REVIEW

CrossMark

A note on "Implicit iterative process with errors"

Esra Yolacan[*]

*Correspondence:
yolacanesra@gmail.com
Department of Mathematics,
Faculty of Science, Ataturk
University, Erzurum 25240,
Turkey

Abstract

In this paper, I introduce an implicit iterative process with mixed errors for two finite family of total asymptotically nonexpansive mappings in a uniformly convex Banach space and prove strong convergence theorems under some conditions. My results improved and extended many know results existing in the literature.

Keywords: Total asymptotically nonexpansive mappings, Common fixed points, Implicit iterative process with errors

Mathematics Subject Classification: 47H09, 47H10, 46B20

Background

Let E is a normed space and K be a nonempty subset of E. I also assume that $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ strictly increasing continuous function with $\phi(0) = 0$. Let $T : K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T if and only if $Tx = x$. I will denote by nonexpansive if $F(T)$ the set of fixed points of T, that is, $F(T) := \{x \in K : Tx = x\}$. T is said to be *nonexpansive* if

$$\left\| Tx - Ty \right\| \leq \left\| x - y \right\|$$

for all $x, y \in K$. T is called *asymptotically nonexpansive* if for a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$,

$$\left\| T^n x - T^n y \right\| \leq k_n \left\| x - y \right\|$$

for all $x, y \in K$ and $n \geq 1$. T is said to be *total asymptotically nonexpansive* (see, e.g., Albert et al. 2006) if

$$\left\| T^n x - T^n y \right\| \leq \left\| x - y \right\| + \mu_n \phi(\left\| x - y \right\|) + l_n$$

for all $x, y \in K, n \geq 1$ where $\{\mu_n\}$ and $\{l_n\}$ nonnegative real sequences such that $\mu_n, l_n \to 0$ as $n \to \infty$. From the definition, I see that the class of total asymptotically nonexpansive mappings include the class of asymptotically nonexpansive mappings as a special case; see also Albert et al. (2006) for more details.

Proposition 1 *Let K be a nonempty subset of E, $\{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N : K \to K$ be $2N$ total asymptotically nonexpansive mappings. Then there exist nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}, n \geq 1$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in K$,*

$$\left\| T_i^n x - T_i^n y \right\| \leq \left\| x - y \right\| + \mu_n \phi\left(\left\| x - y \right\|\right) + l_n, \quad n \geq 1, \tag{1}$$

$$\left\| S_i^n x - S_i^n y \right\| \leq \left\| x - y \right\| + \mu_n \phi\left(\left\| x - y \right\|\right) + l_n, \quad n \geq 1, \quad \text{for } i = 1, 2, \ldots, N. \tag{2}$$

Proof Since $T_i : K \to K$ is a total asymptotically nonexpansive mappings for $i = 1, 2, \ldots, N$, there exist nonnegative real sequences $\{\mu_{in}\}, \{l_{in}\}, n \geq 1$ with $\mu_{in}, l_{in} \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_i(0) = 0$ such that for all $x, y \in K$,

$$\left\| T_i^n x - T_i^n y \right\| \leq \left\| x - y \right\| + \mu_{in} \phi_i\left(\left\| x - y \right\|\right) + l_{in}, \quad n \geq 1.$$

Setting

$$\mu_n = \max \{\mu_{1n}, \mu_{2n}, \ldots, \mu_{Nn}\}, \quad l_n = \max \{l_{1n}, l_{2n}, \ldots, l_{Nn}\},$$
$$\phi(a) = \max \{\phi_1(a), \phi_2(a), \ldots, \phi_N(a)\} \text{ for } a \geq 0,$$

then I get that there exist nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}, n \geq 1$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that

$$\begin{aligned} \left\| T_i^n x - T_i^n y \right\| &\leq \left\| x - y \right\| + \mu_{in} \phi_i\left(\left\| x - y \right\|\right) + l_{in} \\ &\leq \left\| x - y \right\| + \mu_n \phi\left(\left\| x - y \right\|\right) + l_n, \quad n \geq 1, \end{aligned}$$

for all $x, y \in K$, and each $i = 1, 2, \ldots, N$.

In a way similar to the above, I can also prove (2). □

Recently, fixed point problems based on implicit iterative processes have been considered by many authors, (see, for example, Chang et al. 2006; Cianciaruso et al. 2010; Sun 2003; Gu 2006; Qin et al. 2008; Xu and Ori 2001). In Hao (2010) established weak and strong convergence theorems of the implicit iteration process for a finite family of uniformly Lipschitz total asymptotically nonexpansive mappings in a real Hilbert space. In Hao et al. (2012) studied weak and strong convergence theorems for common fixed points of two finite family of asymptotically nonexpansive mappings in a uniformly convex Banach space.

Note the convergence problems of an implicit (an explicit) iterative process to a common fixed point, for total asymptotically nonexpansive (or asymptotically nonexpansive) in Banach space have been obtained by a number of authors (see more details, Mukhamedov and Saburov 2010a, b, 2011, 2012a, b).

Inspired and motivated by this facts, I introduce an implicit iterative process with mixed errors for two finite family of total asymptotically nonexpansive mappings in a uniformly convex Banach space. The results of this paper can be viewed as an improvement and extension of the corresponding results of Chang et al. (2006), Cianciaruso et al. (2010), Sun (2003), Hao et al. (2012), Hao (2010) and others.

Definition 1 Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N : E \to E$ be $2N$ total asymptotically nonexpansive mappings. Define the sequence $\{x_n\}$ as follows: $x_0 \in K$, and

$$x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n,$$
$$y_n = (1 - \beta_n - \delta_n)x_n + \beta_n S_{i(n)}^{k(n)} x_n + \delta_n v_n, n \geq 1 \tag{3}$$

where $n = (k(n) - 1)N + i(n)$, $i(n) \in 1, 2, \ldots, N$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are four real sequences in $[0, 1]$ satisfying $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences.

The purpose of this paper is to study the strong convergence of implicit iterative process with mixed errors for two finite family of total asymptotically nonexpansive mappings in Banach spaces.

Preliminaries

Let E be a Banach space with dimension $E \geq 2$. The modulus of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x\| = \|y\| = 1, \ \varepsilon = \|x - y\| \right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Recall that a mapping $T : K \to K$ is $semi - compact$ ($orhemi - compact$) if any sequence $\{x_n\}$ in K such that $\{x_n - Tx_n\} \to 0$ as $n \to \infty$ has a convergent subsequence.

Lemma 1 (Tan and Xu 1993) *Let* $\{a_n\}, \{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1,$$

if $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} \delta_n < \infty$, *then*

(i) $\lim_{n\to\infty} a_n$ *exists;*
(ii) *In particular, if* $\{a_n\}$ *has a subsequence* $\{a_{n_k}\}$ *converging to* 0, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2 (Schu 1991) *Let* E *be a uniformly convex Banach space,* $\{t_n\}_{n\geq 1} \subseteq [b, c] \subset (0, 1)$, $\{x_n\}_{n\geq 1}$ *and* $\{y_n\}_{n\geq 1}$ *be sequences in* E. *If* $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ *and* $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = a$ *for some constant* $a \geq 0$, *then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Main results

Lemma 3 *Let* E *be a real Banach space, let* K *be a nonempty, closed and convex subset of* E *and* $\{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N : K \to K$ *be* $2N$ *total asymptotically nonexpansive mappings* $\{\mu_n\}$, $\{l_n\}$ *defined by* (1) *and* (2) *such that*

$$\sum_{n=1}^{\infty} \mu_n < \infty, \quad \sum_{n=1}^{\infty} l_n < \infty \tag{4}$$

and $\mathcal{F} : \bigcap_{i=1}^{N} F(T_i) \cap F(S_i) \neq \varnothing$. *Assume that there exist* $M, M^* > 0$ *such that* $\phi(\lambda) \leq M^* \lambda$ *for all* $\lambda \geq M$, $i \in \{i = 1, 2, \ldots, N\}$. *Let* $\{u_n\}$ *and* $\{v_n\}$ *are two bounded sequences in* K. *Let* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\delta_n\}$ *be four real sequences in* $[0, 1]$ *satisfying the following conditions:*

(i) $\alpha_n + \gamma_n \leq 1$ *and* $\beta_n + \delta_n \leq 1$ *for all* $n \geq 1$;
(ii) $\limsup_{n \to \infty} \alpha_n < 1$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$.

Starting from an arbitrary $x_0 \in K$, *define the sequence* $\{x_n\}$ *by recursion* (3). *Then,* $\lim_{n \to \infty} \|x_n - p\|$ *exists for all* $p \in \mathcal{F}$.

Proof   Let $p \in \mathcal{F}$. Since $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K, I have

$$K = \max \left\{ \sup_{n \geq 1} \|u_n - p\|, \sup_{n \geq 1} \|v_n - p\| \right\}.$$

Since $S_1, S_2, \ldots, S_N$ are total asymptotically nonexpansive mappings, it follows from (2) that

$$
\begin{aligned}
\left\| y_n - p \right\| &= \left\| (1 - \beta_n - \delta_n)(x_n - p) + \beta_n \left(S_{i(n)}^{k(n)} x_n - p \right) + \delta_n (v_n - p) \right\| \\
&\leq (1 - \beta_n - \delta_n)\|x_n - p\| + \beta_n \left\| S_{i(n)}^{k(n)} x_n - p \right\| + \delta_n \|v_n - p\| \\
&\leq (1 - \beta_n - \delta_n)\|x_n - p\| + \beta_n [\|x_n - p\| + \mu_n \phi(\|x_n - p\|) + l_n] + \delta_n K \\
&\leq \|x_n - p\| + \beta_n \mu_n \phi(\|x_n - p\|) + \beta_n l_n + \varphi_{(1)}^n,
\end{aligned}
\tag{5}
$$

where $\varphi_{(1)}^n = \delta_n K$. Since $\sum_{n=1}^{\infty} \delta_n < \infty$, I can see that $\sum_{n=1}^{\infty} \varphi_{(1)}^n < \infty$. Note that ϕ is an increasing function, it follows that $\phi(\lambda) \leq \phi(M)$ whenever $\lambda \leq M$ and (by hypothesis) $\phi(\lambda) \leq M^* \lambda$ if $\lambda \geq M$. In either case, I have

$$\phi(\lambda) \leq \phi(M) + M^* \lambda, \tag{6}$$

for some $M, M^* > 0$. Thus, from (5) and (6), I have

$$
\begin{aligned}
\left\| y_n - p \right\| &\leq \|x_n - p\| + \beta_n \mu_n \left[\phi(M) + M^* \|x_n - p\| \right] + \beta_n l_n + \varphi_{(1)}^n \\
&\leq \left(1 + M^* \mu_n \right) \|x_n - p\| + R_1 (\mu_n + l_n) + \varphi_{(1)}^n,
\end{aligned}
\tag{7}
$$

for some constant $R_1 > 0$. It follows from (6) and (7)

$$
\begin{aligned}
\|x_n - p\| &= \left\| (1 - \alpha_n - \gamma_n)(x_{n-1} - p) + \alpha_n \left(T_{i(n)}^{k(n)} y_n - p \right) + \gamma_n(u_n - p) \right\| \\
&\leq (1 - \alpha_n - \gamma_n)\|x_{n-1} - p\| + \alpha_n \left\| T_{i(n)}^{k(n)} y_n - p \right\| + \gamma_n \|u_n - p\| \\
&\leq (1 - \alpha_n - \gamma_n)\|x_{n-1} - p\| + \alpha_n \left[\|y_n - p\| + \mu_n \phi(\|y_n - p\|) + l_n \right] \\
&\quad + \gamma_n \|u_n - p\| \\
&\leq (1 - \alpha_n - \gamma_n)\|x_{n-1} - p\| \\
&\quad + \alpha_n \left[(1 + M^*\mu_n)\|x_n - p\| + R_1(\mu_n + l_n) + \varphi_{(1)}^n \right] \\
&\quad + \alpha_n \mu_n \left[\phi(M) + M^* \|y_n - p\| \right] + \alpha_n l_n + \gamma_n K \\
&\leq (1 - \alpha_n - \gamma_n)\|x_{n-1} - p\| + \alpha_n \|x_n - p\| \\
&\quad + \alpha_n M^* (2 + M^* \mu_n)\mu_n \|x_n - p\| \\
&\quad + \alpha_n R_1(\mu_n + l_n) + \alpha_n \varphi_{(1)}^n + \alpha_n \mu_n \phi(M) + \alpha_n l_n + \gamma_n K \\
&\quad + \alpha_n \mu_n M^* R_1(\mu_n + l_n) + \alpha_n \mu_n M^* \varphi_{(1)}^n \\
&\leq (1 - \alpha_n)\|x_{n-1} - p\| + \alpha_n(1 + M_2\mu_n)\|x_n - p\| \\
&\quad R_2(\mu_n + l_n) + \alpha_n \varphi_{(1)}^n + \gamma_n K + \alpha_n \mu_n M^* \varphi_{(1)}^n,
\end{aligned}
\tag{8}
$$

for some constants $M_2, R_2 > 0$. I note that

$$
\begin{aligned}
\|x_n - p\| &\leq (1 - \alpha_n)\|x_{n-1} - p\| + \alpha_n(1 + M_2\mu_n)\|x_n - p\| \\
&\quad + R_2(\mu_n + l_n) + \varphi_{(2)}^n,
\end{aligned}
\tag{9}
$$

where $\varphi_{(2)}^n = \alpha_n \varphi_{(1)}^n + \gamma_n K + \alpha_n \mu_n M^* \varphi_{(1)}^n$. Since $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \varphi_{(1)}^n < \infty$, I can see that $\sum_{n=1}^{\infty} \varphi_{(2)}^n < \infty$. This implies that

$$
\begin{aligned}
\|x_n - p\| &\leq \frac{1 - \alpha_n}{1 - \alpha_n(1 + M_2\mu_n)}\|x_{n-1} - p\| + \frac{R_2(\mu_n + l_n) + \varphi_{(2)}^n}{1 - \alpha_n(1 + M_2\mu_n)} \\
&\leq \left(1 + \frac{\alpha_n M_2 \mu_n}{1 - \alpha_n(1 + M_2\mu_n)} \right)\|x_{n-1} - p\| + \frac{R_2(\mu_n + l_n) + \varphi_{(2)}^n}{1 - \alpha_n(1 + M_2\mu_n)}.
\end{aligned}
\tag{10}
$$

By hypothesis (ii), it follows that there exists $\lambda < 1$, such that $\alpha_n \leq \lambda$ for big n. It follows that

$$
\frac{\alpha_n M_2}{1 - \alpha_n(1 + M_2\mu_n)} \leq \frac{\lambda M_2}{1 - \lambda(1 + M_2\mu_n)}.
$$

From $\lim_{n \to \infty} \mu_n = 0$, it derives that $\lim_{n \to \infty} \frac{\lambda M_2}{1 - \lambda(1 + M_2\mu_n)} = \frac{\lambda M_2}{1 - \lambda}$. Then there exists a real constant $L_1 > 0$ such that

$$
\frac{\lambda M_2}{1 - \lambda(1 + M_2\mu_n)} \leq L_1, \quad \forall n \geq 1.
$$

It follows from the hypothesis that $\sum_{n \geq 1} \frac{\alpha_n M_2}{1 - \alpha_n(1 + M_2\mu_n)} < \infty$. Similarly, I can prove that

$$
\sum_{n \geq 1} \frac{R_2(\mu_n + l_n) + \varphi_{(2)}^n}{1 - \alpha_n(1 + M_2\mu_n)} < \infty.
\tag{11}
$$

Besides, I can write

$$
\|x_n - p\| \leq (1 + L_1\mu_n)\|x_{n-1} - p\| + \frac{R_2(\mu_n + l_n) + \varphi_{(2)}^n}{1 - \alpha_n(1 + M_2\mu_n)},
$$

where a real constant $L_1 > 0$. Thus, I obtain from it that

$$\|x_n - p\| \leq (1 + \delta_n)\|x_{n-1} - p\| + b_n, \quad \forall n \geq n_0, \tag{12}$$

where $\delta_n = L_1\mu_n$ and $b_n = \frac{R_2(\mu_n + l_n) + \varphi_{(2)}^n}{1 - \alpha_n(1 + M_2\mu_n)}$ and using the condition (iii) and (11), it is easy to see that $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. In view of Lemma 1, I find that $\lim_{n\to\infty} \|x_n - p\|$ exist for all $p \in \mathcal{F}$. $\qquad\square$

Theorem 1 *Let E be a real uniformly convex Banach space, let K be a nonempty,closed and convex subset of E and $\{S_i\}_{i=1}^{N}$, $\{T_i\}_{i=1}^{N} : K \to K$ be 2N total asymptotically nonexpansive mappings $\{\mu_n\}$, $\{l_n\}$ defined by (1) and (2) such that*

$$\sum_{n=1}^{\infty} \mu_n < \infty, \quad \sum_{n=1}^{\infty} l_n < \infty \tag{13}$$

and $\mathcal{F} : \bigcap_{i=1}^{N} F(T_i) \cap F(S_i) \neq \varnothing$. Assume that there exist M, $M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$, $i \in \{i = 1, 2, \ldots, N\}$. Let $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences in $[0, 1]$ satisfying the following conditions:

(i)   *$\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$;*
(ii) *$\limsup_{n\to\infty} \alpha_n < 1$;*
(iii) *$\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$. Then the implicit iterative sequence $\{x_n\}$ by (3) converges strongly to a common fixed point in $\mathcal{F}$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, where $d(x, \mathcal{F})$ denotes the distance of x to set $\mathcal{F}$, i.e., $d(x, \mathcal{F}) = \inf_{y \in \mathcal{F}} d(x, y)$.*

Proof   It suffices to show that $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$ implies that $\{x_n\}$ converges to a common fixed point of $\mathcal{F}$.

Necessity. Since (12) holds for all $p \in \mathcal{F}$, I obtain from it that

$$d(x_n, \mathcal{F}) \leq (1 + \delta_n)d(x_{n-1}, \mathcal{F}) + b_n, \quad \forall n \geq n_0,$$

Lemma 1 that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists and so $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Sufficiency. Now, I show that $\{x_n\}$ is a Cauchy sequence in E. For any positive integers $m, n > n \geq n_0$, from $1 + t \leq e^t$ for all $t > 0$ and (12), I have

$$\begin{aligned}
\|x_m - p\| &\leq (1 + \delta_m)\|x_{m-1} - p\| + b_m \\
&\leq e^{\delta_m}\|x_{m-1} - p\| + b_m \\
&\leq e^{\delta_m}\left(e^{\delta_{m-1}}\|x_{m-2} - p\| + b_{m-1}\right) + b_m \\
&\vdots \\
&\leq e^{\sum_{i=n+1}^{m} \delta_i}\|x_n - p\| + \sum_{k=n+1}^{m-1} b_k e^{\sum_{i=k+1}^{m} \delta_i} + b_m \\
&\leq Q\|x_n - p\| + Q\sum_{k=n+1}^{\infty} b_k + b_m,
\end{aligned}$$

where $Q = e^{\sum_{n=1}^{\infty} \delta_n}$. Thus for any $p \in \mathcal{F}$, I have

$$\|x_n - x_m\| \leq \|x_n - p\| + \|x_m - p\|$$

$$\leq (1 + Q)\|x_n - p\| + Q \sum_{k=n+1}^{\infty} b_k + b_m.$$

Taking the infimum over all $p \in \mathcal{F}$, I obtain that

$$\|x_n - x_m\| \leq (1 + Q)d(x_n, \mathcal{F}) + Q \sum_{k=n+1}^{\infty} b_k + b_m.$$

It follows from $\sum_{n=1}^{\infty} b_n < \infty$ and $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ that $\{x_n\}$ is a Cauchy sequence. Since K is a closed subset of E and so it is complete. Hence, there exists a $p \in K$ such that $x_n \to p$ as $n \to \infty$.

Finally, I have to prove that $p \in \mathcal{F}$. By contradiction, i assume that p is not in $\mathcal{F} : \bigcap_{i=1}^{N} F(T_i) \cap F(S_i) \neq \varnothing$. Since $\mathcal{F}$ is a closed set, $d(p, \mathcal{F}) > 0$. Thus for all $p \in \mathcal{F}$, I have that

$$\|p - p_1\| \leq \|p - x_n\| + \|x_n - p_1\|. \tag{14}$$

This implies that

$$d(p, \mathcal{F}) \leq \|p - x_n\| + d(x_n, \mathcal{F}). \tag{15}$$

From (14) and (15) $(n \to \infty)$, I have that $d(p, \mathcal{F}) \leq 0$. This is a contradiction. Thus, $p \in \mathcal{F}$. This completes the proof. $\qquad\square$

Lemma 4 *Let E be a uniformly convex Banach space, let K be a nonempty,closed and convex subset of E and $\{S_i\}_{i=1}^{N}$, $\{T_i\}_{i=1}^{N} : K \to K$ be $2N$ uniformly $L_i - Lipschitz$ total asymptotically nonexpansive mappings $\{\mu_n\}, \{l_n\}$ defined by (1) and (2) such that*

$$\sum_{n=1}^{\infty} \mu_n < \infty, \quad \sum_{n=1}^{\infty} l_n < \infty \tag{16}$$

and $\mathcal{F} : \bigcap_{i=1}^{N} F(T_i) \cap F(S_i) \neq \varnothing$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$, $i \in \{i = 1, 2, \ldots, N\}$. Let $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences in $\left[\frac{L-1}{L}, a\right]$, where $L = \max_{1 \leq i \leq N} \{L_i\} > 1$ and a is some constant in $(0, 1)$ satisfying the following conditions:

(i) $\alpha_n + \gamma_n \leq 1$ *and* $\beta_n + \delta_n \leq 1$ *for all* $n \geq 1$;
(ii) $\limsup_{n\to\infty} \alpha_n < 1$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$.*Let the sequence $\{x_n\}$ and $\{y_n\}$ be defined by (3). Then*

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0,$$
$$\lim_{n\to\infty} \|x_n - S_l x_n\| = 0,$$
$$\lim_{n\to\infty} \|S_l x_n - T_l x_n\| = 0, \quad \forall l = 1, 2, \ldots, N.$$

Proof For all $p \in \mathcal{F}$, it follows from Lemma 3 that $\lim_{n \to \infty} \|x_n - p\|$ exists. Let $\lim_{n \to \infty} \|x_n - p\| = r$ for some $r \geq 0$. It follows from (7) that

$$\|y_n - p\| \leq (1 + M^* \mu_n) \|x_n - p\| + R_1(\mu_n + l_n) + \varphi_{(1)}^n, \tag{17}$$

where $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty$ and $\sum_{n=1}^{\infty} \varphi_{(1)}^n < \infty$. Taking $\limsup_{n \to \infty}$ in both sides, I obtain

$$\limsup_{n \to \infty} \|y_n - p\| \leq \limsup_{n \to \infty} \left[\begin{array}{c} (1 + M^* \mu_n) \|x_n - p\| \\ + R_1(\mu_n + l_n) + \varphi_{(1)}^n \end{array} \right] \tag{18}$$
$$\leq r,$$

and by (16) and (18)

$$\limsup_{n \to \infty} \left\| T_{i(n)}^{k(n)} y_n - p \right\| \leq \limsup_{n \to \infty} \left[\begin{array}{c} \|y_n - p\| \\ + \mu_n \phi(\|y_n - p\|) + l_n \end{array} \right] \tag{19}$$
$$\leq r.$$

Notice that

$$\left\| T_{i(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1}) \right\| \leq \left\| T_{i(n)}^{k(n)} y_n - p \right\| + \gamma_n \|u_n - x_{n-1}\|. \tag{20}$$

It follows that

$$\limsup_{n \to \infty} \left\| T_{i(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1}) \right\| \leq r, \tag{21}$$

and

$$\|x_{n-1} - p + \gamma_n(u_n - x_{n-1})\| \leq \|x_{n-1} - p\| + \gamma_n \|u_n - x_{n-1}\|. \tag{22}$$

These imply that

$$\limsup_{n \to \infty} \|x_{n-1} - p + \gamma_n(u_n - x_{n-1})\| \leq r, \tag{23}$$

and I have that

$$x_n - p = \alpha_n \left(T_{i(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1}) \right) + (1 - \alpha_n)(x_{n-1} - p + \gamma_n(u_n - x_{n-1})). \tag{24}$$

Hence,

$$r = \lim_{n \to \infty} \|x_n - p\|$$
$$= \lim_{n \to \infty} \left\| \begin{array}{c} \alpha_n \left(T_{i(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1}) \right) \\ + (1 - \alpha_n)(x_{n-1} - p + \gamma_n(u_n - x_{n-1})) \end{array} \right\|. \tag{25}$$

Using (19), (23), (25) and Lemma 2, I find

$$\lim_{n \to \infty} \left\| T_{i(n)}^{k(n)} y_n - x_{n-1} \right\| = 0. \tag{26}$$

Notice that

$$\left\| x_n - T_{i(n)}^{k(n)} y_n \right\| \le \| x_n - x_{n-1} \| + \left\| x_{n-1} - T_{i(n)}^{k(n)} y_n \right\|$$
$$\le (1 + \alpha_n) \left\| x_{n-1} - T_{i(n)}^{k(n)} y_n \right\| + \gamma_n \| u_n - x_{n-1} \|.$$

It follows from (26) that

$$\lim_{n \to \infty} \left\| x_n - T_{i(n)}^{k(n)} y_n \right\| = 0. \tag{27}$$

Notice that

$$\| x_n - p \| \le \left\| x_n - T_{i(n)}^{k(n)} y_n \right\| + \left\| T_{i(n)}^{k(n)} y_n - p \right\|$$
$$\le \left\| x_n - T_{i(n)}^{k(n)} y_n \right\|$$
$$+ \left\| y_n - p \right\| + \mu_n \phi \left(\left\| y_n - p \right\| \right) + l_n.$$

By using (16) and (27), I obtain

$$r = \lim_{n \to \infty} \| x_n - p \| \le \lim \inf_{n \to \infty} \left\| y_n - p \right\|.$$

It follows that

$$r \le \lim \inf_{n \to \infty} \left\| y_n - p \right\| \le \lim \sup_{n \to \infty} \left\| y_n - p \right\| \le r.$$

This implies that

$$\lim_{n \to \infty} \left\| y_n - p \right\| = r. \tag{28}$$

Since $\lim_{n \to \infty} \| x_n - p \| = r$ and (16), I see that

$$\lim \sup_{n \to \infty} \left\| S_{i(n)}^{k(n)} x_n - p \right\| \le \lim \sup_{n \to \infty} \left[\begin{array}{c} \| x_n - p \| \\ + \mu_n \phi(\| x_n - p \|) + l_n \end{array} \right] \tag{29}$$
$$\le r.$$

Notice that

$$\left\| S_{i(n)}^{k(n)} x_n - p + \delta_n (v_n - x_n) \right\| \le \left\| S_{i(n)}^{k(n)} x_n - p \right\| + \delta_n \| v_n - x_n \|.$$

It follows that

$$\lim \sup_{n \to \infty} \left\| S_{i(n)}^{k(n)} x_n - p + \delta_n (v_n - x_n) \right\| \le r, \tag{30}$$

and

$$\| x_n - p + \delta_n (v_n - x_n) \| \le \| x_n - p \| + \delta_n \| v_n - x_n \|, \tag{31}$$

which implies that

$$\limsup_{n \to \infty} \|x_n - p + \delta_n(v_n - x_n)\| \le r, \tag{32}$$

and I have that

$$y_n - p = \beta_n\left(S_{i(n)}^{k(n)}x_n - p + \delta_n(v_n - x_n)\right) + (1 - \beta_n)(x_n - p + \delta_n(v_n - x_n)). \tag{33}$$

Hence,

$$\begin{aligned} r &= \lim_{n \to \infty} \|y_n - p\| \\ &= \lim_{n \to \infty} \left\| \begin{array}{l} \beta_n\left(S_{i(n)}^{k(n)}x_n - p + \delta_n(v_n - x_n)\right) \\ +(1 - \beta_n)(x_n - p + \delta_n(v_n - x_n)) \end{array} \right\|. \end{aligned} \tag{34}$$

Using (30), (32), (34) and Lemma 2, I find

$$\lim_{n \to \infty} \left\| x_n - S_{i(n)}^{k(n)} x_n \right\| = 0. \tag{35}$$

Notice that

$$\begin{aligned} \left\| T_{i(n)}^{k(n)} x_n - x_n \right\| &\le \left\| T_{i(n)}^{k(n)} x_n - y_n \right\| + \left\| T_{i(n)}^{k(n)} y_n - x_n \right\| \\ &\le \left\| x_n - y_n \right\| + \mu_n \phi\left(\left\| x_n - y_n \right\|\right) + l_n + \left\| T_{i(n)}^{k(n)} y_n - x_n \right\| \\ &\le \beta_n \left\| x_n - S_{i(n)}^{k(n)} x_n \right\| + \delta_n \|v_n - x_n\| \\ &\quad + \mu_n \phi\left(\left\| x_n - y_n \right\|\right) + l_n + \left\| T_{i(n)}^{k(n)} y_n - x_n \right\|. \end{aligned}$$

It follows from (16), (27) and (35) that

$$\lim_{n \to \infty} \left\| x_n - T_{i(n)}^{k(n)} x_n \right\| = 0. \tag{36}$$

Notice that

$$\begin{aligned} \left\| y_n - x_n \right\| &= \left\| \beta_n\left(S_{i(n)}^{k(n)}x_n - x_n\right) + \delta_n(v_n - x_n) \right\| \\ &\le \beta_n \left\| S_{i(n)}^{k(n)} x_n - x_n \right\| + \delta_n \|v_n - x_n\|. \end{aligned}$$

In view of (35), I see from the restriction (iii) that

$$\lim_{n \to \infty} \left\| y_n - x_n \right\| = 0. \tag{37}$$

Notice that

$$\begin{aligned} \left\| x_{n-1} - T_{i(n)}^{k(n)} x_n \right\| &\le \left\| x_{n-1} - T_{i(n)}^{k(n)} y_n \right\| + \left\| T_{i(n)}^{k(n)} y_n - T_{i(n)}^{k(n)} x_n \right\| \\ &\le \left\| x_{n-1} - T_{i(n)}^{k(n)} y_n \right\| + \left\| y_n - x_n \right\| + \mu_n \phi\left(\left\| y_n - x_n \right\|\right) + l_n. \end{aligned}$$

It follows that (16), (26) and (37) that

$$\lim_{n\to\infty} \left\| x_{n-1} - T_{i(n)}^{k(n)} x_n \right\| = 0. \tag{38}$$

Notice that

$$\|x_n - x_{n-1}\| \le \alpha_n \left\| T_{i(n)}^{k(n)} y_n - x_{n-1} \right\| + \gamma_n \|u_n - x_{n-1}\|.$$

In view of (26), I see from the restriction (iii) that

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{39}$$

Notice that

$$\left\| T_{i(n)}^{k(n)} x_n - S_{i(n)}^{k(n)} x_n \right\| \le \left\| T_{i(n)}^{k(n)} x_n - x_n \right\| + \left\| x_n - S_{i(n)}^{k(n)} x_n \right\|.$$

It follows from (35) and (36) that

$$\lim_{n\to\infty} \left\| T_{i(n)}^{k(n)} x_n - S_{i(n)}^{k(n)} x_n \right\| = 0.$$

From (39), I also have

$$\lim_{n\to\infty} \left\| x_n - x_{n+j} \right\| = 0, \quad \forall j = 1, 2, \dots, N. \tag{40}$$

For any positive $n > N$, it can be rewritten as $n = (k(n) - 1)N + i(n), i(n) \in \{1, 2, \dots, N\}$. Note that

$$\begin{aligned}
\|x_{n-1} - T_n x_n\| &\le \left\| x_{n-1} - T_{i(n)}^{k(n)} x_n \right\| + \left\| T_{i(n)}^{k(n)} x_n - T_n x_n \right\| \\
&\le \left\| x_{n-1} - T_{i(n)}^{k(n)} x_n \right\| + L \left\| T_{i(n)}^{k(n)-1} x_n - x_n \right\| \\
&\le \left\| x_{n-1} - T_{i(n)}^{k(n)} x_n \right\| + L \left\{ \left\| T_{i(n)}^{k(n)-1} x_n - T_{i(n-N)}^{k(n)-1} x_{n-N} \right\| \right. \\
&\quad \left. + \left\| T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{(n-N)-1} \right\| + \left\| x_{(n-N)-1} - x_n \right\| \right\}.
\end{aligned} \tag{41}$$

On the other hand, I have $n - N = ((k(n) - 1) - 1)N + i(n) = ((k(n) - 1) - 1)N + i(n - N)$, i.e.,

$$k(n - N) = k(n) - 1 \text{ and } i(n - N) = i(n).$$

It follows that

$$\left\| T_{i(n)}^{k(n)-1} x_n - T_{i(n-N)}^{k(n)-1} x_{n-N} \right\| \le L \|x_n - x_{n-N}\|, \tag{42}$$

and

$$\left\| T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{(n-N)-1} \right\| = \left\| T_{i(n-N)}^{k(n-N)} x_{n-N} - x_{(n-N)-1} \right\|. \tag{43}$$

Combining (38), (40) with (41), I see that

$$\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0. \tag{44}$$

On the other hand, I have that

$$\|x_n - T_n x_n\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|.$$

From (39) and (44), I obtain that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{45}$$

Consequently, for any $j = 1, 2, \ldots, N$, I see that

$$\begin{aligned}
\|x_n - T_{n+j} x_n\| &\le \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| \\
&\quad + \|T_{n+j} x_{n+j} - T_{n+j} x_n\| \\
&\le (1 + L)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\|.
\end{aligned}$$

From (40) and (45), I arrive at

$$\lim_{n\to\infty} \|x_n - T_{n+j} x_n\| = 0.$$

Therefore, for $\forall i \in \{1, 2, \ldots, N\}$, there exists some $e \in \{1, 2, \ldots, N\}$ such that $n + e = i(\mathrm{mod} N)$. It follows that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = \lim_{n\to\infty} \|x_n - T_{n+e} x_n\| = 0. \tag{46}$$

Similarly, by using the same argument as in the proof above, I have

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - S_{n+e} x_n\| = 0. \tag{47}$$

Since

$$\|S_i x_n - T_i x_n\| \le \|S_i x_n - x_n\| + \|T_i x_n - x_n\|,$$

I find from (46) and (47) that

$$\lim_{n\to\infty} \|S_i x_n - T_i x_n\| = 0, \quad \forall i \in \{1, 2, \ldots, N\}. \tag{48}$$

This completes the proof. $\qquad\square$

Theorem 2 *Let E be a real uniformly convex Banach space, K be a nonempty,closed and convex subset of E. Let $\{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N : K \to K$ be 2N uniformly L_i-Lipschitz total asymptotically nonexpansive mappings $\{\mu_n\}, \{l_n\}$ defined by (1) and (2) such that*

$$\sum_{n=1}^{\infty} \mu_n < \infty, \quad \sum_{n=1}^{\infty} l_n < \infty \tag{49}$$

and $\mathcal{F} : \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \varnothing$. Assume that there exist at least a $T_i (i \in I)$ which is $semi-compact$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^* \lambda$ for all $\lambda \geq M$, $i \in \{i = 1, 2, \ldots, N\}$. Let $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences in $\left[\frac{L-1}{L}, a\right]$, where $L = \max_{1 \leq i \leq N} \{L_i\} > 1$ and a is some constant in $(0, 1)$ satisfying the following conditions:

(i) $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$;
(ii) $\limsup_{n \to \infty} \alpha_n < 1$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$.

Then the sequence $\{x_n\}$ be defined by (3) *converges strongly to a common fixed point of* $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$ *in E.*

Proof   Without loss of generality, I may assume that T_1 is semicompact. It follows from (46) that

$$\lim_{n \to \infty} \|x_n - T_1 x_n\| = 0. \tag{50}$$

By the semicompactness of T_1, I have there exists a subsequence $\left\{x_{n_i}\right\}$ of $\{x_n\}$ such that $x_{n_i} \to q \in K$ strongly. From (46), I have

$$\lim_{n \to \infty} \left\|x_{n_i} - T_l x_{n_i}\right\| = \|q - T_l q\| = 0,$$

for all $l = 1, 2, \ldots, N$. Also, it follows from (48) and (50), I have

$$\lim_{n \to \infty} \left\|x_{n_i} - S_l x_{n_i}\right\| = \|q - S_l q\| = 0,$$

for any $l \in I$. This implies that $q \in \mathcal{F}$. From Lemma 3 , I know that $\lim_{n \to \infty} \|x_n - q\|$ exists for all $q \in \mathcal{F}$. It follows from $x_{n_i} \to q$ that $\lim_{n \to \infty} \|x_n - q\| = 0$. Thus, the iterative sequence $\{x_n\}$ defined by (3) converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$ in E. This completes the proof. $\square$

Corollary 1   *Let E be a real uniformly convex Banach space, K be a nonempty,closed and convex subset of E. Let $\{T_i\}_{i=1}^N : K \to K$ be N uniformly $L_i - Lipschitz$ total asymptotically nonexpansive mappings $\{\mu_n\}, \{l_n\}$ defined by* (1) *such that*

$$\sum_{n=1}^{\infty} \mu_n < \infty, \quad \sum_{n=1}^{\infty} l_n < \infty \tag{51}$$

and $\mathcal{F} : \bigcap_{i=1}^N F(T_i) \neq \varnothing$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^* \lambda$ for all $\lambda \geq M, i \in \{i = 1, 2, \ldots, N\}$. Let $\{x_n\}$ be the sequence defined by

$$\begin{cases} n = (k(n) - 1) + i(n) \in \{1, \ldots, N\}, \\ x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n, \\ y_n = (1 - \beta_n - \delta_n)x_n + \beta_n T_{i(n)}^{k(n)} x_n + \delta_n v_n, \ n \geq 1, \end{cases}$$

where $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences in $\left[\frac{L-1}{L}, a\right]$, where $L = \max_{1 \le i \le N}\{L_i\} > 1$ and a is some constant in $(0, 1)$ satisfying the following conditions:

(i) *$\alpha_n + \gamma_n \le 1$ and $\beta_n + \delta_n \le 1$ for all $n \ge 1$;*
(ii) *$\limsup_{n \to \infty} \alpha_n < 1$;*
(iii) *$\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$.*

Then

(a) *The implicit iterative sequence $\{x_n\}$ converges strongly to a common fixed point in $\mathcal{F}$ if and only if $\liminf_{n \to \infty} d(x_n, \mathcal{F}) = 0$, where $d(x, \mathcal{F})$ denotes the distance of x to set $\mathcal{F}$, i.e., $d(x, \mathcal{F}) = \inf_{y \in \mathcal{F}} d(x, y)$.*
(b) *If one of the mappings in $\{T_1, \ldots, T_N\}$ is $semi-compact$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.*

Corollary 2 *Let E be a real uniformly convex Banach space, K be a nonempty,closed and convex subset of E. Let $\{T_i\}_{i=1}^{N} : K \to K$ be N uniformly $L_i - Lipschitz$ total asymptotically nonexpansive mappings $\{\mu_n\}, \{l_n\}$ defined by (1) such that*

$$\sum_{n=1}^{\infty} \mu_n < \infty, \quad \sum_{n=1}^{\infty} l_n < \infty \tag{52}$$

and $\mathcal{F} : \bigcap_{i=1}^{N} F(T_i) \ne \varnothing$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \le M^* \lambda$ for all $\lambda \ge M, i \in \{i = 1, 2, \ldots, N\}$. Let $\{x_n\}$ be the sequence defined by

$$\begin{cases} n = (k(n) - 1) + i(n) \in \{1, \ldots, N\}, \\ x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_n + \gamma_n u_n, \ n \ge 1, \end{cases}$$

where $\{u_n\}$ is a bounded sequence in K. Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two real sequences in $\left[\frac{L-1}{L}, a\right]$, where $L = \max_{1 \le i \le N}\{L_i\} > 1$ and a is some constant in $(0, 1)$ satisfying the following conditions:

(i) *$\alpha_n + \gamma_n \le 1$ and $\beta_n + \delta_n \le 1$ for all $n \ge 1$;*
(ii) *$0 < \liminf_{n \to \infty} \alpha_n < \limsup_{n \to \infty} \alpha_n < 1$;*
(iii) *$\sum_{n=1}^{\infty} \gamma_n < \infty$. Then*

(a) *The implicit iterative sequence $\{x_n\}$ converges strongly to a common fixed point in $\mathcal{F}$ if and only if $\liminf_{n \to \infty} d(x_n, \mathcal{F}) = 0$, where $d(x, \mathcal{F})$ denotes the distance of x to set $\mathcal{F}$, i.e., $d(x, \mathcal{F}) = \inf_{y \in \mathcal{F}} d(x, y)$.*
(b) *If one of the mappings in $\{T_1, \ldots, T_N\}$ is $semi - compact$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.*

Remark 1 Since total asymptotically nonexpansive mappings include asymptotically nonexpansive mappings, Theorem 2 improves and generalizes Theorem 3.7 in Cianciaruso et al. (2010) and Theorem 3.7 in Hao et al. (2012).

Conclusion 1

1. *My theorems and corolaries which include the corresponding results announced in* Xu and Ori (2001), Sun (2003), Chang et al. (2006), Gu (2006) *as special cases fundamentally improve and generalize the results of* Cianciaruso et al. (2010) *and* Hao et al. (2012) *in the following sense.*

   (i)   *Extend the mappings from the class of asimptotically nonexpansive mappings to the class of total asimptotically nonexpansive mappings.*

   (ii) *Extend the mappings from $\{T_i\}_{i=1}^{N} : K \to K$ be N mappings to $\{S_i\}_{i=1}^{N}$, $\{T_i\}_{i=1}^{N} : K \to K$ be 2N mappings.*

2. *I considered an Ishikawa-type iterative algorithm for the class of total asimptotically nonexpansive mappings. But, my results are also available in the Mann-type iterative algorithm for the class of total asimptotically nonexpansive mappings.*

Example 1   Let E is the real line with the usual norm $|.|$, $K = (-1, 1)$. Assume that $Tx = \sin x$ and $Sx = \sin(-x)$ for $x \in K$. Let ϕ be a strictly increasing continuous function such that $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$. Let $\{\mu_n\}_{n \geq 1}$ and $\{l_n\}_{n \geq 1}$ in $\mathbb{R}$ be two sequences defined by $\mu_n = \frac{1}{n}$ and $l_n = \frac{1}{n+1}$, for all $n \geq 1$ ($\lim_{n \to \infty} \mu_n = \lim_{n \to \infty} \frac{1}{n} = 0$, $\lim_{n \to \infty} l_n = \frac{1}{n+1} = 0$). Since $Tx = \sin x$ for $x \in K$, I have

$$\left| T^n x - T^n y \right| \leq |x - y|.$$

For all $x, y \in K$, I obtain

$$\begin{aligned}
\left| T^n x - T^n y \right| &- |x - y| - \mu_n \phi(|x - y|) - l_n \\
&\leq |x - y| - |x - y| - \mu_n \phi(|x - y|) - l_n \\
&\leq 0
\end{aligned}$$

for all $n = 1, 2, \ldots$, $\{\mu_n\}_{n \geq 1}$ and $\{l_n\}_{n \geq 1}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and so T is a total asymptoticaly nonexpansive mapping. Also, $Sx = \sin(-x)$ for $x \in K$, I have

$$\left| S^n x - S^n y \right| \leq |x - y|.$$

For all $x, y \in K$, I obtain

$$\begin{aligned}
\left| S^n x - S^n y \right| &- |x - y| - \mu_n \phi(|x - y|) - l_n \\
&\leq |x - y| - |x - y| - \mu_n \phi(|x - y|) - l_n \\
&\leq 0
\end{aligned}$$

for all $n = 1, 2, \ldots$, $\{\mu_n\}_{n \geq 1}$ and $\{l_n\}_{n \geq 1}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and so S is a total asymptoticaly nonexpansive mapping. Clearly, $F := F(T) \cap F(S) = \{0\}$. Set

$$\alpha_n = \beta_n = \frac{n}{2n+1}, \ \delta_n = \gamma_n = \frac{n^3}{6n^3+1} \text{ and } u_n = v_n = \frac{1}{n+1}$$

for $n \geq 1$. In order to easily calculate, I modifed my iteration scheme for $n = 1$. This scheme (53) is defined as follows:

$$\begin{aligned}
x_1 &= (1 - \alpha_1 - \gamma_1)x_0 + \alpha_1 T_{i(1)}^{k(1)} y_1 + \gamma_1 u_1, \\
y_1 &= (1 - \beta_1 - \delta_1)x_1 + \beta_1 S_{i(1)}^{k(1)} x_1 + \delta_1 v_1.
\end{aligned} \tag{53}$$

The numerical experiment outcome obtained by using Scientific WorkPlace 5.5 show that as $x_0 = 0$, the computation of $y_1 = 9.0853 \times 10^{-2}$. This example illustrates the efficiency of approximation of common fixed points of total asymptotically nonexpansive mappings.

Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N : E \to E$ be $2N$ total asymptotically nonexpansive mappings; assuming existence of common fixed points of these operators, our theorems and method of proof easily carry over to this class of mappings using the following implicit iterative scheme $\{x_n\}$ with errors:

$$\begin{cases} x_0 \in K, \\ x_n = a_n x_{n-1} + b_n T_{i(n)}^{k(n)} y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n S_{i(n)}^{k(n)} x_n + c'_n v_n, \ n \geq 1 \end{cases} \tag{54}$$

where $n = (k(n) - 1)N + i(n)$, $i(n) \in 1, 2, \ldots, N$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are six real sequences in $[0, 1]$ satisfying $a_n = b_n = c_n = 1 = a'_n = b'_n = c'_n$ for all $n \geq 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences.

In order not to enlarge this note unnecessarily, I only include total asymptotically nonexpansive mappings. But, in accordance with the above proof of theorem, one can easily prove in total asymptotically quasi nonexpansive mappings.

If $S = I$, then (54) transform to implicit iterative scheme defined by Mukhamedov and Saburov (see, more details Mukhamedov and Saburov 2012a). My theorems and corolaries also improve and generalize the mappings from the class of a finite family of quasi-asymptotically nonexpansive mappings to the class of a finite family of total quasi-asimptotically nonexpansive mappings.

If $S, T : E \to E$ be two total asymptotically nonexpansive mappings and take $S = I$, then (54) reduce to implicit iterative scheme defined by Mukhamedov and Saburov (see, more details Mukhamedov and Saburov 2011). My theorems and corolaries also improve and generalize the mappings from the class of quasi-asimptotically nonexpansive mappings to the class of total quasi-asimptotically nonexpansive mappings.

References
Albert Ya I, Chidume CE, Zegeye H (2006) Approximating fixed points of total asymptotically nonexpansive mappings. Fixed Point Theory Appl. 10673–1–10673–20. doi:10.1155/FPTA/2006/10673
Chang SS, Tan KK, Lee HWJ, Chan CK (2006) On the convergence of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings. J Math Anal Appl 313:273–283
Cianciaruso F, Marino G, Wang X (2010) Weak and strong convergence of the Ishikawa iterative process for a finite family of asymptotically nonexpansive mappings. Appl Math Comput 216:3558–3567
Gu F (2006) Convergence of implicit iterative process with errors for a finite family of asymptotically nonexpansive mappings. Acta Math Sci 26:1131–1143
Hao Y (2010) Convergence theorems for total asymptotically nonexpansive mappings. An Şt Univ Ovidius Constanta 18(1):163–180

Hao Y, Wang X, Tong A (2012) Weak and strong convergence theorems for two finite families of asymptotically nonexpansive mappings in Banach Spaces. Adv Fixed Point Theory 2(4):417–432

Mukhamedov F, Saburov M (2010a) Weak and strong convergence of an implicit iteration process for an asymptotically quasi-I-nonexpansive mapping in Banach Space. Fixed Point Theory and Appl. 2010:719631–1–719631–13. doi:10.1155/2010/719631

Mukhamedov F, Saburov M (2010b) Strong convergence of an explicit iteration process for a totally asymptotically I-nonexpansive mapping in Banach Space. Appl Math Lett 23(12):1473–1478

Mukhamedov F, Saburov M (2011) Weak convergence of an implicit iterative process with errors for an asymptotically quasi I-non-expansive mapping in Banach Space. Asian Eur J Math 4(2):309–319

Mukhamedov F, Saburov M (2012a) On convergence theorems of an implicit iterative process with errors for a finite family of asymptotically quasi I non-expansive mappings. Aust J Math Anal Appl (AJMAA) 9(1):1–17

Mukhamedov F, Saburov M (2012b) On unification of the strong convergence theorems for a finite family of total asymptotically non-expansive mappings in Banach Space. J Appl Math. 2012:281383–1–281383–21 doi:10.1155/2012/281383

Qin X, Su Y, Shang M (2008) On the convergence of implicit iteration process for a finite family of k-strictly asymptotically pseudocontractive mappings. Kochi J Math 3:67–76

Schu J (1991) Weak and strong convergence of a fixed points of asymptotically nonexpansive mappings. Bull Aust Math Soc 43:153–159

Sun ZH (2003) Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings. J Math Anal Appl 286:351–358

Tan KK, Xu HK (1993) Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. J Math Anal Appl 178:301–308

Xu HK, Ori RG (2001) An implicit iteration process for nonexpansive mappings. Numer Funct Anal Optim 22:767–773