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Curvature of curves parameterized by a time scale

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Abstract

Curvature is a fundamental characteristic of curves in differential geometry, as well as in discrete geometry. In this paper we present time scales analogy of the curvature defined by the concept of symmetric derivative on time scales. The goal of our paper is to define this intrinsic characteristic accurately. For this purpose, we consider tangent spaces via symmetric differentiation.

Keywords: symmetric differentiation; time scales calculus; curvature

1 Introduction

Curvature of a curve measures how sharply a curve bends; that is the second order amount by which a curve deviates from being a straight line. In classical differential geometry, curvature of an arc-length parameterized curve $\alpha : I \to \mathbb{R}^3$ can be computed by $\kappa(s) = \|\alpha''(s)\|$ [1]. However, in a discrete case there are several approaches to define the curvature of a discrete curve. A discrete curve α_d is composed of a series of sequential discrete points, see [2], or can be defined as the line segments $|v_{i-1}v_i|$ given by an ordered list of points $v_0, \dots, v_N \in \mathbb{R}^2$, *i.e.*, polygons [3]. The most direct definition of the curvature of a discrete curve as a polygon is the turning angle curvature. Turning angle curvature κ_a can be computed by

$$\kappa_a(t) = \operatorname{sgn} \frac{2|\angle(\vec{v}_{i-1}, \vec{v}_i)|}{\|\vec{v}_{i-1}\| + \|\vec{v}_i\|},$$

where the vertice vector $\vec{v}_i = \vec{v}_{i+1} - \vec{v}_i$, $i \in \{0, \dots, N\}$, see [3]. A curvature model for a discrete curve as a composition of a series of sequential discrete points can be computed as

$$\kappa_s(s) = \operatorname{sgn} \frac{\|\alpha(s) - 2\alpha(s-\varepsilon) + \alpha(s-2\varepsilon)\|}{\varepsilon^2},$$

where ε is the sampling parameter on underlying continuous curve to determine the sequential discrete points. This approach is mostly concerned with second order finite differences.

The time scales calculus, which is introduced by Hilger [4], is the theory to unify discrete and continuous calculus. Geometric aspect of the theory of time scales has been extensively studied after the introduction of partial derivatives on time scales [5–9]. However, an intrinsic characteristic such as curvature of a curve parameterized by a time scale is still an open question. In this paper, we present the concept of curvature via symmetric derivative on time scales. This approach involves both characteristics of discrete and classical differential geometry, and it is accurately applicable to globally discrete settings.

In [10], authors briefly introduced the symmetric derivative on time scales and its relation to forward and backward dynamic derivatives. The main purpose of that study is to aim differentiability of the functions such as $f(t) = |t|$ at $t = 0$. Beside the differentiability, this calculus comes up with a more accurate definition for tangent lines of curves parameterized by time scales. As it is stated in [11], delta (and respectively nabla) derivatives of functions lead us to the concept of 'complete differentiability'. To speak of one variable case, σ-complete differentiability needs the equality of right- and left-hand side derivatives, which makes the strong geometric restrictions for curves involving left dense-right scattered or right dense-left scattered points. Besides, tangent lines are not well defined at isolated points. The main disadvantage of this approach can be seen in [12], where the curvature can only be defined at dense and scattered points separately.

This paper is organized as follows. In Section 2, we introduce symmetric partial differentiation on time scales. We also present the relationship between symmetric differentiation and delta-nabla differentiation. Since the change of tangent spaces is a fundamental characteristic to define curvature, we present tangent lines and tangent planes of curves and surfaces parameterized by time scales in Section 3. In this section, the accuracy of a new tangent space definition via symmetric differentiation can be seen throughout the illustrative examples. Finally, in Section 4, we study the curvature of curves parameterized by an arbitrary time scale. Throughout the study we use the notion such as $f^{\sigma}(t) = f(\sigma(t))$ and $f^{\sigma\sigma}(t) = f(\sigma(\sigma(t)))$ to increase the readability of the paper.

2 Symmetric partial derivative on time scales

Let n be fixed and for all $\mathbb{T}_i$ be time scales where $i \in I = \{1, 2, \dots, n\}$. An n-dimensional time scale can be defined by the Cartesian product as follows:

$$\Lambda^n = \mathbb{T}_1 \times \cdots \times \mathbb{T}_n = \left\{(t_1, \dots, t_n) \mid t_i \in \mathbb{T}_i, \forall i \in I\right\}.$$

For $u \in \mathbb{T}_i$, the forward and backward jump operators can be defined as $\sigma_i(u) = \inf\{v \in \mathbb{T}_i \mid v > u\}$ and $\rho_i(u) = \sup\{v \in \mathbb{T} \mid v < u\}$, respectively. If $\mathbb{T}_i$ has a left scattered maximum M and right scattered minimum m, then $(\mathbb{T}_i)^{\kappa}_{\kappa} = \mathbb{T}_i \backslash \{M, m\}$, $(\mathbb{T}_i)^{\kappa}_{\kappa} = \mathbb{T}_i$; otherwise, see [11].

Let $f : \Lambda^n \to \mathbb{R}$ be a real-valued function. The symmetric partial derivative of f can be defined as

$$\lim_{\substack{s_i \to t_i \\ \rho_i(t_i) \neq s_i \\ \sigma_i(t_i) \neq s_i}} \frac{f^{\sigma_i}(s_i) - f(t_1, \dots, t_i, \dots, t_n) + f(t_1, \dots, 2s_i - t_i, \dots, t_n) - f^{\rho_i}(s_i)}{\sigma(t_i) + 2s_i - 2t_i - \rho(t_i)}$$

existing as a finite number, and is denoted by $f^{\Diamond_i}(t)$ or $\frac{\partial f(t)}{\Diamond_i t_i}$, where $f^{\sigma_i}(s_i) = f(t_1, \dots, \sigma_i(s_i), \dots, t_n)$, $f^{\rho_i}(s_i) = f(t_1, \dots, \rho_i(s_i), \dots, t_n)$, and $t = (t_1, \dots, t_n) \in (\mathbb{T}_1)^{\kappa}_{\kappa} \times \cdots \times (\mathbb{T}_n)^{\kappa}_{\kappa}$.

Definition 1 A function $f : \Lambda^n \to \mathbb{R}$ is symmetric differentiable at a point $t^0 = (t^0_1, \dots, t^0_n) \in (\mathbb{T}_1)^{\kappa}_{\kappa} \times \cdots \times (\mathbb{T}_n)^{\kappa}_{\kappa}$ if there exist numbers $A_1, \dots, A_n$ independent of $t = (t_1, \dots, t_n) \in \Lambda^n$ such that for all $t \in U_\delta(t^0)$ and $i \in \{1,\dots,n\}$,

$$\begin{aligned} &f\big(t_1^0,\dots,\sigma_i\big(t_i^0\big),\dots,t_n^0\big) - f(t_1,\dots,t_n) + f\big(2t_1^0 - t_1,\dots,2t_i^0 - t_i,\dots,2t_n^0 - t_n\big) \\ &\quad - f\big(t_1^0,\dots,\rho_i\big(t_i^0\big),\dots,t_n^0\big) \\ &\quad = \sum_{i=1}^n A_i\big[\sigma_i\big(t_i^0\big) + 2t_i^0 - 2t_i - \rho_i\big(t_i^0\big)\big] + \sum_{i=1}^n \alpha_i\big[\sigma_i\big(t_i^0\big) + 2t_i^0 - 2t_i - \rho_i\big(t_i^0\big)\big], \end{aligned}$$

where δ is a sufficiently small positive number, $U_\delta(t^0)$ is the δ-neighborhood of t^0, and $\alpha_i = \alpha_i(t^0, t)$ are defined on $U_\delta(t^0)$ such that it is equal to zero for $t = t^0$ and $\lim_{t\to t^0} \alpha_i = 0$ for all $i \in \{1,\dots,n\}$.

If $\mathbb{T}_1 = \cdots = \mathbb{T}_n = \mathbb{R}$, then Definition 1 coincides with the classical symmetric differentiability, see [13, 14]. To point out why we do not need a restriction such as 'complete' in the definition of symmetric differentiation, let us consider a one-dimensional case. If t_0 is left dense and right scattered, *i.e.*, $\sigma(t_0) > t_0$ and $\rho(t_0) = t_0$, then

$$f\big(\sigma(t_0)\big) - f(t) + f(2t_0 - t) - f(t_0) = A\big[\sigma(t_0) + t_0 - 2t\big] + \alpha\big[\sigma(t_0) + t_0 - 2t\big].$$

This equation leads us to

$$A = f^\diamondsuit = \gamma(t) f^\Delta(t) + \big(1 - \gamma(t)\big) f'_-(t),$$

where $\lim_{t\to t^0} \alpha = 0$ and $\gamma(t) = \lim_{t\to t_0} \frac{\sigma(t_0)-t}{\sigma(t_0)+t_0-2t}$ as in [10, Proposition 4.2]. For right dense and left scattered and isolated points, the A values become

$$A = f^\diamondsuit = \gamma(t) f'_+ + \big(1 - \gamma(t)\big) f^\nabla(t)$$

and

$$A = f^\diamondsuit = \gamma(t) f^\Delta(t) + \big(1 - \gamma(t)\big) f^\nabla(t),$$

respectively.

Since we restrict our interest to curves and surfaces on time scales, we will consider a two-dimensional case, *i.e.*, $n = 2$, throughout the study. Interested readers can simply extend this idea to a higher-dimensional case.

Definition 2 Let $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be a real-valued function and $(t_0, s_0) \in (\mathbb{T}_1)^\kappa_\kappa \times (\mathbb{T}_2)^\kappa_\kappa$. For all $\varepsilon_1 > 0$, there is an open (relative to the topology of $\mathbb{T}_1 \times \mathbb{T}_2$) neighborhood U_1 of (t_0, s) such that for all $(t, s) \in U_1$,

$$\begin{aligned} &\big|\big[f\big(\sigma_1(t_0), s\big) - f(t,s) + f(2t_0 - t, s) - f\big(\rho_1(t_0), s\big)\big] - f^{\diamondsuit_1}\big[\sigma_1(t_0) + 2t_0 - 2t - \rho_1(t_0)\big]\big| \\ &\quad \le \varepsilon_1 \big|\sigma_1(t_0) + 2t_0 - 2t - \rho_1(t_0)\big|. \end{aligned}$$

Definition 3 Let $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be a real-valued function and $(t_0, s_0) \in (\mathbb{T}_1)^\kappa_\kappa \times (\mathbb{T}_2)^\kappa_\kappa$. For all $\varepsilon_2 > 0$, there is an open (relative to the topology of $\mathbb{T}_1 \times \mathbb{T}_2$) neighborhood U_2 of

(t,s_0) such that for all $(t,s)\in U_2$,

$$\begin{aligned}&\big|\big[f\big(t,\sigma_2(s_0)\big)-f(t,s)+f(t,2s_0-s)-f\big(t,\rho_2(s_0)\big)\big]-f^{\Diamond_2}\big[\sigma_2(s_0)+2s_0-2s-\rho_2(s_0)\big]\big|\\&\quad\le\varepsilon_2\big|\sigma_2(s_0)+2s_0-2s-\rho_2(s_0)\big|.\end{aligned}$$

Note that higher order or mixed symmetric partial derivatives of a function defined on $\mathbb{T}_1\times\mathbb{T}_2$ can be defined in the same sense as

$$\frac{\partial^2 f(t_0,s_0)}{\Diamond_1 t^2}=\frac{\partial}{\Diamond_1 t}\left(\frac{\partial f(t_0,s_0)}{\Diamond_1 t}\right),\qquad\frac{\partial^2 f(t_0,s_0)}{\Diamond_2 s^2}=\frac{\partial}{\Diamond_2 s}\left(\frac{\partial f(t_0,s_0)}{\Diamond_2 s}\right)$$

and

$$\frac{\partial^2 f(t_0,s_0)}{\Diamond_1 t\Diamond_2 s}=\frac{\partial}{\Diamond_2 s}\left(\frac{\partial f(t_0,s_0)}{\Diamond_1 t}\right),\qquad\frac{\partial^2 f(t_0,s_0)}{\Diamond_2 s\Diamond_1 t}=\frac{\partial}{\Diamond_1 t}\left(\frac{\partial f(t_0,s_0)}{\Diamond_2 s}\right).$$

Proposition 4 *If $f:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$ is delta and nabla differentiable, then f is symmetric differentiable for each $(t,s)\in(\mathbb{T}_1)^\kappa_\kappa\times(\mathbb{T}_2)^\kappa_\kappa$ with*

$$\frac{\partial f(t,s)}{\Diamond_1 t}=\gamma_1(t_0)\frac{\partial f(t_0,s_0)}{\Delta_1 t}+\big(1-\gamma_1(t_0)\big)\frac{\partial f(t_0,s_0)}{\nabla_1 t}$$

and

$$\frac{\partial f(t,s)}{\Diamond_2 s}=\gamma_2(s_0)\frac{\partial f(t_0,s_0)}{\Delta_2 s}+\big(1-\gamma_2(s_0)\big)\frac{\partial f(t_0,s_0)}{\nabla_2 s},$$

where

$$\gamma_1(t_0)=\lim_{t\to t_0}\frac{\sigma_1(t_0)-t}{\sigma_1(t_0)+2t_0-2t-\rho_1(t_0)}\quad\textit{and}\quad\gamma_2(s_0)=\lim_{s\to s_0}\frac{\sigma_2(s_0)-s}{\sigma_1(s_0)+2s_0-2s-\rho_1(s_0)}.$$

Proof

$$\begin{aligned}\frac{\partial f}{\Diamond_1 t}&=\lim_{(t,s)\to(t_0,s)}\frac{f(\sigma_1(t_0),s)-f(t,s)+f(2t_0-t,s)-f(\rho_1(t_0),s)}{\sigma_1(t_0)+2t_0-2t-\rho_1(t_0)}\\&=\lim_{(t,s)\to(t_0,s)}\frac{\sigma_1(t_0)-t}{\sigma_1(t_0)+2t_0-2t-\rho_1(t_0)}\frac{f(\sigma_1(t_0),s)-f(t,s)}{\sigma_1(t_0)-t}\\&\quad+\lim_{(t,s)\to(t_0,s)}\frac{2t_0-t-\rho_1(t_0)}{\sigma_1(t_0)+2t_0-2t-\rho_1(t_0)}\frac{f(2t_0-t,s)-f(\rho_1(t_0),s)}{2t_0-t-\rho_1(t_0)}\\&=\lim_{(t,s)\to(t_0,s)}\frac{\sigma_1(t_0)-t}{\sigma_1(t_0)+2t_0-2t-\rho_1(t_0)}\frac{\partial f(t_0,s_0)}{\Delta_1 t}\\&\quad+\lim_{(t,s)\to(t_0,s)}\frac{2t_0-t-\rho_1(t_0)}{\sigma_1(t_0)+2t_0-2t-\rho_1(t_0)}+\frac{\partial f(t_0,s_0)}{\nabla_1 t}\\&=\gamma_1(t_0)\frac{\partial f(t_0,s_0)}{\Delta_1 t}+\bar{\gamma}_1(t_0)+\frac{\partial f(t_0,s_0)}{\nabla_1 t},\end{aligned}$$

where $\gamma_1(t_0)=\lim_{t\to t_0}\frac{\sigma_1(t_0)-t}{\sigma_1(t_0)+2t_0-2t-\rho_1(t_0)}$ and $\bar{\gamma}_1(t_0)=\lim_{t\to t_0}\frac{2t_0-t-\rho_1(t_0)}{\sigma_1(t_0)+2t_0-2t-\rho_1(t_0)}$ for all $t_0\in\mathbb{T}_1$. It is straightforward that $\gamma_1+\bar{\gamma}_1:=1$.

If t_0 is left dense and right scattered, then $\gamma_1(t_0) = \frac{\sigma_1(t_0)-t_0}{\sigma_1(t_0)-t_0} = 1$ and $f^{\Diamond_1}(t_0,s_0) = f^{\Delta_1}(t_0,s_0)$.

If t_0 is right dense and left scattered, then $\sigma_1(t_0) = t_0$ implies $\gamma_1(t_0) = 0$ and therefore $f^{\Diamond_1}(t_0,s_0) = f^{\nabla_1}(t_0,s_0)$. Also if t_0 is dense $\gamma_1(t_0) = \lim_{t\to t_0} \frac{t_0-t}{2t_0-t} = \frac{1}{2}$ and $f^{\Diamond_1}(t_0,s_0) = \frac{1}{2}f^{\Delta_1}(t_0,s_0) + \frac{1}{2}f^{\nabla_1}(t_0,s_0)$.

Moreover, if t_0 is an isolated point, then $\gamma_1(t_0) = \frac{\sigma_1(t_0)-t_0}{\sigma_1(t_0)-\rho_1(t_0)}$ and $f^{\Diamond_1}(t_0,s_0) = \frac{\sigma_1(t_0)-t_0}{\sigma_1(t_0)-\rho_1(t_0)} \times f^{\Delta_1}(t_0,s_0) + \frac{t_0-\rho_1(t_0)}{\sigma_1(t_0)-\rho_1(t_0)} f^{\nabla_1}(t_0,s_0)$.

The same procedure can be followed to obtain a similar result $\frac{\partial f(t,s)}{\Diamond_2 s} = \gamma_2(s_0)\frac{\partial f(t_0,s_0)}{\Delta_2 s} + (1 - \gamma_2(s_0))\frac{\partial f(t_0,s_0)}{\nabla_2 s}$. □

3 Tangent spaces

The geometric theory of the curves and surfaces parameterized by time scales and their analysis with the delta derivative can be found in [8, 9, 15]. It is possible to define curves and surfaces with the idea of symmetric differentiation in the same fashion. One can also extend the previous results to the symmetric differentiation.

Definition 5 A $\Diamond$-regular curve α is defined as a vector-valued mapping from $[a,b] \subset \mathbb{T}$ to $\mathbb{R}^3$ with the non-zero norm $\|\alpha^{\Diamond}(t_0)\|$ for all $t_0 \in [a,b]$. Moreover, if $\|\alpha^{\Delta}(t_0)\| = 1$ for all $t_0 \in [a,b]$, then α is called 'arc length parameterized curve'.

Definition 6 Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^3$. $\mathcal{S}$ is a surface if for each point P in $\mathcal{S}$, there is a neighborhood A of P and a function $\varphi : U \to \mathcal{S}$ where U is a closed set in $\mathbb{R}^2$ and an open set in time scale topology satisfying the following conditions:

(i) $\varphi : U \to \mathbb{R}^3$ is $\Diamond$-differentiable and for all $(t,s) \in U$

$$\frac{\partial\varphi(t,s)}{\Diamond_1 t} \times \frac{\partial\varphi(t,s)}{\Diamond_2 s} \neq 0,$$

i.e., φ is $\Diamond$-regular.

(ii) $\varphi(U) = \mathcal{S} \cap A$ and $\varphi : U \to \varphi(U)$ is a homeomorphism.

The function $\varphi : U \to \mathcal{S}$ is called a surface patch. $\mathcal{S}$ is called a smooth surface if, for all points P in $\mathcal{S}$, there exists a surface patch such that $P \in \varphi(U)$.

Definition 7 Let $\alpha : \mathbb{T} \to \mathbb{R}^n$ be a $\Diamond$-regular curve and $t_0 \in \mathbb{T}^{\kappa}_{\kappa}$. The line with the slope $\alpha^{\Diamond}(t_0)$ passing at the point $\alpha(t_0)$ is called the diamond-tangent line of α at t_0.

Remark 8 By this definition, it is clear that we do not need to have the equality of the left and right side dynamic derivatives. Hence, the tangent line defined with the symmetric derivative on time scales is more sensitive to the geometric change of the curve. In Figure 1 and Figure 2, $\mathcal{L}_0$ is the diamond-tangent line of the α at t_0 with the slope $\alpha^{\Diamond}(t_0)$. It is clear that the translations on the right or left continuous arcs of the curves will lead to new tangent lines. Hence, geometric changes in the curve affect the tangent line.

Definition 9 Let $\mathcal{S}$ be a surface with the patch $\varphi : U \to \mathcal{S}$, where $U \subset \mathbb{T}_1 \times \mathbb{T}_2$ and $(t_0,s_0) \in {\mathbb{T}_1}^{\kappa}_{\kappa} \times {\mathbb{T}_2}^{\kappa}_{\kappa}$. The plane with the normal vector $\varphi^{\Diamond_1} \times \varphi^{\Diamond_2}$ passing at the point $\varphi(t_0,s_0)$ is called a diamond-tangent plane of $\mathcal{S}$ at (t_0,s_0).

Remark 10 As in the definition of diamond-tangent line for a curve, diamond-tangent plane for a surface is also defined on isolated points. In Figure 3, Ω_0 is the diamond-

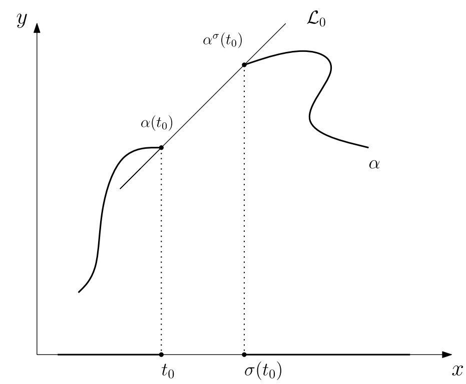

Figure 1 Diamond-tangent line at the left dense and right scattered point of the curve α. Therefore, $\gamma(t_0) = 1$ and $\alpha^{\diamond}(t_0) = \alpha^{\Delta}(t_0)$.

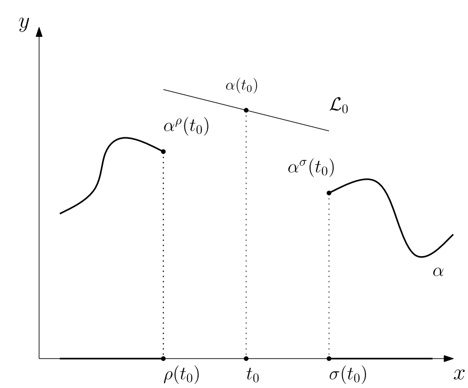

Figure 2 Diamond-tangent line at the isolated point of the curve α. Therefore, $\gamma(t_0) = \frac{\sigma(t_0)-t_0}{\sigma(t_0)-\rho(t_0)}$, and $\alpha^{\diamond}(t_0) = \frac{\sigma(t_0)-t_0}{\sigma(t_0)-\rho(t_0)}\alpha^{\Delta}(t_0) + \frac{t_0-\rho(t_0)}{\sigma(t_0)-\rho(t_0)}\alpha^{\nabla}(t_0)$.

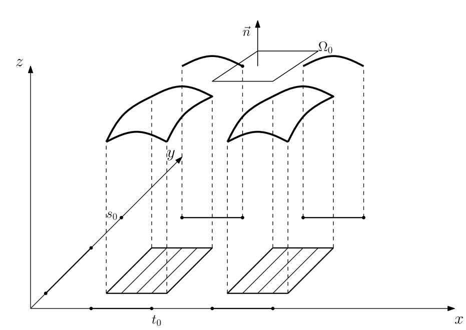

Figure 3 Diamond-tangent plane with the normal vector $\varphi^{\diamond_1}(t_0,s_0) \times \varphi^{\diamond_2}(t_0,s_0)$ at the point $\varphi(t_0,s_0)$ of the surface $\mathcal{S}$.

tangent plane of the $\mathcal{S}$ at (t_0,s_0) with the normal $\varphi^{\diamond_1}(t_0,s_0) \times \varphi^{\diamond_2}(t_0,s_0)$. It is clear that the geometric changes in the surface lead to new tangent planes.

4 Curvature of curves on time scales

Definition 11 Let $\alpha : \mathbb{T} \to \mathbb{R}^n$ be an arc length parameterized regular curve. The curvature of α at $t_0 \in \mathbb{T}^\kappa_\kappa$ is the norm of the second order symmetric derivative of α, *i.e.*,

$$\kappa_{\diamond}(t_0) = \left\|\alpha^{\diamond\diamond}(t_0)\right\|.$$

Remark 12 If $\mathbb{T} = \mathbb{R}$, then

$$\kappa_{\diamond}(t_0) = \left\|\alpha^{\diamond\diamond}(t_0)\right\| = \left\|\alpha''(t_0)\right\| = \kappa(t_0).$$

If the time scale $\mathbb{T}$ is completely discrete, then

$$\alpha^{\diamond\diamond}(t_0) = \frac{1}{(\sigma^2(t_0) - t_0)(\sigma(t_0) - \rho(t_0))}\bigl(\alpha^{\sigma\sigma}(t_0) - \alpha(t_0)\bigr) - \frac{1}{(t_0 - \rho^2(t_0))(\sigma(t_0) - \rho(t_0))}\bigl(\alpha(t_0) - \alpha^{\rho\rho}(t_0)\bigr).$$

Hence $\kappa_{\diamond}(t_0) = \|\alpha^{\diamond\diamond}(t_0)\|$ can be computed very accurately. Moreover, if $\mathbb{T}$ is with a nonzero constant graininess h

$$\begin{aligned}\alpha^{\diamond\diamond}(t_0) &= \frac{1}{4h^2}\bigl(\alpha(t+2h) - 2\alpha(t) + \alpha(t-2h)\bigr)\\ &= \frac{1}{\varepsilon^2}\bigl(\alpha(s) - 2\alpha(s-\varepsilon) + \alpha(s-2\varepsilon)\bigr)\end{aligned}$$

and the norm of the second order symmetric derivative leads us to $\kappa_s(t_0)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SPA and ÖA worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

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