Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 520795, 5 pages
http://dx.doi.org/10.1155/2014/520795

Research Article

Maps Preserving Schatten *p*-Norms of Convex Combinations

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Received 18 October 2013; Accepted 30 December 2013; Published 14 January 2014

Academic Editor: Antonio M. Peralta

We study maps ϕ of positive operators of the Schatten p-classes $(1 < p < +\infty)$, which preserve the p-norms of convex combinations, that is, $\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p$, $\forall \rho, \sigma \in \mathcal{S}_p^+(H)_1$, $t \in [0,1]$. They are exactly those carrying the form $\phi(\rho) = U\rho U^*$ for a unitary or antiunitary U. In the case $p = 2$, we have the same conclusion whenever it just holds $\|\rho + \sigma\|_2 = \|\phi(\rho) + \phi(\sigma)\|_2$ for all the positive Hilbert-Schmidt class operators ρ, σ of norm 1. Some examples are demonstrated.

1. Introduction

The Mazur-Ulam theorem states that every bijective distance preserving map Φ from a Banach space onto another is affine; that is,

$$\Phi(tx + (1-t)y) = t\Phi(x) + (1-t)\Phi(y), \quad \forall x, y, 0 \le t \le 1. \tag{1}$$

After translation, we can assume that $\Phi(0) = 0$ and Φ is indeed a surjective real linear isometry. Let us consider another version of this statement. Suppose that Φ is a bijective map from a Hilbert space H onto H and Φ preserves norm of convex combinations:

$$\|t\Phi(x) + (1-t)\Phi(y)\| = \|tx + (1-t)y\|, \quad \forall x, y \in H,\ 0 \le t \le 1. \tag{2}$$

Let us further relax the assumption that (2) holds for just one fixed t in $(0,1)$. By letting $y = x$ in (2), we see that $\|\Phi(x)\| = \|x\|$ for all x in H. Squaring both sides of (2), we will see that the real parts of the inner products coincide; that is,

$$\operatorname{Re}\langle x, y\rangle = \operatorname{Re}\langle \Phi(x), \Phi(y)\rangle, \quad \forall x, y \in H. \tag{3}$$

Then the classical Wigner theorem (see, e.g., [1, Theorem 3]) ensures that there is a surjective real linear isometry $U : H \to H$ such that $\Phi(x) = Ux$ for all x in H.

Characterizing isometries, linear or not, of spaces of operators under various norms has been a fruitful area of research for a long time. See, for example, [2, 3] for good surveys. In particular, the spaces $\mathcal{S}_p(H)$ of the Schatten p-class operators on a (complex) Hilbert space H $(1 \le p < +\infty)$ are important objects in both analysis and physics. They are widely used in operator theory and quantum mechanics, for example.

Let $\mathcal{S}_p^+(H)$ be the set of all positive operators in $\mathcal{S}_p(H)$, and let $\mathcal{S}_p^+(H)_1$ be the set of all positive operators in $\mathcal{S}_p^+(H)$ of p-norm one. Recall that an affine automorphism (or S-automorphism in [4] or Kadison automorphism in [5]) is a bijective affine map $\phi : \mathcal{S}_1^+(H)_1 \to \mathcal{S}_1^+(H)_1$; that is,

$$\phi(t\rho + (1-t)\sigma) = t\phi(\rho) + (1-t)\phi(\sigma), \quad \forall \rho, \sigma \in \mathcal{S}_1^+(H)_1,\ t \in [0,1]. \tag{4}$$

It is known (see, e.g., [6]) that affine automorphisms are exactly those carrying the form $\phi(\rho) = U\rho U^*$ for a unitary or antiunitary U on H.

Recently, Nagy [7] established a Mazur-Ulam-type result for the Schatten p-class operators. Suppose that $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ $(1 < p < +\infty)$ is a bijective map preserving the distance induced by the norm $\|\cdot\|_p$. Then ϕ is implemented by a unitary or an antiunitary operator U such that $\phi(\rho) = U\rho U^*$. In this paper, we will establish a counterpart of Nagy's result similar to the one demonstrated in the first paragraph. More precisely, we will characterize those maps $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ satisfying

$$\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1,\ t \in [0,1]. \tag{5}$$

We will show that they are implemented by a unitary or an antiunitary operator.

Our main theorem follows.

Theorem 1. *Let H be a separable complex Hilbert space of finite or infinite dimension. Let $1 < p < +\infty$. Suppose that ϕ is a map from $\mathcal{S}_p^+(H)_1$ into $\mathcal{S}_p^+(H)_1$, which will be assumed to be surjective when* $\dim H = +\infty$. *The following conditions are equivalent.*

(1) *ϕ preserves the Schatten p-norms of convex combinations; that is,*

$$\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1,\ t \in [0,1]. \tag{6}$$

(2) *ϕ preserves the pairings; that is, for all $\rho, \sigma \in \mathcal{S}_p^+(H)_1$, one has $\sigma^{p-1}\rho \in \mathcal{S}_1(H)$, and*

$$\operatorname{tr}\left(\sigma^{p-1}\rho\right) = \operatorname{tr}\left(\phi(\sigma)^{p-1}\phi(\rho)\right). \tag{7}$$

(3) *There exists a unitary or antiunitary operator U on H such that*

$$\phi(\rho) = U\rho U^*, \quad \forall \rho \in \mathcal{S}_p^+(H)_1. \tag{8}$$

We note that condition (6) becomes a tautology when $p = 1$. On the other hand, the conclusion of Theorem 1 holds again if we replace $\mathcal{S}_p^+(H)_1$ by $\mathcal{S}_p^+(H)$ everywhere. In this case, setting $\sigma = \rho$ in (6), we see that ϕ does map $\mathcal{S}_p^+(H)_1$ into $\mathcal{S}_p^+(H)_1$.

The proof of Theorem 1 is given in Section 2. When $p = 2$, we see in Section 3 that for ϕ carrying the expected form stated in Theorem 1(3) it suffices to say that condition (6) held for only one fixed t in $(0,1)$. Finally, we demonstrate some examples in Section 4.

2. Proof of the Main Theorem

In what follows, we fix some notation and definitions used throughout the paper. Let H stand for a separable complex Hilbert space of finite dimension or infinite dimension. Let $B(H)$ denote the algebra of all bounded linear operators on H. For a compact operator T in $B(H)$, let $s_1(T) \geq s_2(T) \geq \cdots \geq 0$ denote the singular values of T, that is, the eigenvalues of $|T| = (TT^*)^{1/2}$ arranged in their decreasing order (repeating according to multiplicity). A compact operator T belongs to the Schatten p-classes $\mathcal{S}_p(H)$ $(1 \leq p < +\infty)$ if

$$\|T\|_p := \left(\sum_{i=1}^{\infty} s_i(T)^p\right)^{1/p} = \left(\operatorname{tr}|T|^p\right)^{1/p} < +\infty, \tag{9}$$

where tr denotes the trace functional. We call $\|T\|_p$ the Schatten p-norm of T. In particular, $\mathcal{S}_1(H)$ is the trace class and $\mathcal{S}_2(H)$ is the Hilbert-Schmidt class. The cone of positive operators in $\mathcal{S}_p(H)$ is denoted by $\mathcal{S}_p^+(H)$, and the set of rank one projections in $\mathcal{S}_p^+(H)$ is denoted by $P_1(H)$.

Recall that the norm of a normed space is Fréchet differentiable at $x \neq 0$ if $\lim_{t\to 0}((\|x+ty\| - \|x\|)/t)$ exists and uniform for all norm one vectors y.

Lemma 2 (see [8, Theorem 2.3]). *Let $1 < p < +\infty$ and ρ in $\mathcal{S}_p^+(H)$ be nonzero. The norm of $\mathcal{S}_p^+(H)$ is Fréchet differentiable at ρ. For any σ in $\mathcal{S}_p^+(H)$, one has*

$$\left.\frac{d\|\rho + t\sigma\|_p}{dt}\right|_{t=0} = \operatorname{tr}\left(\frac{\rho^{p-1}\sigma}{\|\rho\|_p^{p-1}}\right). \tag{10}$$

Lemma 3. *Suppose $\rho, \sigma \in \mathcal{S}_p^+(H)$ $(1 < p < +\infty)$. The following conditions are equivalent.*

(1) *$\rho = \sigma$.*

(2) *$\|t\rho + (1-t)P\|_p = \|t\sigma + (1-t)P\|_p$ for all P in $P_1(H)$ and all t in $[0,1]$.*

(3) *$\operatorname{tr}(P\rho) = \operatorname{tr}(P\sigma)$ for all P in $P_1(H)$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3): Differentiating both sides of $\|t\rho + (1-t)P\|_p = \|t\sigma + (1-t)P\|_p$ at $t = 0^+$, we have $\operatorname{tr} P\rho = \operatorname{tr} P^{p-1}\rho = \operatorname{tr} P^{p-1}\sigma = \operatorname{tr} P\sigma$ by Lemma 2.

(3) $\Rightarrow$ (1): Since ρ and σ are positive, $\rho - \sigma$ is Hermitian. There exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of H such that $\rho - \sigma = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$. Choosing $P_i = e_i \otimes e_i$, we have $\lambda_i = \operatorname{tr}(P_i(\rho - \sigma)) = 0$ for all $i = 1, 2, \ldots$. It follows that $\rho - \sigma = 0$. □

We say that two self-adjoint operators ρ, σ in $B(H)$ are orthogonal if $\rho\sigma = 0$, which is equivalent to the property that they have mutually orthogonal ranges.

Lemma 4. *Suppose that $\rho, \sigma \in \mathcal{S}_p^+(H)$ for $1 < p < +\infty$. The following conditions are equivalent.*

(1) *ρ, σ are orthogonal; that is, $\rho\sigma = 0$.*

(2) *$\|\alpha\rho + (1-\alpha)\sigma\|_p^p = \alpha^p\|\rho\|_p^p + (1-\alpha)^p\|\sigma\|_p^p$ for any (and thus all) α in $(0,1)$.*

(3) *$\operatorname{tr}(\rho\sigma) = 0$.*

(4) *$\|\rho + t\sigma\|_p \geq \|\rho\|_p$ for all t in $\mathbb{R}$; that is, $\rho \perp \sigma$ in Birkhoff's sense.*

(5) *$\operatorname{tr}(\rho^{p-1}\sigma) = 0$.*

Proof. (1) $\Leftrightarrow$ (2): From [9, Lemma 2.6], we know that for any two positive operators A, B in $\mathcal{S}_p^+(H)$, we have

$$\operatorname{tr}(A+B)^p \geq \operatorname{tr} A^p + \operatorname{tr} B^p. \tag{11}$$

Here, the equality holds if and only if $AB = 0$. Setting $A = \alpha\rho$ and $B = (1-\alpha)\sigma$, we get

$$\begin{aligned}\rho\sigma = 0 &\Longleftrightarrow (\alpha\rho)((1-\alpha)\sigma) = 0\\ &\Longleftrightarrow \operatorname{tr}(\alpha\rho + (1-\alpha)\sigma)^p = \operatorname{tr}(\alpha\rho)^p + \operatorname{tr}((1-\alpha)\sigma)^p\\ &\Longleftrightarrow \|\alpha\rho + (1-\alpha)\sigma\|_p^p = \alpha^p\|\rho\|_p^p + (1-\alpha)^p\|\sigma\|_p^p.\end{aligned} \tag{12}$$

(1) $\Leftrightarrow$ (3): One direction is obvious. For the other, because ρ, σ are positive,

$$\begin{aligned}&\operatorname{tr}\left[\left(\rho^{1/2}\sigma^{1/2}\right)\left(\rho^{1/2}\sigma^{1/2}\right)^*\right]\\ &\quad = \operatorname{tr}\left(\rho^{1/2}\sigma^{1/2}\sigma^{1/2}\rho^{1/2}\right) = \operatorname{tr}(\rho\sigma) = 0.\end{aligned} \tag{13}$$

This forces $\rho^{1/2}\sigma^{1/2} = 0$, and thus $\rho\sigma = \rho^{1/2}(\rho^{1/2}\sigma^{1/2})\sigma^{1/2} = 0$.

(1) $\Rightarrow$ (4): Since $\rho\sigma = 0$, there exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of H such that $\rho = \sum_{i=1}^{\infty}\lambda_i e_i \otimes e_i$, $\sigma = \sum_{i=1}^{\infty}\mu_i e_i \otimes e_i$, $\lambda_i \geq 0$, $\mu_i \geq 0$, and $\lambda_i\mu_i = 0$ for all $i = 1, 2, \ldots$. Hence,

$$\begin{aligned}\|\rho + t\sigma\|_p^p &= \operatorname{tr}|\rho + t\sigma|^p\\ &= \sum_{i=1}^{\infty}(\lambda_i + |t|\mu_i)^p \geq \sum_{i=1}^{\infty}\lambda_i^p = \|\rho\|_p^p.\end{aligned} \tag{14}$$

(4) $\Rightarrow$ (5): Without loss of generality, we can assume that $\rho \neq 0$. Define $f(t) = \|\rho + t\sigma\|_p \geq \|\rho\|_p$. Then $f(t)$ is differentiable and attains its minimum at $t = 0$. From Lemma 2,

$$0 = \left.\frac{d\|\rho + t\sigma\|_p}{dt}\right|_{t=0} = \operatorname{tr}\left(\frac{\rho^{p-1}\sigma}{\|\rho\|_p^{p-1}}\right), \tag{15}$$

and assertion (5) follows.

(5) $\Rightarrow$ (1): As in proving (3) $\Rightarrow$ (1), we have $\rho^{p-1}\sigma = 0$. Then, there exists an orthonormal basis $\{e_i'\}_{i=1}^{\infty}$ of H such that $\rho^{p-1} = \sum_{i=1}^{\infty}\xi_i e_i' \otimes e_i'$, $\sigma = \sum_{i=1}^{\infty}\eta_i e_i' \otimes e_i'$, with $\xi_i \geq 0$, $\eta_i \geq 0$, and $\xi_i\mu_i = 0$ for all $i = 1, 2, \ldots$. Thus, $\operatorname{tr}(\rho\sigma) = \sum_{i=1}^{\infty}\xi_i^{1/(p-1)}\eta_i = 0$. □

Lemma 5. *Let $1 < p < +\infty$. Suppose that ϕ is a map from $\mathcal{S}_p^+(H)_1$ into $\mathcal{S}_p^+(H)_1$ preserving the Schatten p-norms of convex combinations; that is,* (6) *holds. Then, one has*

$$\operatorname{tr}\left(\sigma^{p-1}\rho\right) = \operatorname{tr}\left(\phi(\sigma)^{p-1}\phi(\rho)\right). \tag{16}$$

Proof. Differentiating both sides of (6) with respect to t and evaluating at $t = 0$, we have

$$\begin{aligned}\left.\frac{d\|t\rho + (1-t)\sigma\|_p}{dt}\right|_{t=0} &= \left.\frac{d\|\sigma + t(\rho-\sigma)\|_p}{dt}\right|_{t=0}\\ &= \operatorname{tr}\left(\frac{\sigma^{p-1}(\rho-\sigma)}{\|\sigma\|_p^{p-1}}\right)\\ &= \frac{\operatorname{tr}\left(\sigma^{p-1}\rho\right)}{\|\sigma\|_p^{p-1}} - \|\sigma\|_p\\ &= \operatorname{tr}\left(\sigma^{p-1}\rho\right) - 1,\\ \left.\frac{d\|t\phi(\rho) + (1-t)\phi(\sigma)\|_p}{dt}\right|_{t=0} &= \frac{\operatorname{tr}\left(\phi(\sigma)^{p-1}\rho\right)}{\|\phi(\sigma)\|_p^{p-1}} - \|\phi(\sigma)\|_p\\ &= \operatorname{tr}\left(\phi(\sigma)^{p-1}\rho\right) - 1.\end{aligned} \tag{17}$$

Since (6) holds for t in $(0, 1]$, these derivatives agree. Therefore, $\operatorname{tr}(\sigma^{p-1}\rho) = \operatorname{tr}(\phi(\sigma)^{p-1}\phi(\rho))$. □

Proposition 6. *Suppose that $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ satisfies*

$$\operatorname{tr}\left(\sigma^{p-1}\rho\right) = \operatorname{tr}\left(\phi(\sigma)^{p-1}\phi(\rho)\right), \quad \forall\rho, \sigma \in \mathcal{S}_p^+(H)_1. \tag{18}$$

Then the following assertions hold.

(1) *ϕ preserves orthogonality in both directions; that is*

$$\rho\sigma = 0 \Longleftrightarrow \phi(\rho)\phi(\sigma) = 0, \quad \forall\rho, \sigma \in \mathcal{S}_p^+(H)_1. \tag{19}$$

(2) *When* $\dim H < +\infty$, *ϕ maps rank-one projections to rank-one projections. This also holds when* $\dim H = +\infty$ *and ϕ is surjective.*

(3) *When* $\dim H < +\infty$, *one has*

$$\operatorname{tr} PQ = \operatorname{tr}\phi(P)\phi(Q), \quad \forall P, Q \in P_1(H). \tag{20}$$

This also holds when $\dim H = +\infty$ *and ϕ is surjective.*

Proof. (1) follows from Lemma 4.

(2) First, we assume that $\dim H = n < +\infty$. Suppose ρ is a rank-one projection. We can find $n-1$ pairwise orthogonal rank-one projections $\rho_1, \ldots, \rho_{n-1}$ such that $\rho\rho_i = 0$ for $1 \leq i \leq n-1$. From (1), we know that $\phi(\rho), \phi(\rho_1), \ldots, \phi(\rho_{n-1})$ are nonzero and pairwise orthogonal. This forces that $\phi(\rho)$ has rank one since $\dim H = n$. By (18), taking $\sigma = \rho$, we see that $\operatorname{tr}\phi(\rho)^p = \operatorname{tr}\rho^p = \operatorname{tr}\rho = 1$. Therefore, the rank-one positive operator $\phi(\rho)$ is a projection.

Next, we consider the case $\dim H = +\infty$ and ϕ is surjective. Suppose that there exists a rank-one projection ρ in $\mathcal{S}_p^+(H)$ such that $\phi(\rho)$ has rank greater than one. Then, there are two nonzero orthogonal operators T_1 and T_2 in $\mathcal{S}_p^+(H)$ such that $\phi(\rho) = T_1 + T_2$. Since ϕ is surjective and preserves orthogonality in both directions, there are two nonzero orthogonal operators ρ_1 and ρ_2 in $\mathcal{S}_p^+(H)_1$ such that $\phi(\rho_1) = T_1/\|T_1\|_p$ and $\phi(\rho_2) = T_1/\|T_2\|_p$. For any σ in $\mathcal{S}_p^+(H)$ with $\sigma\rho = 0$, we have

$$\begin{aligned}\phi(\sigma)\left(\|T_1\|_p\phi(\rho_1) + \|T_2\|_p\phi(\rho_2)\right) \\ = \phi(\sigma)(T_1 + T_2) = \phi(\sigma)\phi(\rho) = 0.\end{aligned} \tag{21}$$

It forces that

$$\|T_1\|_p\phi(\sigma)\phi(\rho_1)\phi(\sigma) = -\|T_2\|_p\phi(\sigma)\phi(\rho_2)\phi(\sigma) = 0, \tag{22}$$

and hence $\phi(\sigma)\phi(\rho_1) = \phi(\sigma)\phi(\rho_2) = 0$, because $\phi(\sigma)$, $\phi(\rho_1)$, and $\phi(\rho_2)$ are all positive. This implies $\sigma\rho_1 = \sigma\rho_2 = 0$. Therefore, $\rho_1 = \lambda_1\rho$ and $\rho_2 = \lambda_2\rho$ for some nonzero λ_1, λ_2. This contradicts the fact that $\rho_1\rho_2 = 0$.

(3) From (2), we know that $\phi(P)$, $\phi(Q)$ are rank-one projections in $P_1(H)$. Therefore, $P^{p-1} = P$, $\phi(P)^{p-1} = \phi(P)$. Using (18) with $\sigma = P$, $\rho = Q$, we have

$$\operatorname{tr} PQ = \operatorname{tr}\left(P^{p-1}Q\right) = \operatorname{tr}\left(\phi(P)^{p-1}\phi(Q)\right) = \operatorname{tr}\phi(P)\phi(Q). \tag{23}$$

□

Proof of Theorem 1. (1) ⇒ (2) follows from Lemma 5.

(3) ⇒ (1) is obvious.

(2) ⇒ (3): From Proposition 6, we obtain that $\phi|_{P_1(H)} : P_1(H) \to P_1(H)$ satisfies $\operatorname{tr} PQ = \operatorname{tr}\phi(P)\phi(Q)$ for all rank-one projections P, Q in $P_1(H)$. From a nonsurjective version of Wigner's theorem, cf. [6, Theorem 2.1.4], there exists an isometry or anti-isometry U on H such that

$$\phi(P) = UPU^*, \quad \forall P \in P_1(H). \tag{24}$$

Note that U is indeed surjective even when H is of infinite dimension, since ϕ is assumed to be surjective in this case.

For any rank-one projection P in $P_1(H)$, setting $\sigma = P$ in (7), we have

$$\begin{aligned}\operatorname{tr}(P\rho) &= \operatorname{tr}\left(P^{p-1}\rho\right) = \operatorname{tr}\left(\phi(P)^{p-1}\phi(\rho)\right) = \operatorname{tr}(\phi(P)\phi(\rho)) \\ &= \operatorname{tr}(UPU^*\phi(\rho)U) = \operatorname{tr}(PU^*\phi(\rho)U).\end{aligned} \tag{25}$$

We have $U^*\phi(\rho)U = \rho$ by Lemma 3. This gives $\phi(\rho) = U\rho U^*$. □

3. Maps Preserving Norms of Just a Special Convex Combination

A careful look at the proof of Lemma 5 tells us that the condition $\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p$ suffices to hold for the members of any sequence in $(0, 1]$ converging to 0 rather than for any point t in $[0, 1]$. Indeed, in order to get some good properties of ϕ stated in the previous section, we only need to assume that ϕ preserves the Schatten p-norm of convex combination with a given system of coefficients.

Proposition 7. *Let $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ $(1 < p < +\infty)$. Let α in $(0, 1)$ be arbitrary but fixed. Suppose*

$$\begin{aligned}\|\alpha\rho + (1-\alpha)\sigma\|_p = \|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p, \\ \forall\rho, \sigma \in \mathcal{S}_p^+(H)_1.\end{aligned} \tag{26}$$

The following properties are satisfied.

(1) *ϕ is injective.*

(2) *ϕ preserves orthogonality in both directions.*

(3) *When $\dim H < +\infty$, ϕ maps rank-one projections to rank-one projections. This also holds when $\dim H = +\infty$ and ϕ is surjective.*

Proof. (1) Assume $\phi(\rho) = \phi(\sigma)$. We have $\|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p = 1$. From (26), we get $\|\alpha\rho + (1-\alpha)\sigma\|_p = 1$. Hence,

$$\|\alpha\rho + (1-\alpha)\sigma\|_p = \alpha\|\rho\|_p + (1-\alpha)\|\sigma\|_p. \tag{27}$$

This forces $\rho = \sigma$ since the norm $\|\cdot\|_p$ is strictly convex for $1 < p < +\infty$.

(2) Assume $\rho\sigma = 0$. From Lemma 4, we have

$$\begin{aligned}\|\alpha\rho + (1-\alpha)\sigma\|_p^p &= \alpha^p\|\rho\|^p + (1-\alpha)^p\|\sigma\|^p \\ &= \alpha^p\|\phi(\rho)\|^p + (1-\alpha)^p\|\phi(\sigma)\|^p.\end{aligned} \tag{28}$$

Together with (26), we have

$$\begin{aligned}&\|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p^p \\ &= \alpha^p\|\phi(\rho)\|^p + (1-\alpha)^p\|\phi(\sigma)\|^p.\end{aligned} \tag{29}$$

Hence, we have $\phi(\rho)\phi(\sigma) = 0$ from Lemma 4 again. The other implication follows similarly.

(3) The proof is similar to that of Proposition 6(2). □

When $p = 2$, we get an improvement of Theorem 1.

Theorem 8. *Let H be a separable complex Hilbert space. Suppose that $\phi : \mathcal{S}_2^+(H)_1 \to \mathcal{S}_2^+(H)_1$, which needs to be surjective when $\dim H = +\infty$. The following conditions are equivalent.*

(1) *ϕ preserves the Hilbert-Schmidt norms of all convex combinations; that is,*

$$\begin{aligned}\|t\rho + (1-t)\sigma\|_2 = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_2, \\ \forall\rho, \sigma \in \mathcal{S}_2^+(H)_1,\ t \in [0, 1].\end{aligned} \tag{30}$$

(2) *For any (and thus all) α in $(0, 1)$ one has*

$$\begin{aligned}\|\alpha\rho + (1-\alpha)\sigma\|_2 = \|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_2, \\ \forall\rho, \sigma \in \mathcal{S}_2^+(H)_1.\end{aligned} \tag{31}$$

A special case states that

$$\|\rho + \sigma\|_2 = \|\phi(\rho) + \phi(\sigma)\|_2, \quad \forall\rho, \sigma \in \mathcal{S}_2^+(H)_1. \tag{32}$$

(3) $\operatorname{tr}(\rho\sigma) = \operatorname{tr}(\phi(\rho)\phi(\sigma))$ *for all* ρ, σ *in* $\mathcal{S}_2^+(H)_1$.

(4) *There exists a unitary or antiunitary operator* U *such that*

$$\phi(\rho) = U\rho U^*, \quad \forall \rho \in \mathcal{S}_2^+(H)_1. \tag{33}$$

Proof. We prove (2) ⇒ (3) only. Observe

$$\begin{aligned} \|\alpha\rho + (1-\alpha)\sigma\|_2^2 &= \operatorname{tr}(\alpha\rho + (1-\alpha)\sigma)^2 \\ &= \alpha^2 \operatorname{tr}\rho^2 + 2\alpha(1-\alpha)\operatorname{tr}(\rho\sigma) \\ &\quad + (1-\alpha)^2 \operatorname{tr}\sigma^2, \\ \|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_2^2 &= \alpha^2 \operatorname{tr}\phi(\rho)^2 \\ &\quad + 2\alpha(1-\alpha)\operatorname{tr}(\phi(\rho)\phi(\sigma)) \\ &\quad + (1-\alpha)^2 \operatorname{tr}\phi(\sigma)^2. \end{aligned} \tag{34}$$

We have $\operatorname{tr}(\rho\sigma) = \operatorname{tr}(\phi(\rho)\phi(\sigma))$. □

4. Examples

We remark that all results in previous sections hold for a map $\phi : \mathcal{S}_p^+(H) \to \mathcal{S}_p^+(H)$ which satisfies instead of (6) the condition

$$\begin{gathered} \|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \\ \forall \rho, \sigma \in \mathcal{S}_p^+(H), \ t \in [0,1]. \end{gathered} \tag{35}$$

The proofs go in exactly the same ways.

The following example shows that a norm preserver of $\mathcal{S}_p^+(H)$ might not be affine.

Example 1. Let H be a finite dimensional Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^n$. Let $1 < p < +\infty$. Define a map ϕ from $\mathcal{S}_p^+(H)$ into itself by

$$\phi(\rho) = \begin{cases} 0, & \text{if } \rho = 0, \\ \dfrac{\|\rho\|_p}{\left\|\sum_{i=1}^n P_i\rho P_i\right\|_p} \displaystyle\sum_{i=1}^n P_i\rho P_i, & \text{if } \rho \neq 0, \end{cases} \tag{36}$$

where $P_i = e_i \otimes e_i$ is a rank-one projection for $i = 1, \ldots, n$. Obviously, $\phi(\rho)$ is positive and $\|\phi(\rho)\|_p = \|\rho\|_p$ for all ρ in $\mathcal{S}_p^+(H)$. However, ϕ does not preserve the Schatten p-norms of convex combinations, as the eigenvalues of ρ and $\phi(\rho)$ can be different from each other.

Our theorems are about the Schatten p-norms for $1 < p < +\infty$. Here is an example of a map of $\mathcal{S}_1^+(H)$ which preserves trace norms of convex combinations. However, it is not implemented by a unitary or antiunitary.

Example 2. Consider Example 1 in the case where $p = 1$. In this case,

$$\phi(\rho) = \sum_{i=1}^n P_i\rho P_i. \tag{37}$$

It is easy to see that the map ϕ satisfies the condition

$$\begin{gathered} \|t\rho + (1-t)\sigma\|_1 = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_1, \\ \forall \rho, \sigma \in \mathcal{S}_1^+(H), \ t \in [0,1]. \end{gathered} \tag{38}$$

But there does not exist a unitary or antiunitary U such that $\phi(\rho) = U\rho U^*$ for all ρ in $\mathcal{S}_1^+(H)$.

Example 3. Let H be a separable Hilbert space of infinite dimension, and let $\{e_n : n = 1, 2, \ldots\}$ be a basis of H. Let S be the unilateral shift on H defined by $Se_n = e_{n+1}$ for $n = 1, 2, \ldots$. Let ϕ be defined by $\phi(\rho) = S\rho S^*$ for ρ in $\mathcal{S}_p^+(H)$. The map ϕ is not surjective, as $e_1 \otimes e_1$ is not in its range. It is easy to see that $\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p$ holds for all ρ, σ in $\mathcal{S}_p^+(H)$ and t in $[0,1]$. However, ϕ is not implemented by a unitary or antiunitary.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research is supported partially by the Aim for the Top University Plan of NSYSU, the NSC Grant (102-2115-M-110-002-MY2), and the NSFC Grant (no. 11171126).

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