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Research Article

The Rate of Convergence of Lupas q-Analogue of the Bernstein Operators

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We discuss the rate of convergence of the Lupas q-analogues of the Bernstein operators $R_{n,q}(f;x)$ which were given by Lupas in 1987. We obtain the estimates for the rate of convergence of $R_{n,q}(f)$ by the modulus of continuity of f, and show that the estimates are sharp in the sense of order for Lipschitz continuous functions.

1. Introduction

In 1912, Bernstein (see [1]) defined the Bernstein polynomials. Later, it was found that the Bernstein polynomials possess many remarkable properties, which made them an area of intensive research. Due to the development of q-calculus, generalizations of Bernstein polynomials connected with q-calculus have emerged. The first person to make progress in this direction was Lupas, who introduced a q-analogue of the Bernstein operator $R_{n,q}(f;x)$ and investigated its approximating and shape-preserving properties in 1987 (see [2]). If $q = 1$, then $\{R_{n,1}(f;x)\}$ are the classical Bernstein polynomials. For $q \neq 1$, the operators $R_{n,q}(f;x)$ are rational functions rather than polynomials. Other generalizations of the Bernstein polynomials, for example, the q-Bernstein polynomials (see [3]), the two-parametric generalization of q-Bernstein polynomials (see [4]), and the q-Bernstein-Durrmeyer operator (see [5]), had also been considered in recent years. Among these generalizations, q-Bernstein polynomials proposed by Phillips attracted the most attention and were studied widely by a number of authors (see [3, 6–15]). The Lupas q-analogues of the Bernstein operators $\{R_{n,q}(f;x)\}$ are less known; see [2, 16–21]. However, they have an advantage of generating positive linear operators for all $q > 0$, whereas q-Bernstein polynomials generate positive linear operators only if $q \in (0,1)$.

In this paper, we will study the rate of convergence of the Lupas q-analogues of the Bernstein operators $\{R_{n,q}(f;x)\}$. We will obtain the estimates for the rate of convergence of $R_{n,q}(f)$ by the modulus of continuity of f, and show that the estimates are sharp in the sense of order for Lipschitz continuous functions. Our results demonstrate that the estimates for the rate of convergence of $\{R_{n,q}(f;x)\}$ are essentially different from those for the classical Bernstein polynomials; however, they are very similar to those for the q-Bernstein polynomials in the case $q \in (0,1)$.

Throughout the paper, we always assume that f is a continuous real function on $[0,1]$, $q > 0$, $q \neq 1$. Denote by $C[0,1]$ (or $C^n[0,1]$, $1 \le n \le \infty$) the space of all continuous (correspondingly, n times continuously differentiable) real-valued functions on $[0,1]$ equipped with the uniform norm $\|\cdot\|$. The expression $A(n) \asymp B(n)$ means that $A(n) \ll B(n)$ and $A(n) \gg B(n)$, and $A(n) \ll B(n)$ means that there exists a positive constant c independent of n such that $A(n) \le cB(n)$.

To formulate our results, we need the following definitions.

Let $q > 0$. For each nonnegative integer k, the q-integer $[k]$ and the q-factorial $[k]!$ are defined by

$$[k] := [k]_q := \begin{cases} \dfrac{(1-q^k)}{(1-q)}, & q \neq 1 \\ k, & q = 1, \end{cases}$$

$$[k]! := \begin{cases} [k]\,[k-1]\cdots[1], & k \geq 1 \\ 1, & k = 0. \end{cases} \tag{1}$$

For integers $0 \leq k \leq n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]!\,[n-k]!}. \tag{2}$$

In [2], Lupas proposed the q-analogue of the Bernstein operator $R_{n,q}(f;x)$: for each positive integer n, and $f \in C[0,1]$,

$$R_{n,q}(f,x) := \begin{cases} \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) r_{n,k}(q,x), & 0 \leq x < 1 \\ f(1), & x = 1, \end{cases} \tag{3}$$

where

$$\begin{aligned} r_{n,k}(q;x) &:= \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2} x^k (1-x)^{n-k}}{(1-x+qx)\cdots(1-x+q^{n-1}x)} \\ &= \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2} (x/(1-x))^k}{\prod_{j=0}^{n-1}\left(1+q^j (x/(1-x))\right)}. \end{aligned} \tag{4}$$

In [19], Ostrovska proved that, for each $f \in C[0,1]$ and $q \in (0,1)$, the sequence $\{R_{n,q}(f,x)\}$ converges to the limit operator $R_{\infty,q}(f,x)$ uniformly on $[0,1]$ as $n \to \infty$, where

$$R_{\infty,q}(f,x) = \begin{cases} \sum_{k=0}^{\infty} f\left(1-q^k\right) r_{\infty k}(q;x), & 0 \leq x < 1 \\ f(1), & x = 1, \end{cases} \tag{5}$$

$$r_{\infty,k}(q;x) := \frac{q^{k(k-1)/2} (x/(1-x))^k}{(1-q)^k [k]! \prod_{j=0}^{\infty}\left(1+q^j (x/(1-x))\right)}. \tag{6}$$

When $q > 1$, the following relations (see [19]) allow us to reduce to the case $q \in (0,1)$:

$$\begin{aligned} R_{n,q}(f;x) &= R_{n,1/q}(g;1-x), \\ R_{\infty,q}(f;x) &= R_{\infty,1/q}(g;1-x), \end{aligned} \tag{7}$$

where $g(x) = f(1-x) \in C[0,1]$.

The problem to find the rate of convergence occurs naturally and this paper deals with the problem of finding estimates for the rate of convergence for a sequence of the q-analogue of the Bernstein operator $R_{n,q}(f;x)$ for $0 < q < 1$. For $f \in C[0,1]$, $t > 0$, the modulus of continuity $\omega(f,t)$ and the second modulus of smoothness $\omega_2(f,t)$ are defined as follows:

$$\begin{aligned} \omega(f;t) &:= \sup_{\substack{|x-y|\leq t \\ x,y\in[0,1]}} |f(x)-f(y)|; \\ \omega_2(f,t) &:= \sup_{0<h\leq t}\ \sup_{x\in[0,1-2h]} |f(x+2h)-2f(x+h)+f(x)|. \end{aligned} \tag{8}$$

The main results of the paper are as follows.

Theorem 1. *Let $q \in (0,1)$ and let $f \in C[0,1]$. Then*

$$\left\| R_{n,q}(f) - R_{\infty,q}(f) \right\| \leq C_q \omega(f;q^n), \tag{9}$$

where $C_q = 2 + 6/(1-q)$. This estimate is sharp in the following sense of order: for each α, $0 < \alpha \leq 1$, there exists a function $f_\alpha(x)$ which belongs to the Lipschitz class Lip $\alpha := \{f \in C[0,1] \mid \omega(f;t) \ll t^\alpha\}$ such that

$$\left\| R_{n,q}(f_\alpha) - R_{\infty,q}(f_\alpha) \right\| \asymp q^{n\alpha}. \tag{10}$$

Theorem 2. *Let $0 < q < 1$. Then*

$$\left\| R_{n,q}(f) - R_{\infty,q}(f) \right\| \leq c\omega_2\left(f;\sqrt{q^n}\right). \tag{11}$$

Furthermore,

$$\sup_{0<q<1} \left| R_{n,q}(f) - R_{\infty,q}(f) \right| \leq c\omega_2\left(f;n^{-1/2}\right), \tag{12}$$

where c is an absolute constant.

Remark 3. From (12), it follows that, for each $f \in C[0,1]$,

$$\lim_{n\to\infty} R_{n,q}(f;x) = R_{\infty,q}(f;x) \tag{13}$$

uniformly not only in $x \in [0,1]$, and but also in $q \in (0,1]$, which generalizes the Ostrovska's result in [19].

Remark 4. It should be emphasized that Theorem 1 cannot be obtained in a way similar to the proof of the Popoviciu Theorem for the classical Bernstein polynomials (see [22]). It requires different estimation techniques due to the infinite product involved. Also, the proof in the paper is more difficult than the one used for q-Bernstein polynomials (see [14]), since the Lupas q-analogue of Bernstein operators has the singular nature at the point $x = 1$ and needs a new method (when $x \to 1$, $x/(1-x) \to \infty$).

Remark 5. Results similar to Theorems 1 and 2 for q-Bernstein polynomials were obtained in [14] and [12], respectively. Note that when $f(x) = x^2$, for $q \in (0,1)$, we have (see (46))

$$\begin{aligned} &\left\| R_{n,q}(f;x) - R_{\infty,q}(f;x) \right\| \\ &\quad = \left\| \frac{q^n x(1-x)}{(1-x+qx)[n]} \right\| \asymp q^n \asymp \omega_2\left(f;\sqrt{q^n}\right). \end{aligned} \tag{14}$$

Hence, the estimate (11) is sharp in the following sense: the sequence $\sqrt{q^n}$ in (11) cannot be replaced by any other sequence decreasing to zero more rapidly as $n \to \infty$. However, (11) is not sharp for the Lipschitz class Lip α ($\alpha \in (0,1]$) in the sense of order. This, combining with Theorem 1, shows that in the case $0 < q < 1$ the modulus of continuity is more appropriate to describe the rate of convergence for the Lupas q-analogue Berstein operators than the second modulus of smoothness. This is different from that in the case $q = 1$.

Remark 6. The numbers c in (11) and C_q in (9) are both the constants independent of f and n. However, while c in (11) does not depend on q, the constant C_q in (9) depends on q and tends to $+\infty$ as $q \to 1-$. Hence, (11) does not follow from (9).

Let $f \in C[0,1]$ and $g(x) = f(1-x)$. Using (7) and the relations

$$\omega(f,t) = \omega(g,t); \qquad \omega_2(f,t) = \omega_2(g,t), \tag{15}$$

we have the following corollaries.

Corollary 7. *Let $f \in C[0,1]$. Then for any $q \in (1,\infty)$,*

$$\left\| R_{n,q}(f) - R_{\infty,q}(f) \right\| \le C_q \omega\left(f; \frac{1}{q^n}\right), \tag{16}$$

where C_q is a constant independent of f and n.

Corollary 8. *Let $f \in C[0,1]$. Then for any $q \in (1,\infty)$,*

$$\left\| R_{n,q}(f) - R_{\infty,q}(f) \right\| \le c\omega_2\left(f; \sqrt{\frac{1}{q^n}}\right). \tag{17}$$

Furthermore,

$$\sup_{q>0}\left| R_{n,q}(f) - R_{\infty,q}(f) \right| \le c\omega_2\left(f; n^{-1/2}\right), \tag{18}$$

where c is an absolute constant.

2. Proofs of Theorems 1 and 2

For the proofs of Theorems 1 and 2, we need the following lemmas.

Lemma 9 (see [2]). *The following equalities are true:*

$$\begin{gathered} R_{n,q}(1;x) = R_{\infty,q}(1;x) = 1, \\ R_{n,q}(t;x) = R_{\infty,q}(t;x) = x, \end{gathered} \tag{19}$$

$$R_{n,q}\left(t^2;x\right) = x^2 + \frac{x(1-x)}{[n]} - \frac{x^2(1-x)(1-q)}{1-x+xq}\left(1 - \frac{1}{[n]}\right). \tag{20}$$

Lemma 10. *With the definitions of $r_{n,k}(q;x)$ and $r_{\infty,k}(q;x)$, we have*

$$\sum_{k=0}^{n} q^k r_{n,k}(q;x) = 1 - x + q^n x, \qquad \sum_{k=0}^{\infty} q^k r_{\infty,k}(q;x) = 1 - x. \tag{21}$$

Proof. Using (19) and (3), we get

$$\begin{aligned} &\sum_{k=0}^{n} q^k r_{n,k}(q;x) \\ &\quad = (q^n - 1)\sum_{k=0}^{n}\frac{q^k - 1}{q^n - 1} r_{n,k}(q;x) + \sum_{k=0}^{n} r_{n,k}(q;x) \\ &\quad = (q^n - 1)\sum_{k=0}^{n}\frac{[k]}{[n]} r_{n,k}(q;x) + 1 \\ &\quad = (q^n - 1) R_{n,q}(t;x) + 1 \\ &\quad = 1 - x + q^n x. \end{aligned} \tag{22}$$

Similarly, using (19) and (5), we have

$$\begin{aligned} &\sum_{k=0}^{\infty} q^k r_{\infty,k}(q;x) \\ &\quad = \sum_{k=0}^{\infty}\left(q^k - 1\right) r_{\infty,k}(q;x) + \sum_{k=0}^{\infty} r_{\infty,k}(q;x) \\ &\quad = -\left(\sum_{k=0}^{\infty}\left(1 - q^k\right) r_{\infty,k}(q;x)\right) + \sum_{k=0}^{\infty} r_{\infty,k}(q;x) \\ &\quad = -R_{\infty,q}(t;x) + 1 = 1 - x. \end{aligned} \tag{23}$$

The proof of Lemma 10 is complete. □

For integers n, k, and $q \in (0,1)$, $x \in [0,1]$, we have

$$\begin{aligned} &r_{n,k}(q;x) - r_{\infty k}(q;x) \\ &\quad = \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2}(x/(1-x))^k}{\prod_{s=0}^{n-1}\left(1 + q^s(x/(1-x))\right)} \\ &\qquad - \frac{q^{k(k-1)/2}(x/(1-x))^k}{(1-q)^k [k]! \prod_{s=0}^{\infty}\left(1 + q^s(x/(1-x))\right)} \\ &\quad = \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2}(x/(1-x))^k}{\prod_{s=0}^{n-1}\left(1 + q^s(x/(1-x))\right)} \\ &\qquad \times \left(1 - \frac{1}{\prod_{s=n}^{\infty}\left(1 + q^s(x/(1-x))\right)}\right) \\ &\qquad + \frac{q^{k(k-1)/2}(x/(1-x))^k}{\prod_{s=0}^{\infty}\left(1 + q^s(x/(1-x))\right)}\left(\begin{bmatrix} n \\ k \end{bmatrix} - \frac{1}{(1-q)^k [k]!}\right) \\ &\quad = r_{n,k}(q;x)\left(1 - \frac{1}{\prod_{s=n}^{\infty}\left(1 + q^s(x/(1-x))\right)}\right) \\ &\qquad - r_{\infty,k}(q;x)\left(1 - \prod_{s=n-k+1}^{n}\left(1 - q^s\right)\right) \\ &\quad = r_{n,k}(q;x) J_1 - r_{\infty,k}(q;x) J_2, \end{aligned} \tag{24}$$

where

$$J_1 := 1 - \frac{1}{\prod_{s=n}^{\infty}\left(1 + q^s\left(x/\left(1-x\right)\right)\right)},$$
$$J_2 := 1 - \prod_{s=n-k+1}^{n}\left(1 - q^s\right). \tag{25}$$

We will prove the following lemma.

Lemma 11. *Let* $0 < q < 1$. *Then for integers n, k and for* $0 < x < 1/(1+q^n)$,

$$\sum_{k=0}^{n} q^k \left| r_{n,k}\left(q;x\right) - r_{\infty,k}\left(q;x\right)\right| \le \frac{3q^n}{1-q}. \tag{26}$$

Proof. It is easy to prove by induction that

$$0 \le J_2 := 1 - \prod_{s=n-k+1}^{n}\left(1-q^s\right) \le \sum_{s=n-k+1}^{n} q^s \le \sum_{s=n-k}^{\infty} q^s = \frac{q^{n-k}}{1-q}. \tag{27}$$

Since $1 - \exp(-x) \le x$ and $\ln(1+x) \le x$ for all $x \in [0,\infty)$, we obtain

$$\begin{aligned} 0 \le J_1 &= 1 - \exp\left(-\sum_{s=n}^{\infty}\ln\left(1 + q^s\frac{x}{1-x}\right)\right) \\ &\le \sum_{s=n}^{\infty}\ln\left(1 + q^s\frac{x}{1-x}\right) \\ &\le \sum_{s=n}^{\infty} q^s\frac{x}{1-x} = \frac{q^n x}{\left(1-q\right)\left(1-x\right)}. \end{aligned} \tag{28}$$

Hence,

$$\begin{aligned} \left| r_{n,k}\left(q;x\right) - r_{\infty k}\left(q;x\right)\right| \le{}& \frac{q^n x}{\left(1-q\right)\left(1-x\right)} r_{n,k}\left(q;x\right) \\ &+ \frac{q^{n-k}}{1-q} r_{\infty,k}\left(q;x\right), \end{aligned} \tag{29}$$

and therefore, by (21) and (19) we get

$$\begin{aligned} &\sum_{k=0}^{n} q^k \left| r_{n,k}\left(q;x\right) - r_{\infty,k}\left(q;x\right)\right| \\ &\quad\le \frac{q^n x}{\left(1-q\right)\left(1-x\right)} \sum_{k=0}^{n} q^k r_{n,k}\left(q;x\right) + \frac{q^n}{1-q}\sum_{k=0}^{n} r_{\infty,k}\left(q;x\right) \\ &\quad\le \frac{q^n x}{\left(1-q\right)\left(1-x\right)}\left(1 - x + q^n x\right) + \frac{q^n}{1-q}. \end{aligned} \tag{30}$$

Since $0 < x < 1/(1+q^n) < 1$, it follows that $0 < x/(1-x) < 1/q^n$ and thence

$$\sum_{k=0}^{n} q^k \left| r_{n,k}\left(q;x\right) - r_{\infty,k}\left(q;x\right)\right| \le \frac{3q^n}{1-q}. \tag{31}$$

This completes the proof of Lemma 11. □

Proof of Theorem 1. It follows from the definition of $R_{n,q}(f;x)$ and $R_{\infty,q}(f;x)$ that both of them possess the end point interpolation property; in other words,

$$\begin{aligned} R_{n,q}\left(f;0\right) &= R_{\infty,q}\left(f;0\right) = f\left(0\right), \\ R_{n,q}\left(f;1\right) &= R_{\infty,q}\left(f;1\right) = f\left(1\right). \end{aligned} \tag{32}$$

It follows from the definition of $r_{n,k}(q;x)$ and $r_{\infty,k}(q;x)$ that $r_{n,k}(q;x) \ge 0$ and $r_{\infty,k}(q;x) \ge 0$ for $0 \le x < 1$. If $x \to 1$, then $x/(1-x) \to \infty$. So, the Lupas q-analogue of Bernstein operators has the singular nature at the point $x = 1$ and the rate of convergence near the point 1 needs to be considered independently. First we suppose $x \in (1/(1+q^n), 1)$; that is, $1 - x < q^n/(1+q^n) < q^n$. Then

$$\begin{aligned} I &= \left| R_{n,q}\left(f;x\right) - R_{\infty,q}\left(f;x\right)\right| \\ &= \left| \sum_{k=0}^{n}\left(f\left(\frac{[k]}{[n]}\right) - f\left(1\right)\right) r_{n,k}\left(q;x\right) \right. \\ &\qquad \left. - \sum_{k=0}^{\infty}\left(f\left(1-q^k\right) - f\left(1\right)\right) r_{\infty,k}\left(q;x\right)\right| \\ &\le \sum_{k=0}^{n}\left| f\left(\frac{[k]}{[n]}\right) - f\left(1\right)\right| r_{n,k}\left(q;x\right) \\ &\quad + \sum_{k=0}^{\infty}\left| f\left(1-q^k\right) - f\left(1\right)\right| r_{\infty,k}\left(q;x\right). \end{aligned} \tag{33}$$

Since

$$\begin{aligned} &\left|\frac{[k]}{[n]} - 1\right| = \left|\frac{1-q^k}{1-q^n} - 1\right| \le \frac{q^k\left(1-q^{n-k}\right)}{1-q^n} \le q^k, \\ &\qquad\qquad \left(0 \le k \le n\right), \\ &\omega\left(f;\lambda t\right) \le \left(1+\lambda\right)\omega\left(f;t\right), \quad \lambda > 0, \end{aligned} \tag{34}$$

we get

$$\begin{aligned} I &\le \sum_{k=0}^{n}\omega\left(f;q^k\right) r_{n,k}\left(q;x\right) + \sum_{k=0}^{\infty}\omega\left(f;q^k\right) r_{\infty,k}\left(q;x\right) \\ &\le \sum_{k=0}^{n}\omega\left(f,q^n\right)\left(1 + \frac{q^k}{q^n}\right) r_{n,k}\left(q;x\right) \\ &\quad + \sum_{k=0}^{\infty}\omega\left(f;q^n\right)\left(1 + \frac{q^k}{q^n}\right) r_{\infty,k}\left(q;x\right) \\ &\le 2\omega\left(f;q^n\right) + \frac{\omega\left(f,q^n\right)}{q^n}\sum_{k=0}^{n} q^k r_{n,k}\left(q;x\right) \\ &\quad + \frac{\omega\left(f,q^n\right)}{q^n}\sum_{k=0}^{\infty} q^k r_{\infty,k}\left(q;x\right). \end{aligned} \tag{35}$$

By Lemma 10 and $1-x<q^n$, $x<1$, we have

$$I \leq 2\omega(f;q^n)+\frac{\omega(f,q^n)}{q^n}(1-x+q^n x) + \frac{\omega(f,q^n)}{q^n}(1-x) \leq 5\omega(f;q^n). \tag{36}$$

Next, we assume that $0<x<1/(1+q^n)$. Then $0\leq x/(1-x)\leq 1/q^n$. We have

$$\begin{aligned} I &= \left|R_{n,q}(f;x)-R_{\infty,q}(f;x)\right| \\ &= \left|\sum_{k=0}^{n}\left(f\left(\frac{[k]}{[n]}\right)-f\left(1-q^k\right)\right)r_{n,k}(q;x)\right. \\ &\quad+\sum_{k=0}^{n}\left(f\left(1-q^k\right)-f(1)\right)\left(r_{n,k}(q;x)-r_{\infty,k}(q;x)\right) \\ &\quad\left.-\sum_{k=n+1}^{\infty}\left(f\left(1-q^k\right)-f(1)\right)r_{\infty,k}(q;x)\right| \\ &\leq \sum_{k=0}^{n}\left|f\left(\frac{[k]}{[n]}\right)-f\left(1-q^k\right)\right|r_{n,k}(q;x) \\ &\quad+\sum_{k=0}^{n}\left|f\left(1-q^k\right)-f(1)\right|\left|r_{n,k}(q;x)-r_{\infty,k}(q;x)\right| \\ &\quad+\sum_{k=n+1}^{\infty}\left|f\left(1-q^k\right)-f(1)\right|r_{\infty,k}(q;x) \\ &=: \delta_1+\delta_2+\delta_3. \end{aligned} \tag{37}$$

First we estimate δ_1 and δ_3. Since

$$\begin{gathered} \left|\frac{[k]}{[n]}-\left(1-q^k\right)\right| = \left|\frac{1-q^k}{1-q^n}-\left(1-q^k\right)\right| = \frac{q^n\left(1-q^k\right)}{1-q^n}\leq q^n, \quad (0\leq k\leq n) \\ \left|1-\left(1-q^k\right)\right| = q^k\leq q^n, \quad (k\geq n+1), \end{gathered} \tag{38}$$

we get

$$\delta_1 \leq \omega(f,q^n)\sum_{k=0}^{n} r_{n,k}(q;x)=\omega(f,q^n), \tag{39}$$

$$\delta_3 \leq \omega(f,q^n)\sum_{k=n+1}^{\infty} r_{\infty,k}(q;x)\leq\omega(f,q^n). \tag{40}$$

Now we estimate δ_2. Since $\omega(f,\lambda t)\leq(1+\lambda)\omega(f,t)$, by Lemma 11 we get

$$\begin{aligned} \delta_2 &\leq \sum_{k=0}^{n}\omega\left(f,q^k\right)\left|r_{n,k}(q;x)-r_{\infty,k}(q;x)\right| \\ &\leq \sum_{k=0}^{n}\omega(f,q^n)\left(1+\frac{q^k}{q^n}\right)\left|r_{n,k}(q;x)-r_{\infty,k}(q;x)\right| \\ &\leq \frac{2\omega(f;q^n)}{q^n}\sum_{k=0}^{n}q^k\left|r_{n,k}(q;x)-r_{\infty,k}(q;x)\right|\leq\frac{6\omega(f;q^n)}{1-q}. \end{aligned} \tag{41}$$

From (39)–(41), we have for $0\leq x\leq 1/(1+q^n)$,

$$I\leq\left(2+\frac{6}{1-q}\right)\omega(f;q^n). \tag{42}$$

Hence from (36) and (42), we conclude that, for $q\in(0,1)$,

$$\left\|R_{n,q}(f;x)-R_{\infty,q}(f;x)\right\|\leq C_q\ \omega(f;q^n), \tag{43}$$

where $C_q=2+6/(1-q)$.

At last we show that the estimate (9) is sharp. For each α, $0<\alpha\leq 1$, suppose that $f_\alpha(x)$ is a continuous function, which is equal to zero in $[0,1-q]$ and $[1-q^2,1]$, equal to $(x-(1-q))^\alpha$ in $[1-q,1-q+q(1-q)/2]$, and linear in the rest of $[0,1]$. It is obvious that $\omega(f_\alpha;t)\leq ct^\alpha$, and

$$\left\|R_{n,q}(f_\alpha)-R_{\infty,q}(f_\alpha)\right\|\approx q^{n\alpha}\left|r_{n,1}(q;\cdot)\right|\asymp q^{n\alpha}. \tag{44}$$

The proof of Theorem 1 is complete. □

In order to prove Theorem 2, we need the following result.

Theorem A (see [12]). *Let the sequence $\{L_n\}$ of positive linear operators on $C[0,1]$ satisfy the following conditions.*

(A) *The sequence $\{L_n(e_2)\}$ converges to a function $L_\infty(e_2)$ in $C[0,1]$, where $e_i(x)=x^i$, $i=0,1,2$.*

(B) *The sequence $\{L_n(f,x)\}_{n\geq1}$ is nonincreasing for any convex function f and for any $x\in[0,1]$.*

Then there exists an operator L_∞ on $C[0,1]$ such that $\|L_n(f)-L_\infty(f)\|\to 0$ for any $f\in C[0,1]$. Furthermore,

$$\left|L_n(f,x)-L_\infty(f,x)\right|\leq c\omega_2\left(f;\sqrt{\lambda_n(x)}\right), \tag{45}$$

where $\lambda_n(x)=L_n(e_2,x)-L_\infty(e_2,x)$ and c is a constant which depends only on $\|L_1(e_0)\|$.

Proof of Theorem 2. From [2], we know that the Lupas q-analogues of the Bernstein operators satisfy Condition (B).

It follows from [19] that, for $q \in (0,1)$, $\{R_{n,q}(f;x)\}$ converges to $R_{\infty,q}(f;x)$ uniformly in $x \in [0,1]$ as $n \to \infty$; and

$$\begin{aligned} 0 \le \lambda_n(x) &= R_{n,q}\left(t^2, x\right) - R_{\infty,q}\left(t^2, x\right) \\ &= R_{n,q}\left(t^2, x\right) - \lim_{n\to\infty} R_{n,q}\left(t^2; x\right) \\ &= \frac{x(1-x)}{[n]} - \frac{x^2(1-x)(1-q)}{1-x+xq}\left(1-\frac{1}{[n]}\right) \\ &\quad - x(1-x)(1-q) + \frac{x^2(1-x)(1-q)q}{1-x+xq} \\ &= x(1-x)\left(\frac{1}{[n]} - (1-q)\right) \\ &\quad + \frac{x^2(1-x)(1-q)}{1-x+xq}\left(\frac{1}{[n]} - (1-q)\right) \\ &= \frac{x(1-x)}{1-x+xq}\frac{(1-q)q^n}{1-q^n} \le q^n. \end{aligned} \tag{46}$$

Theorem 2 follows from (46) and Theorem A. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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