

## Research Article

# Error Analysis for a Noisy Lacunary Cubic Spline Interpolation and a Simple Noisy Cubic Spline Quasi Interpolation

Feng-Gong Lang and Xiao-Ping Xu

School of Mathematical Sciences, Ocean University of China, Qingdao, Shandong 266100, China

Correspondence should be addressed to Feng-Gong Lang; fenggonglang@sina.com

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We mainly present the error analysis for two new cubic spline based methods; one is a lacunary interpolation method and the other is a very simple quasi interpolation method. The new methods are able to reconstruct a function and its first two derivatives from noisy function data. The explicit error bounds for the methods are given and proved. Numerical tests and comparisons are performed. Numerical results verify the efficiency of our methods.

## 1. Introduction

Cubic spline, as the most commonly used spline in practice, is a fundamental approximation tool [1–11]. Nowadays, it has been widely used in many fields such as numerical analysis, computer aided geometric design, mathematical modeling, and engineering problems. Essentially, cubic spline is a twice differentiable piecewise cubic polynomial defined over a partitioned interval.

Mathematically, cubic spline interpolation is often introduced as follows. Let  $y(x)$  be a function defined over  $[a, b]$ , let

$$y_j = y(x_j), \quad j = 0, 1, \dots, n, \quad (1)$$

be a set of given function data at the nodes

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b, \quad (2)$$

and let

$$y'_0 = y'(x_0), \quad y'_n = y'(x_n) \quad (3)$$

be two boundary derivatives. Then, there exists a unique cubic spline  $s(x)$  satisfying

$$\begin{aligned} s'(x_0) &= y'_0, & s'(x_n) &= y'_n, \\ s(x_j) &= y_j, & j &= 0, 1, \dots, n. \end{aligned} \quad (4)$$

However, we often meet two troubles in the practical applications of cubic spline interpolation. The first trouble is that we cannot obtain the precise function values in (1). They generally involve some unavoidable measurement noise. The second trouble is that it often lacks the boundary derivatives in (3).

To deal with the troubles, in this paper, we give two new effective cubic spline based methods for reconstructing  $y(x)$ ,  $y'(x)$ , and  $y''(x)$  from the given noisy data

$$\tilde{y}_j = y(x_j) + \varepsilon_j, \quad j = 0, 1, \dots, n, \quad (5)$$

where  $\varepsilon_j$  is the measurement noise. The first one is a noisy lacunary interpolation method (*Method I*) and the second one is a very simple noisy quasi interpolation method (*Method II*). The error bounds of the methods, which have not been studied before and are important and useful for the users of cubic spline, are mainly studied in this paper.

We organize the remainder of this paper as follows. In Section 2, we present some useful preliminaries; in Section 3, we give the new methods; in Section 4, we present the theoretical results of the errors; in Section 5, we perform some numerical tests to verify the error analysis; finally, we conclude this paper in Section 6.



TABLE 1: The values of  $B_i^{(k)}(x)$  ( $i = -1, 0, \dots, n+1; k = 0, 1, 2$ ) at the nodes.

	$x_{i-1}$	$x_i$	$x_{i+1}$	Else
$B_i(x)$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	0
$B_i'(x)$	$\frac{1}{2h}$	0	$-\frac{1}{2h}$	0
$B_i''(x)$	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0

TABLE 2: The approximate boundary derivatives and their errors (I).

	Two-point results	Three-point results
$\tilde{y}'_0$	$\frac{\tilde{y}_1 - \tilde{y}_0}{h}$	$\frac{-3\tilde{y}_0 + 4\tilde{y}_1 - \tilde{y}_2}{2h}$
$\tilde{\varepsilon}'_0$	$\frac{h}{2}y''(\xi_1) + \frac{\varepsilon_1 - \varepsilon_0}{h}$	$-\frac{h^2}{3}y'''(\xi_3) + \frac{-3\varepsilon_0 + 4\varepsilon_1 - \varepsilon_2}{2h}$
$\tilde{y}'_n$	$\frac{\tilde{y}_n - \tilde{y}_{n-1}}{h}$	$\frac{\tilde{y}_{n-2} - 4\tilde{y}_{n-1} + 3\tilde{y}_n}{2h}$
$\tilde{\varepsilon}'_n$	$-\frac{h}{2}y''(\xi_2) + \frac{\varepsilon_n - \varepsilon_{n-1}}{h}$	$-\frac{h^2}{3}y'''(\xi_4) + \frac{\varepsilon_{n-2} - 4\varepsilon_{n-1} + 3\varepsilon_n}{2h}$

TABLE 3: The approximate boundary derivatives and their errors (II).

	Five-point results
$\tilde{y}'_0$	$\frac{-25\tilde{y}_0 + 48\tilde{y}_1 - 36\tilde{y}_2 + 16\tilde{y}_3 - 3\tilde{y}_4}{12h}$
$\tilde{\varepsilon}'_0$	$-\frac{h^4}{5}y^{(5)}(\xi_5) + \frac{-25\varepsilon_0 + 48\varepsilon_1 - 36\varepsilon_2 + 16\varepsilon_3 - 3\varepsilon_4}{12h}$
$\tilde{y}'_n$	$\frac{3\tilde{y}_{n-4} - 16\tilde{y}_{n-3} + 36\tilde{y}_{n-2} - 48\tilde{y}_{n-1} + 25\tilde{y}_n}{12h}$
$\tilde{\varepsilon}'_n$	$-\frac{h^4}{5}y^{(5)}(\xi_6) + \frac{3\varepsilon_{n-4} - 16\varepsilon_{n-3} + 36\varepsilon_{n-2} - 48\varepsilon_{n-1} + 25\varepsilon_n}{12h}$

3.2. *Method II.* By using the given function data, we can directly get a cubic spline

$$\tilde{s}(x) = \sum_{i=-1}^{n+1} \tilde{y}_i B_i(x), \quad (13)$$

where  $\tilde{y}_{-1} = y(x_0 - h) + \varepsilon_{-1}$  and  $\tilde{y}_{n+1} = y(x_n + h) + \varepsilon_{n+1}$ . We can also use  $\tilde{s}(x)$ ,  $\tilde{s}'(x)$ , and  $\tilde{s}''(x)$  to approximate  $y(x)$ ,  $y'(x)$ , and  $y''(x)$ , respectively.

The method is very simple and effective method for noisy data because it avoids using approximate boundary derivatives and also avoids solving the linear system (12).

## 4. Main Results

4.1. *Error Analysis for Method I.* We denote (12) by  $A\tilde{C} = \tilde{Y}$ .

**Lemma 1.** *A is invertible and  $\|A^{-1}\|_{\infty} \leq 2$ .*

*Proof.* Add column one to column three and also add column  $n+3$  to column  $n+1$ , and we get a strictly diagonally dominant matrix

$$A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 4 & 2 \\ & 1 & 4 & 1 \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 4 & 1 \\ & & & & 2 & 4 & 1 \\ & & & & & 0 & 0 & 1 \end{pmatrix}_{(n+3) \times (n+3)}. \quad (14)$$

Obviously,  $A_1$  is invertible and  $A_1 = AP$ , where

$$P = \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \\ & & & \ddots \\ & & & & 1 \\ & & & & & 0 & 1 \\ & & & & & & 1 & 0 & 1 \end{pmatrix}_{(n+3) \times (n+3)}. \quad (15)$$

We have

$$A^{-1} = PA_1^{-1}, \quad \|A_1^{-1}\|_{\infty} \leq 1, \quad \|P\|_{\infty} = 2. \quad (16)$$

Hence, we have  $\|A^{-1}\|_{\infty} = \|PA_1^{-1}\|_{\infty} \leq \|P\|_{\infty} \|A_1^{-1}\|_{\infty} \leq 2$ .  $\square$

Let  $s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$  be the cubic spline determined by (4),

$$\begin{aligned} C &= (c_{-1}, c_0, \dots, c_n, c_{n+1})^T, \\ Y &= (2hy'_0, 6y_0, \dots, 6y_n, 2hy'_n)^T, \end{aligned} \quad (17)$$

and then we have  $AC = Y$ . Let

$$\begin{aligned} \varepsilon' &= \max \{ |\varepsilon'_0|, |\varepsilon'_n| \}, \\ \varepsilon &= \max_{0 \leq i \leq n} \{ |\varepsilon_i| \}, \end{aligned} \quad (18)$$

and we have

$$\|\tilde{Y} - Y\|_{\infty} \leq \max \{ 2h\varepsilon', 6\varepsilon \}; \quad (19)$$

see Table 4 for the results, where  $M_5 = \|y^{(5)}(x)\|_{\infty} = \max_{a \leq x \leq b} |y^{(5)}(x)|$  and  $M_4, M_3$ , and  $M_2$  are defined similarly.

**Lemma 2.** *Consider  $\|\tilde{C} - C\|_{\infty} \leq 2\|\tilde{Y} - Y\|_{\infty}$ .*

*Proof.* Consider  $\|\tilde{C} - C\|_{\infty} = \|A^{-1}(\tilde{Y} - Y)\|_{\infty} \leq \|A^{-1}\|_{\infty} \|\tilde{Y} - Y\|_{\infty} \leq 2\|\tilde{Y} - Y\|_{\infty}$ .  $\square$

**Lemma 3.** *Consider  $\sum_{i=-1}^{n+1} |B_i(x)| \equiv 1$ ,  $\sum_{i=-1}^{n+1} |B'_i(x)| \leq 3/2h$ , and  $\sum_{i=-1}^{n+1} |B''_i(x)| \leq 4/h^2$ .*

TABLE 4: The bounds of  $\varepsilon'$  and  $\|\tilde{Y} - Y\|_\infty$ .

	Two-point results	Three-point results	Five-point results
$\varepsilon' \leq$	$\frac{1}{2}M_2h + \frac{2\varepsilon}{h}$	$\frac{1}{3}M_3h^2 + \frac{4\varepsilon}{h}$	$\frac{1}{5}M_5h^4 + \frac{32\varepsilon}{3h}$
$\ \tilde{Y} - Y\ _\infty \leq$	$\max\{M_2h^2 + 4\varepsilon, 6\varepsilon\}$	$\frac{2}{3}M_3h^3 + 8\varepsilon$	$\frac{2}{5}M_5h^5 + \frac{64\varepsilon}{3}$

*Proof.* Because of property (8), we only need to check them over a typical subinterval  $[x_j, x_{j+1}]$ . By differentiating (6), for a general  $i$ , we have

$$B'_i(x) = \frac{1}{2h^3} \begin{cases} (x-x_{i-2})^2, & \text{if } x \in [x_{i-2}, x_{i-1}] \\ (x-x_{i-2})^2 - 4(x-x_{i-1})^2, & \text{if } x \in [x_{i-1}, x_i] \\ -(x_{i+2}-x)^2 + 4(x_{i+1}-x)^2, & \text{if } x \in [x_i, x_{i+1}] \\ -(x_{i+2}-x)^2, & \text{if } x \in [x_{i+1}, x_{i+2}] \\ 0, & \text{else,} \end{cases}$$

$$B''_i(x) = \frac{1}{h^3} \begin{cases} x - x_{i-2}, & \text{if } x \in [x_{i-2}, x_{i-1}] \\ -3x - x_{i-2} + 4x_{i-1}, & \text{if } x \in [x_{i-1}, x_i] \\ x_{i+2} - 4x_{i+1} + 3x, & \text{if } x \in [x_i, x_{i+1}] \\ x_{i+2} - x, & \text{if } x \in [x_{i+1}, x_{i+2}] \\ 0, & \text{else.} \end{cases} \quad (20)$$

All of them are locally supported over four adjacent subintervals.

- (i) By the nonnegativity and partition of unity of cubic B-splines, for  $x \in [x_j, x_{j+1}]$ , we have

$$\sum_{i=-1}^{n+1} |B_i(x)| = \sum_{i=-1}^{n+1} B_i(x) = \sum_{i=j-1}^{j+2} B_i(x) \equiv 1. \quad (21)$$

- (ii) For  $x \in [x_j, x_{j+1}]$ , we have

$$\begin{aligned} & \sum_{i=-1}^{n+1} |B'_i(x)| \\ &= |B'_{j-1}(x)| + |B'_j(x)| + |B'_{j+1}(x)| + |B'_{j+2}(x)| \\ &= \frac{1}{2h^3} \left[ (x-x_{j+1})^2 + \left| 4(x-x_{j+1})^2 - (x-x_{j+2})^2 \right| \right. \\ & \quad \left. + \left| (x-x_{j-1})^2 - 4(x-x_j)^2 \right| + (x-x_j)^2 \right] \\ &= \frac{1}{2h^3} \left[ -4(x-x_j)^2 + 4h(x-x_j) + 2h^2 \right] \leq \frac{3}{2h}. \end{aligned} \quad (22)$$

- (iii) For  $x \in [x_j, x_{j+1}]$ , we have

$$\begin{aligned} & \sum_{i=-1}^{n+1} |B''_i(x)| \\ &= |B''_{j-1}(x)| + |B''_j(x)| + |B''_{j+1}(x)| + |B''_{j+2}(x)| \end{aligned}$$

$$\begin{aligned} &= \frac{1}{h^3} \left[ (x_{j+1}-x) + |3x+x_{j+2}-4x_{j+1}| \right. \\ & \quad \left. + |-3x-x_{j-1}+4x_j| + (x-x_j) \right] \\ &= \frac{1}{h^3} \begin{cases} 4h-6(x-x_j), & x \in \left[ x_j, x_j + \frac{h}{3} \right] \\ 2h, & x \in \left[ x_j + \frac{h}{3}, x_j + \frac{2h}{3} \right] \\ 6(x-x_j)-2h, & x \in \left[ x_j + \frac{2h}{3}, x_{j+1} \right] \end{cases} \\ &\leq \frac{4}{h^2}. \quad \square \end{aligned} \quad (23)$$

**Lemma 4.** Let  $s(x)$  and  $\tilde{s}(x)$  be the cubic spline interpolants of  $y(x)$  determined by (4) and (11), respectively. Then we have

$$\begin{aligned} |s(x) - \tilde{s}(x)| &\leq 2\|\tilde{Y} - Y\|_\infty, \\ |s'(x) - \tilde{s}'(x)| &\leq \frac{3}{h}\|\tilde{Y} - Y\|_\infty, \\ |s''(x) - \tilde{s}''(x)| &\leq \frac{8}{h^2}\|\tilde{Y} - Y\|_\infty. \end{aligned} \quad (24)$$

*Proof.* First of all, for  $k = 0, 1, 2$ , we have

$$\begin{aligned} |s^{(k)}(x) - \tilde{s}^{(k)}(x)| &= \left| \sum_{i=-1}^{n+1} (c_i - \tilde{c}_i) B_i^{(k)}(x) \right| \\ &\leq \|\bar{C} - C\|_\infty \sum_{i=-1}^{n+1} |B_i^{(k)}(x)|. \end{aligned} \quad (25)$$

And then using Lemmas 2 and 3, we get these results.  $\square$

**Theorem 5.** Let  $\tilde{s}(x)$  be the noisy lacunary cubic spline interpolant of  $y(x)$  determined by (11). Then one has

$$\begin{aligned} |\tilde{s}(x) - y(x)| &\leq \frac{5}{384}M_4h^4 + 2\|\tilde{Y} - Y\|_\infty, \\ |\tilde{s}'(x) - y'(x)| &\leq \frac{1}{24}M_4h^3 + \frac{3}{h}\|\tilde{Y} - Y\|_\infty, \\ |\tilde{s}''(x) - y''(x)| &\leq \frac{3}{8}M_4h^2 + \frac{8}{h^2}\|\tilde{Y} - Y\|_\infty. \end{aligned} \quad (26)$$

*Proof.* These results follow from the traditional cubic spline interpolation error theory [1, 3, 7], Lemma 4, and the following triangle inequality

$$\begin{aligned} \left| \tilde{s}^{(k)}(x) - y^{(k)}(x) \right| &\leq \left| \tilde{s}^{(k)}(x) - s^{(k)}(x) \right| \\ &+ \left| s^{(k)}(x) - y^{(k)}(x) \right|, \quad k = 0, 1, 2. \end{aligned} \quad (27)$$

□

4.2. *Error Analysis for Method II.* From (13) and Table 1, for  $j = 0, 1, \dots, n$ , we have

$$\tilde{s}(x_j) = \frac{\tilde{y}_{j-1} + 4\tilde{y}_j + \tilde{y}_{j+1}}{6}, \quad (28)$$

$$\tilde{s}'(x_j) = \frac{\tilde{y}_{j+1} - \tilde{y}_{j-1}}{2h}, \quad (29)$$

$$\tilde{s}''(x_j) = \frac{\tilde{y}_{j-1} - 2\tilde{y}_j + \tilde{y}_{j+1}}{h^2}. \quad (30)$$

It is a surprise to find that  $\tilde{s}'(x_j)$  and  $\tilde{s}''(x_j)$  are the same as the well-known central numerical differentiation formulae.

**Lemma 6.** *Let  $\tilde{s}(x)$  be the noisy cubic spline quasi interpolant of  $y(x)$  determined by (13). Then for  $j = 0, 1, \dots, n$ , one has*

$$\left| \tilde{s}(x_j) - y(x_j) \right| \leq \frac{1}{6}M_2h^2 + \varepsilon, \quad (31)$$

$$\left| \tilde{s}'(x_j) - y'(x_j) \right| \leq \frac{1}{6}M_3h^2 + \frac{\varepsilon}{h}, \quad (32)$$

$$\left| \tilde{s}''(x_j) - y''(x_j) \right| \leq \frac{1}{12}M_4h^2 + \frac{4\varepsilon}{h^2}. \quad (33)$$

*Proof.* By (30), we have

$$\begin{aligned} &\left| \tilde{s}''(x_j) - y''(x_j) \right| \\ &= \left| \frac{\tilde{y}_{j-1} - 2\tilde{y}_j + \tilde{y}_{j+1}}{h^2} - y''(x_j) \right| \\ &= \left| \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} - y''(x_j) + \frac{\varepsilon_{j-1} - 2\varepsilon_j + \varepsilon_{j+1}}{h^2} \right| \\ &\leq \frac{1}{12}M_4h^2 + \frac{4\varepsilon}{h^2}. \end{aligned} \quad (34)$$

The proofs of (31) and (32) are similar, which are omitted. □

**Theorem 7.** *Let  $\tilde{\tilde{s}}(x)$  be the noisy cubic spline quasi interpolant of  $y(x)$  determined by (13). Then we have*

$$\left| \tilde{\tilde{s}}(x) - y(x) \right| \leq \frac{1}{384}M_4h^4 + \frac{1}{24}M_3h^3 + \frac{1}{6}M_2h^2 + \frac{5}{4}\varepsilon, \quad (35)$$

$$\left| \tilde{\tilde{s}}'(x) - y'(x) \right| \leq \frac{5}{48}M_4h^3 + \frac{1}{6}M_3h^2 + \frac{3}{h}\varepsilon, \quad (36)$$

$$\left| \tilde{\tilde{s}}''(x) - y''(x) \right| \leq \frac{5}{24}M_4h^2 + \frac{4}{h^2}\varepsilon. \quad (37)$$

*Proof.* We first prove (37).  $\tilde{\tilde{s}}(x)$  is a cubic spline; hence  $\tilde{\tilde{s}}''(x)$  is a piecewise continuous linear function over  $[a, b]$  with respect to the partition  $\Delta$ . Let  $L(x)$  be the piecewise linear interpolant to  $y''(x)$  with respect to  $\Delta$ . For  $j = 1, 2, \dots, n$ , let  $\tilde{\tilde{s}}_j''(x)$  and  $L_j(x)$  be the restriction of  $\tilde{\tilde{s}}''(x)$  and  $L(x)$  over  $[x_{j-1}, x_j]$ . Then we have

$$\tilde{\tilde{s}}_j''(x) = \tilde{\tilde{s}}_j''(x_{j-1}) \frac{x_j - x}{h} + \tilde{\tilde{s}}_j''(x_j) \frac{x - x_{j-1}}{h}, \quad (38)$$

$$L_j(x) = y''(x_{j-1}) \frac{x_j - x}{h} + y''(x_j) \frac{x - x_{j-1}}{h}.$$

For  $j = 1, 2, \dots, n$ , by (38) and (33), we get

$$\begin{aligned} \left| \tilde{\tilde{s}}_j''(x) - L_j(x) \right| &= \left| \left( \tilde{\tilde{s}}_j''(x_{j-1}) - y''(x_{j-1}) \right) \frac{x_j - x}{h} \right. \\ &\quad \left. + \left( \tilde{\tilde{s}}_j''(x_j) - y''(x_j) \right) \frac{x - x_{j-1}}{h} \right| \\ &\leq \frac{1}{12}M_4h^2 + \frac{4}{h^2}\varepsilon. \end{aligned} \quad (39)$$

Hence, for all  $x \in [a, b]$ , we have

$$\left| \tilde{\tilde{s}}''(x) - L(x) \right| \leq \frac{1}{12}M_4h^2 + \frac{4}{h^2}\varepsilon. \quad (40)$$

Moreover, by the piecewise linear polynomial interpolation theory [1, 3, 7], for all  $x \in [a, b]$ , we have

$$\left| y''(x) - L(x) \right| \leq \frac{1}{8}M_4h^2. \quad (41)$$

Then (37) follows immediately from (40) and (41).

Next, we prove (36). For  $j = 1, 2, \dots, n$ , let  $\tilde{\tilde{s}}_j'(x)$  be the restriction of  $\tilde{\tilde{s}}'(x)$  over  $[x_{j-1}, x_j]$ . Then for  $x \in [x_{j-1}, x_j]$ , by (32) and (37), we have

$$\left| \tilde{\tilde{s}}_j'(x) - y'(x) \right| = \begin{cases} \left| \int_{x_{j-1}}^x (\tilde{\tilde{s}}_j''(t) - y''(t)) dt + (\tilde{\tilde{s}}_j'(x_{j-1}) - y'(x_{j-1})) \right|, & x \in \left[ x_{j-1}, x_{j-1} + \frac{h}{2} \right] \\ \left| \int_x^{x_j} (\tilde{\tilde{s}}_j''(t) - y''(t)) dt + (\tilde{\tilde{s}}_j'(x_j) - y'(x_j)) \right|, & x \in \left[ x_{j-1} + \frac{h}{2}, x_j \right] \end{cases}$$

$$\begin{aligned} & \leq \begin{cases} \int_{x_{j-1}}^{x_{j-1}+h/2} |\tilde{s}_j''(t) - y''(t)| dt + |\tilde{s}_j'(x_{j-1}) - y'(x_{j-1})|, & x \in \left[ x_{j-1}, x_{j-1} + \frac{h}{2} \right] \\ \int_{x_{j-1}+h/2}^{x_j} |\tilde{s}_j''(t) - y''(t)| dt + |\tilde{s}_j'(x_j) - y'(x_j)|, & x \in \left[ x_{j-1} + \frac{h}{2}, x_j \right] \end{cases} \\ & \leq \frac{5}{48} M_4 h^3 + \frac{1}{6} M_3 h^2 + \frac{3}{h} \varepsilon. \end{aligned} \quad (42)$$

Finally, we prove (35). For every subinterval  $[x_{j-1}, x_j]$ ,  $j = 1, 2, \dots, n$ , we give

$$\begin{aligned} \alpha_{j1}(x) &= \frac{2(x - x_{j-1})^3 - 3h(x - x_{j-1})^2 + h^3}{h^3}, \\ \alpha_{j2}(x) &= \frac{-2(x - x_{j-1})^3 + 3h(x - x_{j-1})^2}{h^3}, \\ \beta_{j1}(x) &= \frac{(x - x_j)^2 (x - x_{j-1})}{h^2}, \\ \beta_{j2}(x) &= \frac{(x - x_j)(x - x_{j-1})^2}{h^2}. \end{aligned} \quad (43)$$

They are very useful in cubic Hermite interpolation, and we also have

$$\begin{aligned} \alpha_{j1}(x) &\geq 0, & \alpha_{j2}(x) &\geq 0, \\ \beta_{j1}(x) &\geq 0, & \beta_{j2}(x) &\leq 0, \\ \alpha_{j1}(x) + \alpha_{j2}(x) &= 1, \end{aligned} \quad (44)$$

$$\beta_{j1}(x) - \beta_{j2}(x) = \frac{(x_{j-1} - x)(x - x_j)}{h} \leq \frac{h}{4}.$$

Let  $\tilde{s}_j(x) = \tilde{s}(x)|_{[x_{j-1}, x_j]}$  be the restriction of  $\tilde{s}(x)$  over  $[x_{j-1}, x_j]$ ; then it can also be written as

$$\begin{aligned} \tilde{s}_j(x) &= \tilde{s}(x_{j-1}) \alpha_{j1}(x) + \tilde{s}(x_j) \alpha_{j2}(x) \\ &\quad + \tilde{s}'(x_{j-1}) \beta_{j1}(x) + \tilde{s}'(x_j) \beta_{j2}(x). \end{aligned} \quad (45)$$

Let

$$\begin{aligned} H_j(x) &= y(x_{j-1}) \alpha_{j1}(x) + y(x_j) \alpha_{j2}(x) \\ &\quad + y'(x_{j-1}) \beta_{j1}(x) + y'(x_j) \beta_{j2}(x) \end{aligned} \quad (46)$$

be the cubic Hermite interpolant of  $y(x)$  over  $[x_{j-1}, x_j]$ ; then for  $x \in [x_{j-1}, x_j]$ , by using (31), (32), (44), (45), and (46), we have

$$\begin{aligned} |y(x) - H_j(x)| &\leq \frac{1}{384} M_4 h^4, \\ |\tilde{s}_j(x) - H_j(x)| &= |(\tilde{s}(x_{j-1}) - y(x_{j-1})) \alpha_{j1}(x) \\ &\quad + (\tilde{s}(x_j) - y(x_j)) \alpha_{j2}(x) \\ &\quad + (\tilde{s}'(x_{j-1}) - y'(x_{j-1})) \beta_{j1}(x) \\ &\quad + (\tilde{s}'(x_j) - y'(x_j)) \beta_{j2}(x)| \\ &\leq \left( \frac{1}{6} M_2 h^2 + \varepsilon \right) + \frac{h}{4} \left( \frac{1}{6} M_3 h^2 + \frac{\varepsilon}{h} \right) \\ &= \frac{1}{24} M_3 h^3 + \frac{1}{6} M_2 h^2 + \frac{5}{4} \varepsilon. \end{aligned} \quad (47)$$

By the triangle inequality, we get (35).  $\square$

## 5. Numerical Tests and Discussions

*5.1. Numerical Tests.* In this section, we perform numerical tests by Matlab. The following examples

$$\begin{aligned} f_1(x) &= \frac{5}{1+x^2}, \quad x \in [-1, 1], \\ f_2(x) &= e^x, \quad x \in [-1, 1], \end{aligned} \quad (48)$$

are considered.

In every numerical test, the mesh size  $h$  and the measurement noise bound  $\varepsilon$  are both given. Because the measurement noises  $\varepsilon_i$  ( $i = 0, 1, \dots, n$ ) are random, we let  $\varepsilon_i = \varepsilon \cdot r_i$ , where  $r_i$  ( $i = 0, 1, \dots, n$ ) are random numbers and satisfy  $-1 \leq r_i \leq 1$ .

In Tables 5 and 6, Methods I-1, I-2, and I-3 represent Method I with two-point, three-point, and five-point approximate boundary derivatives, respectively. CSM represents the cubic spline method in [11].  $E_0$ ,  $E_1$ , and  $E_2$  are the maximum absolute error of the function, the first order derivative, and the second order derivative, respectively.

TABLE 5: Numerical results of  $f_1(x)$ .

		Method	I-1	I-2	I-3	II	CSM [11]
$\varepsilon = 10^{-4}$	$h = 0.2$	$E_0$	$7.8 \times 10^{-3}$	$1.6 \times 10^{-3}$	$5.9 \times 10^{-4}$	$6.0 \times 10^{-2}$	$4.4 \times 10^{-3}$
		$E_1$	$2.4 \times 10^{-1}$	$4.7 \times 10^{-2}$	$1.3 \times 10^{-2}$	$1.4 \times 10^{-1}$	$2.1 \times 10^{-2}$
		$E_2$	4.16	0.75	0.41	0.38	0.45
$\varepsilon = 10^{-4}$	$h = 0.1$	$E_0$	$2.0 \times 10^{-3}$	$1.3 \times 10^{-4}$	$1.2 \times 10^{-4}$	$1.6 \times 10^{-2}$	$3.4 \times 10^{-4}$
		$E_1$	$1.2 \times 10^{-1}$	$4.7 \times 10^{-3}$	$3.0 \times 10^{-3}$	$3.8 \times 10^{-2}$	$3.1 \times 10^{-3}$
		$E_2$	4.29	0.15	0.14	0.10	0.15
$\varepsilon = 10^{-3}$	$h = 0.2$	$E_0$	$8.0 \times 10^{-3}$	$1.8 \times 10^{-3}$	$1.3 \times 10^{-3}$	$6.0 \times 10^{-2}$	$4.5 \times 10^{-3}$
		$E_1$	$2.4 \times 10^{-1}$	$4.7 \times 10^{-2}$	$1.8 \times 10^{-2}$	$1.5 \times 10^{-1}$	$2.3 \times 10^{-2}$
		$E_2$	4.17	0.75	0.43	0.39	0.45
$\varepsilon = 10^{-3}$	$h = 0.1$	$E_0$	$2.2 \times 10^{-3}$	$1.0 \times 10^{-3}$	$9.1 \times 10^{-4}$	$1.6 \times 10^{-2}$	$9.4 \times 10^{-4}$
		$E_1$	$1.2 \times 10^{-1}$	$1.6 \times 10^{-2}$	$1.0 \times 10^{-2}$	$4.0 \times 10^{-2}$	$1.1 \times 10^{-2}$
		$E_2$	4.29	0.65	0.42	0.29	0.44

TABLE 6: Numerical results of  $f_2(x)$ .

		Method	I-1	I-2	I-3	II	CSM [11]
$\varepsilon = 10^{-5}$	$h = 0.2$	$E_0$	$8.0 \times 10^{-3}$	$9.8 \times 10^{-4}$	$1.7 \times 10^{-5}$	$1.3 \times 10^{-2}$	$8.2 \times 10^{-5}$
		$E_1$	$6.8 \times 10^{-2}$	$8.3 \times 10^{-3}$	$2.3 \times 10^{-4}$	$1.4 \times 10^{-2}$	$1.1 \times 10^{-4}$
		$E_2$	1.17	0.13	$6.8 \times 10^{-3}$	$7.3 \times 10^{-3}$	$7.3 \times 10^{-3}$
$\varepsilon = 10^{-5}$	$h = 0.1$	$E_0$	$2.0 \times 10^{-3}$	$1.3 \times 10^{-4}$	$1.0 \times 10^{-5}$	$3.9 \times 10^{-3}$	$1.2 \times 10^{-5}$
		$E_1$	$3.5 \times 10^{-2}$	$2.2 \times 10^{-3}$	$2.7 \times 10^{-4}$	$4.1 \times 10^{-3}$	$1.1 \times 10^{-4}$
		$E_2$	1.21	$7.5 \times 10^{-2}$	$8.3 \times 10^{-3}$	$3.6 \times 10^{-3}$	$8.4 \times 10^{-3}$
$\varepsilon = 10^{-4}$	$h = 0.2$	$E_0$	$8.0 \times 10^{-3}$	$9.8 \times 10^{-4}$	$1.0 \times 10^{-4}$	$1.3 \times 10^{-2}$	$1.2 \times 10^{-4}$
		$E_1$	$6.8 \times 10^{-2}$	$8.3 \times 10^{-3}$	$1.4 \times 10^{-3}$	$1.4 \times 10^{-2}$	$4.8 \times 10^{-4}$
		$E_2$	1.17	0.13	0.019	0.010	0.021
$\varepsilon = 10^{-4}$	$h = 0.1$	$E_0$	$2.0 \times 10^{-3}$	$1.5 \times 10^{-4}$	$7.3 \times 10^{-5}$	$3.9 \times 10^{-3}$	$8.5 \times 10^{-5}$
		$E_1$	$3.5 \times 10^{-2}$	$2.6 \times 10^{-3}$	$2.9 \times 10^{-3}$	$4.0 \times 10^{-3}$	$1.0 \times 10^{-3}$
		$E_2$	1.22	0.08	0.08	0.03	0.09

5.2. *Discussions.* Generally, the maximum absolute errors  $E_0$ ,  $E_1$ , and  $E_2$  vary if one of  $\varepsilon$  and  $h$  does. If  $h$  is fixed and  $\varepsilon$  decreases, then the maximum absolute errors  $E_0$ ,  $E_1$ , and  $E_2$  will decrease. But if  $\varepsilon$  is fixed while  $h$  decreases, the errors will not decrease necessarily; they maybe increase sometimes. See the theoretical results in Theorem 5 and Theorem 7 and the numerical results in Tables 5 and 6.

When  $h$  and  $\varepsilon$  are both fixed in a specific test, it is easy to find that  $E_0$  and  $E_1$  of Method I-2 and Method I-3 are better than those of Method I-1 and Method II, while  $E_2$  of Method I-3 and Method II are better than Method I-1 and Method I-2. See Tables 5 and 6.

It is very reasonable to compare our methods with the cubic spline method (CSM) in [11] because our methods are also based on cubic spline. From Tables 5 and 6, we find that the errors of Method I-3 are overall better than CSM in [11]. At the same time,  $E_2$  of Method II are better than CSM in [11]. In summary, when approximating a function, we advise using Method I-3, Method I-2, and CSM [11]; when approximating its first order derivative, we advise using Method I-3 and CSM [11]; when approximating its second order derivative, we advise using Method I-3, Method II, and CSM [11].

When  $\varepsilon = 0$ , Method I-1 and Method II are  $O(h^2)$  methods, Method I-2 is an  $O(h^3)$  method, and Method I-3 is an  $O(h^4)$  method. The cubic spline method (CSM) [11] is also an  $O(h^4)$  method, while the method in [12] is an  $O(h^2)$  method; the method in [13] is an  $O(h^{2.5} \log h)$  method conditionally, only if the shape parameter  $c = O(h)$  therein. Obviously, the approximation orders of Method I-2 and Method I-3 are higher than the methods in [12, 13], the approximation orders of Method I-1 and Method II equal that of the method in [12], and the approximation order of Method I-3 equals that of CSM [11]. Undoubtedly, our methods are full of approximation ability. Furthermore, [12, 13] have not studied first order and second order derivative approximations. At the same time, our methods are more suitable for noisy data than the methods in [12, 13]. Hence, Method I-2, Method I-3, and CSM [11] are more preferable than others.

### 6. Conclusions

The explicit error bounds for a noisy lacunary cubic spline interpolation and a simple noisy cubic spline quasi interpolation are well studied in this paper; see Theorems 5 and 7.

These new results are very useful in numerical approximation and related practical fields. Moreover, these results are also verified by some numerical examples. In a word, both theoretical analysis and numerical tests show that our methods are well behaved. We end the paper with the following remarks.

- (i) The main contributions of the paper include (i) studying two new methods to approximate a function and its first order and second order derivatives from the given noisy data and (ii) analyzing the explicit error bounds for the methods.
- (ii) The main advantages of our new methods include the following: (i) they are very simple; (ii) they are not only applicable to noisy data but also applicable to exact data; (iii) Method I-2 and Method I-3 have better performance in function approximation and first order derivative approximation than other methods; Method I-3 and Method II have better performance in second order derivative approximation than other methods.

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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