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# Periodic boundary value problems for second-order impulsive integro-differential equations with integral jump conditions

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available at the end of the article**Abstract**

This paper is concerned with the existence of extremal solutions of periodic boundary value problems for second-order impulsive integro-differential equations with integral jump conditions. We introduce a new definition of lower and upper solutions with integral jump conditions and prove some new maximum principles. The method of lower and upper solutions and the monotone iterative technique are used.

**MSC:** 34B37; 34K10; 34K45**Keywords:** impulsive integro-differential equation; lower and upper solutions; periodic boundary value problem; monotone iterative technique

## 1 Introduction

Differential equations which have impulse effects describe many evolution processes that abruptly change their state at a certain moment. In recent years, impulsive differential equations have become more important tools in some mathematical models of real processes and phenomena studied in physics, biotechnology, chemical technology, population dynamics and economics; see [1–5]. Many papers have been published about existence analysis of periodic boundary value problems of first and second order for impulsive ordinary or functional or integro-differential equations. We refer the readers to the papers [6–29]. More recent works on existence results of impulsive problems with integral boundary conditions can be found in [30–35] and the reference therein. This literature has lead to significant development of a general theory for impulsive differential equations.

The monotone iterative technique coupled with the method of upper and lower solutions has been used to study the existence of extremal solutions of periodic boundary value problems for second-order impulsive equations; see, for example, [36–41]. This method has been also used to study abstract nonlinear problems; see [42]. However, in most of these papers concerned with applications of the monotone iterative technique to second-order periodic boundary value problems with impulses, the authors assume that the jump conditions at impulse point  $t_k$  of solution values and the derivative of solution values depend on the left-hand limits of solutions or the slope of solutions themselves, such as  $\Delta x(t_k) = I_k(x(t_k^-))$ ,  $\Delta x(t_k) = I_k(x'(t_k^-))$ ,  $\Delta x'(t_k) = I_k(x(t_k^-))$ ,  $\Delta x'(t_k) = I_k(x'(t_k^-))$ .

In this paper, we consider the periodic boundary value problem for second-order impulsive integro-differential equation (PBVP) with integral jump conditions:

$$\begin{cases} x''(t) = f(t, x(t), (Kx)(t), (Sx)(t)), & t \in J = [0, T], t \neq t_k, \\ \Delta x(t_k) = I_k \left( \int_{t_k - \delta_k}^{t_k - \varepsilon_k} x'(s) ds \right), & k = 1, 2, \dots, m, \\ \Delta x'(t_k) = I_k^* \left( \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds \right), & k = 1, 2, \dots, m, \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \quad (1.1)$$

where  $0 = t_0 < t_1 < t_2 < \dots < t_k < \dots < t_m < t_{m+1} = T$ ,  $f : J \times R^3 \rightarrow R$  is continuous everywhere except at  $\{t_k\} \times R^3$ ,  $f(t_k^+, x, y, z)$ ,  $f(t_k^-, x, y, z)$  exist,  $f(t_k^-, x, y, z) = f(t_k, x, y, z)$ ,  $I_k \in C(R, R)$ ,  $I_k^* \in C(R, R)$ ,  $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ ,  $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$ ,  $0 \leq \varepsilon_k \leq \delta_k \leq t_k - t_{k-1}$ ,  $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$ ,  $k = 1, 2, \dots, m$ ,

$$(Kx)(t) = \int_0^t k(t, s)x(s) ds, \quad (Sx)(t) = \int_0^T h(t, s)x(s) ds,$$

$k(t, s) \in C(D, R^+)$ ,  $h(t, s) \in C(J \times J, R^+)$ ,  $D = \{(t, s) \in R^2, 0 \leq s \leq t \leq T\}$ ,  $R^+ = [0, +\infty)$ ,  $k_0 = \max\{k(t, s) : (t, s) \in D\}$ ,  $h_0 = \max\{h(t, s) : (t, s) \in J \times J\}$ .

In [43, 44], the authors discussed some kinds of first-order impulsive problems with the integral jump condition

$$\Delta x(t_k) = I_k \left( \int_{t_k - \tau_k}^{t_k} x(s) ds - \int_{t_{k-1}}^{t_{k-1} + \sigma_{k-1}} x(s) ds \right), \quad (1.2)$$

where  $0 < \sigma_{k-1} \leq (t_k - t_{k-1})/2$ ,  $0 \leq \tau_{k-1} \leq (t_k - t_{k-1})/2$ ,  $k = 1, 2, \dots, m$ . We note that the jump condition (1.2) depends on functionals of path history before impulse points  $t_k$  and after the past impulse points  $t_{k-1}$ . The aim of our research is to deal with the integral jump conditions

$$\Delta x(t_k) = I_k \left( \int_{t_k - \delta_k}^{t_k - \varepsilon_k} x'(s) ds \right), \quad \Delta x'(t_k) = I_k^* \left( \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds \right), \quad (1.3)$$

where  $0 \leq \varepsilon_k \leq \delta_k \leq t_k - t_{k-1}$ ,  $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$ ,  $k = 1, 2, \dots, m$ . The integral jump condition (1.3) means that a sudden change of solution values and the derivative of solution values at impulse point  $t_k$  depend on the area under the curves of  $x'(t)$  and  $x(t)$  between  $t = t_k - \delta_k$  to  $t = t_k - \varepsilon_k$  and  $t = t_k - \tau_k$  to  $t = t_k - \sigma_k$ , respectively. It should be noticed that the impulsive effects of PBVP (1.1) have memory of the past states.

This paper is organized as follows. Firstly, we introduce a new concept of lower and upper solutions. After that, we establish some new comparison principles and discuss the existence and uniqueness of the solutions for second-order impulsive integro-differential equations with integral jump conditions. By using the method of upper and lower solutions and the monotone iterative technique, we obtain the existence of an extreme solution of PBVP (1.1). Finally, we give an example to illustrate the obtained results.

## 2 Preliminaries

Let  $J^- = J \setminus \{t_1, t_2, \dots, t_m\}$ ,  $J_0 = [t_0, t_1]$ ,  $J_k = (t_k, t_{k+1}]$  for  $k = 1, 2, \dots, m$ . Let  $PC(J, R) = \{x : J \rightarrow R; x(t)$  is continuous everywhere except for some  $t_k$  at which  $x(t_k^+)$  and  $x(t_k^-)$  exist and  $x(t_k^-) = x(t_k), k = 1, 2, \dots, m\}$ , and  $PC^1(J, R) = \{x \in PC(J, R); x'(t)$  is continuous everywhere except for some  $t_k$  at which  $x'(t_k^+)$  and  $x'(t_k^-)$  exist and  $x'(t_k^-) = x'(t_k), k = 1, 2, \dots, m\}$ .  $PC(J,$

$R$ ) and  $PC^1(J, R)$  are Banach spaces with the norms  $\|x\|_{PC} = \sup\{x(t) : t \in J\}$  and  $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$ . Let  $E = PC^1(J, R) \cap C^2(J^-, R)$ . A function  $x \in E$  is called a solution of PBVP (1.1) if it satisfies (1.1).

**Definition 2.1** We say that the functions  $\alpha_0, \beta_0 \in E$  are lower and upper solutions of PBVP (1.1), respectively, if there exist  $M > 0, N \geq 0, L \geq 0, L_k \geq 0, L_k^* \geq 0, 0 \leq \varepsilon_k \leq \delta_k \leq t_k - t_{k-1}, 0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$ , such that

$$\begin{cases} \alpha_0''(t) \geq f(t, \alpha_0(t), (K\alpha_0)(t), (S\alpha_0)(t)) + a(t), & t \in J^-, \\ \Delta\alpha_0(t_k) = I_k(\int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_0'(s) ds) + m_k, & k = 1, 2, \dots, m, \\ \Delta\alpha_0'(t_k) \geq I_k^*(\int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_0(s) ds) + l_k, & k = 1, 2, \dots, m, \\ \alpha_0(0) = \alpha_0(T), \end{cases}$$

where

$$\begin{aligned} a(t) &= \begin{cases} 0 & \text{if } \alpha_0'(0) \geq \alpha_0'(T), \\ \frac{[\alpha_0'(T) - \alpha_0'(0)]}{T} [2 + M[tT - t^2] + N \int_0^t k(t, s)(sT - s^2) ds \\ \quad + L \int_0^T h(t, s)(sT - s^2) ds] & \text{if } \alpha_0'(0) < \alpha_0'(T), \end{cases} \\ m_k &= \begin{cases} 0 & \text{if } \alpha_0'(0) \geq \alpha_0'(T), \\ \frac{L_k[\alpha_0'(T) - \alpha_0'(0)]}{T} \int_{t_k-\delta_k}^{t_k-\varepsilon_k} (T - 2s) ds & \text{if } \alpha_0'(0) < \alpha_0'(T), \end{cases} \\ l_k &= \begin{cases} 0 & \text{if } \alpha_0'(0) \geq \alpha_0'(T), \\ \frac{L_k^*[\alpha_0'(T) - \alpha_0'(0)]}{T} \int_{t_k-\tau_k}^{t_k-\sigma_k} (sT - s^2) ds & \text{if } \alpha_0'(0) < \alpha_0'(T), \end{cases} \end{aligned}$$

and

$$\begin{cases} \beta_0''(t) \leq f(t, \beta_0(t), (K\beta_0)(t), (S\beta_0)(t)) - b(t), & t \in J^-, \\ \Delta\beta_0(t_k) = I_k(\int_{t_k-\delta_k}^{t_k-\varepsilon_k} \beta_0'(s) ds) - m_k^*, & k = 1, 2, \dots, m, \\ \Delta\beta_0'(t_k) \leq I_k^*(\int_{t_k-\tau_k}^{t_k-\sigma_k} \beta_0(s) ds) - l_k^*, & k = 1, 2, \dots, m, \\ \beta_0(0) = \beta_0(T), \end{cases}$$

where

$$\begin{aligned} b(t) &= \begin{cases} 0 & \text{if } \beta_0'(0) \leq \beta_0'(T), \\ \frac{[\beta_0'(0) - \beta_0'(T)]}{T} [2 + M[tT - t^2] + N \int_0^t k(t, s)(sT - s^2) ds \\ \quad + L \int_0^T h(t, s)(sT - s^2) ds] & \text{if } \beta_0'(0) > \beta_0'(T), \end{cases} \\ m_k^* &= \begin{cases} 0 & \text{if } \beta_0'(0) \leq \beta_0'(T), \\ \frac{L_k[\beta_0'(0) - \beta_0'(T)]}{T} \int_{t_k-\delta_k}^{t_k-\varepsilon_k} (T - 2s) ds & \text{if } \beta_0'(0) > \beta_0'(T), \end{cases} \\ l_k^* &= \begin{cases} 0 & \text{if } \beta_0'(0) \leq \beta_0'(T), \\ \frac{L_k^*[\beta_0'(0) - \beta_0'(T)]}{T} \int_{t_k-\tau_k}^{t_k-\sigma_k} (sT - s^2) ds & \text{if } \beta_0'(0) > \beta_0'(T). \end{cases} \end{aligned}$$

Now we are in the position to establish some new comparison principles which play an important role in the monotone iterative technique.

**Lemma 2.1** Assume that  $x \in E$  satisfies

$$\begin{cases} x''(t) \geq Mx(t) + N \int_0^t k(t,s)x(s) ds + L \int_0^T h(t,s)x(s) ds, & t \in J^-, \\ \Delta x(t_k) = L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} x'(s) ds, & k = 1, 2, \dots, m, \\ \Delta x'(t_k) \geq L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds, & k = 1, 2, \dots, m, \\ x(0) = x(T), \quad x'(0) \geq x'(T), \end{cases} \quad (2.1)$$

where  $M > 0, N \geq 0, L \geq 0, L_k \geq 0, L_k^* \geq 0$  are constants and  $0 \leq \varepsilon_k \leq \delta_k \leq t_k - t_{k-1}, 0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}, k = 1, 2, \dots, m$ , and they satisfy

$$\left[ \sum_{k=1}^m L_k(\delta_k - \varepsilon_k) + T \right] \left[ \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)T \right] \leq 1. \quad (2.2)$$

Then  $x(t) \leq 0, t \in J$ .

*Proof* Suppose, to the contrary, that  $x(t) > 0$  for some  $t \in J$ . We divide the proof into two cases:

Case (i). There exists a  $\tilde{t} \in J$  such that  $x(\tilde{t}) > 0$  and  $x(t) \geq 0$  for all  $t \in J$ .

From (2.1), we have  $x''(t) \geq 0$  for  $t \in J^-$ . Since  $\Delta x'(t_k) \geq L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds \geq 0$ , then  $x'(t)$  is nondecreasing in  $t \in J$  and so  $x'(0) \leq x'(T)$ . However, by (2.1)  $x'(0) \geq x'(T)$ , then  $x'(0) = x'(T)$ , which implies  $x'(t) = \text{constant}$  for all  $t \in J$ . Thus,  $0 = x''(t) \geq Mx(\tilde{t}) > 0$ , a contradiction.

Case (ii). There exists  $t^*, t_* \in J$  such that  $x(t^*) > 0, x(t_*) < 0$ .

Let  $\inf_{t \in J} x(t) = -\lambda < 0$ , then there exists  $t_* \in J_i$ , for some  $i \in \{0, 1, \dots, m\}$ , such that  $x(t_*) = -\lambda$  or  $x(t_i^+) = -\lambda$ . Without loss of generality, we only consider  $x(t_*) = -\lambda$ . For the case  $x(t_i^+) = -\lambda$  the proof is similar. It follows that

$$\begin{aligned} x''(t) &\geq Mx(t) + N \int_0^t k(t,s)x(s) ds + L \int_0^T h(t,s)x(s) ds \\ &\geq -\lambda(M + Nk_0T + Lh_0T), \quad t \in J^-, \\ \Delta x'(t_k) &\geq L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds \geq -\lambda L_k^*(\tau_k - \sigma_k), \quad k = 1, 2, \dots, m. \end{aligned}$$

If  $x'(t) > 0$  for all  $t \in J$ , then  $\Delta x(t_k) = L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} x'(s) ds \geq 0, k = 1, 2, \dots, m$ . Hence,  $x(t)$  is strictly increasing on  $J$ , which contradicts  $x(0) = x(T)$ . Then there exists a  $\bar{t} \in J$  such that  $x'(\bar{t}) \leq 0$ .

Let  $\bar{t} \in J_j, j \in \{0, 1, \dots, m\}$ . By mean value theorem, we have

$$\begin{aligned} x'(t_j^-) - \lambda L_j^*(\tau_j - \sigma_j) - x'(\bar{t}) &\leq x'(t_j^+) - x'(\bar{t}) = -x''(s_j)(\bar{t} - t_j^+) \\ &\leq \lambda(M + Nk_0T + Lh_0T)(\bar{t} - t_j^+), \quad s_j \in (t_j, \bar{t}), \\ x'(t_{j-1}^-) - \lambda L_{j-1}^*(\tau_{j-1} - \sigma_{j-1}) - x'(t_j) &\leq x'(t_{j-1}^+) - x'(t_j) = -x''(s_{j-1})(t_j - t_{j-1}^+) \\ &\leq \lambda(M + Nk_0T + Lh_0T)(t_j - t_{j-1}^+), \quad s_{j-1} \in (t_{j-1}, t_j), \end{aligned}$$

⋮

$$\begin{aligned} x'(t_1^-) - \lambda L_1^*(\tau_1 - \sigma_1) - x'(t_2) &\leq x'(t_1^+) - x'(t_2) = -x''(s_1)(t_2 - t_1^+) \\ &\leq \lambda(M + Nk_0T + Lh_0T)(t_2 - t_1^+), \quad s_1 \in (t_1, t_2), \\ x'(0) - x'(t_1) &= -x''(s_0)(t_1 - t_0) \leq \lambda(M + Nk_0T + Lh_0T)(t_1 - t_0), \quad s_0 \in (t_0, t_1). \end{aligned}$$

Summing up the above inequalities, we obtain

$$\begin{aligned} x'(0) &\leq x'(\bar{t}) + \lambda \left[ \sum_{k=1}^j L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)(\bar{t} - t_0) \right] \\ &\leq \lambda \left[ \sum_{k=1}^j L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)\bar{t} \right]. \end{aligned} \tag{2.3}$$

Let  $t \in J_h$ ,  $h \in \{0, 1, \dots, m\}$ . If  $t \leq \bar{t}$  by using the method to get (2.3), then we have

$$\begin{aligned} x'(t) &\leq x'(\bar{t}) + \lambda \left[ \sum_{k=h+1}^j L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)(\bar{t} - t) \right] \\ &\leq \lambda \left[ \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)T \right]. \end{aligned}$$

If  $t > \bar{t}$ , then the above method together with (2.1), (2.3) implies that

$$\begin{aligned} x'(t) &\leq x'(T) + \lambda \left[ \sum_{k=h+1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)(T - t) \right] \\ &\leq x'(0) + \lambda \left[ \sum_{k=h+1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)(T - t) \right] \\ &\leq \lambda \left[ \sum_{k=1}^j L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)\bar{t} \right] \\ &\quad + \lambda \left[ \sum_{k=h+1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)(T - t) \right] \\ &\leq \lambda \left[ \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)T \right]. \end{aligned}$$

Thus,

$$x'(t) \leq \lambda \left[ \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)T \right], \quad t \in J^-.$$

Let  $t^* \in J_r$  for some  $r \in \{0, 1, \dots, m\}$ . We first assume that  $t^* < t_r^+$ , then  $i < r$ . By the mean value theorem, we have

$$\begin{aligned} x(t^*) - x(t_r) &= x(t^*) - x(t_r^+) + L_r \int_{t_r - \delta_r}^{t_r - \varepsilon_r} x'(s) ds \\ &= x'(s_r)(t^* - t_r^+) + L_r \int_{t_r - \delta_r}^{t_r - \varepsilon_r} x'(s) ds \end{aligned}$$

$$\begin{aligned} &\leq ((t^* - t_r) + L_r(\delta_r - \varepsilon_r)) \\ &\quad \times \lambda \left[ \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)T \right], \quad s_r \in (t_r, t^*), \\ x(t_r) - x(t_{r-1}) &= x(t_r) - x(t_{r-1}^+) + L_{r-1} \int_{t_{r-1} - \delta_{r-1}}^{t_{r-1} - \varepsilon_{r-1}} x'(s) ds \\ &= x'(s_{r-1})(t_r - t_{r-1}^+) + L_{r-1} \int_{t_{r-1} - \delta_{r-1}}^{t_{r-1} - \varepsilon_{r-1}} x'(s) ds \\ &\leq ((t_r - t_{r-1}) + L_{r-1}(\delta_{r-1} - \varepsilon_{r-1})) \\ &\quad \times \lambda \left[ \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)T \right], \quad s_{r-1} \in (t_{r-1}, t_r), \\ &\quad \vdots \\ x(t_{i+1}) - x(t^*) &\leq ((t_{i+1} - t_i) + L_{i+1}(\delta_{i+1} - \varepsilon_{i+1})) \\ &\quad \times \lambda \left[ \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)T \right]. \end{aligned}$$

Summing up, we get

$$0 < x(t^*) \leq -\lambda + \lambda \left[ \sum_{k=1}^m L_k(\delta_k - \varepsilon_k) + T \right] \left[ \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)T \right].$$

Hence,

$$\left[ \sum_{k=1}^m L_k(\delta_k - \varepsilon_k) + T \right] \left[ \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)T \right] > 1,$$

which contradicts (2.2).

For the case  $t^* < t^*$ , the proof is similar, and thus we omit it. This completes the proof.  $\square$

**Lemma 2.2** Assume that  $x \in E$  satisfies

$$\begin{aligned} x''(t) &\geq Mx(t) + N \int_0^t k(t,s)x(s) ds + L \int_0^T h(t,s)x(s) ds + \frac{[x'(T) - x'(0)]}{T} \\ &\quad \times \left[ 2 + M[tT - t^2] + N \int_0^t k(t,s)(sT - s^2) ds \right. \\ &\quad \left. + L \int_0^T h(t,s)(sT - s^2) ds \right], \quad t \in J^-, \\ \Delta x(t_k) &= L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} x'(s) ds + \frac{L_k[x'(T) - x'(0)]}{T} \int_{t_k - \delta_k}^{t_k - \varepsilon_k} (T - 2s) ds, \quad k = 1, 2, \dots, m, \\ \Delta x'(t_k) &\geq L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds + \frac{L_k^*[x'(T) - x'(0)]}{T} \int_{t_k - \tau_k}^{t_k - \sigma_k} (sT - s^2) ds, \quad k = 1, 2, \dots, m, \\ x(0) &= x(T), \quad x'(0) < x'(T), \end{aligned}$$

where  $M > 0, N \geq 0, L \geq 0, L_k \geq 0, L_k^* \geq 0$  are constants and  $0 \leq \varepsilon_k \leq \delta_k \leq t_k - t_{k-1}, 0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}, k = 1, 2, \dots, m$ , and they satisfy (2.2). Then  $x(t) \leq 0$  for all  $t \in J$ .

*Proof* Let  $u(t) = [\frac{tT-t^2}{T}][x'(T) - x'(0)], t \in J$ , and define

$$w(t) = x(t) + u(t) = x(t) + \left[ \frac{tT - t^2}{T} \right] [x'(T) - x'(0)].$$

Note that  $u(0) = u(T), u(t) \geq 0$  for  $t \in J$ . If we prove that  $w \leq 0$ , then  $x(t) \leq x(t) + u(t) \leq 0$  and the proof is complete. Since  $u'(t) = [\frac{T-2t}{T}][x'(T) - x'(0)]$ , then we get

$$\begin{aligned} w(0) &= x(0) + u(0) = x(T) + u(T) = w(T), \\ w'(0) &= x'(0) + u'(0) = x'(0) + x'(T) - x'(0) = x'(T), \\ w'(T) &= x'(T) + u'(T) = x'(T) - x'(T) + x'(0) = x'(0). \end{aligned}$$

Hence,  $w'(0) > w'(T)$ . Indeed, for  $k = 1, 2, \dots, m$ ,

$$\begin{aligned} \Delta w(t_k) &= \Delta x(t_k) + \Delta u(t_k) \\ &= L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} x'(s) ds + \frac{L_k [x'(T) - x'(0)]}{T} \int_{t_k - \delta_k}^{t_k - \varepsilon_k} T - 2s ds \\ &= L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} x'(s) + u'(s) ds \\ &= L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} w'(s) ds, \end{aligned}$$

and

$$\begin{aligned} \Delta w'(t_k) &= \Delta x'(t_k) + \Delta u'(t_k) \\ &\geq L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds + \frac{L_k^* [x'(T) - x'(0)]}{T} \int_{t_k - \tau_k}^{t_k - \sigma_k} sT - s^2 ds \\ &\geq L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) + u(s) ds \\ &\geq L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} w(s) ds. \end{aligned}$$

Meanwhile, for  $t \neq t_k, t \in J$ ,

$$\begin{aligned} w''(t) &- Mw(t) - N \int_0^t k(t,s)w(s) ds - L \int_0^T h(t,s)w(s) ds \\ &= x''(t) - Mx(t) - \frac{Mt}{T} [x'(T) - x'(0)] - N \int_0^t k(t,s)x(s) ds - L \int_0^T h(t,s)x(s) ds \\ &\quad - \frac{[x'(T) - x'(0)]}{T} \left[ 2 + M[tT - t^2] + N \int_0^t k(t,s)(sT - s^2) ds \right. \\ &\quad \left. + L \int_0^T h(t,s)(sT - s^2) ds \right] \geq 0. \end{aligned}$$

Then by Lemma 2.1, we get  $w(t) \leq 0$  for all  $t \in J$ , which implies that  $x(t) \leq 0, t \in J$ . □

Consider the linear PBVP

$$\begin{cases} x''(t) = Mx(t) + N \int_0^t k(t,s)x(s) ds + L \int_0^T h(t,s)x(s) ds - g(t), & t \in J^-, \\ \Delta x(t_k) = L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s) ds + \gamma_k, & k = 1, 2, \dots, m, \\ \Delta x'(t_k) = L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s) ds + \lambda_k, & k = 1, 2, \dots, m, \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \quad (2.4)$$

where constants  $M > 0, N \geq 0, L \geq 0, L_k \geq 0, L_k^* \geq 0, \gamma_k, \lambda_k$  are constants and  $g \in PC(J, R)$ ,  $0 \leq \varepsilon_k \leq \delta_k \leq t_k - t_{k-1}, 0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}, k = 1, 2, \dots, m$ .

**Lemma 2.3**  $x \in E$  is a solution of (2.4) if and only if  $x \in PC^1(J, R)$  is a solution of the following impulsive integral equation:

$$\begin{aligned} x(t) = & \int_0^T G_1(t,s) \left[ g(s) - N \int_0^s k(s,r)x(r) dr - L \int_0^T h(s,r)x(r) dr \right] ds \\ & + \sum_{k=1}^m \left[ -G_1(t,t_k) \left( L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s) ds + \lambda_k \right) \right. \\ & \left. + G_2(t,t_k) \left( L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s) ds + \gamma_k \right) \right], \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} G_1(t,s) = & [2\sqrt{M}(e^{\sqrt{M}T} - 1)]^{-1} \begin{cases} e^{\sqrt{M}(T-t+s)} + e^{\sqrt{M}(t-s)}, & 0 \leq s < t \leq T, \\ e^{\sqrt{M}(T+t-s)} + e^{\sqrt{M}(s-t)}, & 0 \leq t \leq s \leq T, \end{cases} \\ G_2(t,s) = & [2(e^{\sqrt{M}T} - 1)]^{-1} \begin{cases} e^{\sqrt{M}(T-t+s)} - e^{\sqrt{M}(t-s)}, & 0 \leq s < t \leq T, \\ -e^{\sqrt{M}(T+t-s)} + e^{\sqrt{M}(s-t)}, & 0 \leq t \leq s \leq T. \end{cases} \end{aligned}$$

This proof is similar to the proof of Lemma 2.1 in [36], and we omit it.

**Lemma 2.4** Let  $M > 0, N \geq 0, L \geq 0, L_k \geq 0, L_k^* \geq 0$  are constants and  $0 \leq \varepsilon_k \leq \delta_k \leq t_k - t_{k-1}, 0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}, k = 1, 2, \dots, m$ . If

$$\begin{aligned} \psi = & \frac{1 + e^{\sqrt{M}T}}{2\sqrt{M}(e^{\sqrt{M}T} - 1)} \left[ \int_0^T \left[ N \int_0^s k(s,r) dr + L \int_0^T h(s,r) dr \right] ds + \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) \right] \\ & + \frac{1}{2} \sum_{k=1}^m L_k(\delta_k - \varepsilon_k) < 1, \end{aligned} \quad (2.6)$$

$$\begin{aligned} \mu = & \frac{1}{2} \left[ \int_0^T \left[ N \int_0^s k(s,r) dr + L \int_0^T h(s,r) dr \right] ds + \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) \right] \\ & + \frac{\sqrt{M}(1 + e^{\sqrt{M}T})}{2(e^{\sqrt{M}T} - 1)} \sum_{k=1}^m L_k(\delta_k - \varepsilon_k) < 1, \end{aligned} \quad (2.7)$$

then (2.4) has a unique solution  $x$  in  $E$ .



*Proof* For any  $x \in E$ , we define an operator  $F$  by

$$\begin{aligned} (Fx)(t) = & \int_0^T G_1(t, s) \left[ g(s) - N \int_0^s k(s, r)x(r) dr - L \int_0^T h(s, r)x(r) dr \right] ds \\ & + \sum_{k=1}^m \left[ -G_1(t, t_k) \left( L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds + \lambda_k \right) \right. \\ & \left. + G_2(t, t_k) \left( L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} x'(s) ds + \gamma_k \right) \right], \end{aligned}$$

where  $G_1, G_2$  are given by Lemma 2.3. Then  $Fx \in PC^1(J, R)$  and

$$\begin{aligned} (Fx)'(t) = & - \int_0^T G_2(t, s) \left[ g(s) - N \int_0^s k(s, r)x(r) dr - L \int_0^T h(s, r)x(r) dr \right] ds \\ & + \sum_{k=1}^m \left[ G_2(t, t_k) \left( L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds + \lambda_k \right) \right. \\ & \left. - MG_1(t, t_k) \left( L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} x'(s) ds + \gamma_k \right) \right]. \end{aligned}$$

By computing directly, we have

$$\max_{(t,s) \in J \times J} \{G_1(t, s)\} = \frac{1 + e^{\sqrt{MT}}}{2\sqrt{M}(e^{\sqrt{MT}} - 1)}$$

and

$$\max_{(t,s) \in J \times J} \{G_2(t, s)\} = \frac{1}{2}.$$

On the other hand, for  $x, y \in PC^1(J, R)$ , we have

$$\begin{aligned} \|(Fx) - (Fy)\|_{PC} &= \sup_{t \in J} |(Fx)(t) - (Fy)(t)| \\ &= \sup_{t \in J} \left| \int_0^T G_1(t, s) \left[ -N \int_0^s k(s, r)[x(r) - y(r)] dr \right. \right. \\ &\quad \left. \left. - L \int_0^T h(s, r)[x(r) - y(r)] dr \right] ds \right. \\ &\quad \left. + \sum_{k=1}^m \left[ -G_1(t, t_k) \left( L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} [x(s) - y(s)] ds \right) \right. \right. \\ &\quad \left. \left. + G_2(t, t_k) \left( L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} [x'(s) - y'(s)] ds \right) \right] \right| \\ &\leq \sup_{t \in J} \int_0^T G_1(t, s) \left[ N|x(s) - y(s)| \int_0^s k(s, r) dr \right. \\ &\quad \left. + L|x(s) - y(s)| \int_0^T h(s, r) dr \right] ds \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^m [G_1(t, t_k)L_k^*(\tau_k - \sigma_k)|x(t) - y(t)| \\
 & + G_2(t, t_k)L_k(\delta_k - \varepsilon_k)|x'(t) - y'(t)|] \\
 \leq & \|x - y\|_{PC^1} \left[ \sup_{t \in J} \int_0^T G_1(t, s) \left[ N \int_0^s k(s, r) dr \right. \right. \\
 & \left. \left. + L \int_0^T h(s, r) dr \right] ds + \sum_{k=1}^m [G_1(t, t_k)L_k^*(\tau_k - \sigma_k) \right. \\
 & \left. + G_2(t, t_k)L_k(\delta_k - \varepsilon_k)] \right] \\
 \leq & \psi \|x - y\|_{PC^1}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \|(Fx)' - (Fy)'\|_{PC} & = \sup_{t \in J} \left| - \int_0^T G_2(t, s) \left[ -N \int_0^s k(s, r) [x(r) - y(r)] dr \right. \right. \\
 & \left. \left. - L \int_0^T h(s, r) [x(r) - y(r)] dr \right] ds \right. \\
 & \left. + \sum_{k=1}^m \left[ G_2(t, t_k) \left( L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} [x(s) - y(s)] ds \right) \right. \right. \\
 & \left. \left. - MG_1(t, t_k) \left( L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} [x'(s) - y'(s)] ds \right) \right] \right| \\
 \leq & \sup_{t \in J} \int_0^T G_2(t, s) \left[ N |x(s) - y(s)| \int_0^s k(s, r) dr \right. \\
 & \left. + L |x(s) - y(s)| \int_0^T h(s, r) dr \right] ds \\
 & + \sum_{k=1}^m [G_2(t, t_k)L_k^*(\tau_k - \sigma_k)|x(t) - y(t)| \\
 & + MG_1(t, t_k)L_k(\delta_k - \varepsilon_k)|x'(s) - y'(s)|] \\
 \leq & \|x - y\|_{PC^1} \left[ \sup_{t \in J} \int_0^T G_2(t, s) \left[ N \int_0^s k(s, r) dr \right. \right. \\
 & \left. \left. + L \int_0^T h(s, r) dr \right] ds + \sum_{k=1}^m [G_2(t, t_k)L_k^*(\tau_k - \sigma_k) \right. \\
 & \left. + MG_1(t, t_k)L_k(\delta_k - \varepsilon_k)] \right] \\
 \leq & \mu \|x - y\|_{PC^1}.
 \end{aligned}$$

Thus,

$$\|(Fx) - (Fy)\|_{PC^1} \leq \max\{\psi, \mu\} \|x - y\|_{PC^1}.$$

By the Banach fixed-point theorem,  $F$  has a unique fixed point  $x^* \in PC^1(J, R)$ , and by Lemma 2.3,  $x^*$  is also the unique solution of (2.4). This completes the proof.  $\square$

### 3 Main results

In this section, we establish existence criteria for solutions of PBVP (1.1) by the method of lower and upper solutions and the monotone iterative technique. For  $\alpha_0, \beta_0 \in E$ , we write  $\alpha_0 \leq \beta_0$  if  $\alpha_0(t) \leq \beta_0(t)$  for all  $t \in J$ . In such a case, we denote  $[\alpha_0, \beta_0] = \{x \in E : \alpha_0(t) \leq x(t) \leq \beta_0(t), t \in J\}$ .

**Theorem 3.1** *Suppose that the following conditions hold:*

- (H<sub>1</sub>)  $\alpha_0$  and  $\beta_0$  are lower and upper solutions for PBVP (1.1), respectively, such that  $\alpha_0 \leq \beta_0$ .
- (H<sub>2</sub>) The function  $f$  satisfies

$$f(t, x_2, y_2, z_2) - f(t, x_1, y_1, z_1) \leq M(x_2 - x_1) + N(y_2 - y_1) + L(z_2 - z_1),$$

for all  $t \in J$ ,  $\alpha_0(t) \leq x_1 \leq x_2 \leq \beta_0(t)$ ,  $(K\alpha_0)(t) \leq y_1 \leq y_2 \leq (K\beta_0)(t)$ ,  $(S\alpha_0)(t) \leq z_1 \leq z_2 \leq (S\beta_0)(t)$ .

- (H<sub>3</sub>)  $M > 0$ ,  $N \geq 0$ ,  $L \geq 0$ ,  $L_k \geq 0$ ,  $L_k^* \geq 0$  are constants, and  $0 \leq \varepsilon_k \leq \delta_k \leq t_k - t_{k-1}$ ,  $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$ ,  $k = 1, 2, \dots, m$ , and they satisfy (2.2), (2.6) and (2.7).
- (H<sub>4</sub>) The functions  $I_k, I_k^*$  satisfy

$$I_k \left( \int_{t_k - \delta_k}^{t_k - \varepsilon_k} x'(s) ds \right) - I_k \left( \int_{t_k - \delta_k}^{t_k - \varepsilon_k} y'(s) ds \right) = L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} x'(s) - y'(s) ds,$$

$$I_k^* \left( \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds \right) - I_k^* \left( \int_{t_k - \tau_k}^{t_k - \sigma_k} y(s) ds \right) \leq L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) - y(s) ds,$$

where  $\int_{t_k - \tau_k}^{t_k - \sigma_k} \alpha_0(s) ds \leq \int_{t_k - \tau_k}^{t_k - \sigma_k} y(s) ds \leq \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds \leq \int_{t_k - \tau_k}^{t_k - \sigma_k} \beta_0(s) ds$ ,  $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$ ,  $0 \leq \varepsilon_k \leq \delta_k \leq t_k - t_{k-1}$ ,  $k = 1, 2, \dots, m$ .

Then there exist monotone sequences  $\{\alpha_n\}, \{\beta_n\} \subset E$  which converge in  $E$  to the extreme solutions of PBVP (1.1) in  $[\alpha_0, \beta_0]$ , respectively.

*Proof* For any  $\eta \in [\alpha_0, \beta_0]$ , we consider linear PBVP (2.4) with

$$g(t) = f(t, \eta(t), (K\eta)(t), (S\eta)(t)) - M\eta(t) - N \int_0^t k(t, s)\eta(s) ds - L \int_0^T h(t, s)\eta(s) ds,$$

$$\gamma_k = I_k \left( \int_{t_k - \delta_k}^{t_k - \varepsilon_k} \eta'(s) ds \right) - L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} \eta'(s) ds, \quad k = 1, 2, \dots, m,$$

$$\lambda_k = I_k^* \left( \int_{t_k - \tau_k}^{t_k - \sigma_k} \eta(s) ds \right) - L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} \eta(s) ds, \quad k = 1, 2, \dots, m.$$

By Lemma 2.4, PBVP (2.4) has a unique solution  $x \in E$ . We define an operator  $A$  from  $[\alpha_0, \beta_0]$  to  $E$  by  $x(t) = A\eta(t)$ . We complete the proof in four steps.

*Step 1.* We claim that  $\alpha_0 \leq A\alpha_0$  and  $A\beta_0 \leq \beta_0$ . We only prove  $\alpha_0 \leq A\alpha_0$  since the second inequality can be proved in a similar manner.

Let  $\alpha_1 = A\alpha_0$  and  $p = \alpha_0 - \alpha_1$ . Then  $\alpha_1$  satisfies

$$\begin{aligned} &\alpha_1''(t) - M\alpha_1(t) - N(K\alpha_1)(t) - L(S\alpha_1)(t) \\ &= f(t, \alpha_0(t), (K\alpha_0)(t), (S\alpha_0)(t)) - M\alpha_0(t) - N(K\alpha_0)(t) \\ &\quad - L(S\alpha_0)(t), \quad t \in J^-, \\ \Delta\alpha_1(t_k) &= L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_1'(s) ds + I_k \left( \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_0'(s) ds \right) \\ &\quad - L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_0'(s) ds, \quad k = 1, 2, \dots, m, \\ \Delta\alpha_1'(t_k) &= L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_1(s) ds + I_k^* \left( \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_0(s) ds \right) \\ &\quad - L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_0(s) ds, \quad k = 1, 2, \dots, m, \\ \alpha_1(0) &= \alpha_1(T), \quad \alpha_1'(0) = \alpha_1'(T). \end{aligned}$$

We finish Step 1 in two cases.

*Case 1.*  $\alpha_0'(0) \geq \alpha_0'(T)$ , which implies that

$$a(t) = 0, \quad \alpha_0''(t) \geq f(t, \alpha_0(t), (K\alpha_0)(t), (S\alpha_0)(t)).$$

As  $\alpha_0$  is a lower solution of PBVP (1.1), then for  $t \in J^-$ ,

$$\begin{aligned} &p''(t) - Mp(t) - N(Kp)(t) - L(Sp)(t) \\ &= \alpha_0''(t) - M\alpha_0(t) - N(K\alpha_0)(t) - L(S\alpha_0)(t) \\ &\quad - \alpha_1''(t) + M\alpha_1(t) + N(K\alpha_1)(t) + L(S\alpha_1)(t) \\ &\geq f(t, \alpha_0(t), (K\alpha_0)(t), (S\alpha_0)(t)) - M\alpha_0(t) - N(K\alpha_0)(t) \\ &\quad - L(S\alpha_0)(t) - f(t, \alpha_0(t), (K\alpha_0)(t), (S\alpha_0)(t)) + M\alpha_0(t) \\ &\quad + N(K\alpha_0)(t) + L(S\alpha_0)(t) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \Delta p(t_k) &= \Delta\alpha_0(t_k) - \alpha_1(t_k) \\ &= I_k \left( \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_0'(s) ds \right) - L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_1'(s) ds - I_k \left( \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_0'(s) ds \right) \\ &\quad + L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_0'(s) ds \\ &= L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_0'(s) - \alpha_1'(s) ds = L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} p'(s) ds, \quad k = 1, 2, \dots, m, \\ \Delta p'(t_k) &= \Delta\alpha_0'(t_k) - \alpha_1'(t_k) \end{aligned}$$

$$\begin{aligned}
 &\geq I_k^* \left( \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_0(s) ds \right) - L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_1(s) ds - I_k^* \left( \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_0(s) ds \right) \\
 &\quad + L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_0(s) ds \\
 &= L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_0(s) - \alpha_1(s) ds = L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} p(s) ds, \quad k = 1, 2, \dots, m, \\
 p(0) &= \alpha_0(0) - \alpha_1(0) = \alpha_0(T) - \alpha_1(T) = p(T), \\
 p'(0) &= \alpha'_0(0) - \alpha'_1(0) \geq \alpha'_0(T) - \alpha'_1(T) = p'(T).
 \end{aligned}$$

Then by Lemma 2.1,  $p(t) \leq 0$ , which implies that  $\alpha_0(t) \leq A\alpha_0(t)$ , i.e.,  $\alpha_0 \leq A\alpha_0$ .

Case 2.  $\alpha'_0(0) < \alpha'_0(T)$ , which implies that

$$\begin{aligned}
 a(t) &= \frac{[\alpha'_0(T) - \alpha'_0(0)]}{T} \left[ 2 + M[tT - t^2] + N \int_0^t k(t,s)(sT - s^2) ds \right. \\
 &\quad \left. + L \int_0^T h(t,s)(sT - s^2) ds \right].
 \end{aligned}$$

Hence,

$$\begin{aligned}
 &p''(t) - Mp(t) - N(Kp)(t) - L(Sp)(t) - \frac{[p'(T) - p'(0)]}{T} \left[ 2 + M[tT - t^2] \right. \\
 &\quad \left. + N \int_0^t k(t,s)(sT - s^2) ds + L \int_0^T h(t,s)(sT - s^2) ds \right] \\
 &= \alpha''_0(t) - M\alpha_0(t) - N(K\alpha_0)(t) - L(S\alpha_0)(t) - \frac{[\alpha'_0(T) - \alpha'_0(0)]}{T} \left[ 2 + M[tT - t^2] \right. \\
 &\quad \left. + N \int_0^t k(t,s)(sT - s^2) ds + L \int_0^T h(t,s)(sT - s^2) ds \right] \\
 &\quad - \alpha''_1(t) + M\alpha_1(t) + N(K\alpha_1)(t) + L(S\alpha_1)(t) \\
 &\geq f(t, \alpha_0(t), (K\alpha_0)(t), (S\alpha_0)(t)) - M\alpha_0(t) - N(K\alpha_0)(t) \\
 &\quad - L(S\alpha_0)(t) - f(t, \alpha_0(t), (K\alpha_0)(t), (S\alpha_0)(t)) \\
 &\quad + M\alpha_0(t) + N(K\alpha_0)(t) + L(S\alpha_0)(t) = 0,
 \end{aligned}$$

and

$$\begin{aligned}
 \Delta p(t_k) &= \Delta \alpha_0(t_k) - \Delta \alpha_1(t_k) \\
 &= I_k \left( \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha'_0(s) ds \right) + \frac{L_k[\alpha'_0(T) - \alpha'_0(0)]}{T} \int_{t_k-\delta_k}^{t_k-\varepsilon_k} T - 2s ds \\
 &\quad - L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha'_1(s) ds - I_k \left( \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha'_0(s) ds \right) + L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha'_0(s) ds \\
 &= L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} p'(s) ds + \frac{L_k[p'(T) - p'(0)]}{T} \int_{t_k-\delta_k}^{t_k-\varepsilon_k} T - 2s ds, \quad k = 1, 2, \dots, m,
 \end{aligned}$$

and

$$\begin{aligned} \Delta p'(t_k) &= \Delta \alpha'_0(t_k) - \alpha'_1(t_k) \\ &\geq L_k^* \left( \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_0(s) ds \right) + \frac{L_k^* [\alpha'_0(T) - \alpha'_0(0)]}{T} \int_{t_k-\tau_k}^{t_k-\sigma_k} sT - s^2 ds \\ &\quad - L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_1(s) ds - I_k^* \left( \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_0(s) ds \right) + L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_0(s) ds \\ &= L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} p(s) ds + \frac{L_k^* [p'(T) - p'(0)]}{T} \int_{t_k-\tau_k}^{t_k-\sigma_k} sT - s^2 ds, \end{aligned}$$

$k = 1, 2, \dots, m$ , and

$$p(0) = p(T), \quad p'(0) = \alpha'_0(0) - \alpha'_1(0) < \alpha'_0(T) - \alpha'_1(T) = p'(T).$$

Then by Lemma 2.2,  $p(t) \leq 0$ , which implies  $\alpha_0(t) \leq A\alpha_0(t)$ , i.e.,  $\alpha_0 \leq A\alpha_0$ .

*Step 2.* We prove that if  $\alpha_0 \leq \eta_1 \leq \eta_2 \leq \beta_0$ , then  $A\eta_1 \leq A\eta_2$ .

Let  $\eta_1^* = A\eta_1$ ,  $\eta_2^* = A\eta_2$ , and  $p = \eta_1^* - \eta_2^*$ , then for  $t \in J^-$ , and by (H<sub>2</sub>), we obtain

$$\begin{aligned} p''(t) - Mp(t) - N(Kp)(t) - L(Sp)(t) \\ = f(t, \eta_1(t), (K\eta_1)(t), (S\eta_1)(t)) - M\eta_1(t) - N(K\eta_1)(t) \\ \quad - L(S\eta_1)(t) - f(t, \eta_2(t), (K\eta_2)(t), (S\eta_2)(t)) \\ \quad + M\eta_2(t) + N(K\eta_2)(t) + L(S\eta_2)(t) \\ \geq 0 \quad (\text{by (H}_2\text{)}). \end{aligned}$$

From (H<sub>3</sub>), we obtain

$$\begin{aligned} \Delta p(t_k) &= L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} p'(s) ds, \quad k = 1, 2, \dots, m, \\ \Delta p'(t_k) &\geq L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} p(s) ds, \quad k = 1, 2, \dots, m, \\ p(0) &= p(T), \quad p'(0) = p'(T). \end{aligned}$$

Applying Lemma 2.1, we get  $p(t) \leq 0$ , which implies  $A\eta_1 \leq A\eta_2$ .

*Step 3.* We show that PBVP (1.1) has solutions.

Let  $\alpha_n = A\alpha_{n-1}$ ,  $\beta_n = A\beta_{n-1}$ ,  $n = 1, 2, \dots$ . Following the first two steps, we have

$$\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n \leq \dots \leq \beta_n \leq \dots \leq \beta_1 \leq \beta_0, \quad \forall n \in N.$$

Obviously, each  $\alpha_i, \beta_i$  ( $i = 1, 2, \dots$ ) satisfies

$$\begin{aligned} \alpha_i''(t) - M\alpha_i(t) - N(K\alpha_i)(t) - L(S\alpha_i)(t) \\ = f(t, \alpha_{i-1}(t), (K\alpha_{i-1})(t), (S\alpha_{i-1})(t)) - M\alpha_{i-1}(t) - N(K\alpha_{i-1})(t) \\ \quad - L(S\alpha_{i-1})(t), \quad t \in J^-, \end{aligned}$$

$$\begin{aligned} \Delta\alpha_i(t_k) &= L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha'_i(s) ds + I_k \left( \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha'_{i-1}(s) ds \right) \\ &\quad - L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha'_{i-1}(s) ds, \quad k = 1, 2, \dots, m, \\ \Delta\alpha'_i(t_k) &= L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_i(s) ds + I_k^* \left( \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_{i-1}(s) ds \right) \\ &\quad - L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_{i-1}(s) ds, \quad k = 1, 2, \dots, m, \\ \alpha_i(0) &= \alpha_i(T), \quad \alpha'_i(0) = \alpha'_i(T), \end{aligned}$$

and

$$\begin{aligned} &\beta_i''(t) - M\beta_i(t) - N(K\beta_i)(t) - L(S\beta_i)(t) \\ &= f(t, \beta_{i-1}(t), (K\beta_{i-1})(t), (S\beta_{i-1})(t)) - M\beta_{i-1}(t) - N(K\beta_{i-1})(t) \\ &\quad - L(S\beta_{i-1})(t), \quad t \in J^-, \\ \Delta\beta_i(t_k) &= L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \beta'_i(s) ds + I_k \left( \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \beta'_{i-1}(s) ds \right) \\ &\quad - L_k \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \beta'_{i-1}(s) ds, \quad k = 1, 2, \dots, m, \\ \Delta\beta'_i(t_k) &= L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} \beta_i(s) ds + I_k^* \left( \int_{t_k-\tau_k}^{t_k-\sigma_k} \beta_{i-1}(s) ds \right) \\ &\quad - L_k^* \int_{t_k-\tau_k}^{t_k-\sigma_k} \beta_{i-1}(s) ds, \quad k = 1, 2, \dots, m, \\ \beta_i(0) &= \beta_i(T), \quad \beta'_i(0) = \beta'_i(T). \end{aligned}$$

Thus, there exist  $x^*$  and  $x^\circ$  such that

$$\lim_{i \rightarrow \infty} \alpha_i(t) = x^*(t), \quad \lim_{i \rightarrow \infty} \beta_i(t) = x^\circ(t), \quad \text{uniformly on } t \in J.$$

Clearly,  $x^*$ ,  $x^\circ$  satisfy PBVP (1.1).

*Step 4.* We show that  $x^*$ ,  $x^\circ$  are extreme solutions of PBVP (1.1).

Let  $x(t)$  be any solution of PBVP (1.1), which satisfies  $\alpha_0(t) \leq x(t) \leq \beta_0(t)$ ,  $t \in J$ . Suppose that there exists a positive integer  $n$  such that for  $t \in J$ ,  $\alpha_n(t) \leq x(t) \leq \beta_n(t)$ . Setting  $p(t) = \alpha_{n+1}(t) - x(t)$ , then for  $t \in J^-$ ,

$$\begin{aligned} p''(t) &= \alpha''_{n+1} - x''(t) \\ &= M\alpha_{n+1}(t) + N(K\alpha_{n+1})(t) + L(S\alpha_{n+1})(t) \\ &\quad + f(t, \alpha_n(t), (K\alpha_n)(t), (S\alpha_n)(t)) - M\alpha_n(t) \\ &\quad - N(K\alpha_n)(t) - L(S\alpha_n)(t) \\ &\quad - f(t, x(t), (Kx)(t), (Sx)(t)) \\ &= M\alpha_{n+1}(t) + N(K\alpha_{n+1})(t) + L(S\alpha_{n+1})(t) - Mx(t) \end{aligned}$$

$$\begin{aligned}
 & -N(Kx)(t) - L(Sx)(t) + f(t, \alpha_n(t), (K\alpha_n)(t), (S\alpha_n)(t)) \\
 & -f(t, x(t), (Kx)(t), (Sx)(t)) - M(\alpha_n(t) - x(t)) \\
 & -N(K(\alpha_n)(t) - (Kx)(t)) - L((S\alpha_n)(t) - (Sx)(t)) \\
 & \geq Mp(t) + N(Kp)(t) + L(Sp)(t),
 \end{aligned}$$

and

$$\Delta p(t_k) = \Delta \alpha_{n+1}(t_k) - \Delta x(t_k) = L_k \int_{t_k - \delta_k}^{t_k - \varepsilon_k} p'(s) ds, \quad k = 1, 2, \dots, m,$$

and

$$\begin{aligned}
 \Delta p'(t_k) &= \Delta \alpha'_{n+1}(t_k) - \Delta x'(t_k) \\
 &= L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} \alpha_{n+1}(s) ds + I_k^* \left( \int_{t_k - \tau_k}^{t_k - \sigma_k} \alpha_n(s) ds \right) \\
 &\quad - L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} \alpha_n(s) ds - I_k^* \left( \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds \right) \\
 &\geq L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} \alpha_{n+1}(s) ds + L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} \alpha_n(s) ds \\
 &\quad - L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} \alpha_n(s) ds - L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds \\
 &= L_k^* \int_{t_k - \tau_k}^{t_k - \sigma_k} p(s) ds, \quad k = 1, 2, \dots, m,
 \end{aligned}$$

and

$$p(0) = p(T), \quad p'(0) = p'(T).$$

Still by Lemma 2.1, we have for all  $t \in J$ ,  $p(t) \leq 0$ , i.e.,  $\alpha_{n+1}(t) \leq x(t)$ . Similarly, we can prove that  $x(t) \leq \beta_{n+1}(t)$ ,  $t \in J$ . Therefore,  $\alpha_{n+1}(t) \leq x(t) \leq \beta_{n+1}(t)$ , for all  $t \in J$ , which implies  $x^*(t) \leq x(t) \leq x^s(t)$ . The proof is complete.  $\square$

#### 4 An example

In this section, in order to illustrate our results, we consider an example.

**Example 4.1** Consider the following PBVP:

$$\begin{cases}
 u''(t) = \frac{1}{4}t^3(u(t) - 2) + \frac{5}{18} \left[ \int_0^t t^2 s^4 u(s) ds \right]^2 \\
 \quad + \frac{1}{8} \left[ \int_0^1 t^3 s^2 u(s) ds \right]^2, \quad t \in J = [0, 1], t \neq \frac{1}{2}, \\
 \Delta u(\frac{1}{2}) = \frac{1}{2} \int_{\frac{1}{6}}^{\frac{3}{10}} u'(s) ds, \quad k = 1, \\
 \Delta u'(\frac{1}{2}) = \frac{1}{3} \int_{\frac{1}{10}}^{\frac{3}{10}} u(s) ds, \quad k = 1, \\
 u(0) = u(1), \quad u'(0) = u'(1).
 \end{cases} \tag{4.1}$$



Set  $k(t, s) = t^2 s^4$ ,  $h(t, s) = t^3 s^2$ ,  $m = 1$ ,  $t_1 = \frac{1}{2}$ ,  $\delta_1 = \frac{1}{3}$ ,  $\varepsilon_1 = \frac{1}{5}$ ,  $\tau_1 = \frac{2}{5}$ ,  $\sigma_1 = \frac{1}{5}$ ,  $T = 1$ . Obviously,  $\alpha_0 = 0$ ,  $\beta_0 = 3$  are lower and upper solutions for (4.1), respectively, and  $\alpha_0 \leq \beta_0$ .

Let

$$f(t, x_1, y_1, z_1) = \frac{1}{4}t^3(x_1 - 2) + \frac{5}{18}y_1^2 + \frac{1}{8}z_1^2,$$

we have

$$f(t, x_2, y_2, z_2) - f(t, x_1, y_1, z_1) \leq \frac{1}{4}(x_2 - x_1) + \frac{1}{3}(y_2 - y_1) + \frac{1}{4}(z_2 - z_1),$$

where  $\alpha(t) \leq x_1 \leq x_2 \leq \beta(t)$ ,  $(K\alpha)(t) \leq y_1 \leq y_2 \leq (K\beta)(t)$ ,  $(S\alpha)(t) \leq z_1 \leq z_2 \leq (S\beta)(t)$ ,  $t \in J$ .

It is easy to see that

$$I_1 \left( \int_{\frac{1}{6}}^{\frac{3}{10}} x'(s) ds \right) - I_1 \left( \int_{\frac{1}{6}}^{\frac{3}{10}} y'(s) ds \right) = \frac{1}{2} \int_{\frac{1}{6}}^{\frac{3}{10}} x'(s) - y'(s) ds,$$

and

$$I_1^* \left( \int_{\frac{1}{10}}^{\frac{3}{10}} x(s) ds \right) - I_1^* \left( \int_{\frac{1}{10}}^{\frac{3}{10}} y(s) ds \right) = \frac{1}{3} \int_{\frac{1}{10}}^{\frac{3}{10}} x(s) - y(s) ds,$$

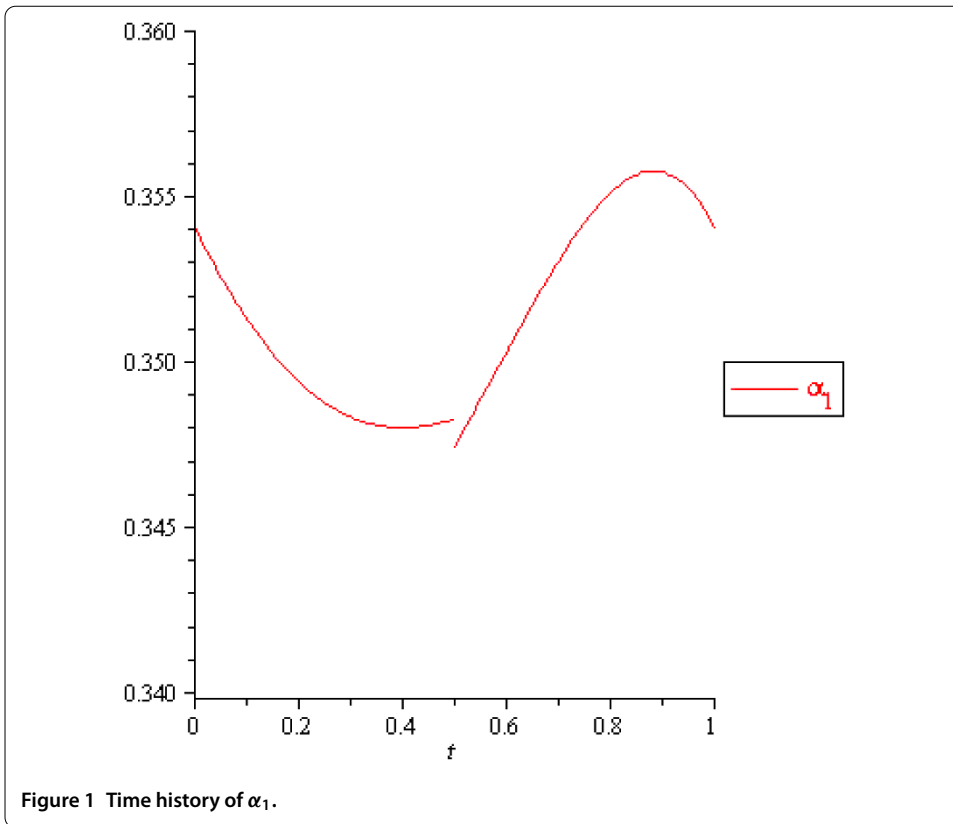
whenever  $\int_{t_k - \tau_k}^{t_k - \sigma_k} \alpha(s) ds \leq \int_{t_k - \tau_k}^{t_k - \sigma_k} y(s) ds \leq \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s) ds \leq \int_{t_k - \tau_k}^{t_k - \sigma_k} \beta(s) ds$ ,  $k = 1$ .

Taking  $M = \frac{1}{4}$ ,  $N = \frac{1}{3}$ ,  $L = \frac{1}{4}$ ,  $L_1 = \frac{1}{2}$ ,  $L_1^* = \frac{1}{3}$ , it follows that

$$\begin{aligned} & \left[ \sum_{k=1}^m L_k(\delta_k - \varepsilon_k) + T \right] \left[ \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) + (M + Nk_0T + Lh_0T)T \right] \\ &= \left[ \frac{1}{2} \left( \frac{1}{3} - \frac{1}{5} \right) + 1 \right] \left[ \frac{1}{3} \left( \frac{2}{5} - \frac{1}{5} \right) + \left( \frac{1}{4} + \left( \frac{1}{3} \right) (1)(1) + \left( \frac{1}{4} \right) (1)(1) \right) 1 \right] \\ &= \frac{24}{25} \leq 1, \end{aligned}$$

and

$$\begin{aligned} \psi &= \frac{1 + e^{\sqrt{MT}}}{2\sqrt{M}(e^{\sqrt{MT}} - 1)} \\ &\times \left[ \int_0^T \left[ N \int_0^s k(s, r) dr + L \int_0^T h(s, r) dr \right] ds + \sum_{k=1}^m L_k^*(\tau_k - \sigma_k) \right] \\ &+ \frac{1}{2} \sum_{k=1}^m L_k(\delta_k - \varepsilon_k) \\ &= \frac{1 + e^{\frac{1}{2}}}{2(\frac{1}{2})(e^{\frac{1}{2}} - 1)} \left[ \int_0^1 \left[ \frac{1}{3} \int_0^s s^2 r^4 dr + \frac{1}{4} \int_0^1 s^3 r^2 dr \right] ds + \frac{1}{3} \left( \frac{2}{5} - \frac{1}{5} \right) \right] \\ &+ \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{3} - \frac{1}{5} \right) \\ &\approx 0.4246196990 < 1, \end{aligned}$$

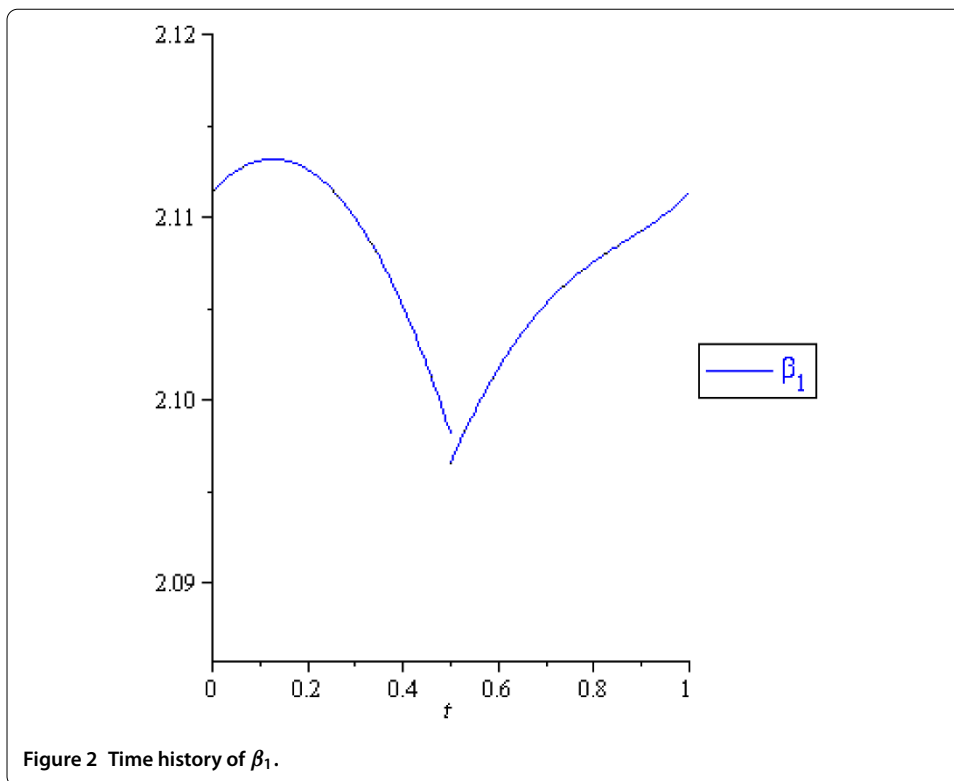


$$\begin{aligned} \mu &= \frac{1}{2} \left[ \int_0^T \left[ N \int_0^s k(s,r) dr + L \int_0^T h(s,r) dr \right] ds + \sum_{k=1}^m L_k^* (\tau_k - \sigma_k) \right] \\ &\quad + \frac{\sqrt{M}(1 + e^{\sqrt{MT}})}{2(e^{\sqrt{MT}} - 1)} \sum_{k=1}^m L_k (\delta_k - \varepsilon_k) \\ &= \frac{1}{2} \left[ \int_0^1 \left[ \frac{1}{3} \int_0^s s^2 r^4 dr + \frac{1}{4} \int_0^1 s^3 r^2 dr \right] ds + \frac{1}{3} \left( \frac{2}{5} - \frac{1}{5} \right) \right] \\ &\quad + \frac{\frac{1}{2}(1 + e^{\frac{1}{2}})}{2(e^{\frac{1}{2}} - 1)} \left( \frac{1}{2} \right) \left( \frac{1}{3} - \frac{1}{5} \right) \\ &\approx 0.1159664694 < 1. \end{aligned}$$

Therefore, (4.1) satisfies all the conditions of Theorem 3.1. So, PBVP (4.1) has minimal and maximal solutions in the segment  $[\alpha_0, \beta_0]$ .

Substituting  $\alpha_0, \beta_0$  into monotone iterative scheme, we obtain

$$\begin{cases} \alpha_1''(t) - \frac{1}{4}\alpha_1(t) - \frac{1}{3} \int_0^t t^2 s^4 \alpha_1(s) ds - \frac{1}{4} \int_0^1 t^3 s^2 \alpha_1(s) ds \\ \quad = -\frac{1}{2}t^3, \quad J = [0, 1], t \neq \frac{1}{2}, \\ \Delta \alpha_1(\frac{1}{2}) = \frac{1}{2} \int_{\frac{1}{6}}^{\frac{3}{10}} \alpha_1'(s) ds, \quad k = 1, \\ \Delta \alpha_1'(\frac{1}{2}) = \frac{1}{3} \int_{\frac{1}{10}}^{\frac{3}{10}} \alpha_1(s) ds, \quad k = 1, \\ \alpha_1(0) = \alpha_1(T), \quad \alpha_1'(0) = \alpha_1'(T), \end{cases} \quad (4.2)$$



and

$$\begin{cases}
 \beta_1''(t) - \frac{1}{4}\beta_1(t) - \frac{1}{3}\int_0^t t^2 s^4 \beta_1(s) ds - \frac{1}{4}\int_0^1 t^3 s^2 \beta_1(s) ds \\
 = \frac{1}{10}t^{14} - \frac{1}{5}t^7 + \frac{1}{8}t^6 - \frac{3}{4}, \quad J = [0, 1], t \neq \frac{1}{2}, \\
 \Delta\beta_1(\frac{1}{2}) = \frac{1}{2}\int_{\frac{1}{6}}^{\frac{3}{10}} \beta_1'(s) ds, \quad k = 1, \\
 \Delta\beta_1'(\frac{1}{2}) = \frac{1}{3}\int_{\frac{1}{10}}^{\frac{3}{10}} \beta_1(s) ds, \quad k = 1, \\
 \beta_1(0) = \beta_1(T), \quad \beta_1'(0) = \beta_1'(T).
 \end{cases} \tag{4.3}$$

After using the variational iteration method [45] for (4.2), (4.3), the approximate solutions for  $\alpha_1$  and  $\beta_1$  can be illustrated as Figure 1 and Figure 2, respectively.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally in this article. They read and approved the final manuscript.

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