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Research Article

Spectral Theory for a Mathematical Model of the Weak Interaction—Part I: The Decay of the Intermediate Vector Bosons $W^{\pm}$

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We consider a Hamiltonian with cutoffs describing the weak decay of spin 1 massive bosons into the full family of leptons. The Hamiltonian is a self-adjoint operator in an appropriate Fock space with a unique ground state. We prove a Mourre estimate and a limiting absorption principle above the ground state energy and below the first threshold for a sufficiently small coupling constant. As a corollary, we prove the absence of eigenvalues and absolute continuity of the energy spectrum in the same spectral interval.

1. Introduction

In this article, we consider a mathematical model of the weak interaction as patterned according to the Standard Model in Quantum Field Theory (see [1, 2]). We choose the example of the weak decay of the intermediate vector bosons $W^{\pm}$ into the full family of leptons.

The mathematical framework involves fermionic Fock spaces for the leptons and bosonic Fock spaces for the vector bosons. The interaction is described in terms of annihilation and creation operators together with kernels which are square integrable with respect to momenta. The total Hamiltonian, which is the sum of the free energy of the particles and antiparticles and of the interaction, is a self-adjoint operator in the Fock space for the leptons and the vector bosons and it has an unique ground state in the Fock space for a sufficiently small coupling constant.

The weak interaction is one of the four fundamental interactions known up to now. But the weak interaction is the only one which does not generate bound states. As it is well known, it is not the case for the strong, electromagnetic, and gravitational interactions. Thus we are expecting that the spectrum of the Hamiltonian associated with every model of weak decays is absolutely continuous above the energy of the ground state, and this article is a first step towards a proof of such a statement. Moreover a scattering theory has to be established for every such Hamiltonian.

In this paper we establish a Mourre estimate and a limiting absorption principle for any spectral interval above the energy of the ground state and below the mass of the electron for a small coupling constant.

Our study of the spectral analysis of the total Hamiltonian is based on the conjugate operator method with a self-adjoint conjugate operator. The methods used in this article are taken largely from [3, 4] and are based on [5, 6]. Some of the results of this article have been announced in [7].

For other applications of the conjugate operator method see [8–19].

For related results about models in Quantum Field Theory see [20, 21] in the case of the Quantum Electrodynamics and [22] in the case of the weak interaction.

The paper is organized as follows. In Section 2, we give a precise definition of the model we consider. In Section 3, we state our main results and in the following sections, together with the appendix, detailed proofs of the results are given.

2. The Model

The weak decay of the intermediate bosons W^+ and W^- involves the full family of leptons together with the bosons themselves, according to the Standard Model (see [1, formula (4.139)] and [2]).

The full family of leptons involves the electron e^- and the positron e^+, together with the associated neutrino ν_e and antineutrino $\overline{\nu}_e$, the muons μ^- and μ^+ together with the associated neutrino ν_μ and antineutrino $\overline{\nu}_\mu$, and the tau leptons τ^- and τ^+ together with the associated neutrino ν_τ and antineutrino $\overline{\nu}_\tau$.

It follows from the Standard Model that neutrinos and antineutrinos are massless particles. Neutrinos are left handed, that is, neutrinos have helicity $-1/2$ and antineutrinos are right handed, that is, antineutrinos have helicity $+1/2$.

In what follows, the mathematical model for the weak decay of the vector bosons W^+ and W^- that we propose is based on the Standard Model, but we adopt a slightly more general point of view because we suppose that neutrinos and antineutrinos are both massless particles with helicity $\pm 1/2$. We recover the physical situation as a particular case. We could also consider a model with massive neutrinos and antineutrinos built upon the Standard Model with neutrino mixing [23].

Let us sketch how we define a mathematical model for the weak decay of the vector bosons $W^\pm$ into the full family of leptons.

The energy of the free leptons and bosons is a self-adjoint operator in the corresponding Fock space (see below), and the main problem is associated with the interaction between the bosons and the leptons. Let us consider only the interaction between the bosons and the electrons, the positrons, and the corresponding neutrinos and antineutrinos. Other cases are strictly similar. In the Schrödinger representation the interaction is given by (see [1, page 159, equation (4.139)] and [2, page 308, equation (21.3.20)])

$$I = \int \mathrm{d}^3x\overline{\Psi_e}(x)\gamma^\alpha(1-\gamma_5)\Psi_{\nu_e}(x)W_\alpha(x) + \int \mathrm{d}^3x\overline{\Psi_{\nu_e}}(x)\gamma^\alpha(1-\gamma_5)\Psi_e(x)W_\alpha(x)^*, \tag{2.1}$$

where γ^α, $\alpha = 0,1,2,3$ and γ_5 are the Dirac matrices and $\Psi_\cdot(x)$ and $\overline{\Psi_\cdot}(x)$ are the Dirac fields for e_-, e_+, ν_e, and $\overline{\nu}_e$.

We have

$$\Psi_e(x) = \left(\frac{1}{2\pi}\right)^{3/2} \sum_{s=\pm 1/2} \int \mathrm{d}^3 p \left(b_{e,+}(p,s) \frac{u(p,s)}{\sqrt{p_0}} \mathrm{e}^{ip\cdot x} + b^*_{e,-}(p,s) \frac{v(p,s)}{\sqrt{p_0}} \mathrm{e}^{-ip\cdot x} \right),$$
$$\overline{\Psi_e}(x) = \Psi_e(x)^\dagger \gamma^0. \tag{2.2}$$

Here $p_0 = (|p|^2 + m_e^2)^{1/2}$ where $m_e > 0$ is the mass of the electron, and $u(p,s)$ and $v(p,s)$ are the normalized solutions to the Dirac equation (see [1, Appendix]).

The operators $b_{e,+}(p,s)$ and $b^*_{e,+}(p,s)$ (resp., $b_{e,-}(p,s)$ and $b^*_{e,-}(p,s)$) are the annihilation and creation operators for the electrons (resp., the positrons) satisfying the anticommutation relations (see below).

Similarly we define $\Psi_{\nu_e}(x)$ and $\overline{\Psi_{\nu_e}}(x)$ by substituting the operators $c_{\nu_e,\pm}(p,s)$ and $c^*_{\nu_e,\pm}(p,s)$ for $b_{e,\pm}(p,s)$ and $b^*_{e,\pm}(p,s)$ with $p_0 = |p|$. The operators $c_{\nu_e,+}(p,s)$ and $c^*_{\nu_e,+}(p,s)$ (resp., $c_{\nu_e,-}(p,s)$ and $c^*_{\nu_e,-}(p,s)$) are the annihilation and creation operators for the neutrinos associated with the electrons (resp., the antineutrinos).

For the W_α fields we have (see [24, Section 5.3])

$$W_\alpha(x) = \left(\frac{1}{2\pi}\right)^{3/2} \sum_{\lambda=-1,0,1} \int \frac{\mathrm{d}^3 k}{\sqrt{2k_0}} \left(\epsilon_\alpha(k,\lambda) a_+(k,\lambda) \mathrm{e}^{ik\cdot x} + \epsilon^*_\alpha(k,\lambda) a^*_-(k,\lambda) \mathrm{e}^{-ik\cdot x} \right). \tag{2.3}$$

Here $k_0 = (|k|^2 + m_W^2)^{1/2}$ where $m_W > 0$ is the mass of the bosons $W^\pm$. W^+ is the antiparticule of W^-. The operators $a_+(k,\lambda)$ and $a^*_+(k,\lambda)$ (resp., $a_-(k,\lambda)$ and $a^*_-(k,\lambda)$) are the annihilation and creation operators for the bosons W^- (resp., W^+) satisfying the canonical commutation relations. The vectors $\epsilon_\alpha(k,\lambda)$ are the polarizations of the massive spin 1 bosons $W^\pm$ (see [24, Section 5.2]).

The interaction (2.1) is a formal operator and, in order to get a well-defined operator in the Fock space, one way is to adapt what Glimm and Jaffe have done in the case of the Yukawa Hamiltonian (see [25]). For that sake, we have to introduce a spatial cutoff $g(x)$ such that $g \in L^1(\mathbb{R}^3)$, together with momentum cutoffs $\chi(p)$ and $\rho(k)$ for the Dirac fields and the W_μ fields, respectively.

Thus when one develops the interaction I with respect to products of creation and annihilation operators, one gets a finite sum of terms associated with kernels of the form

$$\chi(p_1)\chi(p_2)\rho(k)\widehat{g}(p_1 + p_2 - k), \tag{2.4}$$

where $\widehat{g}$ is the Fourier transform of g. These kernels are square integrable.

In what follows, we consider a model involving terms of the above form but with more general square integrable kernels.

We follow the convention described in [24, Section 4.1] that we quote: "The state-vector will be taken to be symmetric under interchange of any bosons with each other, or any bosons with any fermions, and antisymmetric with respect to interchange of any two fermions with each other, in all cases, whether the particles are of the same species or not."

Thus, as it follows from [24, Section 4.2], fermionic creation and annihilation operators of different species of leptons will always anticommute.

Concerning our notations, from now on, $\ell \in \{1,2,3\}$ denotes each species of leptons. $\ell = 1$ denotes the electron e^- the positron e^+ and the neutrinos ν_e, $\overline{\nu}_e$. $\ell = 2$ denotes the muons μ^-, μ^+ and the neutrinos ν_μ and $\overline{\nu}_\mu$, and $\ell = 3$ denotes the tau-leptons and the neutrinos ν_τ and $\overline{\nu}_\tau$.

Let $\xi_1 = (p_1, s_1)$ be the quantum variables of a massive lepton, where $p_1 \in \mathbb{R}^3$ and $s_1 \in \{-1/2, 1/2\}$ is the spin polarization of particles and antiparticles. Let $\xi_2 = (p_2, s_2)$ be the quantum variables of a massless lepton where $p_2 \in \mathbb{R}^3$ and $s_2 \in \{-1/2, 1/2\}$ is the helicity of particles and antiparticles, and, finally, let $\xi_3 = (k, \lambda)$ be the quantum variables of the spin 1 bosons W^+ and W^- where $k \in \mathbb{R}^3$ and $\lambda \in \{-1, 0, 1\}$ is the polarization of the vector bosons (see [24, Section 5]). We set $\Sigma_1 = \mathbb{R}^3 \times \{-1/2, 1/2\}$ for the leptons and $\Sigma_2 = \mathbb{R}^3 \times \{-1, 0, 1\}$ for the bosons. Thus $L^2(\Sigma_1)$ is the Hilbert space of each lepton and $L^2(\Sigma_2)$ is the Hilbert space of each boson. The scalar product in $L^2(\Sigma_j)$, $j = 1, 2$ is defined by

$$(f, g) = \int_{\Sigma_j} \overline{f(\xi)} g(\xi) \mathrm{d}\xi, \quad j = 1, 2. \tag{2.5}$$

Here

$$\int_{\Sigma_1} \mathrm{d}\xi = \sum_{s=+1/2,-1/2} \int \mathrm{d}p, \quad \int_{\Sigma_2} \mathrm{d}\xi = \sum_{\lambda=0,1,-1} \int \mathrm{d}k, \quad \left(p, k \in \mathbb{R}^3\right). \tag{2.6}$$

The Hilbert space for the weak decay of the vector bosons W^+ and W^- is the Fock space for leptons and bosons that we now describe.

Let $\mathfrak{S}$ be any separable Hilbert space. Let $\bigotimes_a^n \mathfrak{S}$ (resp., $\bigotimes_s^n \mathfrak{S}$) denote the antisymmetric (resp., symmetric) nth tensor power of $\mathfrak{S}$. The fermionic (resp., bosonic) Fock space over $\mathfrak{S}$, denoted by $\mathfrak{F}_a(\mathfrak{S})$ (resp., $\mathfrak{F}_s(\mathfrak{S})$), is the direct sum

$$\mathfrak{F}_a(\mathfrak{S}) = \bigoplus_{n=0}^{\infty} \bigotimes_a^n \mathfrak{S} \quad \left(\text{resp., } \mathfrak{F}_s(\mathfrak{S}) = \bigoplus_{n=0}^{\infty} \bigotimes_s^n \mathfrak{S}\right), \tag{2.7}$$

where $\bigotimes_a^0 \mathfrak{S} = \bigotimes_s^0 \mathfrak{S} \equiv \mathbb{C}$. The state $\Omega = (1, 0, 0, \ldots, 0, \ldots)$ denotes the vacuum state in $\mathfrak{F}_a(\mathfrak{S})$ and in $\mathfrak{F}_s(\mathfrak{S})$.

For every ℓ, $\mathfrak{F}_\ell$ is the fermionic Fock space for the corresponding species of leptons including the massive particle and antiparticle together with the associated neutrino and antineutrino, that is,

$$\mathfrak{F}_\ell = \bigotimes^4 \mathfrak{F}_a\left(L^2(\Sigma_1)\right) \quad \ell = 1, 2, 3. \tag{2.8}$$

We have

$$\mathfrak{F}_\ell = \bigoplus_{q_\ell \geq 0, \overline{q}_\ell \geq 0, r_\ell \geq 0, \overline{r}_\ell \geq 0} \mathfrak{F}_\ell^{(q_\ell, \overline{q}_\ell, r_\ell, \overline{r}_\ell)} \tag{2.9}$$

with

$$\mathfrak{F}_\ell^{(q_\ell,\overline{q}_\ell,r_\ell,\overline{r}_\ell)} = \left(\bigotimes_a^{q_\ell} L^2(\Sigma_1)\right) \otimes \left(\bigotimes_a^{\overline{q}_\ell} L^2(\Sigma_1)\right) \otimes \left(\bigotimes_a^{r_\ell} L^2(\Sigma_1)\right) \otimes \left(\bigotimes_a^{\overline{r}_\ell} L^2(\Sigma_1)\right). \tag{2.10}$$

Here q_ℓ (resp., $\overline{q}_\ell$) is the number of massive fermionic particle (resp., antiparticles) and r_ℓ (resp., $\overline{r}_\ell$) is the number of neutrinos (resp., antineutrinos). The vector Ω_ℓ is the associated vacuum state. The fermionic Fock space denoted by $\mathfrak{F}_L$ for the leptons is then

$$\mathfrak{F}_L = \bigotimes_{\ell=1}^{3} \mathfrak{F}_\ell, \tag{2.11}$$

and $\Omega_L = \bigotimes_{\ell=1}^3 \Omega_\ell$ is the vacuum state.

The bosonic Fock space for the vector bosons W^+ and W^-, denoted by $\mathfrak{F}_W$, is then

$$\mathfrak{F}_W = \mathfrak{F}_s\left(L^2(\Sigma_2)\right) \otimes \mathfrak{F}_s\left(L^2(\Sigma_2)\right) \simeq \mathfrak{F}_s\left(L^2(\Sigma_2) \oplus L^2(\Sigma_2)\right). \tag{2.12}$$

We have

$$\mathfrak{F}_W = \bigoplus_{t\geq 0, \bar{t}\geq 0} \mathfrak{F}_W^{(t,\bar{t})}, \tag{2.13}$$

where $\mathfrak{F}_W^{(t,\bar{t})} = (\bigotimes_s^t L^2(\Sigma_2)) \otimes (\bigotimes_s^{\bar{t}} L^2(\Sigma_2))$. Here t (resp., $\bar{t}$) is the number of bosons W^- (resp., W^+). The vector Ω_W is the corresponding vacuum.

The Fock space for the weak decay of the vector bosons W^+ and W^-, denoted by $\mathfrak{F}$, is thus

$$\mathfrak{F} = \mathfrak{F}_L \otimes \mathfrak{F}_W, \tag{2.14}$$

and $\Omega = \Omega_L \otimes \Omega_W$ is the vacuum state.

For every $\ell \in \{1,2,3\}$ let $\mathfrak{D}_\ell$ denote the set of smooth vectors $\psi_\ell \in \mathfrak{F}_\ell$ for which $\psi_\ell^{(q_\ell,\overline{q}_\ell,r_\ell,\overline{r}_\ell)}$ has a compact support and $\psi_\ell^{(q_\ell,\overline{q}_\ell,r_\ell,\overline{r}_\ell)} = 0$ for all but finitely many $(q_\ell, \overline{q}_\ell, r_\ell, \overline{r}_\ell)$. Let

$$\mathfrak{D}_L = \widehat{\bigotimes}_{\ell=1}^{3} \mathfrak{D}_\ell. \tag{2.15}$$

Here $\widehat{\bigotimes}$ is the algebraic tensor product.

Let $\mathfrak{D}_W$ denote the set of smooth vectors $\phi \in \mathfrak{F}_W$ for which $\phi^{(t,\bar{t})}$ has a compact support and $\phi^{(t,\bar{t})} = 0$ for all but finitely many $(t, \bar{t})$.

Let

$$\mathfrak{D} = \mathfrak{D}_L \widehat{\otimes} \mathfrak{D}_W. \tag{2.16}$$

The set $\mathfrak{D}$ is dense in $\mathfrak{F}$.

Let A_ℓ be a self-adjoint operator in $\mathfrak{F}_\ell$ such that $\mathfrak{D}_\ell$ is a core for A_ℓ. Its extension to $\mathfrak{F}_L$ is, by definition, the closure in $\mathfrak{F}_L$ of the operator $A_1 \otimes 1_2 \otimes 1_3$ with domain $\mathfrak{D}_L$ when $\ell = 1$, of the operator $1_1 \otimes A_2 \otimes 1_3$ with domain $\mathfrak{D}_L$ when $\ell = 2$, and of the operator $1_1 \otimes 1_2 \otimes A_3$ with domain $\mathfrak{D}_L$ when $\ell = 3$. Here 1_ℓ is the operator identity on $\mathfrak{F}_\ell$.

The extension of A_ℓ to $\mathfrak{F}_L$ is a self-adjoint operator for which $\mathfrak{D}_L$ is a core and it can be extended to $\mathfrak{F}$. The extension of A_ℓ to $\mathfrak{F}$ is, by definition, the closure in $\mathfrak{F}$ of the operator $\widetilde{A}_\ell \otimes 1_W$ with domain $\mathfrak{D}$, where $\widetilde{A}_\ell$ is the extension of A_ℓ to $\mathfrak{F}_L$. The extension of A_ℓ to $\mathfrak{F}$ is a self-adjoint operator for which $\mathfrak{D}$ is a core.

Let B be a self-adjoint operator in $\mathfrak{F}_W$ for which $\mathfrak{D}_W$ is a core. The extension of the self-adjoint operator $A_\ell \otimes B$ is, by definition, the closure in $\mathfrak{F}$ of the operator $A_1 \otimes 1_2 \otimes 1_3 \otimes B$ with domain $\mathfrak{D}$ when $\ell = 1$, of the operator $1_1 \otimes A_2 \otimes 1_3 \otimes B$ with domain $\mathfrak{D}$ when $\ell = 2$, and of the operator $1_1 \otimes 1_2 \otimes A_3 \otimes B$ with domain $\mathfrak{D}$ when $\ell = 3$. The extension of $A_\ell \otimes B$ to $\mathfrak{F}$ is a self-adjoint operator for which $\mathfrak{D}$ is a core.

We now define the creation and annihilation operators.

For each $\ell = 1, 2, 3$, $b_{\ell,\epsilon}(\xi_1)$ (resp., $b^*_{\ell,\epsilon}(\xi_1)$) is the fermionic annihilation (resp., fermionic creation) operator for the corresponding species of massive particle when $\epsilon = +$ and for the corresponding species of massive antiparticle when $\epsilon = -$. The operators $b_{\ell,\epsilon}(\xi_1)$ and $b^*_{\ell,\epsilon}(\xi_1)$ are defined as usually (see, e.g., [20, 26]; see also the detailed definitions in [27]).

Similarly, for each $\ell = 1, 2, 3$, $c_{\ell,\epsilon}(\xi_2)$ (resp., $c^*_{\ell,\epsilon}(\xi_2)$) is the fermionic annihilation (resp., fermionic creation) operator for the corresponding species of neutrino when $\epsilon = +$ and for the corresponding species of antineutrino when $\epsilon = -$. The operators $c_{\ell,\epsilon}(\xi_2)$ and $c^*_{\ell,\epsilon}(\xi_2)$ are defined in a standard way, but with the additional requirements that for any ℓ, ℓ', ϵ and ϵ', the operators $b^\sharp_{\ell,\epsilon}(\xi_1)$ and $c^\sharp_{\ell',\epsilon'}(\xi_2)$ anticommutes, where $\sharp$ stands either for a $*$ or for no symbol (see the detailed definitions in [27]).

The operator $a_\epsilon(\xi_3)$ (resp., $a^*_\epsilon(\xi_3)$) is the bosonic annihilation (resp., bosonic creation) operator for the boson W^- when $\epsilon = +$ and for the boson W^+ when $\epsilon = -$ (see, e.g., [20, 26], or [27]). Note that $a^\sharp(\xi_3)$ commutes with $b^\sharp_{\ell,\epsilon}(\xi_1)$ and $c^\sharp_{\ell',\epsilon'}(\xi_2)$.

The following canonical anticommutation and commutation relations hold:

$$\begin{gathered}
\left\{b_{\ell,\epsilon}(\xi_1), b^*_{\ell',\epsilon'}(\xi'_1)\right\} = \delta_{\ell\ell'}\delta_{\epsilon\epsilon'}\delta(\xi_1 - \xi'_1), \\
\left\{c_{\ell,\epsilon}(\xi_2), c^*_{\ell',\epsilon'}(\xi'_2)\right\} = \delta_{\ell\ell'}\delta_{\epsilon\epsilon'}\delta(\xi_2 - \xi'_2), \\
\left[a_\epsilon(\xi_3), a^*_{\epsilon'}(\xi'_3)\right] = \delta_{\epsilon\epsilon'}\delta(\xi_3 - \xi'_3), \\
\left\{b_{\ell,\epsilon}(\xi_1), b_{\ell',\epsilon'}(\xi'_1)\right\} = \left\{c_{\ell,\epsilon}(\xi_2), c_{\ell',\epsilon'}(\xi'_2)\right\} = 0, \\
\left[a_\epsilon(\xi_3), a_{\epsilon'}(\xi'_3)\right] = 0, \\
\left\{b_{\ell,\epsilon}(\xi_1), c_{\ell',\epsilon'}(\xi_2)\right\} = \left\{b_{\ell,\epsilon}(\xi_1), c^*_{\ell',\epsilon'}(\xi_2)\right\} = 0, \\
\left[b_{\ell,\epsilon}(\xi_1), a_{\epsilon'}(\xi_3)\right] = \left[b_{\ell,\epsilon}(\xi_1), a^*_{\epsilon'}(\xi_3)\right] = \left[c_{\ell,\epsilon}(\xi_2), a_{\epsilon'}(\xi_3)\right] = \left[c_{\ell,\epsilon}(\xi_2), a^*_{\epsilon'}(\xi_3)\right] = 0,
\end{gathered} \tag{2.17}$$

where we used the notation $\delta(\xi_j - \xi'_j) = \delta_{\lambda\lambda'}\delta(k - k')$.

We recall that the following operators, with $\varphi \in L^2(\Sigma_1)$,

$$\begin{aligned} b_{\ell,\epsilon}(\varphi) &= \int_{\Sigma_1} b_{\ell,\epsilon}(\xi)\overline{\varphi(\xi)}\mathrm{d}\xi, \qquad c_{\ell,\epsilon}(\varphi) = \int_{\Sigma_1} c_{\ell,\epsilon}(\xi)\overline{\varphi(\xi)}\mathrm{d}\xi, \\ b^*_{\ell,\epsilon}(\varphi) &= \int_{\Sigma_1} b^*_{\ell,\epsilon}(\xi)\varphi(\xi)\mathrm{d}\xi, \qquad c^*_{\ell,\epsilon}(\varphi) = \int_{\Sigma_1} c^*_{\ell,\epsilon}(\xi)\varphi(\xi)\mathrm{d}\xi \end{aligned} \tag{2.18}$$

are bounded operators in $\mathfrak{F}$ such that

$$\left\| b^\sharp_{\ell,\epsilon}(\varphi) \right\| = \left\| c^\sharp_{\ell,\epsilon}(\varphi) \right\| = \|\varphi\|_{L^2}, \tag{2.19}$$

where $b^\sharp$ (resp., $c^\sharp$) is b (resp., c) or b^* (resp., c^*).

The operators $b^\sharp_{\ell,\epsilon}(\varphi)$ and $c^\sharp_{\ell,\epsilon}(\varphi)$ satisfy similar anticommutaion relations (see, e.g., [28]).

The free Hamiltonian H_0 is given by

$$\begin{aligned} H_0 &= H_0^{(1)} + H_0^{(2)} + H_0^{(3)} \\ &= \sum_{\ell=1}^{3}\sum_{\epsilon=\pm} \int w_\ell^{(1)}(\xi_1) b^*_{\ell,\epsilon}(\xi_1) b_{\ell,\epsilon}(\xi_1)\mathrm{d}\xi_1 + \sum_{\ell=1}^{3}\sum_{\epsilon=\pm} \int w_\ell^{(2)}(\xi_2) c^*_{\ell,\epsilon}(\xi_2) c_{\ell,\epsilon}(\xi_2)\mathrm{d}\xi_2 \\ &\quad + \sum_{\epsilon=\pm} \int w^{(3)}(\xi_3) a^*_\epsilon(\xi_3) a_\epsilon(\xi_3)\mathrm{d}\xi_3, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} w_\ell^{(1)}(\xi_1) &= \left(|p_1|^2 + m_\ell^2\right)^{1/2}, \quad \text{with } 0 < m_1 < m_2 < m_3, \\ w_\ell^{(2)}(\xi_2) &= |p_2|, \\ w^{(3)}(\xi_3) &= \left(|k|^2 + m_W^2\right)^{1/2}, \end{aligned} \tag{2.21}$$

where m_W is the mass of the bosons W^+ and W^- such that $m_W > m_3$.

The spectrum of H_0 is $[0,\infty)$ and 0 is a simple eigenvalue with Ω as eigenvector. The set of thresholds of H_0, denoted by T, is given by

$$T = \left\{ pm_1 + qm_2 + rm_3 + sm_W;\ (p,q,r,s) \in \mathbb{N}^4,\ p+q+r+s \geq 1 \right\}, \tag{2.22}$$

and each set $[t,\infty)$, $t \in T$, is a branch of absolutely continuous spectrum for H_0.

The interaction, denoted by H_I, is given by

$$H_I = \sum_{\alpha=1}^{2} H_I^{(\alpha)}, \tag{2.23}$$

where

$$
\begin{aligned}
H_I^{(1)} &= \sum_{\ell=1}^{3} \sum_{\epsilon \neq \epsilon'} \int G^{(1)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3) b^*_{\ell,\epsilon}(\xi_1) c^*_{\ell,\epsilon'}(\xi_2) a_\epsilon(\xi_3) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \\
&\quad + \sum_{\ell=1}^{3} \sum_{\epsilon \neq \epsilon'} \int \overline{G^{(1)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3)} a^*_\epsilon(\xi_3) c_{\ell,\epsilon'}(\xi_2) b_{\ell,\epsilon}(\xi_1) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3, \\
H_I^{(2)} &= \sum_{\ell=1}^{3} \sum_{\epsilon \neq \epsilon'} \int G^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3) b^*_{\ell,\epsilon}(\xi_1) c^*_{\ell,\epsilon'}(\xi_2) a^*_\epsilon(\xi_3) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \\
&\quad + \sum_{\ell=1}^{3} \sum_{\epsilon \neq \epsilon'} \int \overline{G^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3)} a_\epsilon(\xi_3) c_{\ell,\epsilon'}(\xi_2) b_{\ell,\epsilon}(\xi_1) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3.
\end{aligned}
\tag{2.24}
$$

The kernels $G^{(2)}_{\ell,\epsilon,\epsilon'}(\cdot,\cdot,\cdot)$, $\alpha = 1, 2$, are supposed to be functions.

The total Hamiltonian is then

$$
H = H_0 + gH_I, \quad g > 0, \tag{2.25}
$$

where g is a coupling constant.

The operator $H_I^{(1)}$ describes the decay of the bosons W^+ and W^- into leptons. Because of $H_I^{(2)}$ the bare vacuum will not be an eigenvector of the total Hamiltonian for every $g > 0$ as we expect from the physics.

Every kernel $G_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3)$, computed in theoretical physics, contains a δ-distribution because of the conservation of the momentum (see [1] and [24, Section 4.4]). In what follows, we approximate the singular kernels by square integrable functions.

Thus, from now on, the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ are supposed to satisfy the following hypothesis.

Hypothesis 2.1. For $\alpha = 1, 2$, $\ell = 1, 2, 3$, $\epsilon, \epsilon' = \pm$, we assume

$$
G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3) \in L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2). \tag{2.26}
$$

Remark 2.2. A similar model can be written down for the weak decay of pions π^- and π^+ (see [1, Section 6.2]).

Remark 2.3. The total Hamiltonian is more general than the one involved in the theory of weak interactions because, in the Standard Model, neutrinos have helicity $-1/2$ and antineutrinos have helicity $1/2$.

In the physical case, the Fock space, denoted by $\mathfrak{F}'$, is isomorphic to $\mathfrak{F}'_L \otimes \mathfrak{F}_W$, with

$$
\begin{aligned}
\mathfrak{F}'_L &= \bigotimes_{\ell=1}^{3} \mathfrak{F}'_\ell, \\
\mathfrak{F}'_\ell &= \left(\bigotimes_a^2 L^2(\Sigma_1) \right) \otimes \left(\bigotimes_a^2 L^2\left(\mathbb{R}^3\right) \right).
\end{aligned}
\tag{2.27}
$$

The free Hamiltonian, now denoted by H_0', is then given by

$$H_0' = \sum_{\ell=1}^{3}\sum_{\epsilon=\pm}\int w_\ell^{(1)}(\xi_1)b_{\ell,\epsilon}^*(\xi_1)b_{\ell,\epsilon}(\xi_1)d\xi_1 + \sum_{\ell=1}^{3}\sum_{\epsilon=\pm}\int_{\mathbb{R}^3}|p_2|c_{\ell,\epsilon}^*(p_2)c_{\ell,\epsilon}(p_2)dp_2 + \sum_{\epsilon=\pm}\int w^{(3)}(\xi_3)a_\epsilon^*(\xi_3)a_\epsilon(\xi_3)d\xi_3, \tag{2.28}$$

and the interaction, now denoted by H_I', is the one obtained from H_I by supposing that $G^{(\alpha)}(\xi_1,(p_2,s_2),\xi_3) = 0$ if $s_2 = \epsilon(1/2)$. The total Hamiltonian, denoted by H', is then given by $H' = H_0' + gH_I'$. The results obtained in this paper for H hold true for H' with obvious modifications.

Under Hypothesis 2.1 a well-defined operator on $\mathfrak{D}$ corresponds to the formal interaction H_I as it follows.

The formal operator

$$\int G_{\ell,\epsilon,\epsilon'}^{(1)}(\xi_1,\xi_2,\xi_3)b_{\ell,\epsilon}^*(\xi_1)c_{\ell,\epsilon'}^*(\xi_2)a_\epsilon(\xi_3)d\xi_1 d\xi_2 d\xi_3 \tag{2.29}$$

is defined as a quadratic form on $(\mathfrak{D}_\ell\otimes\mathfrak{D}_W)\times(\mathfrak{D}_\ell\otimes\mathfrak{D}_W)$ as

$$\int\left(c_{\ell,\epsilon'}(\xi_2)b_{\ell,\epsilon}(\xi_1)\psi, G_{\ell,\epsilon,\epsilon'}^{(1)}a_\epsilon(\xi_3)\phi\right)d\xi_1 d\xi_2 d\xi_3, \tag{2.30}$$

where $\psi,\phi\in\mathfrak{D}_\ell\otimes\mathfrak{D}_W$.

By mimicking the proof of [29, Theorem X.44], we get a closed operator, denoted by $H_{I,\ell,\epsilon,\epsilon'}^{(1)}$, associated with the quadratic form such that it is the unique operator in $\mathfrak{F}_\ell\otimes\mathfrak{F}_W$ such that $\mathfrak{D}_\ell\otimes\mathfrak{D}_W\subset\mathfrak{D}(H_{I,\ell,\epsilon,\epsilon'}^{(1)})$ is a core for $H_{I,\ell,\epsilon,\epsilon'}^{(1)}$ and

$$H_{I,\ell,\epsilon,\epsilon'}^{(1)} = \int G_{\ell,\epsilon,\epsilon'}^{(1)}(\xi_1,\xi_2,\xi_3)b_{\ell,\epsilon}^*(\xi_1)c_{\ell,\epsilon'}^*(\xi_2)a_\epsilon(\xi_3)d\xi_1 d\xi_2 d\xi_3 \tag{2.31}$$

as quadratic forms on $(\mathfrak{D}_\ell\otimes\mathfrak{D}_W)\times(\mathfrak{D}_\ell\otimes\mathfrak{D}_W)$.

Similarly for the operator $(H_{I,\ell,\epsilon,\epsilon'}^{(1)})^*$, we have the equality as quadratic forms

$$\left(H_{I,\ell,\epsilon,\epsilon'}^{(1)}\right)^* = \int\overline{G_{\ell,\epsilon,\epsilon'}^{(1)}(\xi_1,\xi_2,\xi_3)}a_\epsilon^*(\xi_3)c_{\ell,\epsilon'}(\xi_2)b_{\ell,\epsilon}(\xi_1)d\xi_1 d\xi_2 d\xi_3. \tag{2.32}$$

Again, there exists two closed operators $H_{I,\ell,\epsilon,\epsilon'}^{(2)}$ and $(H_{I,\ell,\epsilon,\epsilon'}^{(2)})^*$ such that $\mathfrak{D}_\ell\otimes\mathfrak{D}_W\subset\mathfrak{D}(H_{I,\ell,\epsilon,\epsilon'}^{(2)})$, $\mathfrak{D}_\ell\otimes\mathfrak{D}_W\subset\mathfrak{D}((H_{I,\ell,\epsilon,\epsilon'}^{(2)})^*)$, and $\mathfrak{D}_\ell\otimes\mathfrak{D}_W$ is a core for $H_{I,\ell,\epsilon,\epsilon'}^{(2)}$ and $(H_{I,\ell,\epsilon,\epsilon'}^{(2)})^*$ and such that

$$H^{(2)}_{I,\ell,\epsilon,\epsilon'} = \int G^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) b^*_{\ell,\epsilon}(\xi_1) c^*_{\ell,\epsilon'}(\xi_2) a^*_{\epsilon}(\xi_3) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3,$$
$$\left(H^{(2)}_{I,\ell,\epsilon,\epsilon'}\right)^* = \int G^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) a_{\epsilon}(\xi_3) c_{\ell,\epsilon'}(\xi_2) b_{\ell,\epsilon}(\xi_1) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \tag{2.33}$$

as quadratic forms on $(\mathfrak{D}_\ell \otimes \mathfrak{D}_W) \times (\mathfrak{D}_\ell \otimes \mathfrak{D}_W)$.

We will still denote by $H^{(\alpha)}_{I,\ell,\epsilon,\epsilon'}$ and $(H^{(\alpha)}_{I,\ell,\epsilon,\epsilon'})^*$ $(\alpha = 1,2)$ their extensions to $\mathfrak{F}$. The set $\mathfrak{D}$ is then a core for $H^{(\alpha)}_{I,\ell,\epsilon,\epsilon'}$ and $(H^{(\alpha)}_{I,\ell,\epsilon,\epsilon'})^*$.

Thus

$$H = H_0 + g \sum_{\alpha=1,2} \sum_{\ell=1}^{3} \sum_{\epsilon \neq \epsilon'} \left(H^{(\alpha)}_{I,\ell,\epsilon,\epsilon'} + \left(H^{(2)}_{I,\ell,\epsilon,\epsilon'}\right)^* \right) \tag{2.34}$$

is a symmetric operator defined on $\mathfrak{D}$.

We now want to prove that H is essentially self-adjoint on $\mathfrak{D}$ by showing that $H^{(\alpha)}_{I,\ell,\epsilon,\epsilon'}$ and $(H^{(\alpha)}_{I,\ell,\epsilon,\epsilon'})^*$ are relatively H_0-bounded.

Once again, as above, for almost every $\xi_3 \in \Sigma_2$, there exists closed operators in $\mathfrak{F}_L$, denoted by $B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3)$ and $(B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3))^*$ such that

$$B^{(1)}_{\ell,\epsilon,\epsilon'}(\xi_3) = -\int \overline{G^{(1)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)} b_{\ell,\epsilon}(\xi_1) c_{\ell,\epsilon'}(\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2,$$
$$\left(B^{(1)}_{\ell,\epsilon,\epsilon'}(\xi_3)\right)^* = \int G^{(1)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) b^*_{\ell,\epsilon}(\xi_1) c^*_{\ell,\epsilon'}(\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2,$$
$$B^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_3) = \int G^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) b^*_{\ell,\epsilon}(\xi_1) c^*_{\ell,\epsilon'}(\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2,$$
$$\left(B^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_3)\right)^* = -\int \overline{G^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)} b_{\ell,\epsilon}(\xi_1) c_{\ell,\epsilon'}(\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \tag{2.35}$$

as quadratic forms on $\mathfrak{D}_\ell \times \mathfrak{D}_\ell$.

We have that $\mathfrak{D}_\ell \subset \mathfrak{D}(B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3))$ (resp., $\mathfrak{D}_\ell \subset \mathfrak{D}((B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3))^*)$ is a core for $B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3)$ (resp., for $(B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3))^*$). We still denote by $B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3))$ and $(B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3))^*)$ their extensions to $\mathfrak{F}_L$.

It then follows that the operator H_I with domain $\mathfrak{D}$ is symmetric and can be written in the following form:

$$H_I = \sum_{\alpha=1,2} \sum_{\ell=1}^{3} \sum_{\epsilon \neq \epsilon'} \left(H^{(\alpha)}_{I,\ell,\epsilon,\epsilon'} + \left(H^{(\alpha)}_{I,\ell,\epsilon,\epsilon'}\right)^* \right) = \sum_{\alpha=1,2} \sum_{\ell=1}^{3} \sum_{\epsilon \neq \epsilon'} \int B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3) \otimes a^*_{\epsilon}(\xi_3) \mathrm{d}\xi_3$$
$$+ \sum_{\alpha=1,2} \sum_{\ell=1}^{3} \sum_{\epsilon \neq \epsilon'} \int \left(B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3)\right)^* \otimes a_{\epsilon}(\xi_3) \mathrm{d}\xi_3. \tag{2.36}$$

Let N_ℓ denote the operator number of massive leptons ℓ in $\mathfrak{F}_\ell$, that is,

$$N_\ell = \sum_\epsilon \int b^*_{\ell,\epsilon}(\xi_1) b_{\ell,\epsilon}(\xi_1) \mathrm{d}\xi_1. \tag{2.37}$$

The operator N_ℓ is a positive self-adjoint operator in $\mathfrak{F}_\ell$. We still denote by N_ℓ its extension to $\mathfrak{F}_L$. The set $\mathfrak{D}_L$ is a core for N_ℓ.

We then have the following.

Proposition 2.4. *For almost every $\xi_3 \in \Sigma_2$, $\mathfrak{D}(B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3))$, $\mathfrak{D}((B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3))^*) \supset \mathfrak{D}(N_\ell^{1/2})$, and for $\Phi \in \mathfrak{D}(N_\ell^{1/2}) \subset \mathfrak{F}_L$ one has*

$$\begin{aligned} \left\| B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3)\Phi \right\|_{\mathfrak{F}_L} &\leq \left\| G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\cdot,\cdot,\xi_3) \right\|_{L^2(\Sigma_1\times\Sigma_1)} \left\| N_\ell^{1/2}\Phi \right\|_{\mathfrak{F}_L}, \\ \left\| \left(B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3)\right)^*\Phi \right\|_{\mathfrak{F}_L} &\leq \left\| G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\cdot,\cdot,\xi_3) \right\|_{L^2(\Sigma_1\times\Sigma_1)} \left\| N_\ell^{1/2}\Phi \right\|_{\mathfrak{F}_L}. \end{aligned} \tag{2.38}$$

Proof. The estimates (2.38) are examples of N_τ estimates (see [25]). The proof is quite similar to the proof of [20, Proposition 3.7]. Details can be found in [27] but are omitted here. □

Let

$$H^{(3)}_{0,\epsilon} = \int w^{(3)}(\xi_3) a^*_\epsilon(\xi_3) a_\epsilon(\xi_3) \mathrm{d}\xi_3. \tag{2.39}$$

Then $H^{(3)}_{0,\epsilon}$ is a self-adjoint operator in $\mathfrak{F}_W$, and $\mathfrak{D}_W$ is a core for $H^{(3)}_{0,\epsilon}$.

We get the following.

Proposition 2.5. *One has*

$$\begin{aligned} &\left\| \int \left(B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3)\right)^* \otimes a_\epsilon(\xi_3) \mathrm{d}\xi_3 \Psi \right\|^2 \\ &\quad \leq \left(\int_{\Sigma_1\times\Sigma_1\times\Sigma_2} \frac{\left|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)\right|^2}{w^{(3)}(\xi_3)} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \right) \left\| (N_\ell + 1)^{1/2} \otimes \left(H^{(3)}_{0,\epsilon}\right)^{1/2} \Psi \right\|^2, \end{aligned} \tag{2.40}$$

$$\begin{aligned} &\left\| \int B^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_3) \otimes a^*_\epsilon(\xi_3) \mathrm{d}\xi_3 \Psi \right\|^2 \\ &\quad \leq \left(\int_{\Sigma_1\times\Sigma_1\times\Sigma_2} \frac{\left|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)\right|^2}{w^{(3)}(\xi_3)} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \right) \left\| (N_\ell + 1)^{1/2} \otimes \left(H^{(3)}_{0,\epsilon}\right)^{1/2} \Psi \right\|^2 \\ &\qquad + \left(\int_{\Sigma_1\times\Sigma_1\times\Sigma_2} \left|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)\right|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \right) \left(\eta \left\| (N_\ell + 1)^{1/2} \otimes 1\Psi \right\|^2 + \frac{1}{4\eta} \|\Psi\|^2 \right) \end{aligned} \tag{2.41}$$

for every $\Psi \in \mathfrak{D}(H_0)$ and every $\eta > 0$.

Proof. Suppose that $\Psi \in \mathfrak{D}(N_\ell^{1/2})\widehat{\otimes}\mathfrak{D}((H_{0,\epsilon}^{(3)})^{1/2})$. Let

$$\Psi_\epsilon(\xi_3) = w^{(3)}(\xi_3)^{1/2}\Big((N_\ell + 1)^{1/2} \otimes a_\epsilon(\xi_3)\Big)\Phi. \tag{2.42}$$

We have

$$\int \Big(B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3)\Big)^* \otimes a_\epsilon(\xi_3)\mathrm{d}\xi_3\Psi = \int_{\Sigma_2} \frac{1}{\left(w^{(3)}(\xi_3)\right)^{1/2}}\Big(\Big(B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3)\Big)^*(N_\ell + 1)^{-1/2} \otimes 1\Big)\Psi_\epsilon(\xi_3)\mathrm{d}\xi_3. \tag{2.43}$$

Therefore, for $\Psi \in \mathfrak{D}(N_\ell^{1/2})\widehat{\otimes}\mathfrak{D}((H_{0,\epsilon}^{(3)})^{1/2})$, (2.40) follows from Proposition 2.4.

We now have

$$\begin{aligned}
&\left\|\int B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3) \otimes a_\epsilon^*(\xi_3)\Psi\mathrm{d}\xi_3\right\|_{\mathfrak{F}}^2 \\
&\quad = \int\Big(B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3) \otimes a_\epsilon(\xi_3')\Psi, B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3') \otimes a_\epsilon(\xi_3)\Psi\Big)\mathrm{d}\xi_3\mathrm{d}\xi_3' + \int\left\|\Big(B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3) \otimes 1\Big)\Psi\right\|^2\mathrm{d}\xi_3,
\end{aligned} \tag{2.44}$$

$$\begin{aligned}
&\int_{\Sigma_2\times\Sigma_2}\Big(B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3) \otimes a_\epsilon(\xi_3')\Psi, B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3') \otimes a_\epsilon(\xi_3)\Psi\Big)\mathrm{d}\xi_3\mathrm{d}\xi_3' \\
&\quad = \int_{\Sigma_2\times\Sigma_2} \frac{1}{w^{(3)}(\xi_3)^{1/2}w^{(3)}(\xi_3')^{1/2}} \\
&\qquad \times\Big(\Big(B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3)(N_\ell + 1)^{-1/2} \otimes 1\Big)\Psi_\epsilon(\xi_3'), \Big(B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3')(N_\ell + 1)^{-1/2} \otimes 1\Big)\Psi_\epsilon(\xi_3)\Big)\mathrm{d}\xi_3\mathrm{d}\xi_3' \\
&\quad \le \left(\int_{\Sigma_2} \frac{1}{w^{(3)}(\xi_3)^{1/2}}\left\|B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3)(N_\ell + 1)^{-1/2}\right\|_{\mathfrak{F}_L}\|\Psi_\epsilon(\xi_3)\|\mathrm{d}\xi_3\right)^2 \\
&\quad \le \left(\int_{\Sigma_1\times\Sigma_1\times\Sigma_2} \frac{\left|G^{(\alpha)}(\xi_1,\xi_2,\xi_3)\right|^2}{w^{(3)}(\xi_3)}\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\right)\left\|(N_\ell + 1)^{1/2} \otimes \left(H_{0,\epsilon}^{(3)}\right)^{1/2}\Psi\right\|^2.
\end{aligned} \tag{2.45}$$

Furthermore

$$\begin{aligned}
&\int_{\Sigma_2}\left\|\Big(B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3) \otimes 1\Big)\Psi\right\|^2\mathrm{d}\xi_3 \\
&\quad = \int_{\Sigma_2}\left\|\Big(B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3)(N_\ell + 1)^{-1/2} \otimes 1\Big)\Big((N_\ell + 1)^{1/2} \otimes 1\Big)\Psi\right\|^2\mathrm{d}\xi_3 \\
&\quad \le \left(\int_{\Sigma_1\times\Sigma_1\times\Sigma_2}\left|G_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_1,\xi_2,\xi_3)\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\right)\left(\eta\|(N_\ell + 1)\Psi\|^2 + \frac{1}{4\eta}\|\Psi\|^2\right)
\end{aligned} \tag{2.46}$$

for every $\eta > 0$.

By (2.40), (2.45), and (2.46), we finally get (2.41) for every $\Psi \in \mathfrak{D}(N_\ell^{1/2})\widehat{\otimes}\mathfrak{D}(H_{0,\epsilon}^{(3)})$. It then follows that (2.40) and (2.41) are verified for every $\Psi \in \mathfrak{D}(H_0)$. □

We now prove that H is a self-adjoint operator in $\mathfrak{F}$ for g sufficiently small.

Theorem 2.6. *Let $g_1 > 0$ be such that*

$$\frac{3g_1^2}{m_W}\left(\frac{1}{m_1^2}+1\right)\sum_{\alpha=1,2}\sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'}\left\|G_{\ell,\epsilon,\epsilon'}^{(\alpha)}\right\|_{L^2(\Sigma_1\times\Sigma_1\times\Sigma_2)}^2 < 1. \tag{2.47}$$

Then for every g satisfying $g \le g_1$, H is a self-adjoint operator in $\mathfrak{F}$ with domain $\mathfrak{D}(H) = \mathfrak{D}(H_0)$, and $\mathfrak{D}$ is a core for H.

Proof. Let Ψ be in $\mathfrak{D}$. We have

$$\|H_I\Psi\|^2 \le 12\sum_{\alpha=1,2}\sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'}\left\{\left\|\int\left(B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3)\right)^*\otimes a_\epsilon(\xi_3)\Psi\mathrm{d}\xi_3\right\|^2+\left\|\int\left(B_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_3)\right)\otimes a_\epsilon^*(\xi_3)\Psi\mathrm{d}\xi_3\right\|^2\right\}. \tag{2.48}$$

Note that

$$\begin{gathered}\left\|H_{0,\epsilon}^{(3)}\Psi\right\| \le \left\|H_0^{(3)}\Psi\right\| \le \|H_0\Psi\|,\\ \|N_\ell\Psi\| \le \frac{1}{m_\ell}\|H_{0,\ell}\Psi\| \le \frac{1}{m_1}\|H_{0,\ell}\Psi\| \le \frac{1}{m_1}\|H_0\Psi\|,\end{gathered} \tag{2.49}$$

where

$$H_{0,\ell} = \sum_\epsilon\int w_\ell^{(1)}(\xi_1)b_{\ell,\epsilon}^*(\xi_1)b_{\ell,\epsilon}(\xi_1)\mathrm{d}\xi_1 + \sum_\epsilon\int w_\ell^{(2)}(\xi_2)c_{\ell,\epsilon}^*(\xi_2)c_{\ell,\epsilon}(\xi_2)\mathrm{d}\xi_2. \tag{2.50}$$

We further note that

$$\left\|(N_\ell+1)^{1/2}\otimes\left(H_{0,\epsilon}^{(3)}\right)^{1/2}\Psi\right\|^2 \le \frac{1}{2}\left(\frac{1}{m_1^2}+1\right)\|H_0\Psi\|^2+\frac{\beta}{2m_1^2}\|H_0\Psi\|^2+\left(\frac{1}{2}+\frac{1}{8\beta}\right)\|\Psi\|^2 \tag{2.51}$$

for $\beta > 0$, and

$$\eta\|((N_\ell+1)\otimes 1)\Psi\|^2+\frac{1}{4\eta}\|\Psi\|^2 \le \frac{\eta}{m_1^2}\|H_0\Psi\|^2+\frac{\eta\beta}{m_1^2}\|H_0\Psi\|^2+\eta\left(1+\frac{1}{4\beta}\right)\|\Psi\|^2+\frac{1}{4\eta}\|\Psi\|^2. \tag{2.52}$$

Combining (2.48) with (2.40), (2.41), (2.51), and (2.52) we get for $\eta > 0$, $\beta > 0$

$$
\begin{aligned}
\|H_I\Psi\|^2 \leq 6\Bigg(\sum_{\alpha=1,2}\sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'}\left\|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right\|^2\Bigg) \\
\times\left(\frac{1}{2m_W}\left(\frac{1}{m_1^2}+1\right)\|H_0\Psi\|^2+\frac{\beta}{2m_W m_1^2}\|H_0\Psi\|^2+\frac{1}{2m_W}\left(1+\frac{1}{4\beta}\right)\|\Psi\|^2\right) \\
+12\Bigg(\sum_{\alpha=1,2}\sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'}\left\|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right\|^2\Bigg)\left(\frac{\eta}{m_1^2}(1+\beta)\|H_0\Psi\|^2+\left(\eta\left(1+\frac{1}{4\beta}\right)+\frac{1}{4\eta}\right)\|\Psi\|^2\right),
\end{aligned}
\tag{2.53}
$$

by noting

$$
\int_{\Sigma_1\times\Sigma_1\times\Sigma_2}\frac{|G_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)|^2}{w^{(3)}(\xi_3)}\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \leq \frac{1}{m_W}\left\|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right\|^2. \tag{2.54}
$$

By (2.53) the theorem follows from the Kato-Rellich theorem. □

3. Main Results

In the sequel, we will make the following additional assumptions on the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$.

Hypothesis 3.1. (i) For $\alpha = 1,2$, $\ell = 1,2,3$, $\epsilon,\epsilon' = \pm$,

$$
\int_{\Sigma_1\times\Sigma_1\times\Sigma_2}\frac{\left|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)\right|^2}{|p_2|^2}\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 < \infty. \tag{3.1}
$$

(ii) There exists $C > 0$ such that for $\alpha = 1,2$, $\ell = 1,2,3$, $\epsilon,\epsilon' = \pm$,

$$
\left(\int_{\Sigma_1\times\{|p_2|\leq\sigma\}\times\Sigma_2}\left|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\right)^{1/2} \leq C\sigma^2. \tag{3.2}
$$

(iii) For $\alpha = 1,2$, $\ell = 1,2,3$, $\epsilon,\epsilon' = \pm$, and $i,j = 1,2,3$

$$
\begin{aligned}
&\text{(iii.a)} \quad \int_{\Sigma_1\times\Sigma_1\times\Sigma_2}\left|\left[(p_2\cdot\nabla_{p_2})G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right](\xi_1,\xi_2,\xi_3)\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 < \infty, \\
&\text{(iii.b)} \quad \int_{\Sigma_1\times\Sigma_1\times\Sigma_2}p_{2,i}^2 p_{2,j}^2\left|\frac{\partial^2 G^{(\alpha)}_{\ell,\epsilon,\epsilon'}}{\partial p_{2,i}\partial p_{2,j}}(\xi_1,\xi_2,\xi_3)\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 < \infty.
\end{aligned}
\tag{3.3}
$$

(iv) There exists $\Lambda > m_1$ such that for $\alpha = 1, 2$, $\ell = 1, 2, 3$, $\epsilon, \epsilon' = \pm$,

$$G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3) = 0 \quad \text{if } |p_2| \geq \Lambda. \tag{3.4}$$

Remark 3.2. Hypothesis 3.1(ii) is nothing but an infrared regularization of the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$. In order to satisfy this hypothesis it is, for example, sufficient to suppose that

$$G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3) = |p_2|^{1/2} \widetilde{G}^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3), \tag{3.5}$$

where $\widetilde{G}^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ is a smooth function of (p_1, p_2, p_3) in the Schwartz space.

Hypothesis 3.1(iv), which is a sharp ultraviolet cutoff, is actually not necessary, and can be removed at the expense of some additional technicalities. However, in order to simplify the proof of Proposition 3.5, we will leave it.

Our first result is devoted to the existence of a ground state for H together with the location of the spectrum of H and of its absolutely continuous spectrum when g is sufficiently small.

Theorem 3.3. *Suppose that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypotheses 2.1 and 3.1(i). Then there exists $0 < g_2 \leq g_1$ such that H has a unique ground state for $g \leq g_2$. Moreover*

$$\sigma(H) = \sigma_{\mathrm{ac}}(H) = [\inf \sigma(H), \infty) \tag{3.6}$$

with $\inf \sigma(H) \leq 0$.

According to Theorem 3.3 the ground state energy $E = \inf \sigma(H)$ is a simple eigenvalue of H, and our main results are concerned with a careful study of the spectrum of H above the ground state energy. The spectral theory developed in this work is based on the conjugated operator method as described in [5, 6, 30]. Our choice of the conjugate operator denoted by A is the second quantized dilation generator for the neutrinos.

Let a denote the following operator in $L^2(\Sigma_1)$:

$$a = \frac{1}{2}\left(p_2 \cdot i\nabla_{p_2} + i\nabla_{p_2} \cdot p_2\right). \tag{3.7}$$

The operator a is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$. Its second quantized version $\mathrm{d}\Gamma(a)$ is a self-adjoint operator in $\mathfrak{F}_a(L^2(\Sigma_1))$. From the definition (2.8) of the space $\mathfrak{F}_\ell$, the following operator in $\mathfrak{F}_\ell$

$$A_\ell = 1 \otimes 1 \otimes \mathrm{d}\Gamma(a) \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \mathrm{d}\Gamma(a) \tag{3.8}$$

is essentially self-adjoint on $\mathfrak{D}_L$.

Let now A be the following operator in $\mathfrak{F}_L$:

$$A = A_1 \otimes 1_2 \otimes 1_3 + 1_1 \otimes A_2 \otimes 1_3 + 1_1 \otimes 1_2 \otimes A_3. \tag{3.9}$$

Then A is essentially self-adjoint on $\mathfrak{D}_L$.

We will denote again by A its extension to $\mathfrak{F}$. Thus A is essentially self-adjoint on $\mathfrak{D}$ and we still denote by A its closure.

We also set

$$\langle A \rangle = \left(1 + A^2\right)^{1/2}. \tag{3.10}$$

We then have the following.

Theorem 3.4. *Suppose that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypotheses 2.1 and 3.1. For any $\delta > 0$ satisfying $0 < \delta < m_1$ there exists $0 < g_\delta \le g_2$ such that, for $0 < g \le g_\delta$, the following points are satisfied.*

(i) *The spectrum of H in $(\inf \sigma(H), m_1 - \delta]$ is purely absolutely continuous.*

(ii) *Limiting absorption principle.*

For every $s > 1/2$ and φ, ψ in $\mathfrak{F}$, the limits

$$\lim_{\varepsilon \to 0} \left(\varphi, \langle A \rangle^{-s} (H - \lambda \pm i\varepsilon) \langle A \rangle^{-s} \psi\right) \tag{3.11}$$

exist uniformly for λ in any compact subset of $(\inf \sigma(H), m_1 - \delta]$.

(iii) *Pointwise decay in time.*

Suppose $s \in (1/2, 1)$ and $f \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} f \subset (\inf \sigma(H), m_1 - \delta)$. Then

$$\left\| \langle A \rangle^{-s} e^{-itH} f(H) \langle A \rangle^{-s} \right\| = \mathcal{O}\left(t^{1/2-s}\right) \tag{3.12}$$

as $t \to \infty$.

The proof of Theorem 3.4 is based on a positive commutator estimate, called the Mourre estimate, and on a regularity property of H with respect to A (see [5, 6, 30]). According to [4], the main ingredient of the proof is auxiliary operators associated with infrared cutoff Hamiltonians with respect to the momenta of the neutrinos that we now introduce.

Let $\chi_0(\cdot)$, $\chi_\infty(\cdot) \in C^\infty(\mathbb{R}, [0,1])$ with $\chi_0 = 1$ on $(-\infty, 1]$, $\chi_\infty = 1$ on $[2, \infty)$ and ${\chi_0}^2 + {\chi_\infty}^2 = 1$.

For $\sigma > 0$ we set

$$\begin{aligned} \chi_\sigma(p) &= \chi_0\left(\frac{|p|}{\sigma}\right), \\ \chi^\sigma(p) &= \chi_\infty\left(\frac{|p|}{\sigma}\right), \\ \tilde{\chi}^\sigma(p) &= 1 - \chi_\sigma(p), \end{aligned} \tag{3.13}$$

where $p \in \mathbb{R}^3$.

The operator $H_{I,\sigma}$ is the interaction given by (2.23) and (2.24) and associated with the kernels $\tilde{\chi}^{\sigma}(p_2)G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)$. We then set

$$H_\sigma := H_0 + gH_{I,\sigma}. \tag{3.14}$$

Let

$$\begin{aligned}
\Sigma_{1,\sigma} &= \Sigma_1 \cap \{(p_2,s_2);\, |p_2| < \sigma\},\\
\Sigma_1^\sigma &= \Sigma_1 \cap \{(p_2,s_2);\, |p_2| \geq \sigma\}\\
\mathfrak{F}_{\ell,2,\sigma} &= \mathfrak{F}_a\Big(L^2(\Sigma_{1,\sigma})\Big) \otimes \mathfrak{F}_a\Big(L^2(\Sigma_{1,\sigma})\Big),\\
\mathfrak{F}^\sigma_{\ell,2} &= \mathfrak{F}_a\Big(L^2(\Sigma_1^\sigma)\Big) \otimes \mathfrak{F}_a\Big(L^2(\Sigma_1^\sigma)\Big),\\
\mathfrak{F}_{\ell,2} &= \mathfrak{F}_{\ell,2,\sigma} \otimes \mathfrak{F}^\sigma_{\ell,2},\\
\mathfrak{F}_{\ell,1} &= \bigotimes^2 \mathfrak{F}_a\Big(L^2(\Sigma_1)\Big).
\end{aligned} \tag{3.15}$$

The space $\mathfrak{F}_{\ell,1}$ is the Fock space for the massive leptons ℓ, and $\mathfrak{F}_{\ell,2}$ is the Fock space for the neutrinos and antineutrinos ℓ.

Set

$$\begin{aligned}
\mathfrak{F}^\sigma_\ell &= \mathfrak{F}_{\ell,1} \otimes \mathfrak{F}^\sigma_{\ell,2}\\
\mathfrak{F}_{\ell,\sigma} &= \mathfrak{F}_{\ell,2,\sigma}.
\end{aligned} \tag{3.16}$$

We have

$$\mathfrak{F}_\ell \simeq \mathfrak{F}^\sigma_\ell \otimes \mathfrak{F}_{\ell,\sigma}. \tag{3.17}$$

Set

$$\begin{aligned}
\mathfrak{F}^\sigma_L &= \bigotimes_{\ell=1}^3 \mathfrak{F}^\sigma_\ell\\
\mathfrak{F}_{L,\sigma} &= \bigotimes_{\ell=1}^3 \mathfrak{F}_{\ell,\sigma}.
\end{aligned} \tag{3.18}$$

We have

$$\mathfrak{F}_L \simeq \mathfrak{F}^\sigma_L \otimes \mathfrak{F}_{L,\sigma}. \tag{3.19}$$

Set

$$\mathfrak{F}^{\sigma} = \mathfrak{F}_L^{\sigma} \otimes \mathfrak{F}_W. \tag{3.20}$$

We have

$$\mathfrak{F} \simeq \mathfrak{F}_{L,\sigma} \otimes \mathfrak{F}^{\sigma}. \tag{3.21}$$

Set

$$\begin{aligned}
H_0^{(1)} &= \sum_{\ell=1}^{3} \sum_{\epsilon=\pm} \int w_\ell^{(1)}(\xi_1) b_{\ell,\epsilon}^*(\xi_1) b_{\ell,\epsilon}(\xi_1) \mathrm{d}\xi_1, \\
H_0^{(2)} &= \sum_{\ell=1}^{3} \sum_{\epsilon=\pm} \int w_\ell^{(2)}(\xi_2) c_{\ell,\epsilon}^*(\xi_2) c_{\ell,\epsilon}(\xi_2) \mathrm{d}\xi_2, \\
H_0^{(3)} &= \sum_{\epsilon=\pm} \int w^{(3)}(\xi_3) a_\epsilon^*(\xi_3) a_\epsilon(\xi_3) \mathrm{d}\xi_3, \\
H_0^{(2)\sigma} &= \sum_{\ell=1}^{3} \sum_{\epsilon=\pm} \int_{|p_2|>\sigma} w_\ell^{(2)}(\xi_2) c_{\ell,\epsilon}^*(\xi_2) c_{\ell,\epsilon}(\xi_2) \mathrm{d}\xi_2, \\
H_{0,\sigma}^{(2)} &= \sum_{\ell=1}^{3} \sum_{\epsilon=\pm} \int_{|p_2|\leq\sigma} w_\ell^{(2)}(\xi_2) c_{\ell,\epsilon}^*(\xi_2) c_{\ell,\epsilon}(\xi_2) \mathrm{d}\xi_2.
\end{aligned} \tag{3.22}$$

We have on $\mathfrak{F}^{\sigma} \otimes \mathfrak{F}_{\sigma}$

$$H_0^{(2)} = H_0^{(2)\sigma} \otimes 1_{\sigma} + 1^{\sigma} \otimes H_{0,\sigma}^{(2)}. \tag{3.23}$$

Here, 1^{σ} (resp., 1_{σ}) is the identity operator on $\mathfrak{F}^{\sigma}$ (resp., $\mathfrak{F}_{\sigma}$).

Define

$$H^{\sigma} = H_{\sigma}|_{\mathfrak{F}^{\sigma}}, \qquad H_0^{\sigma} = H_0|_{\mathfrak{F}^{\sigma}}. \tag{3.24}$$

We get

$$\begin{aligned}
H^{\sigma} &= H_0^{(1)} + H_0^{(2)\sigma} + H_0^{(3)} + g H_{I,\sigma} \quad \text{on } \mathfrak{F}^{\sigma}, \\
H_{\sigma} &= H^{\sigma} \otimes 1_{\sigma} + 1^{\sigma} \otimes H_{0,\sigma}^{(2)} \quad \text{on } \mathfrak{F}^{\sigma} \otimes \mathfrak{F}_{\sigma}.
\end{aligned} \tag{3.25}$$

In order to implement the conjugate operator theory, we have to show that H^{σ} has a gap in its spectrum above its ground state.

We now set, for $\beta > 0$ and $\eta > 0$,

$$C_{\beta\eta} = \left(\frac{3}{m_W}\left(1+\frac{1}{m_1{}^2}\right) + \frac{3\beta}{m_W m_1{}^2} + \frac{12\eta}{m_1{}^2}(1+\beta) \right)^{1/2},$$
$$B_{\beta\eta} = \left(\frac{3}{m_W}\left(1+\frac{1}{4\beta}\right) + 12\left(\eta\left(1+\frac{1}{4\beta}\right)+\frac{1}{4\eta}\right) \right)^{1/2}. \tag{3.26}$$

Let

$$G = \left(G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\cdot,\cdot,\cdot)\right)_{\alpha=1,2;\ell=1,2,3;\epsilon,\epsilon'=\pm,\epsilon\neq\epsilon'}, \tag{3.27}$$

and set

$$K(G) = \left(\sum_{\alpha=1,2}\sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'} \left\| G^{(\alpha)}_{\ell,\epsilon,\epsilon'} \right\|^2_{L^2(\Sigma_1\times\Sigma_1\times\Sigma_2)} \right)^{1/2}. \tag{3.28}$$

Let

$$\widetilde{C}_{\beta\eta} = C_{\beta\eta}\left(1 + \frac{g_1 K(G) C_{\beta\eta}}{1-g_1 K(G) C_{\beta\eta}}\right),$$
$$\widetilde{B}_{\beta\eta} = \left(1 + \frac{g_1 K(G) C_{\beta\eta}}{1-g_1 K(G) C_{\beta\eta}}\left(2 + \frac{g_1 K(G) B_{\beta\eta} C_{\beta\eta}}{1-g_1 K(G) C_{\beta\eta}}\right)\right) B_{\beta\eta}. \tag{3.29}$$

Let

$$\widetilde{K}(G) = \left(\sum_{\alpha=1,2}\sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'} \int_{\Sigma_1\times\Sigma_1\times\Sigma_2} \frac{\left| G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)\right|^2}{|p_2|^2} d\xi_1 d\xi_2 d\xi_3 \right)^{1/2}. \tag{3.30}$$

Let $\delta \in \mathbb{R}$ be such that

$$0 < \delta < m_1. \tag{3.31}$$

We set

$$\widetilde{D} = \sup\left(\frac{4\Lambda\gamma}{2m_1-\delta}, 1\right)\widetilde{K}(G)\left(2m_1\widetilde{C}_{\beta\eta} + \widetilde{B}_{\beta\eta}\right), \tag{3.32}$$

where $\Lambda > m_1$ has been introduced in Hypothesis 3.1(iv).

Let us define the sequence $(\sigma_n)_{n\geq 0}$ by

$$\begin{aligned}
&\sigma_0 = \Lambda, \\
&\sigma_1 = m_1 - \frac{\delta}{2}, \\
&\sigma_2 = m_1 - \delta = \gamma\sigma_1, \\
&\sigma_{n+1} = \gamma\sigma_n, \quad n \geq 1,
\end{aligned} \tag{3.33}$$

where $\gamma = 1 - \delta/(2m_1 - \delta)$.

Let $g_\delta^{(1)}$ be such that

$$0 < g_\delta^{(1)} < \inf\left(1, g_1, \frac{\gamma - \gamma^2}{3\widetilde{D}}\right). \tag{3.34}$$

For $0 < g \leq g_\delta^{(1)}$ we have

$$0 < \gamma < \left(1 - \frac{3g\widetilde{D}}{\gamma}\right), \tag{3.35}$$

$$0 < \sigma_{n+1} < \left(1 - \frac{3g\widetilde{D}}{\gamma}\right)\sigma_n, \quad n \geq 1. \tag{3.36}$$

Set

$$\begin{gathered}
H^n = H^{\sigma_n}; \quad H_0^n = H_0^{\sigma_n}, \quad n \geq 0, \\
E^n = \inf \sigma(H^n), \quad n \geq 0.
\end{gathered} \tag{3.37}$$

We then get the following.

Proposition 3.5. *Suppose that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypotheses 2.1, 3.1(i), and 3.1(iv). Then there exists $0 < \tilde{g}_\delta \leq g_\delta^{(1)}$ such that, for $g \leq \tilde{g}_\delta$ and $n \geq 1$, E^n is a simple eigenvalue of H^n and H^n does not have spectrum in $(E^n, E^n + (1 - 3g\widetilde{D}/\gamma)\sigma_n)$.*

The proof of Proposition 3.5 is given in the appendix.

We now introduce the positive commutator estimates and the regularity property of H with respect to A in order to prove Theorem 3.4.

The operator A has to be split into two pieces depending on σ.

Let

$$\begin{aligned}\eta_\sigma(p_2) &= \chi_{2\sigma}(p_2),\\ \eta^\sigma(p_2) &= \chi^{2\sigma}(p_2),\\ a_\sigma &= \eta_\sigma(p_2)a\eta_\sigma(p_2),\\ a^\sigma &= \eta^\sigma(p_2)a\eta^\sigma(p_2).\end{aligned} \tag{3.38}$$

Since $\eta_\sigma^2 + (\eta^\sigma)^2 = 1$, and $[\eta_\sigma, [\eta_\sigma, a]] = 0 = [\eta^\sigma, [\eta^\sigma, a]]$, we obtain (see [4])

$$a = a^\sigma + a_\sigma. \tag{3.39}$$

Note that we also have

$$\begin{aligned}a_\sigma &= \frac{1}{2}\Big(\eta_\sigma(p_2)^2 p_2\cdot i\nabla p_2 + i\nabla p_2\cdot\eta_\sigma(p_2)^2 p_2\Big),\\ a^\sigma &= \frac{1}{2}\Big(\eta^\sigma(p_2)^2 p_2\cdot i\nabla p_2 + i\nabla p_2\cdot\eta^\sigma(p_2)^2 p_2\Big).\end{aligned} \tag{3.40}$$

The operators a, a_σ, and a^σ are essentially self-adjoint on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$ (see [5, Proposition 4.2.3]). We still denote by a, a_σ, and a^σ their closures. If $\tilde{a}$ denotes any of the operator a, a_σ, and a^σ, we have

$$\mathfrak{D}(\tilde{a}) = \Big\{u\in L^2(\Sigma_1);\ \tilde{a}u\in L^2(\Sigma_1)\Big\}. \tag{3.41}$$

The operators $d\Gamma(a)$, $d\Gamma(a^\sigma)$, and $d\Gamma(a_\sigma)$ are self-adjoint operators in $\mathfrak{F}_a(L^2(\Sigma_1))$, and we have

$$d\Gamma(a) = d\Gamma(a^\sigma) + d\Gamma(a_\sigma). \tag{3.42}$$

By (2.8), the following operators in $\mathfrak{F}_\ell$, denoted by A_ℓ^σ and $A_{\sigma\ell}$, respectively,

$$\begin{aligned}A_\ell^\sigma &= 1\otimes 1\otimes d\Gamma(a^\sigma)\otimes 1 + 1\otimes 1\otimes 1\otimes d\Gamma(a^\sigma),\\ A_{\sigma\ell} &= 1\otimes 1\otimes d\Gamma(a_\sigma)\otimes 1 + 1\otimes 1\otimes 1\otimes d\Gamma(a_\sigma)\end{aligned} \tag{3.43}$$

are essentially self-adjoint on $\mathfrak{D}_\ell$.

Let A^σ and A_σ be the following two operators in $\mathfrak{F}_L$:

$$\begin{aligned}A^\sigma &= A_1^\sigma\otimes 1_2\otimes 1_3 + 1_1\otimes A_2^\sigma\otimes 1_3 + 1_1\otimes 1_2\otimes A_3^\sigma,\\ A_\sigma &= A_{\sigma 1}\otimes 1_2\otimes 1_3 + 1_1\otimes A_{\sigma 2}\otimes 1_3 + 1_1\otimes 1_2\otimes A_{\sigma 3}.\end{aligned} \tag{3.44}$$

The operators A^σ and A_σ are essentially self-adjoint on $\mathfrak{D}_L$. Still denoting by A^σ and A_σ their extensions to $\mathfrak{F}$, A^σ and A_σ are essentially self-adjoint on $\mathfrak{D}$ and we still denote by A^σ and A_σ their closures.

We have

$$A = A^\sigma + A_\sigma. \tag{3.45}$$

The operators a, a^σ, and a_σ are associated to the following C^∞-vector fields in $\mathbb{R}^3$, respectively:

$$\begin{aligned} v(p_2) &= p_2, \\ v^\sigma(p_2) &= \eta^\sigma(p_2)^2 p_2, \\ v_\sigma(p_2) &= \eta_\sigma(p_2)^2 p_2. \end{aligned} \tag{3.46}$$

Let $\mathcal{V}(p)$ be any of these vector fields. We have

$$|\mathcal{V}(p)| \le \Gamma |p| \tag{3.47}$$

for some $\Gamma > 0$, and we also have

$$\mathcal{V}(p) = \tilde{v}(|p|)p, \tag{3.48}$$

where the $\tilde{v}$'s are defined by (3.46) and (3.48) and fulfil $|p|^\alpha (\mathrm{d}^\alpha/\mathrm{d}|p|^\alpha)\tilde{v}(|p|)$ bounded for $\alpha = 0, 1, 2$.

Let $\psi_t(\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ be the corresponding flow generated by $\mathcal{V}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(p) &= \mathcal{V}(\psi_t(p)), \\ \psi_0(p) &= p. \end{aligned} \tag{3.49}$$

$\psi_t(p)$ is a C^∞-flow and we have

$$\mathrm{e}^{-\Gamma|t|}|p| \le |\psi_t(p)| \le \mathrm{e}^{\Gamma|t|}|p|. \tag{3.50}$$

$\psi_t(p)$ induces a one-parameter group of unitary operators $U(t)$ in $L^2(\Sigma_1) \simeq L^2(\mathbb{R}^3, \mathbb{C}^2)$ defined by

$$(U(t)f)(p) = f(\psi_t(p))(\det \nabla\psi_t(p))^{1/2}. \tag{3.51}$$

Let $\phi_t(\cdot)$, $\phi_t^\sigma(\cdot)$, and $\phi_{\sigma t}(\cdot)$ be the flows associated with the vector fields $v(\cdot)$, $v^\sigma(\cdot)$, and $v_\sigma(\cdot)$, respectively.

Let $U(t)$, $U^\sigma(t)$, and $U_\sigma(t)$ be the corresponding one-parameter groups of unitary operators in $L^2(\Sigma_1)$. The operators a, a^σ, and a_σ are the generators of $U(t)$, $U^\sigma(t)$, and $U_\sigma(t)$, respectively, that is,

$$\begin{aligned} U(t) &= \mathrm{e}^{-iat}, \\ U^\sigma(t) &= \mathrm{e}^{-ia^\sigma t}, \\ U_\sigma(t) &= \mathrm{e}^{-ia_\sigma t}. \end{aligned} \tag{3.52}$$

Let

$$\begin{aligned} w^{(2)}(\xi_2) &= \left(w_\ell^{(2)}(\xi_2)\right)_{\ell=1,2,3}, \\ \mathrm{d}\Gamma\left(w^{(2)}\right) &= \sum_{\ell=1}^{3}\sum_{\epsilon}\int w_\ell^{(2)}(\xi_2)c_{\ell,\epsilon}^*(\xi_2)c_{\ell\epsilon}(\xi_2)\mathrm{d}\xi_2. \end{aligned} \tag{3.53}$$

Let $V(t)$ be any of the one-parameter groups $U(t)$, $U^\sigma(t)$, and $U_\sigma(t)$. We set

$$V(t)w^{(2)}V(t)^* = \left(V(t)w_\ell^{(2)}V(t)^*\right)_{\ell=1,2,3}, \tag{3.54}$$

and we have

$$V(t)w^{(2)}V(t)^* = w^{(2)}(\varphi_t). \tag{3.55}$$

Here φ_t is the flow associated to $V(t)$.

This yields, for any $\varphi \in \mathfrak{D}$, (see [11, Lemma 2.8])

$$\begin{aligned} \mathrm{e}^{-iAt}H_0\mathrm{e}^{iAt}\varphi - H_0\varphi &= \left(\mathrm{d}\Gamma\left(\mathrm{e}^{-iat}w^{(2)}\mathrm{e}^{iat}\right) - \mathrm{d}\Gamma\left(w^{(2)}\right)\right)\varphi \\ &= \left(\mathrm{d}\Gamma\left(\mathrm{w}^{(2)}\circ\phi_t - w^{(2)}\right)\right)\varphi, \end{aligned} \tag{3.56}$$

$$\begin{aligned} \mathrm{e}^{-iA^\sigma t}H_0\mathrm{e}^{iA^\sigma t}\varphi - H_0\varphi &= \left(\mathrm{d}\Gamma\left(\mathrm{e}^{-ia^\sigma t}w^{(2)}\mathrm{e}^{ia^\sigma t}\right) - \mathrm{d}\Gamma\left(w^{(2)}\right)\right)\varphi \\ &= \left(\mathrm{d}\Gamma\left(w^{(2)}\circ\phi_t^\sigma - w^{(2)}\right)\right)\varphi, \end{aligned} \tag{3.57}$$

$$\begin{aligned} \mathrm{e}^{-iA_\sigma t}H_0\mathrm{e}^{iA_\sigma t}\varphi - H_0\varphi &= \left(\mathrm{d}\Gamma\left(\mathrm{e}^{-ia_\sigma t}w^{(2)}\mathrm{e}^{ia_\sigma t}\right) - \mathrm{d}\Gamma\left(w^{(2)}\right)\right)\varphi \\ &= \left(\mathrm{d}\Gamma\left(w^{(2)}\circ\phi_{\sigma t} - w^{(2)}\right)\right)\varphi. \end{aligned} \tag{3.58}$$

Proposition 3.6. *Suppose that the kernels $G_{\ell,\epsilon,\epsilon'}^{(\alpha)}$ satisfy Hypothesis 2.1.*

For every $t \in \mathbb{R}$ one has, for $g \leq g_1$,

$$\begin{aligned} &\text{(i)} \quad e^{itA}\mathfrak{D}(H_0) = e^{itA}\mathfrak{D}(H) \subset \mathfrak{D}(H_0) = \mathfrak{D}(H), \\ &\text{(ii)} \quad e^{itA^{\sigma}}\mathfrak{D}(H_0) = e^{itA^{\sigma}}\mathfrak{D}(H) \subset \mathfrak{D}(H_0) = \mathfrak{D}(H), \\ &\text{(iii)} \quad e^{itA_{\sigma}}\mathfrak{D}(H_0) = e^{itA_{\sigma}}\mathfrak{D}(H) \subset \mathfrak{D}(H_0) = \mathfrak{D}(H). \end{aligned} \tag{3.59}$$

Proof. We only prove (i), since (ii) and (iii) can be proved similarly. By (3.56) we have, for $\varphi \in \mathfrak{D}$,

$$\mathrm{e}^{-itA} H_0 \mathrm{e}^{itA}\varphi = \left(H_0^{(1)} + H_0^{(3)} + \mathrm{d}\Gamma\left(w^{(2)} \circ \phi_t\right)\right)\varphi. \tag{3.60}$$

It follows from (3.50) and (3.60) that

$$\left\|H_0 \mathrm{e}^{itA}\varphi\right\| \leq \mathrm{e}^{\Gamma|t|}\|H_0\varphi\|. \tag{3.61}$$

This yields (i) because $\mathfrak{D}$ is a core for H_0. Moreover we get

$$\left\|H_0 \mathrm{e}^{itA}(H_0+1)^{-1}\right\| \leq \mathrm{e}^{\Gamma|t|}. \tag{3.62}$$

In view of $\mathfrak{D}(H_0) = \mathfrak{D}(H)$, the operators $H_0(H+i)^{-1}$ and $H(H_0+i)^{-1}$ are bounded, and there exists a constant $C > 0$ such that

$$\left\|H \mathrm{e}^{itA}(H+i)^{-1}\right\| \leq C\mathrm{e}^{\Gamma|t|}. \tag{3.63}$$

Similarly, we also get

$$\begin{aligned} &\left\|H_0 \mathrm{e}^{itA^{\sigma}}(H_0+1)^{-1}\right\| \leq \mathrm{e}^{\Gamma|t|}, \\ &\left\|H_0 \mathrm{e}^{itA_{\sigma}}(H_0+1)^{-1}\right\| \leq \mathrm{e}^{\Gamma|t|}, \\ &\left\|H \mathrm{e}^{itA^{\sigma}}(H+i)^{-1}\right\| \leq C\mathrm{e}^{\Gamma|t|}, \\ &\left\|H \mathrm{e}^{itA_{\sigma}}(H+i)^{-1}\right\| \leq C\mathrm{e}^{\Gamma|t|}. \end{aligned} \tag{3.64}$$

□

Let $H_I(G)$ be the interaction associated with the kernels $G = (G^{(\alpha)}_{\ell,\epsilon,\epsilon'})_{\alpha=1,2;\ell=1,2,3;\epsilon\neq\epsilon'=\pm}$, where the kernels $(G^{(\alpha)}_{\ell,\epsilon,\epsilon'})$ satisfy Hypothesis 2.1.

We set

$$V(t)G = \left(V(t)G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)_{\alpha=1,2;\ell=1,2,3;\epsilon\neq\epsilon'=\pm}. \tag{3.65}$$

We have for $\varphi \in \mathfrak{D}$ (see [11, Lemma 2.7]),

$$\begin{aligned} e^{-iAt}H_I(G)e^{iAt}\varphi &= H_I\left(e^{-iat}G\right)\varphi, \\ e^{-iA^\sigma t}H_I(G)e^{iA^\sigma t}\varphi &= H_I\left(e^{-ia^\sigma t}G\right)\varphi, \\ e^{-iA_\sigma t}H_I(G)e^{iA_\sigma t}\varphi &= H_I\left(e^{-ia_\sigma t}G\right)\varphi. \end{aligned} \tag{3.66}$$

According to [5, 6], in order to prove Theorem 3.4 we must prove that H is locally of class $C^2(A^\sigma)$, $C^2(A_\sigma)$, and $C^2(A)$ in $(-\infty, m_1 - \delta/2)$ and that A and A_σ are locally strictly conjugate to H in $(E, m_1 - \delta/2)$.

Recall that H is locally of class $C^2(A)$ in $(-\infty, m_1 - \delta/2)$ if, for any $\varphi \in C_0^\infty((-\infty, m_1 - \delta/2))$, $\varphi(H)$ is of class $C^2(A)$; that is, $t \to e^{-iAt}\varphi(H)e^{itA}\psi$ is twice continuously differentiable for all $\varphi \in C_0^\infty((-\infty, m_1 - \delta/2))$ and all $\psi \in \mathfrak{F}$.

Thus, one of our main results is the following one.

Theorem 3.7. *Suppose that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypotheses 2.1, 3.1(i) and 3.1(iii).*

(a) *H is locally of class $C^2(A)$, $C^2(A^\sigma)$, and $C^2(A_\sigma)$ in $(-\infty, m_1 - \delta/2)$.*

(b) *H^σ is locally of class $C^2(A^\sigma)$ in $(-\infty, m_1 - \delta/2)$.*

It follows from Theorem 3.7 that $[H, iA]$, $[H, iA_\sigma]$, $[H, iA^\sigma]$, and $[H^\sigma, iA^\sigma]$ are defined as sesquilinear forms on $\bigcup_K E_K(H)\mathfrak{F}$, where the union is taken over all the compact subsets K of $(-\infty, m_1 - \delta/2)$.

Furthermore, by Proposition 3.6, Theorem 3.7 and [4, Lemma 29], we get for all $\varphi \in C_0^\infty((E, m_1 - \delta/2))$ and all $\psi \in \mathfrak{F}$,

$$\begin{aligned} \varphi(H)[H, iA]\varphi(H)\psi &= \lim_{t\to 0}\varphi(H)\left[H, \frac{e^{itA}-1}{t}\right]\varphi(H)\psi, \\ \varphi(H)[H, iA_\sigma]\varphi(H)\psi &= \lim_{t\to 0}\varphi(H)\left[H, \frac{e^{itA_\sigma}-1}{t}\right]\varphi(H)\psi, \\ \varphi(H)[H, iA^\sigma]\varphi(H)\psi &= \lim_{t\to 0}\varphi(H)\left[H, \frac{e^{itA^\sigma}-1}{t}\right]\varphi(H)\psi, \\ \varphi(H^\sigma)[H^\sigma, iA^\sigma]\varphi(H^\sigma)\psi &= \lim_{t\to 0}\varphi(H^\sigma)\left[H^\sigma, \frac{e^{itA^\sigma}-1}{t}\right]\varphi(H^\sigma)\psi. \end{aligned} \tag{3.67}$$

The following proposition allows us to compute $[H, iA]$, $[H, iA^\sigma]$, $[H, iA_\sigma]$, and $[H^\sigma, iA^\sigma]$ as sesquilinear forms. By Hypotheses 2.1 and 3.1(iii.a), the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1, \cdot, \xi_3)$ belong to the domains of a, a^σ, and a_σ.

Proposition 3.8. *Suppose that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypotheses 2.1 and 3.1(iii.a). Then*

(a) *for all $\psi \in \mathfrak{D}(H)$ one has*

(i) $\lim_{t\to 0}[H,(e^{itA}-1)/t]\psi = (d\Gamma(w^{(2)}) + gH_I(-iaG))\psi$,

(ii) $\lim_{t\to 0}[H,(e^{itA^\sigma}-1)/t]\psi = (d\Gamma((\eta^\sigma)^2 w^{(2)}) + gH_I(-ia^\sigma G))\psi$,

(iii) $\lim_{t\to 0}[H,(e^{itA_\sigma}-1)/t]\psi = (d\Gamma((\eta_\sigma)^2 w^{(2)}) + gH_I(-ia_\sigma G))\psi$,

(iv) $\lim_{t\to 0}[H^\sigma,(e^{itA^\sigma}-1)/t]\psi = (d\Gamma((\eta^\sigma)^2 w^{(2)}) + gH_I(-ia^\sigma(\tilde{\chi}^\sigma(p_2)G)))\psi$;

(b) *and*

(i) $\sup_{0<|t|\le 1}\|[H,(e^{itA}-1)/t](H+i)^{-1}\| < \infty$,

(ii) $\sup_{0<|t|\le 1}\|[H,(e^{itA^\sigma}-1)/t](H+i)^{-1}\| < \infty$,

(iii) $\sup_{0<|t|\le 1}\|[H,(e^{itA_\sigma}-1)/t](H+i)^{-1}\| < \infty$,

(iv) $\sup_{0<|t|\le 1}\|[H^\sigma,(e^{itA^\sigma}-1)/t](H+i)^{-1}\| < \infty$.

Proof. Part (b) follows from part (a) by the uniform boundedness principle. For part (a), we only prove (a)(i), since other statements can be proved similarly.

By (3.50), we obtain

$$\frac{1}{|t|}\left|w^{(2)}_\ell(\phi_t(p_2)) - w^{(2)}_\ell(p_2)\right| \le \frac{1}{|t|}\left(e^{\Gamma|t|}-1\right)w^{(2)}_\ell(p_2) \tag{3.68}$$

for $\ell = 1,2,3$.

By (3.56)–(3.58) and Lebesgue's Theorem we then get for all $\psi \in \mathfrak{D}(H_0)$

$$\begin{aligned}
&\lim_{t\to 0}\left[H_0,\frac{e^{itA}-1}{t}\right]\psi = \lim_{t\to 0}\frac{1}{t}\left[e^{-itA}H_0e^{itA} - H_0\right]\psi = d\Gamma\left(w^{(2)}\right)\psi,\\
&\lim_{t\to 0}\left[H_0,\frac{e^{itA^\sigma}-1}{t}\right]\psi = \lim_{t\to 0}\frac{1}{t}\left[e^{-itA^\sigma}H_0e^{itA^\sigma} - H_0\right]\psi = d\Gamma\left(\left(\eta^\sigma\right)^2 w^{(2)}\right)\psi,\\
&\lim_{t\to 0}\left[H_0,\frac{e^{itA_\sigma}-1}{t}\right]\psi = \lim_{t\to 0}\frac{1}{t}\left[e^{-itA_\sigma}H_0e^{itA_\sigma} - H_0\right]\psi = d\Gamma\left(\left(\eta_\sigma\right)^2 w^{(2)}\right)\psi.
\end{aligned} \tag{3.69}$$

By (3.66), we obtain, for all $\psi \in \mathfrak{D}(H)$,

$$\lim_{t\to 0}\left[H_I(G),\frac{e^{itA}-1}{t}\right]\psi = \lim_{t\to 0}\frac{1}{t}\left[e^{-itA}H_I(G)e^{itA} - H_I(G)\right]\psi = H_I(-i(aG))\psi,$$

$$\lim_{t\to 0}\left[H_I(G), \frac{\mathrm{e}^{itA^\sigma}-1}{t}\right]\psi = \lim_{t\to 0}\frac{1}{t}\left[\mathrm{e}^{-itA^\sigma}H_I(G)\mathrm{e}^{itA^\sigma} - H_I(G)\right]\psi = H_I(-i(a^\sigma G))\psi,$$

$$\lim_{t\to 0}\left[H_I(G), \frac{\mathrm{e}^{itA_\sigma}-1}{t}\right]\psi = \lim_{t\to 0}\frac{1}{t}\left[\mathrm{e}^{-itA_\sigma}H_I(G)\mathrm{e}^{itA_\sigma} - H_I(G)\right]\psi = H_I(-i(a_\sigma G))\psi,$$

$$\lim_{t\to 0}\left[H_I(\widetilde{\chi}^\sigma(p_2)G), \frac{\mathrm{e}^{itA^\sigma}-1}{t}\right]\psi = \lim_{t\to 0}\frac{1}{t}\left[\mathrm{e}^{-itA^\sigma}H_I(\widetilde{\chi}^\sigma(p_2)G)\mathrm{e}^{itA^\sigma} - H_I(\widetilde{\chi}^\sigma(p_2)G)\right]\psi$$
$$= H_I\left(-i\left(a^\sigma\left(\widetilde{\chi}^\sigma(p_2)G\right)\right)\right)\psi. \tag{3.70}$$

This concludes the proof of Proposition 3.8. □

Combining (3.67) with Proposition 3.8, we finally get for every $\varphi \in C_0^\infty((-\infty, m_1-\delta/2))$ and every $\psi \in \mathfrak{F}$

$$\varphi(H)[H, iA]\varphi(H)\psi = \varphi(H)\left[\mathrm{d}\Gamma\left(w^{(2)}\right) + gH_I(-i(aG))\right]\varphi(H)\psi, \tag{3.71}$$

$$\varphi(H)[H, iA^\sigma]\varphi(H)\psi = \varphi(H)\left[\mathrm{d}\Gamma\left((\eta^\sigma)^2 w^{(2)}\right) + gH_I(-i(a^\sigma G))\right]\varphi(H)\psi, \tag{3.72}$$

$$\varphi(H)[H, iA_\sigma]\varphi(H)\psi = \varphi(H)\left[\mathrm{d}\Gamma\left((\eta_\sigma)^2 w^{(2)}\right) + gH_I(-i(a_\sigma G))\right]\varphi(H)\psi, \tag{3.73}$$

$$\varphi(H^\sigma)[H^\sigma, iA^\sigma]\varphi(H^\sigma)\psi = \varphi(H^\sigma)\left[\mathrm{d}\Gamma\left((\eta^\sigma)^2 w^{(2)}\right) + gH_I\left(-i\left(a^\sigma\left(\widetilde{\chi}^\sigma G\right)\right)\right)\right]\varphi(H^\sigma)\psi. \tag{3.74}$$

We now introduce the Mourre inequality.

Let N be the smallest integer such that

$$N\gamma \geq 1. \tag{3.75}$$

We have, for $g \leq g_\delta^{(1)}$,

$$\gamma < \gamma + \frac{1}{N}\left(1 - \frac{3g\widetilde{D}}{\gamma} - \gamma\right) < 1 - \frac{3g\widetilde{D}}{\gamma},$$
$$\frac{\gamma}{N} \leq \gamma - \frac{1}{N}\left(1 - \frac{3g\widetilde{D}}{\gamma} - \gamma\right) < \gamma. \tag{3.76}$$

Let

$$\epsilon_\gamma = \frac{1}{2N}\left(1 - \frac{3g_\delta^{(1)}\widetilde{D}}{\gamma} - \gamma\right). \tag{3.77}$$

We choose $f \in C_0^\infty(\mathbb{R})$ such that $1 \geq f \geq 0$ and

$$f(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \left[(\gamma - \epsilon_\gamma)^2, \gamma + \epsilon_\gamma\right], \\ 0 & \text{if } \lambda > \gamma + \dfrac{1}{N}\left(1 - \dfrac{3g_\delta^{(1)}\widetilde{D}}{\gamma} - \gamma\right) = \gamma + 2\epsilon_\gamma, \\ 0 & \text{if } \lambda < \left(\gamma - \dfrac{1}{N}\left(1 - \dfrac{3g_\delta^{(1)}\widetilde{D}}{\gamma} - \gamma\right)\right)^2 = (\gamma - 2\epsilon_\gamma)^2. \end{cases} \tag{3.78}$$

Note that $\gamma + 2\epsilon_\gamma < 1 - 3g\widetilde{D}/\gamma$ for $g \leq g_\delta^{(1)}$ and $\gamma - \epsilon_\gamma > \gamma/N$.

We set, for $n \geq 1$,

$$\mathrm{f}_n(\lambda) = f\left(\frac{\lambda}{\sigma_n}\right). \tag{3.79}$$

Let

$$\begin{aligned} H_n &= H_{\sigma_n}, \\ E_n &= \inf \sigma(H_n), \\ H_{0n}^{(2)} &= H_{0\sigma_n}^{(2)}. \end{aligned} \tag{3.80}$$

Let P^n denote the ground state projection of H^n. It follows from Proposition 3.5 that, for $n \geq 1$ and $g \leq \widetilde{g}_\delta \leq g_\delta^{(1)}$,

$$f_n(H_n - E_n) = P^n \otimes f_n\left(H_{0,n}^{(2)}\right). \tag{3.81}$$

Note that

$$E_n = E^n = \inf \sigma(H^n). \tag{3.82}$$

Set

$$\begin{aligned} a^n &= a^{\sigma_n}, \\ a_n &= a_{\sigma_n}, \\ A^n &= A^{\sigma_n}, \\ A_n &= A_{\sigma_n}, \\ \mathfrak{F}^n &= \mathfrak{F}^{\sigma_n}, \\ \mathfrak{F}_n &= \mathfrak{F}_{\sigma_n}. \end{aligned} \tag{3.83}$$

We have

$$\mathfrak{F} \simeq \mathfrak{F}^n \otimes \mathfrak{F}_n, \qquad A = A^n + A_n. \tag{3.84}$$

We further note that

$$a^n \widetilde{\chi}^{\sigma_n}(p_2) = a^n. \tag{3.85}$$

By (3.72), (3.74), and (3.85), we obtain

$$[H, iA^n] = [H^n, iA^n] \otimes 1 \tag{3.86}$$

as sesquilinear forms with respect to $\mathfrak{F} = \mathfrak{F}^n \otimes \mathfrak{F}_n$.

Furthermore, it follows from the virial theorem (see [6, Proposition 3.2] and Proposition 6.1) that

$$P^n [H^n, \mathrm{i}A^n] P^n = 0. \tag{3.87}$$

By (3.81) and (3.87) we then get, for $g \leq \widetilde{g}_\delta \leq g_\delta^{(1)}$,

$$f_n(H_n - E_n)[H, iA^n] f_n(H_n - E_n) = 0. \tag{3.88}$$

We then have the following.

Proposition 3.9. *Suppose that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypotheses 2.1 and 3.1. Then there exists $\widetilde{C}_\delta > 0$ and $\widetilde{g}_\delta^{(1)} > 0$ such that $\widetilde{g}_\delta^{(1)} \leq \widetilde{g}_\delta$ and*

$$f_n(H_n - E_n)[H, iA_n] f_n(H_n - E_n) \geq \widetilde{C}_\delta \frac{\gamma^2}{N^2} \sigma_n f_n(H_n - E_n)^2 \tag{3.89}$$

for $n \geq 1$ and $g \leq \widetilde{g}_\delta^{(1)}$.

Let $E_\Delta(H - E)$ be the spectral projection for the operator $H - E$ associated with the interval Δ, and let

$$\Delta_n = \left[(\gamma - \epsilon_\gamma)^2 \sigma_n, (\gamma + \epsilon_\gamma)\sigma_n \right], \quad n \geq 1. \tag{3.90}$$

Note that

$$[\sigma_{n+2}, \sigma_{n+1}] \subset \left((\gamma - \epsilon_\gamma)^2 \sigma_n, (\gamma + \epsilon_\gamma)\sigma_n \right), \quad n \geq 1. \tag{3.91}$$

Theorem 3.10. *Suppose that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypotheses 2.1 and 3.1. Then there exists $C_\delta > 0$ and $\widetilde{g}^{(2)}_\delta > 0$ such that $\widetilde{g}^{(2)}_\delta \leq \widetilde{g}^{(1)}_\delta$ and*

$$E_{\Delta_n}(H-E)[H,iA]E_{\Delta_n}(H-E) \geq C_\delta \frac{\gamma^2}{N^2}\sigma_n E_{\Delta_n}(H-E) \tag{3.92}$$

for $n \geq 1$ and $g \leq \widetilde{g}^{(2)}_\delta$.

4. Existence of a Ground State and Location of the Absolutely Continuous Spectrum

We now prove Theorem 3.3. The scheme of the proof is quite well known (see [9, 31]). It follows from Proposition 3.5 that H^n has a unique ground state, denoted by ϕ^n, in $\mathfrak{F}^n$:

$$H^n\phi^n = E^n\phi^n, \quad \phi^n \in \mathfrak{D}(H^n), \quad \left\|\phi^n\right\| = 1, \quad n \geq 1. \tag{4.1}$$

Therefore H_n has an unique normalized ground state in $\mathfrak{F}$, given by $\widetilde{\phi}_n = \phi^n \otimes \Omega_n$, where Ω_n is the vacuum state in $\mathfrak{F}_n$:

$$H_n\widetilde{\phi}_n = E^n\widetilde{\phi}_n, \quad \widetilde{\phi}_n \in \mathfrak{D}(H_n), \quad \left\|\widetilde{\phi}_n\right\| = 1, \quad n \geq 1. \tag{4.2}$$

Since $\|\widetilde{\phi}_n\| = 1$, there exists a subsequence $(n_k)_{k\geq 1}$, converging to ∞ such that $(\widetilde{\phi}_{n_k})_{k\geq 1}$ converges weakly to a state $\widetilde{\phi} \in \mathfrak{F}$. We have to prove that $\widetilde{\phi} \neq 0$. By adapting the proof of Theorem 4.1 in [22] (see also [20]), the key point is to estimate $\|c_{\ell,\epsilon}(\xi_2)\widetilde{\Phi}_n\|_{\mathfrak{F}}$ in order to show that

$$\sum_{\ell=1}^{3}\sum_{\epsilon}\int\left\|c_{\ell,\epsilon}(\xi_2)\widetilde{\phi}_n\right\|^2 d\xi_2 = \mathcal{O}\left(g^2\right), \tag{4.3}$$

uniformly with respect to n.

The estimate (4.3) is a consequence of the so-called "pull-through" formula as it follows.

Let $H_{I,n}$ denote the interaction H_I associated with the kernels $1_{\{|p_2|\geq\sigma_n\}}(p_2)G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$. We thus have

$$\begin{aligned} H_0c_{\ell,\epsilon}(\xi_2)\widetilde{\phi}_n &= c_{\ell,\epsilon}(\xi_2)H_0\widetilde{\phi}_n - w^{(2)}_\ell(\xi_2)c_{\ell,\epsilon}(\xi_2)\widetilde{\phi}_n, \\ gH_{I,n}c_{\ell,\epsilon}(\xi_2)\widetilde{\phi}_n &= c_{\ell,\epsilon}(\xi_2)gH_{I,n}\widetilde{\phi}_n + gV_{\ell,\epsilon,\epsilon'}(\xi_2)\widetilde{\phi}_n \end{aligned} \tag{4.4}$$

with

$$V_{\ell,\epsilon,\epsilon'}(\xi_2) = g\int G^{(1)}_{\ell,\epsilon'\epsilon}(\xi_1,\xi_2,\xi_3)b^*_{\ell,\epsilon'}(\xi_1)a_\epsilon(\xi_3)\mathrm{d}\xi_1\mathrm{d}\xi_3 \\ + g\int G^{(2)}_{\ell,\epsilon'\epsilon}(\xi_1,\xi_2,\xi_3)b^*_{\ell,\epsilon'}(\xi_1)a^*_\epsilon(\xi_3)\mathrm{d}\xi_1\mathrm{d}\xi_3. \tag{4.5}$$

This yields

$$\left(H_n - E_n + w^{(2)}_\ell(\xi_2)\right)c_{\ell,\epsilon}(\xi_2)\widetilde{\phi}_n = V_{\ell,\epsilon,\epsilon'}(\xi_2)\widetilde{\phi}_n. \tag{4.6}$$

By adapting the proof of Propositions 2.4 and 2.5 we easily get

$$\left\|V_{\ell,\epsilon,\epsilon'}\psi\right\|_{\mathfrak{F}} \leq \frac{g}{m_W{}^{1/2}}\left(\sum_{\alpha=1,2}\left\|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\cdot,\xi_2,\cdot)\right\|_{L^2(\Sigma_1\times\Sigma_2)}\right)\left\|H_0^{1/2}\psi\right\| \\ + g\left\|G^{(2)}_{\ell,\epsilon,\epsilon'}(\cdot,\xi_2,\cdot)\right\|_{L^2(\Sigma_1\times\Sigma_2)}\|\psi\|, \tag{4.7}$$

where $\psi \in \mathfrak{D}(H_0)$.

Let us estimate $\|H_0\widetilde{\phi}_n\|$. By (2.53), (2.54), (3.26), and (3.28) we have

$$g\left\|H_{I,n}\widetilde{\phi}_n\right\| \leq gK(G)\left(C_{\beta\eta}\left\|H_0\widetilde{\phi}_n\right\| + B_{\beta\eta}\right), \\ \left\|H_0\widetilde{\phi}_n\right\| \leq |E_n| + g\left\|H_{I,n}\widetilde{\phi}_n\right\|. \tag{4.8}$$

Therefore

$$\left\|H_0\widetilde{\phi}_n\right\| \leq \frac{|E_n|}{1-g_1K(G)C_{\beta\eta}} + \frac{gK(G)B_{\beta\eta}}{1-g_1K(G)C_{\beta\eta}}. \tag{4.9}$$

By (3.82), (A.11), and (4.9), there exists $C > 0$ such that

$$\left\|H_0\widetilde{\phi}_n\right\| \leq C, \tag{4.10}$$

uniformly in n and $g \leq g_1$.

By (4.6), (4.7), and (4.10) we get

$$\left\|c_{\ell,\epsilon}\widetilde{\phi}_n\right\| \leq \frac{g}{|p_2|}\left(C^{1/2}\left(\sum_{\alpha=1}^{2}\left\|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\cdot,\xi_2,\cdot)\right\|_{L^2(\Sigma_1\times\Sigma_2)}\right) + \left\|G^{(2)}_{\ell,\epsilon,\epsilon'}(\cdot,\xi_2,\cdot)\right\|_{L^2(\Sigma_1\times\Sigma_2)}\right). \tag{4.11}$$

By Hypothesis 3.1(i), there exists a constant $C(G) > 0$ depending on the kernels $G = (G^{(\alpha)}_{\ell,\epsilon,\epsilon'})_{\ell=1,2,3;\alpha=1,2;\epsilon\neq\epsilon'=\pm}$ and such that

$$\sum_{\ell=1}^{3}\sum_{\epsilon}\int\left\|c_{\ell,\epsilon}(\xi_2)\widetilde{\phi}_n\right\|^2 d\xi_2 \leq C(G)^2 g^2. \tag{4.12}$$

The existence of a ground state $\widetilde{\phi}$ for H follows by choosing g sufficiently small, that is, $g \leq g_2$, as in [20, 22]. By adapting the method developed in [32] (see [32, Corollary 3.4]), one proves that the ground state of H is unique. We omit here the details.

Statements about $\sigma(H)$ are consequences of the existence of a ground state and follows from the existence of asymptotic Fock representations for the CAR associated with the $c^{\sharp}_{\ell,\epsilon}(\xi_2)$'s. For $f \in L^2(\mathbb{R}^3,\mathbb{C}^2)$, we define on $\mathfrak{D}(H_0)$ the operators

$$c^{\sharp t}_{\ell,\epsilon}(f) = e^{itH}e^{-itH_0}c^{\sharp}_{\ell,\epsilon}(f)e^{itH_0}e^{-itH}. \tag{4.13}$$

By mimicking the proof given in [21, 31] one proves, under the hypothesis of Theorem 3.3 and for $f \in C_0^{\infty}(\mathbb{R}^3,\mathbb{C}^2)$, that the strong limits of $c^{\sharp t}_{\ell,\epsilon}(f)$ when $t \to \pm\infty$ exist for $\psi \in \mathfrak{D}(H_0)$:

$$\lim_{t\to\pm\infty} c^{\sharp t}_{\ell,\epsilon}(f)\psi := c^{\sharp\pm}_{\ell,\epsilon}(f)\psi. \tag{4.14}$$

The operators $c^{\sharp\pm}_{\ell,\epsilon}(f)$ satisfy the CAR and we have

$$c^{\pm}_{\ell,\epsilon}(f)\widetilde{\phi} = 0, \quad f \in C_0^{\infty}\left(\mathbb{R}^3,\mathbb{C}^2\right), \tag{4.15}$$

where $\widetilde{\phi}$ is the ground state of H.

It then follows from (4.14) and (4.15) that the absolutely continuous spectrum of H equals to $[\inf\sigma(H),\infty)$. We omit the details (see [21, 31]).

5. Proof of the Mourre Inequality

We first prove Proposition 3.9. In view of Proposition 3.8(a)(iii) and (3.73), we have, as sesquilinear forms,

$$[H,iA_\sigma] = (1-g)d\Gamma\left((\eta_\sigma)^2 w^{(2)}\right) + g\left(d\Gamma\left((\eta_\sigma)^2 w^{(2)}\right)\right) + gH_I(-i(a_\sigma G)). \tag{5.1}$$

Let $\mathfrak{F}^{(1)}_\ell$ (resp., $\mathfrak{F}^{(2)}_\ell$) be the Fock space for the massive leptons ℓ (resp., the neutrinos and antineutrinos ℓ).

We have

$$\mathfrak{F}_\ell \simeq \mathfrak{F}^{(1)}_\ell \otimes \mathfrak{F}^{(2)}_\ell. \tag{5.2}$$

Let

$$\mathfrak{F}^{(1)} = \mathfrak{F}_W \otimes \left(\bigotimes_{\ell=1}^{3} \mathfrak{F}_\ell^{(1)} \right), \qquad \mathfrak{F}^{(2)} = \bigotimes_{\ell=1}^{3} \mathfrak{F}_\ell^{(2)}. \tag{5.3}$$

We have

$$\mathfrak{F} \simeq \mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(2)}, \tag{5.4}$$

$\mathfrak{F}^{(1)}$ is the Fock space for the massive leptons and the bosons $W^{\pm}$, and $\mathfrak{F}^{(2)}$ is the Fock space for the neutrinos and antineutrinos.

We have, as sesquilinear forms and with respect to (5.4),

$$\begin{aligned}
&\mathrm{d}\Gamma\Big((\eta_\sigma)^2 w^{(2)}\Big) + H_I(-i(a_\sigma G)) \\
&\quad = \sum_{\ell=1}^{3} \sum_{\epsilon} \int \eta_\sigma(p_2)^2 |p_2| c^*_{\ell,\epsilon}(\xi_2) c_{\ell,\epsilon}(\xi_2) \mathrm{d}\xi_2 \\
&\qquad + \sum_{\ell=1}^{3} \sum_{\epsilon \neq \epsilon'} \int |p_2| \left(1_1 \otimes \eta_\sigma(p_2) c^*_{\ell,\epsilon}(\xi_2) + \sum_{\alpha=1,2} \frac{\mathcal{M}^{(\alpha)*}_{\ell,\epsilon,\epsilon',\sigma}(\xi_2)}{|p_2|} \otimes 1_2 \right) \\
&\qquad \times \left(1_1 \otimes \eta_\sigma(p_2) c_{\ell,\epsilon}(\xi_2) + \sum_{\alpha=1,2} \frac{\mathcal{M}^{(\alpha)}_{\ell,\epsilon,\epsilon',\sigma}(\xi_2)}{|p_2|} \otimes 1_2 \right) \mathrm{d}\xi_2 \\
&\qquad - \sum_{\ell=1}^{3} \sum_{\epsilon \neq \epsilon'} \int \left(\sum_{\alpha=1,2} \frac{\mathcal{M}^{(\alpha)*}_{\ell,\epsilon,\epsilon',\sigma}(\xi_2)}{|p_2|^{1/2}} \otimes 1_2 \right) \left(\sum_{\alpha=1,2} \frac{\mathcal{M}^{(\alpha)}_{\ell,\epsilon,\epsilon',\sigma}(\xi_2)}{|p_2|^{1/2}} \otimes 1_2 \right) \mathrm{d}\xi_2,
\end{aligned} \tag{5.5}$$

where

$$\mathcal{M}^{(\alpha)}_{\ell,\epsilon,\epsilon',\sigma}(\xi_2) = i \int \left(\sum_{\alpha=1,2} \left(a\eta_\sigma(p_2) G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3) \right) \right) b^*_{\ell,\epsilon'}(\xi_1) a_{\epsilon'}(\xi_3) \mathrm{d}\xi_1 \mathrm{d}\xi_3, \tag{5.6}$$

and where 1_j is the identity operator in $\mathfrak{F}^{(j)}$.

By mimicking the proofs of Propositions 2.4 and 2.5, we get, for every $\psi \in \mathfrak{D}$,

$$\begin{aligned}
&\sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'}\left(\psi, \int\left(\sum_{\alpha=1,2}\frac{\mathcal{M}^{(\alpha)*}_{\ell,\epsilon,\epsilon',\sigma}(\xi_2)}{|p_2|^{1/2}}\otimes 1_2\right)\left(\sum_{\alpha=1,2}\frac{\mathcal{M}^{(\alpha)}_{\ell,\epsilon,\epsilon',\sigma}(\xi_2)}{|p_2|^{1/2}}\otimes 1_2\right)\psi \mathrm{d}\xi_2\right)\\
&\quad=\sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'}\left\|\int\left(\sum_{\alpha=1,2}\frac{\mathcal{M}^{\alpha}_{\ell,\epsilon,\epsilon',\sigma}(\xi_2)}{|p_2|^{1/2}}\otimes 1_2\right)\psi \mathrm{d}\xi_2\right\|^2\\
&\quad\leq\left(\int\frac{\left|\sum_{\alpha=1,2}\left(a\eta_\sigma(p_2)G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)(\xi_1,\xi_2,\xi_3)\right|^2}{w^{(3)}(\xi_3)|p_2|}\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\right)\left\|\left(H_0^{(3)}\right)^{1/2}\psi\right\|.
\end{aligned}\tag{5.7}$$

Noting that $|(a\eta_\sigma)(p_2)| \leq C$ uniformly with respect to σ, it follows from Hypotheses 2.1 and 3.1 that there exists a constant $C(G) > 0$ such that

$$\int\frac{\left|\sum_{\alpha=1,2}\left(a\eta_\sigma(p_2)G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)(\xi_1,\xi_2,\xi_3)\right|^2}{w^{(3)}(\xi_3)|p_2|}\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \leq C(G)\sigma. \tag{5.8}$$

This yields

$$-\int\left(\sum_{\alpha=1,2}\frac{\mathcal{M}^{(\alpha)*}_{\ell,\epsilon,\epsilon',\sigma}(\xi_2)}{|p_2|^{1/2}}\otimes 1_2\right)\left(\sum_{\alpha=1,2}\frac{\mathcal{M}^{(\alpha)}_{\ell,\epsilon,\epsilon',\sigma}(\xi_2)}{|p_2|^{1/2}}\otimes 1_2\right)\mathrm{d}\xi_2 \geq -C(G)\sigma. \tag{5.9}$$

Combining (5.1), (5.5) with (5.9), we obtain

$$[H, iA_n] \geq (1-g)\mathrm{d}\Gamma\left(\left(\eta_{\sigma_n}\right)^2 w^{(2)}\right) - gC(G)\sigma_n. \tag{5.10}$$

We have

$$\mathrm{d}\Gamma\left(\left(\eta_{\sigma_n}\right)^2 w^{(2)}\right) \geq H_{0n}^{(2)}. \tag{5.11}$$

By (3.76), (3.81), and (5.11) we get

$$\begin{aligned}
&f_n(H_n-E_n)\mathrm{d}\Gamma\left({\eta_{\sigma_n}}^2 w^{(2)}\right)f_n(H_n-E_n)\\
&\quad\geq P_n\otimes f_n\left(H_{0n}^{(2)}\right)H_{0n}^{(2)}f_n\left(H_{0n}^{(2)}\right)\\
&\quad\geq \frac{\gamma^2}{N^2}\sigma_n f_n(H_n-E_n)^2
\end{aligned}\tag{5.12}$$

for $g \leq g_\delta^{(1)}$.

This, together with (5.10), yields for $g \le g_\delta^{(1)}$

$$\begin{aligned} & f_n(H_n - E_n)[H, iA_n] f_n(H_n - E_n) \\ & \quad \ge \left(1 - g_\delta^{(1)}\right) \frac{\gamma^2}{N^2} \sigma_n f_n(H_n - E_n)^2 - gC(G)\sigma_n f_n(H_n - E_n)^2. \end{aligned} \tag{5.13}$$

Setting

$$g_\delta^{(2)} = \inf\left(g_\delta^{(1)}, \frac{1 - g_\delta^{(1)}}{2C(G)} \frac{\gamma^2}{N^2} \right), \tag{5.14}$$

we get

$$f_n(H_n - E_n)[\mathrm{H}, iA_n] f_n(H_n - E_n) \ge \frac{1 - g_\delta^{(1)}}{2} \frac{\gamma^2}{N^2} \sigma_n f_n(H_n - E_n)^2 \tag{5.15}$$

for $g \le g_\delta^{(2)}$.

Proposition 3.9 is proved by setting $\tilde{g}_\delta^{(1)} = g_\delta^{(2)}$ and $\tilde{C}_\delta = (1 - g_\delta^{(1)})/2$.

The proof of Theorem 3.10 is the consequence of the following two lemmas.

Lemma 5.1. *Assume that the kernels $G_{\ell,\epsilon,\epsilon'}^{(\alpha)}$ satisfy Hypotheses 2.1 and 3.1(ii). Then there exists a constant $D > 0$ such that*

$$|E - E_n| \le gD{\sigma_n}^2 \tag{5.16}$$

for $n \ge 1$ and $g \le g^{(2)}$.

Proof. Let ϕ (resp., $\tilde{\phi}_n$) be the unique normalized ground state of H (resp., H_n). We have

$$\begin{aligned} E - E_n &\le \left(\tilde{\phi}_n, (H - H_n)\tilde{\phi}_n\right), \\ E_n - E &\le (\phi, (H_n - H)\phi) \end{aligned} \tag{5.17}$$

with

$$H - H_n = gH_I(\chi_{\sigma_n}(p_2)G). \tag{5.18}$$

Combining (2.53) and (2.54) with (3.26)–(3.28) and (5.18), we get

$$\begin{aligned} \left\|(H - H_n)\tilde{\phi}_n\right\| &\le gK(\chi_{\sigma_n}(p_2)G)\left(C_{\beta\eta}\left\|H_0\tilde{\phi}_n\right\| + B_{\beta\eta}\right), \\ \|(H - H_n)\phi\| &\le gK(\chi_{\sigma_n}(p_2)G)(C_{\beta\eta}\|H_0\phi\| + B_{\beta\eta}). \end{aligned} \tag{5.19}$$

It follows from Hypothesis 3.1(ii), (4.10), and (5.19) that there exists a constant $D > 0$ such that

$$\max\left(\left\|(H - H_n)\tilde{\phi}_n\right\|, \|(H - H_n)\phi\|\right) \leq gD\sigma_n{}^2 \tag{5.20}$$

for $n \geq 1$ and $g \leq g^{(2)}$.

By (5.17), this proves Lemma 5.1. □

Lemma 5.2. *Suppose that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypotheses 2.1 and 3.1(ii). Then there exists a constant $C > 0$ such that*

$$\|f_n(H - E) - f_n(H_n - E_n)\| \leq gC\sigma_n \tag{5.21}$$

for $n \geq 1$ and $g \leq g^{(2)}$.

Proof. Let $\tilde{f}(\cdot)$ be an almost analytic extension of $f(\cdot)$ given by (3.78) satisfying

$$\left|\partial_{\overline{z}}\tilde{f}(x + iy)\right| \leq Cy^2. \tag{5.22}$$

Note that $\tilde{f}(x + iy) \in C_0^\infty(\mathbb{R}^2)$. We thus have

$$f(s) = \int \frac{\mathrm{d}\tilde{f}(z)}{z - s}, \quad \mathrm{d}\tilde{f}(z) = -\frac{1}{\pi}\frac{\partial\tilde{f}}{\partial\overline{z}}\mathrm{d}x\,\mathrm{d}y. \tag{5.23}$$

Using the functional calculus based on this representation of $f(s)$, we get

$$f_n(H - E) - f_n(H_n - E_n) = \sigma_n \int \frac{1}{H - E - z\sigma_n}(H - H_n + E_n - E)\frac{1}{H_n - E_n - z\sigma_n}\mathrm{d}\tilde{f}(z). \tag{5.24}$$

Combining (2.53) and (2.54) with (3.26)–(3.28) and Hypothesis 3.1(ii), we get, for every $\psi \in \mathfrak{D}(H_0)$ and for $g \leq g^{(2)}$,

$$g\|H_I(\chi_{\sigma_n}G)\psi\| \leq 2gC\sigma_n{}^2K(G)\left(C_{\beta\eta}\|(H_0 + 1)\psi\| + (C_{\beta\eta} + B_{\beta\eta})\|\psi\|\right). \tag{5.25}$$

This yields

$$g\left\|H_I(\chi_{\sigma_n}(p_2)G)(H_0 + 1)^{-1}\right\| \leq gC_1\sigma_n{}^2 \tag{5.26}$$

for some constant $C_1 > 0$ and for $g \leq g^{(2)}$.

By mimicking the proof of (A.21) we show that there exists a constant $C_2 > 0$ such that

$$\left\|(H_0+1)(H_n-E_n-z\sigma_n)^{-1}\right\| \le C_2\left(1+\frac{1}{|Im\,z|\sigma_n}\right) \tag{5.27}$$

or $g \le g^{(1)}$.

Combining Lemma 5.1 and (5.24) with (5.25)–(5.27) we obtain

$$\|f_n(H-E)-f_n(H_n-E_n)\| \le gC\sigma_n\int\frac{\left|\left(\partial\tilde{f}/\partial\bar{z}\right)(x+iy)\right|}{y^2}\mathrm{d}x\,\mathrm{d}y \tag{5.28}$$

for some constant $C > 0$ and for $g \le g^{(2)}$.

Using (5.22) and $\tilde{f}(x+iy) \in C_0^\infty(\mathbb{R}^2)$ one concludes the proof of Lemma 5.2. □

We now prove Theorem 3.10.

Proof. It follows from Proposition 3.9 that

$$\begin{aligned} &f_n(H_n-E_n)[H,iA]f_n(H_n-E_n) \\ &\quad= f_n(H_n-E_n)[H,iA_n]f_n(H_n-E_n) \ge \tilde{C}_\delta\frac{\gamma^2}{N^2}\sigma_n f_n(H_n-E_n)^2 \end{aligned} \tag{5.29}$$

for $n \ge 1$ and $g \le \tilde{g}_\delta^{(1)}$.

This yields

$$\begin{aligned} &f_n(H-E)[H,iA_n]f_n(H-E) \\ &\quad\ge \tilde{C}_\delta\frac{\gamma^2}{N^2}\sigma_n f_n(H-E)^2 \\ &\qquad- f_n(H-E)[H,iA]\left(f_n(H_n-E_n)-f_n(H-E)\right) \\ &\qquad- \left(f_n(H_n-E_n)-f_n(H-E)\right)[H,iA]f_n(H_n-E_n) \\ &\qquad+ \tilde{C}_\delta\frac{\gamma^2}{N^2}\sigma_n\left(f_n(H_n-E_n)-f_n(H-E)\right)^2 \\ &\qquad+ \tilde{C}_\delta\frac{\gamma^2}{N^2}\sigma_n f_n(H-E)\left(f_n(H_n-E_n)-f_n(H-E)\right) \\ &\qquad+ \tilde{C}_\delta\frac{\gamma^2}{N^2}\sigma_n\left(f_n(H_n-E_n)-f_n(H-E)\right)f_n(H-E). \end{aligned} \tag{5.30}$$

Combining Proposition 3.8(i) and (5.23) with (5.26) and (5.27) we show that $[H, iA]f_n(H_n - E_n)$ and $f_n(H-E)[H, iA]$ are bounded operators uniformly with respect to n. This, together with Lemma 5.2, yields

$$f_n(H-E)[H, iA]f_n(H-E) \geq \widetilde{C}_\delta \frac{\gamma^2}{N^2}\sigma_n f_n(H-E)^2 - \widetilde{C} g \sigma_n \tag{5.31}$$

for some constant $\widetilde{C} > 0$ and for $g \leq \inf(g^{(2)}, \widetilde{g}_\delta^{(1)})$.

Multiplying both sides of (5.31) with $E_{\Delta_n}(H-E)$ we then get

$$E_{\Delta_n}(H-E)[H, iA]E_{\Delta_n}(H-E) \geq \widetilde{C}_\delta \frac{\gamma^2}{N^2}\sigma_n E_{\Delta_n}(H-E) - \widetilde{C} g \sigma_n E_{\Delta_n}(H-E). \tag{5.32}$$

Setting

$$\widetilde{g}_\delta^{(2)} < \inf\left(\frac{\widetilde{C}_\delta}{\widetilde{C}}\frac{\gamma^2}{N^2}, g^{(2)}, \widetilde{g}_\delta^{(1)}\right), \tag{5.33}$$

Theorem 3.10 is proved with $C_\delta = \widetilde{C}_\delta - \widetilde{C}(N^2/\gamma^2)\widetilde{g}_\delta^{(2)} > 0$. □

6. Proof of Theorem 3.7

We set

$$\begin{aligned}
A_t &= \frac{e^{itA-1}}{t}, \\
\mathrm{ad}_{A_t}\cdot &= [A_t, \cdot], \\
A_t^\sigma &= \frac{e^{itA^\sigma} - 1}{t}, \\
A_{\sigma t} &= \frac{e^{itA_\sigma} - 1}{t}.
\end{aligned} \tag{6.1}$$

The fact that H is of class $C^1(A)$, $C^1(A^\sigma)$, and $C^1(A_\sigma)$ in $(-\infty, m_1 - \delta/2)$ is the consequence of the following proposition.

Proposition 6.1. *Suppose that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypotheses 2.1 and 3.1(iii.a). For every $\varphi \in C_0^\infty((-\infty, m_1 - \delta/2))$ and $g \leq g_1$, one then has*

$$\begin{aligned}
&\sup_{0<|t|\leq 1} \left\| [\varphi(H), A_t] \right\| < \infty, \\
&\sup_{0<|t|\leq 1} \left\| [\varphi(H), A_t^\sigma] \right\| < \infty, \\
&\sup_{0<|t|\leq 1} \left\| [\varphi(H), A_{\sigma t}] \right\| < \infty, \\
&\sup_{0<|t|\leq 1} \left\| [\varphi(H^\sigma), A_t^\sigma] \right\| < \infty.
\end{aligned} \tag{6.2}$$

Proof. We use the representation

$$\varphi(H) = \int d\phi(z)(z-H)^{-1}, \tag{6.3}$$

where $\phi(z)$ is an almost analytic extension of φ with

$$\left|\partial_{\bar{z}}\phi(x+iy)\right| \leq C|y|^2, \qquad d\phi(z) = -\frac{1}{\pi}\frac{\partial}{\partial\bar{z}}\phi(z)dx\,dy. \tag{6.4}$$

Note that $\phi(x+iy) \in C_0^\infty(\mathbb{R}^2)$.

We get

$$\operatorname{ad}_{A_t}\varphi(H) = \int d\phi(z)(z-H)^{-1}[A_t, H](z-H)^{-1}. \tag{6.5}$$

This yields

$$\left\|\operatorname{ad}_{A_t}\varphi(H)\right\| \leq \sup_{0<|t|\leq 1}\left\|[A_t, H](i-H)^{-1}\right\| \int \left|d\phi(z)\right| \left\|(z-H)^{-1}\right\| \left\|(i-H)(z-H)^{-1}\right\|. \tag{6.6}$$

It is easy to prove that

$$\int \left|d\phi(z)\right| \left\|(z-H)^{-1}\right\| \left\|(i-H)(z-H)^{-1}\right\| \leq C\int \frac{\left|d\phi(z)\right|}{|Im\, z|^2} < \infty. \tag{6.7}$$

By Proposition 3.8(b)(i) and (6.7) we finally get, for $g \leq g_1$,

$$\sup_{0<|t|\leq 1}\left\|\operatorname{ad}_{A_t}\varphi(H)\right\| < \infty. \tag{6.8}$$

In a similar way we obtain, for $g \leq g_1$,

$$\begin{aligned} &\sup_{0<|t|\leq 1}\left\|\left[A_t^\sigma, \varphi(H)\right]\right\| < \infty, \\ &\sup_{0<|t|\leq 1}\left\|\left[A_{\sigma t}, \varphi(H)\right]\right\| < \infty, \\ &\sup_{0<|t|\leq 1}\left\|\left[A_t^\sigma, \varphi(H^\sigma)\right]\right\| < \infty. \end{aligned} \tag{6.9}$$

□

The proof of Theorem 3.7 is the consequence of the following proposition.

Proposition 6.2. *Suppose that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypotheses 2.1, 3.1(i) and 3.1(iii). One then has, for $g \le g_1$,*

$$\begin{aligned}
&\sup_{0<|t|\le 1}\left\|[A_t,[A_t,H]](H+i)^{-1}\right\| < \infty,\\
&\sup_{0<|t|\le 1}\left\|[A_t^{\sigma},[A_t^{\sigma},H]](H+i)^{-1}\right\| < \infty,\\
&\sup_{0<|t|\le 1}\left\|[A_{\sigma t},[A_{\sigma t},H]](H+i)^{-1}\right\| < \infty,\\
&\sup_{0<|t|\le 1}\left\|[A_t^{\sigma},[A_t^{\sigma},H^{\sigma}]](H^{\sigma}+i)^{-1}\right\| < \infty.
\end{aligned} \tag{6.10}$$

Proof. We have, for every $\psi \in \mathfrak{D}(H)$,

$$[A_t,[A_t,H]]\psi = \frac{1}{t^2}e^{2itA}\left(e^{-2itA}He^{2itA} - 2e^{-itA}He^{itA} + H\right)\psi. \tag{6.11}$$

By (3.56) we get

$$[A_t,[A_t,H_0]]\psi = \frac{1}{t^2}e^{2itA}\left(d\Gamma\left(w^{(2)}\circ\phi_{2t} - 2w^{(2)}\circ\phi_t + w^{(2)}\right)\right)\psi, \tag{6.12}$$

where, for $\ell = 1,2,3$,

$$\left(w^{(2)}_{\ell}\circ\phi_{2t}\right)(p_2) - 2\left(w^{(2)}_{\ell}\circ\phi_t\right)(p_2) + w^{(2)}_{\ell}(p_2) = |\phi_{2t}(p_2)| - 2|\phi_t(p_2)| + |p_2|. \tag{6.13}$$

We further note that

$$\begin{aligned}
\frac{1}{t^2}\big||\phi_{2t}(p_2)| - 2|\phi_t(p_2)| + |p_2|\big| &\le \sup_{|s|\le 2|t|}\left|\frac{\partial^2}{\partial s^2}|\phi_s(p_2)|\right|,\\
\frac{\partial^2}{\partial s^2}|\phi_s(p_2)| = |\phi_s(p_2)| &\le e^{\Gamma|s|}|p_2|.
\end{aligned} \tag{6.14}$$

Combining (6.12) with (6.13) and (6.14) we get

$$\begin{aligned}
&\left\|[A_t,[A_t,H_0]](H_0+1)^{-1}\right\| \le e^{2\Gamma|t|},\\
&\sup_{0<|t|\le 1}\left\|[A_t,[A_t,H_0]](H_0+1)^{-1}\right\| \le e^{2\Gamma}.
\end{aligned} \tag{6.15}$$

In a similar way we obtain

$$\sup_{0<|t|\le 1}\left\|\left[A_t^{\sigma},\left[A_t^{\sigma},H_0\right]\right](H_0+1)^{-1}\right\|\le C\mathrm{e}^{2\Gamma},$$
$$\sup_{0<|t|\le 1}\left\|\left[A_{\sigma t},\left[A_{\sigma t},H_0\right]\right](H_0+1)^{-1}\right\|\le C\mathrm{e}^{2\Gamma}. \tag{6.16}$$

Here C is a positive constant.

Let us now prove that

$$\sup_{0<|t|\le 1}\left\|\left[A_t,\left[A_t,H_I(G)\right]\right](H+i)^{-1}\right\|<\infty. \tag{6.17}$$

By (3.66) and (6.11) we get, for every $\psi\in\mathfrak{D}(H)$,

$$\begin{aligned}
&[A_t,[A_t,H_I(G)]]\psi\\
&\quad=\sum_{\alpha=1,2}\sum_{\ell=1,2,3}\sum_{\epsilon\neq\epsilon'}\frac{\mathrm{e}^{2itA}}{t^2}\left(\mathrm{e}^{-2itA}H_I\left(G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)\mathrm{e}^{2itA}-2\mathrm{e}^{-itA}H_I\left(G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)\mathrm{e}^{itA}+H_I\left(G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)\right)\psi\\
&\quad=\sum_{\alpha=1,2}\sum_{\ell=1,2,3}\sum_{\epsilon\neq\epsilon'}\frac{\mathrm{e}^{2itA}}{t^2}\left(H_I\left(G^{(\alpha)}_{\ell,\epsilon,\epsilon';2t}\right)-2H_I\left(G^{(\alpha)}_{\ell,\epsilon,\epsilon';t}\right)+H_I\left(G^{(\alpha)}_{\ell,\epsilon,\epsilon';0}\right)\right)\psi,
\end{aligned} \tag{6.18}$$

where

$$\begin{aligned}
G^{(\alpha)}_{\ell,\epsilon,\epsilon';t}(\xi_1,\xi_2,\xi_3)&=\left(D\phi_t(p_2)\right)^{1/2}G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1;\phi_t(p_2),s_2;\xi_3)\\
&=\left(e^{-ita}G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)(\xi_1,\xi_2,\xi_3).
\end{aligned} \tag{6.19}$$

Combining (2.53) and (2.54) with (3.26)–(3.28) and (6.18) we get

$$\left\|[A_t,[A_t,H_I(G)]]\psi\right\|\le gK(G_t)\left(C_{\beta\eta}\left\|(H_0+I)\psi\right\|+\left(C_{\beta\eta}+B_{\beta\eta}\right)\|\psi\|\right). \tag{6.20}$$

Here $K(G_t)>0$ and

$$K(G_t)^2=\sum_{\alpha=1,2}\sum_{\ell=1,2,3}\sum_{\epsilon\neq\epsilon'}\frac{1}{t^2}\left\|G^{(\alpha)}_{\ell,\epsilon,\epsilon';2t}-2G^{(\alpha)}_{\ell,\epsilon,\epsilon';t}+G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right\|^2_{L^2(\Sigma_1\times\Sigma_1\times\Sigma_2)}. \tag{6.21}$$

We further note that, for $0 < |t| \leq 1$,

$$K(G_t) \leq \sup_{0<|s|\leq 2} \left(\sum_{\alpha=1,2} \sum_{\ell=1,2,3} \sum_{\epsilon \neq \epsilon'} \left\| \frac{\partial^2}{\partial s^2} G^{(\alpha)}_{\ell,\epsilon,\epsilon';s} \right\|^2_{L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2)} \right)^{1/2}. \tag{6.22}$$

We get

$$\begin{aligned}
\left(\frac{\partial}{\partial t} G^{(\alpha)}_{\ell,\epsilon,\epsilon';t} \right) &= \frac{3}{2} \left(\mathrm{e}^{-ita} G^{(\alpha)}_{\ell,\epsilon,\epsilon'} \right) + \left(\mathrm{e}^{-ita} \left(p_2 \cdot \nabla_{p_2} G^{(\alpha)}_{\ell,\epsilon,\epsilon'} \right) \right), \\
\left(\frac{\partial^2}{\partial t^2} G^{(\alpha)}_{\ell,\epsilon,\epsilon';t} \right) &= \frac{9}{4} \left(\mathrm{e}^{-ita} G^{(\alpha)}_{\ell,\epsilon,\epsilon'} \right) + \frac{7}{2} \left(\mathrm{e}^{-ita} \left(p_2 \cdot \nabla_{p_2} G^{(\alpha)}_{\ell,\epsilon,\epsilon'} \right) \right) \\
&\quad + \sum_{i,j=1,2,3} \mathrm{e}^{-ita} \left(p_{2,i} p_{2,j} \partial^2_{p_{2,i} p_{2,j}} G^{(\alpha)}_{\ell,\epsilon,\epsilon'} \right).
\end{aligned} \tag{6.23}$$

Recall that e^{-ita} is an one parameter group of unitary operators in $L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2)$.

Combining Hypothesis 3.1(iii.a) and (iii.b) with (6.20)–(6.23) we finally get

$$\sup_{0<|t|\leq 1} \left\| [A_t, [A_t, H_I(G)]] (H_0 + 1)^{-1} \right\| < \infty. \tag{6.24}$$

In view of $\mathfrak{D}(H) = \mathfrak{D}(H_0)$ the operators $H_0(H+i)^{-1}$ and $H(H_0 - 1)^{-1}$ are bounded and we obtain

$$\begin{aligned}
&\sup_{0<|t|\leq 1} \left\| [A_t, [A_t, H_0]] (H + i)^{-1} \right\| < \infty, \\
&\sup_{0<|t|\leq 1} \left\| [A_t, [A_t, H_I(G)]] (H + i)^{-1} \right\| < \infty.
\end{aligned} \tag{6.25}$$

This yields

$$\sup_{0<|t|\leq 1} \left\| [A_t, [A_t, H]] (H + i)^{-1} \right\| < \infty \tag{6.26}$$

for $g \leq g_1$.

Let $V(p_2)$ denote any of the two C^∞-vector fields $v^\sigma(p_2)$ and $v_\sigma(p_2)$ and let $\tilde{a}$ denote the corresponding a^σ and a_σ operators. We get

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial t^2}\left(e^{-i\tilde{a}t}G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)\right)(\xi_1,\xi_2,\xi_3)\\
&\quad= \frac{1}{4}\left(e^{-i\tilde{a}t}\left((\operatorname{div} V(p_2))^2 G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)\right)(\xi_1,\xi_2,\xi_3)\\
&\qquad+ \frac{1}{2}\left(e^{-i\tilde{a}t}\left((\operatorname{div} V(p_2))V(p_2)\cdot\nabla_{p_2}G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)\right)(\xi_1,\xi_2,\xi_3)\\
&\qquad+ \frac{1}{2}\left(e^{-i\tilde{a}t}\left(\sum_{i,j=1}^{3}\left(V_i(p_2)\left(\partial^2_{p_{2,i}p_{2,j}}V_j(p_2)\right)\right)G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)\right)(\xi_1,\xi_2,\xi_3)\\
&\qquad+ \frac{1}{2}\left(e^{-i\tilde{a}t}\left(\sum_{i,j=1}^{3}V_i(p_2)\frac{\partial V_j}{\partial p_{2,i}}(p_2)\frac{\partial}{\partial p_{2,j}}G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)\right)(\xi_1,\xi_2,\xi_3)\\
&\qquad+ \frac{1}{2}\left(e^{-i\tilde{a}t}\left(\sum_{i,j=1}^{3}V_i(p_2)V_j(p_2)\frac{\partial^2}{\partial p_{2,i}\partial p_{2,j}}G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\right)\right)(\xi_1,\xi_2,\xi_3).
\end{aligned}\tag{6.27}$$

Combining the properties of the C^∞ fields $v^\sigma(p_2)$ and $v_\sigma(p_2)$ together with Hypotheses 2.1, 3.1(i) and 3.1(iii) we get, from (6.25) and by mimicking the proof of (6.26),

$$\begin{aligned}
&\sup_{0<|t|\le 1}\left\|\left[A_t^\sigma,\left[A_t^\sigma,H\right]\right](H+i)^{-1}\right\|<\infty,\\
&\sup_{0<|t|\le 1}\left\|\left[A_{\sigma t},\left[A_{\sigma t},H\right]\right](H+i)^{-1}\right\|<\infty
\end{aligned}\tag{6.28}$$

for $g \le g_1$.

Similarly, by mimicking the proof of (6.28), we easily get, for $g \le g_1$,

$$\sup_{0<|t|\le 1}\left\|\left[A_t^\sigma,\left[A_t^\sigma,H^\sigma\right]\right](H^\sigma+i)^{-1}\right\|<\infty. \tag{6.29}$$

This concludes the proof of Proposition 6.2. □

We now prove Theorem 3.7.

Proof of Theorem 3.7. In view of [5, Lemma 6.2.3] (see also [4, Proposition 28]), the proof of Theorem 3.7 will follow from Proposition 6.1 and the following estimates:

$$\sup_{0<|t|\le 1}\left\|\left[A_t,\left[A_t,\varphi(H)\right]\right]\right\|<\infty, \tag{6.30}$$

$$\sup_{0<|t|\leq 1} \| [A_t^\sigma, [A_t^\sigma, \varphi(H)]] \| < \infty,$$
$$\sup_{0<|t|\leq 1} \| [A_{\sigma t}, [A_{\sigma t}, \varphi(H)]] \| < \infty, \tag{6.31}$$
$$\sup_{0<|t|\leq 1} \| [A_t^\sigma, [A_t^\sigma, \varphi(H^\sigma)]] \| < \infty$$

for every $\varphi \in C_0^\infty((-\infty, m_1 - \delta/2))$ and for $g \leq g_1$.

Let us prove (6.30). The estimates (6.31) can be proved similarly.

To this end, let ϕ be an almost analytic extension of φ satisfying

$$\left|\partial_{\overline{z}}\phi(x+iy)\right| \leq C|y|^3,$$
$$\varphi(H) = \int (z-H)^{-1} \mathrm{d}\phi(z), \quad \mathrm{d}\phi(z) = -\frac{1}{\pi}\frac{\partial}{\partial \overline{z}}\phi(z)\mathrm{d}x\,\mathrm{d}y. \tag{6.32}$$

It follows that

$$[A_t[A_t, \varphi(H)]]$$
$$= \int \left((z-H)^{-1}[A_t[A_t, H]](z-H)^{-1} + 2(z-H)^{-1}[A_t, H](z-H)^{-1}[A_t, H](z-H)^{-1}\right)\mathrm{d}\phi(z). \tag{6.33}$$

We note that

$$\left\|(H+i)(H-z)^{-1}\right\| \leq \frac{C}{|Im\, z|}, \quad \text{for } z \in \operatorname{supp}\phi. \tag{6.34}$$

We also have

$$\sup_{0<|t|\leq 1}\left\|\int (z-H)^{-1}[A_t[A_t, H]](z-H)^{-1}\mathrm{d}\phi(z)\right\|$$
$$\leq \sup_{0<|t|\leq 1}\int \left\|[A_t[A_t, H]](H+i)^{-1}\right\|\left\|(H+i)(z-H)^{-1}\right\|\frac{|\mathrm{d}\phi(z)|}{|Im\, z|} \tag{6.35}$$
$$\leq C \sup_{0<|t|\leq 1}\left\|[A_t, [A_t, H]](H+i)^{-1}\right\|\int \frac{|\mathrm{d}\phi(z)|}{|Im\, z|^2}.$$

Therefore, combining Proposition 3.8(b)(i) and (6.34) we obtain

$$\begin{aligned}
&\sup_{0<|t|\le 1}\left\|\int \mathrm{d}\phi(z)(H-z)^{-1}[A_t,H](H-z)^{-1}[A_t,H](H-z)^{-1}\right\| \\
&= \sup_{0<|t|\le 1}\left\|\int (H-z)^{-1}[A_t,H](H+i)^{-1}(H+i)(H-z)^{-1}[A_t,H](H+i)^{-1}(H+i)(H-z)^{-1}\right\|\mathrm{d}\phi(z) \\
&\le C\left(\int \frac{|\mathrm{d}\phi(z)|}{|y|^3}\right)\sup_{0<|t|\le 1}\left\|[A_t,H](H+i)^{-1}\right\|^2 < \infty.
\end{aligned} \tag{6.36}$$

Inequality (6.36) together with (6.35) yields (6.30), and H is locally of class $C^2(A)$ on $(-\infty, m_1-\delta/2)$ for $g \le g_1$.

In a similar way it follows from Propositions 3.8(b), 6.1, and 6.2 that H is locally of class $C^2(A^\sigma)$ and $C^2(A_\sigma)$ in $(-\infty, m_1-\delta/2)$ and that H^σ is locally of class $C^2(A^\sigma)$ in $(-\infty, m_1-\delta/2)$, for $g \le g_1$. This ends the proof of Theorem 3.7. □

7. Proof of Theorem 3.4

By (3.91), $\bigcup_{n\ge 1}((\gamma-\epsilon_\gamma)^2\sigma_n, (\gamma+\epsilon_\gamma)\sigma_n))$ is a covering by open sets of any compact subset of $(E, m_1-\delta]$ and of the interval $(E, m_1-\delta]$ itself. Theorem 3.4(i) and (ii) follow from [6, Theorems 0.1 and 0.2] and Theorems 3.7 and 3.10 above with $g_\delta = \tilde{g}_\delta^{(2)}$, where $\tilde{g}_\delta^{(2)}$ is given in Theorem 3.10. Theorem 3.4(iii) follows from [30, Theorem 25].

Appendix

In this appendix, we will prove Proposition 3.5. We apply the method developed in [3] because every infrared cutoff Hamiltonian that one considers has a ground state energy which is a simple eigenvalue.

Let, for $n \ge 0$,

$$\begin{aligned}
\mathfrak{F}^{\sigma_n} &= \mathfrak{F}^n, \\
\Sigma_{1n}^{n+1} &= \Sigma_1 \cap \{p_2; \sigma_{n+1} \le |p_2| < \sigma_n\}, \\
\mathfrak{F}_{\ell,2,n}^{n+1} &= \mathfrak{F}_a\left(L^2\left(\Sigma_{1n}^{n+1}\right)\right) \otimes \mathfrak{F}_a\left(L^2\left(\Sigma_{1n}^{n+1}\right)\right), \\
\mathfrak{F}_n^{n+1} &= \bigotimes_{\ell=1}^{3} \mathfrak{F}_{\ell,2,n}^{n+1}.
\end{aligned} \tag{A.1}$$

We have

$$\mathfrak{F}^{n+1} \simeq \mathfrak{F}^n \otimes \mathfrak{F}_n^{n+1}. \tag{A.2}$$

Let Ω^n (resp., Ω_n^{n+1}) be the vacuum state in $\mathfrak{F}^n$ (resp., in $\mathfrak{F}_n^{n+1}$). We now set

$$H_{0n}^{n+1} = H_0^{(1)} + H_0^{(3)} + \sum_{\ell=1}^{3} \sum_{\epsilon=\pm} \int_{\sigma_{n+1} \le |p_2| < \sigma_n} w_\ell^{(2)}(\xi_2) c_{\ell,\epsilon}^*(\xi_2) c_{\ell,\epsilon}(\xi_2) \mathrm{d}\xi_2. \tag{A.3}$$

The operator H_{0n}^{n+1} is a self-adjoint operator in $\mathfrak{F}_n^{n+1}$.

Let us denote by H_I^n and H_{In}^{n+1} the interaction H_I given by (2.23) and (2.24) but associated with the following kernels:

$$\widetilde{\chi}^{\sigma_n}(p_2) G_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_1, \xi_2, \xi_3), \tag{A.4}$$

$$\left(\widetilde{\chi}^{\sigma_{n+1}}(p_2) - \widetilde{\chi}^{\sigma_n}(p_2)\right) G_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_1, \xi_2, \xi_3), \tag{A.5}$$

respectively, where $\widetilde{\chi}^{\sigma_{n+1}}$ is defined by (3.13).

Let for $n \ge 0$,

$$\begin{aligned} H_+^n &= H^n - E^n, \\ \widetilde{H}_+^n &= H_+^n \otimes 1_n^{n+1} + 1_n \otimes H_{0n}^{n+1}. \end{aligned} \tag{A.6}$$

The operators H_+^n and $\widetilde{H}_+^n$ are self-adjoint operators in $\mathfrak{F}^n$ and $\mathfrak{F}^{n+1}$, respectively. Here 1^n and 1_n^{n+1} are the identity operators in $\mathfrak{F}^n$ and $\mathfrak{F}_n^{n+1}$, respectively.

Combining (2.53) and (2.54) with (3.26)–(3.28) we obtain for $n \ge 0$,

$$g\left\|H_I^n \psi\right\| \le g K(G) \left(C_{\beta\eta} \left\|H_0 \psi\right\| + B_{\beta\eta} \|\psi\|\right) \tag{A.7}$$

for every $\psi \in \mathfrak{D}(H_0^n) \subset \mathfrak{F}^n$.

It follows from [33, Section V, Theorem 4.11] that

$$\begin{aligned} H^n &\ge -\frac{g K(G) B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}} \ge -\frac{g_1 K(G) B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}, \\ E^n &\ge -\frac{g K(G) B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}. \end{aligned} \tag{A.8}$$

We have

$$(\Omega^n, H^n \Omega^n) = 0. \tag{A.9}$$

Therefore

$$E^n \le 0, \tag{A.10}$$

$$|E^n| \le \frac{g K(G) B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}. \tag{A.11}$$

Let

$$K_n^{n+1}(G) = K\left(1_{\sigma_{n+1}\le|p_2|\le 2\sigma_n} G\right). \tag{A.12}$$

Combining (2.53) and (2.54) with (3.26) and (A.12) we obtain for $n \ge 0$

$$g\left\|H_{In}^{n+1}\psi\right\| \le gK_n^{n+1}(G)\left(C_{\beta\eta}\left\|H_0^{n+1}\psi\right\| + B_{\beta\eta}\|\psi\|\right) \tag{A.13}$$

for $\psi \in \mathfrak{D}(H_0^{n+1}) \subset \mathfrak{F}^{n+1}$, where we remind that $H_0^{n+1} = H_0|_{\mathfrak{F}^{\sigma_{n+1}}}$ as defined in (3.24).

We have, for every $\psi \in \mathfrak{D}(H_0^{n+1})$,

$$H_0^{n+1}\psi = \widetilde{H}_+^n\psi + E^n\psi - g\left(H_I^n \otimes 1_n^{n+1}\right)\psi, \tag{A.14}$$

and by (A.7)

$$g\left\|\left(H_I^n \otimes 1_n^{n+1}\right)\psi\right\| \le gK(G)\left(C_{\beta\eta}\left\|H_0^{n+1}\psi\right\| + B_{\beta\eta}\|\psi\|\right). \tag{A.15}$$

In view of (A.11) and (A.14) it follows from (A.15) that

$$\begin{aligned} &g\left\|\left(H_I^n \otimes 1_n^{n+1}\right)\psi\right\| \\ &\quad\le \frac{gK(G)C_{\beta\eta}}{1-g_1K(G)C_{\beta\eta}}\left\|\widetilde{H}_+^n\psi\right\| + \frac{gK(G)B_{\beta\eta}}{1-g_1K(G)C_{\beta\eta}}\left(1+\frac{gK(G)B_{\beta\eta}}{1-g_1K(G)C_{\beta\eta}}\right)\|\psi\|. \end{aligned} \tag{A.16}$$

By (3.29), (A.13), (A.14), and (A.16) we finally get

$$g\left\|H_{In}^{n+1}\psi\right\| \le gK_n^{n+1}(G)\left(\widetilde{C}_{\beta\eta}\left\|\widetilde{H}_+^n\psi\right\| + \widetilde{B}_{\beta\eta}\|\psi\|\right). \tag{A.17}$$

For $n \ge 0$, a straightforward computation yields

$$K_n^{n+1}(G) \le \sigma_n\widetilde{K}(G) \le \sup\left(\frac{4\Lambda\gamma}{2m_1-\delta}, 1\right)\widetilde{K}(G)\frac{\sigma_{n+1}}{\gamma}. \tag{A.18}$$

Recall that, for $n \ge 0$,

$$\sigma_{n+1} < m_1. \tag{A.19}$$

By (A.17), (A.18), and (A.19), we get, for $\psi \in \mathfrak{D}(H_0)$,

$$g\left\|H_{In}^{n+1}\psi\right\| \le gK_n^{n+1}(G)\left(\widetilde{C}_{\beta\eta}\left\|\left(\widetilde{H}_+^n + \sigma_{n+1}\right)\psi\right\| + \left(\widetilde{C}_{\beta\eta}m_1 + \widetilde{B}_{\beta\eta}\right)\|\psi\|\right), \tag{A.20}$$

and for $\phi \in \mathfrak{F}$,

$$\begin{aligned} g\left\| H_{In}^{n+1}\left(\widetilde{H}_{+}^{n}+\sigma_{n+1}\right)^{-1}\phi\right\| &\leq gK_{n}^{n+1}(G)\left(\widetilde{C}_{\beta\eta}+\frac{m_1\widetilde{C}_{\beta\eta}+\widetilde{B}_{\beta\eta}}{\sigma_{n+1}}\right)\|\phi\| \\ &\leq \frac{g}{\gamma}\sup\left(\frac{4\Lambda\gamma}{2m_1-\delta},1\right)\widetilde{K}(G)\left(2m_1\widetilde{C}_{\beta\eta}+\widetilde{B}_{\beta\eta}\right)\|\phi\|. \end{aligned} \tag{A.21}$$

Thus, by (A.21), the operator $H_{In}^{n+1}(\widetilde{H}_{+}^{n}+\sigma_{n+1})^{-1}$ is bounded and

$$g\left\| H_{In}^{n+1}\left(\widetilde{H}_{+}^{n}+\sigma_{n+1}\right)^{-1}\right\| \leq g\frac{\widetilde{D}}{\gamma}, \tag{A.22}$$

where $\widetilde{D}$ is given by (see (3.32))

$$\widetilde{D}=\sup\left(\frac{4\Lambda\gamma}{2m_1-\delta},1\right)\widetilde{K}(G)\left(2m_1\widetilde{C}_{\beta\eta}+\widetilde{B}_{\beta\eta}\right). \tag{A.23}$$

This yields, for $\psi \in \mathfrak{D}(\widetilde{H}_{+}^{n})$,

$$g\left\| H_{In}^{n+1}\psi\right\| \leq g\frac{\widetilde{D}}{\gamma}\left\|\left(\widetilde{H}_{+}^{n}+\sigma_{n+1}\right)\psi\right\|. \tag{A.24}$$

Hence it follows from [33, Section V, Theorems 4.11 and 4.12] that

$$g\left|\left(H_{In}^{n+1}\psi,\psi\right)\right| \leq g\frac{\widetilde{D}}{\gamma}\left(\left(\widetilde{H}_{+}^{n}+\sigma_{n+1}\right)\psi,\psi\right). \tag{A.25}$$

Let $g_{\delta}^{(2)}>0$ be such that

$$g_{\delta}^{(2)}\frac{\widetilde{D}}{\gamma}<1, \qquad g_{\delta}^{(2)} \leq g_{\delta}^{(1)}. \tag{A.26}$$

By (A.25) we get, for $g \leq g_{\delta}^{(2)}$,

$$H^{n+1}=\widetilde{H}_{+}^{n}+E^{n}+gH_{In}^{n+1} \geq E^{n}-\frac{g\widetilde{D}}{\gamma}\sigma_{n+1}+\left(1-\frac{g\widetilde{D}}{\gamma}\right)\widetilde{H}_{+}^{n}. \tag{A.27}$$

Because $(1-g\widetilde{D}/\gamma)\widetilde{H}_{+}^{n} \geq 0$ we get from (A.27)

$$E^{n+1} \geq E^{n}-\frac{g\widetilde{D}}{\gamma}\sigma_{n+1}, \quad n \geq 0. \tag{A.28}$$

Suppose that $\psi^n \in \mathfrak{F}^n$ satisfies $\|\psi^n\| = 1$ and, for $\epsilon > 0$,

$$\left(\psi^n, H^n \psi^n\right) \leq E^n + \epsilon. \tag{A.29}$$

Let

$$\widetilde{\psi}^{n+1} = \psi^n \otimes \Omega_n^{n+1} \in \mathfrak{F}^{n+1}. \tag{A.30}$$

We obtain

$$E^{n+1} \leq \left(\widetilde{\psi}^{n+1}, H^{n+1} \widetilde{\psi}^{n+1}\right) \leq E^n + \epsilon + g\left(\widetilde{\psi}^{n+1}, H_{In}^{n+1} \widetilde{\psi}^{n+1}\right). \tag{A.31}$$

By (A.25), (A.29), (A.30), and (A.31) we get, for every $\epsilon > 0$,

$$E^{n+1} \leq E^n + \epsilon\left(1 + \frac{g\widetilde{D}}{\gamma}\right) + \frac{g\widetilde{D}}{\gamma} \sigma_{n+1}, \tag{A.32}$$

where $g \leq g_\delta^{(2)}$.

This yields

$$E^{n+1} \leq E^n + \frac{g\widetilde{D}}{\gamma} \sigma_{n+1}, \tag{A.33}$$

and by (A.28), we obtain

$$\left|E^n - E^{n+1}\right| \leq \frac{g\widetilde{D}}{\gamma} \sigma_{n+1}. \tag{A.34}$$

For $n = 0$, since $\sigma_0 = \Lambda$, remind that $H_0^0 = H_0^{n=0} = H_0^{\sigma_0} = H_0|_{\mathfrak{F}^\Lambda}$. Thus, the ground state energy of H_0^0 is 0 and it is a simple isolated eigenvalue of H_0^0 with Ω^0, the vacuum in $\mathfrak{F}^0$, as eigenvector. Moreover, since $\Lambda > m_1$,

$$\inf\left(\sigma\left(H_0^0\right) \setminus \{0\}\right) = m_1, \tag{A.35}$$

thus $(0, m_1)$ belongs to the resolvent set of H_0^0.

By Hypothesis 3.1(iv) we have $H^0 = H_0^0$. Hence $E^0 = \{0\}$ is a simple isolated eigenvalue of H^0 and $H^0 = H_+^0$. We finally get

$$\inf\left(\sigma\left(H_+^0\right) - \{0\}\right) = m_1 > m_1 - \frac{\delta}{2} = \sigma_1. \tag{A.36}$$

We now prove Proposition 3.5 by induction in $n \in \mathbb{N}^*$. Suppose that E^n is a simple isolated eigenvalue of H^n such that

$$\inf(\sigma(H_+^n) \setminus \{0\}) \geq \left(1 - \frac{3g\widetilde{D}}{\gamma}\right)\sigma_n, \quad n \geq 1. \tag{A.37}$$

Since (3.36) gives $\sigma_{n+1} < (1 - 3g\widetilde{D}/\gamma)\sigma_n$ for $g \leq g_\delta^{(2)}$, 0 is also a simple isolated eigenvalue of $\widetilde{H}_+^n$ such that

$$\inf\left(\sigma\left(\widetilde{H}_+^n\right) \setminus \{0\}\right) \geq \sigma_{n+1}. \tag{A.38}$$

We must now prove that E^{n+1} is a simple isolated eigenvalue of H^{n+1} such that

$$\inf\left(\sigma\left(H_+^{n+1}\right) \setminus \{0\}\right) \geq \left(1 - \frac{3g\widetilde{D}}{\gamma}\right)\sigma_{n+1}. \tag{A.39}$$

Let

$$\lambda^{(n+1)} = \sup_{\psi \in \mathfrak{F}^{n+1}; \psi \neq 0} \ \inf_{(\phi,\psi)=0; \phi \in \mathfrak{D}(H^{n+1}); \|\phi\|=1} \left(\phi, H_+^{n+1}\phi\right). \tag{A.40}$$

By (A.27) and (A.33), we obtain, in $\mathfrak{F}^{n+1}$,

$$\begin{aligned} H_+^{n+1} &\geq E^n - E^{n+1} - \frac{g\widetilde{D}}{\gamma}\sigma_{n+1} + \left(1 - \frac{g\widetilde{D}}{\gamma}\right)\widetilde{H}_+^n \\ &\geq \left(1 - \frac{g\widetilde{D}}{\gamma}\right)\widetilde{H}_+^n - \frac{2g\widetilde{D}}{\gamma}\sigma_{n+1}. \end{aligned} \tag{A.41}$$

By (A.30), $\widetilde{\varphi}^{n+1}$ is the unique ground state of $\widetilde{H}_+^n$, and by (A.38) and (A.41), we have, for $g \leq g_\delta^{(2)}$,

$$\begin{aligned} \lambda^{(n+1)} &\geq \inf_{(\phi,\widetilde{\varphi}^{n+1})=0; \phi \in \mathfrak{D}(H^{n+1}); \|\phi\|=1} \left(\phi, H_+^{n+1}\phi\right) \\ &\geq \left(1 - \frac{g\widetilde{D}}{\gamma}\right)\sigma_{n+1} - \frac{2g\widetilde{D}}{\gamma}\sigma_{n+1} = \left(1 - \frac{3g\widetilde{D}}{\gamma}\right)\sigma_{n+1} > 0. \end{aligned} \tag{A.42}$$

This concludes the proof of Proposition 3.5 by choosing $g_\delta = g_\delta^{(2)}$, if one proves that H^1 satisfies Proposition 3.5. By noting that 0 is a simple isolated eigenvalue of $\widetilde{H}_+^0$ such that $\inf(\sigma(\widetilde{H}_+^0) \setminus \{0\}) = \sigma_1$, we prove that E^1 is indeed an isolated simple eigenvalue of H^1 such that $\inf(\sigma(H_+^1) \setminus \{0\}) \geq (1 - 3g\widetilde{D}/\gamma)\sigma_1$ by mimicking the proof given above for H_+^{n+1}.

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