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding to the publication of this paper.

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