CORE

# Some inequalities involving $k$-gamma and $k$-beta functions with applications - II 

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#### Abstract

In this paper, we present some inequalities involving $k$-gamma and $k$-beta functions via some classical inequalities, like Chebyshev's inequality for synchronous (asynchronous) mappings, Grüss', and Ostrowski's inequality. Also, we give applications of $k$-beta function in probability distributions. Most of the inequalities produced in this paper are the $k$-analogs of existing results. If $k=1$, we have the classical one.


Keywords: k-gamma; $k$-beta; inequalities; probability distribution

## 1 Introduction

In this section, we present some fundamental relations for $k$-gamma and $k$-beta functions introduced in [1-7]. In Section 2, we introduce some $k$-analog properties of the mapping $l_{p, q}$, which is helpful in coming sections. Sections 3 to 5 are devoted to the applications of some integral inequalities like Chebyshev's, Grüss', and Ostrowski's inequality for $k$ beta mappings. In the last section, we give the applications of the said function for the probability distribution and the probability density function.
Recently, Diaz and Pariguan [1] introduced the generalized $k$-gamma function as

$$
\begin{equation*}
\Gamma_{k}(x)=\lim _{n \rightarrow \infty} \frac{n!k^{n}(n k)^{\frac{x}{k}-1}}{(x)_{n, k}}, \quad k>0, x \in \mathbb{C} \backslash k \mathbb{Z}^{-} \tag{1}
\end{equation*}
$$

and also gave the properties of the said function. $\Gamma_{k}$ is one parameter deformation of the classical gamma function such that $\Gamma_{k} \rightarrow \Gamma$ as $k \rightarrow 1 . \Gamma_{k}$ is based on the repeated appearance of the expression of the following form:

$$
\begin{equation*}
\alpha(\alpha+k)(\alpha+2 k)(\alpha+3 k) \cdots(\alpha+(n-1) k) . \tag{2}
\end{equation*}
$$

The function of the variable $\alpha$ given by the statement (2), denoted by $(\alpha)_{n, k}$ is called the Pochhammer $k$-symbol. We obtain the usual Pochhammer symbol $(\alpha)_{n}$ by taking $k=1$. The definition given in (1) is the generalization of $\Gamma(x)$ and the integral form of $\Gamma_{k}$ is given by

$$
\begin{equation*}
\Gamma_{k}(x)=\int_{0}^{\infty} t^{x-1} e^{-\frac{t^{k}}{k}} d t, \quad \operatorname{Re}(x)>0 . \tag{3}
\end{equation*}
$$

From (3), we can easily show that

$$
\begin{equation*}
\Gamma_{k}(x)=k^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right) . \tag{4}
\end{equation*}
$$

The same authors defined the $k$-beta function as

$$
\begin{equation*}
\beta_{k}(x, y)=\frac{\Gamma_{k}(x) \Gamma_{k}(y)}{\Gamma_{k}(x+y)}, \quad \operatorname{Re}(x)>0, \operatorname{Re}(y)>0 \tag{5}
\end{equation*}
$$

and the integral form of $\beta_{k}(x, y)$ is

$$
\begin{equation*}
\beta_{k}(x, y)=\frac{1}{k} \int_{0}^{1} t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} d t \tag{6}
\end{equation*}
$$

From the definition of $\beta_{k}(x, y)$ given in (5) and (6), we can easily prove that

$$
\begin{equation*}
\beta_{k}(x, y)=\frac{1}{k} \beta\left(\frac{x}{k}, \frac{y}{k}\right) . \tag{7}
\end{equation*}
$$

Also, the researchers in [2-6] have worked on the generalized $k$-gamma and $k$-beta functions and discussed the following properties:

$$
\begin{align*}
& \Gamma_{k}(x+k)=x \Gamma_{k}(x),  \tag{8}\\
& (x)_{n, k}=\frac{\Gamma_{k}(x+n k)}{\Gamma_{k}(x)},  \tag{9}\\
& \Gamma_{k}(k)=1, \quad k>0,  \tag{10}\\
& \Gamma_{k}(x)=a^{\frac{x}{k}} \int_{0}^{\infty} t^{x-1} e^{-\frac{t^{k}}{k}} d t, \quad a \in \mathbb{R},  \tag{11}\\
& \Gamma_{k}(\alpha k)=k^{\alpha-1} \Gamma(\alpha), \quad k>0, \alpha \in \mathbb{R},  \tag{12}\\
& \Gamma_{k}(n k)=k^{n-1}(n-1)!, \quad k>0, n \in \mathbb{N},  \tag{13}\\
& \Gamma_{k}\left((2 n+1) \frac{k}{2}\right)=k^{\frac{2 n-1}{2}} \frac{(2 n)!\sqrt{\pi}}{2^{n} n!}, \quad k>0, n \in \mathbb{N} . \tag{14}
\end{align*}
$$

Using (5) and (7), we see that, for $x, y>0$ and $k>0$, the following properties of $k$-beta function are valid (see [2,3] and [7]):

$$
\begin{align*}
& \beta_{k}(x+k, y)=\frac{x}{x+y} \beta_{k}(x, y),  \tag{15}\\
& \beta_{k}(x, y+k)=\frac{y}{x+y} \beta_{k}(x, y),  \tag{16}\\
& \beta_{k}(x k, y k)=\frac{1}{k} \beta(x, y),  \tag{17}\\
& \beta_{k}(n k, n k)=\frac{[(n-1)!]^{2}}{k(2 n-1)!}, \quad n \in \mathbb{N},  \tag{18}\\
& \beta_{k}(x, k)=\frac{1}{x}, \quad \beta_{k}(k, y)=\frac{1}{y} . \tag{19}
\end{align*}
$$

Note that when $k \rightarrow 1, \beta_{k}(x, y) \rightarrow \beta(x, y)$.

For more details about the theory of $k$-special functions like the $k$-gamma function, the $k$-polygamma function, the $k$-beta function, the $k$-hypergeometric functions, solutions of $k$-hypergeometric differential equations, contiguous functions relations, inequalities with applications and integral representations with applications involving $k$-gamma and $k$-beta functions, $k$-gamma and $k$-beta probability distributions, and so forth (see [8-15]).

## 2 Main results: some $\boldsymbol{k}$-analog properties of the mapping $I_{p, q}$

For the applications of some integral inequalities involving $k$-gamma and $k$-beta functions, we have to discuss some $k$-analog properties regarding these mappings. For this purpose, consider the mapping $l_{p, q}:[0,1] \rightarrow \mathbb{R}$, defined by

$$
\begin{equation*}
l_{p, q}(x)=x^{\frac{p}{k}}(1-x)^{\frac{q}{k}}, \quad p, q, k>0, \tag{20}
\end{equation*}
$$

and differentiation of above equation gives

$$
\begin{equation*}
l_{p, q}^{\prime}(x)=\frac{1}{k} x^{\frac{p-k}{k}}(1-x)^{\frac{q-k}{k}}[p-(p+q) x] . \tag{21}
\end{equation*}
$$

Here, we see that $l_{p, q}^{\prime}(x)=0$ has the solution $x_{0}=\frac{p}{p+q}$ in the interval $(0,1)$. Also, $l_{p, q}^{\prime}(x)>0$ on $\left(0, x_{0}\right)$ and $l_{p, q}^{\prime}(x)<0$ on $\left(x_{0}, 1\right)$. Thus, we conclude that $x_{0}$ is the maximum point in the interval $(0,1)$ and consequently, we have

$$
\begin{equation*}
m_{p, q}=\inf _{x \in[0,1]} l_{p, q}(x)=0 \tag{22}
\end{equation*}
$$

and

$$
\begin{equation*}
M_{p, q}=\sup _{x \in[0,1]} l_{p, q}(x)=l_{p, q}\left(\frac{p}{p+q}\right)=\frac{p^{\frac{p}{k}} q^{\frac{q}{k}}}{(p+q)^{\frac{(p+q)}{k}}} . \tag{23}
\end{equation*}
$$

Also, we have

$$
\begin{align*}
& \left\|l_{p, q}\right\|_{\infty}=\frac{p^{\frac{p}{k}} q^{\frac{q}{k}}}{(p+q)^{\frac{(p+q)}{k}}}, \quad p, q, k>0,  \tag{24}\\
& \left\|l_{p, q}\right\|_{\infty}=k \beta_{k}(p+k, q+k), \quad p, q, k>0, \tag{25}
\end{align*}
$$

and

$$
\begin{equation*}
\left\|l_{p, q}\right\|_{r}=\left[k \beta_{k}(p r+k, q r+k)\right]^{\frac{1}{r}}, \quad p, q, k>0, r>1 . \tag{26}
\end{equation*}
$$

Further, we observe that

$$
\begin{aligned}
\left\|l_{p, q}^{\prime}(x)\right\| & \leq \frac{1}{k} x^{\frac{p-k}{k}}(1-x)^{\frac{q-k}{k}}|p-(p+q) x| \\
& \leq \max \{p, q\} l_{p-k, q-k}(x), \quad p, q, k>0, x \in(0,1)
\end{aligned}
$$

Now, we have the estimations

$$
\begin{equation*}
\left\|l_{p, q}^{\prime}\right\|_{\infty}=\frac{1}{k} \max \{p, q\} \frac{(p-k)^{\frac{p-k}{k}}(q-k)^{\frac{q-k}{k}}}{(p+q-2 k)^{\frac{(p+q-2 k)}{k}}}, \quad \text { if } p, q>k>0 \tag{27}
\end{equation*}
$$

$$
\begin{equation*}
\left\|l_{p, q}^{\prime}\right\|_{r}=\max \{p, q\}\left[\beta_{k}(r(p-k)+k, r(q-k)+k)\right]^{\frac{1}{r}}, \quad p, q>k>0, r>1, \tag{28}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|l_{p, q}^{\prime}\right\|_{1}=\max \{p, q\} \beta_{k}(p, q), \quad p, q, k>0 . \tag{29}
\end{equation*}
$$

Again, the second derivative of the said mapping gives

$$
\begin{aligned}
l_{p, q}^{\prime \prime}(x) & =\left[l_{p-k, q-k}(x)\right]^{\prime}[p-(p+q) x]-l_{p-k, q-k}(x)(p+q) \\
& =\frac{1}{k} l_{p-2 k, q-2 k}(x)[p-k-(p-k+q-k) x]-l_{p-k, q-k}(x)(p+q) \\
& =\frac{1}{k} l_{p-2 k, q-2 k}(x)\left[(p+q) x^{2}-2(p+q-k) x+p-k\right] .
\end{aligned}
$$

Now, consider the mapping $g_{p, q}:[0,1) \rightarrow \mathbb{R}$, defined by

$$
g_{p, q}(x)=(p+q) x^{2}-2(p+q-k) x+p-k .
$$

Here, we have $g_{p, q}(0)=p-k$ and $g_{p, q}(1)=k-q$. If $p, q>k$, then $g_{p, q}$ has a solution on the interval $(0,1)$ and one solution in the interval $(1, \infty)$. Also, the quadratic function $f(x)=$ $a x^{2}+b x+c$ has a vertex at $x=-\frac{b}{2 a}$. So, the coordinates of the vertex are

$$
x_{v}=\frac{2(p+q-k)}{2(p+q)}=\frac{p+q-k}{p+q}<1
$$

and

$$
y_{v}=-\frac{q^{2}+p q-p k-q k+k^{2}}{p+q}=-\left(q-k+\frac{k^{2}}{p+q}\right) .
$$

Consequently, we have

$$
\left|g_{p, q}(x)\right| \leq \max \left\{g_{p, q},\left|y_{v}\right|\right\}=\max \left\{p-k, q-k+\frac{k^{2}}{p+q}\right\}=\max \left\{p, q+\frac{k^{2}}{p+q}\right\}-k
$$

and then we get

$$
\begin{equation*}
\left\|l_{p, q}^{\prime \prime}(x)\right\| \leq\left[\max \left\{p, q+\frac{k^{2}}{p+q}\right\}-k\right] l_{p-2 k, q-2 k}(x), \quad p, q>k, x \in(0,1) \tag{30}
\end{equation*}
$$

If $p, q>2 k$, we have

$$
\begin{equation*}
\left\|l_{p, q}^{\prime \prime}\right\|_{\infty} \leq\left[\max \left\{p, q+\frac{k^{2}}{p+q}\right\}-k\right] \frac{(p-2 k)^{\frac{p-2 k}{k}}(q-2 k)^{\frac{q-2 k}{k}}}{(p+q-4 k)^{\frac{(p+q-4 k)}{k}}} . \tag{31}
\end{equation*}
$$

From (30), if $p, q>k$, we get

$$
\begin{equation*}
\left\|l_{p, q}^{\prime \prime}\right\|_{1} \leq\left[\max \left\{p, q+\frac{k^{2}}{p+q}\right\}-k\right] \beta_{k}(p-k, q-k) \tag{32}
\end{equation*}
$$

and if $p, q>2 k$

$$
\begin{equation*}
\left\|l_{p, q}^{\prime \prime}\right\|_{r} \leq\left[\max \left\{p, q+\frac{k^{2}}{p+q}\right\}-k\right]\left[\beta_{k}(r(p-2 k)+k, r(q-2 k)+k)\right]^{\frac{1}{r}} . \tag{33}
\end{equation*}
$$

Remark If $k=1$, we have the properties of the mapping $l_{p, q}$ given in [16].

## 3 Chebyshev type inequalities involving $\boldsymbol{k}$-beta and $\boldsymbol{k}$-gamma functions

In this section, we prove some inequalities which involve $k$-gamma and $k$-beta functions by using some natural inequalities [17]. The following result is well known in the literature as Chebyshev's integral inequality for synchronous (asynchronous) functions. Here, we use this result to prove some $k$-analog inequalities.

Lemma 3.1 Letf,g, $h: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be such that $h(x) \geq 0$ for all $x \in I$ and $h, h f g$, $h f$, and $h g$ are integrable on I. Iff, $g$ are synchronous (asynchronous) on I, i.e.,

$$
\begin{equation*}
(f(x)-f(y))(g(x)-g(y)) \geq(\leq)=0 \quad \text { for all } x, y \in I \text {, } \tag{34}
\end{equation*}
$$

then we have the inequality (see $[18,19])$

$$
\begin{equation*}
\int_{I} h(x) d x \int_{I} h(x) f(x) g(x) d x \geq(\leq) \int_{I} h(x) f(x) d x \int_{I} h(x) g(x) d x . \tag{35}
\end{equation*}
$$

Lemma 3.1 can be proved by using Korkine's identity [20],

$$
\begin{align*}
& \int_{I} h(x) d x \int_{I} h(x) f(x) g(x) d x-\int_{I} h(x) f(x) d x \int_{I} h(x) g(x) d x \\
& \quad=\frac{1}{2} \int_{I} \int_{I} h(x) h(y)(f(x)-f(y))(g(x)-g(y)) d x d y \tag{36}
\end{align*}
$$

and an inequality generalizing Chebyshev's inequality is

$$
\begin{align*}
& \left|\int_{I} h(x) d x \int_{I} h(x) f(x) g(x) d x-\int_{I} h(x) f(x) d x \int_{I} h(x) g(x) d x\right| \\
& \quad \leq\left\|f^{\prime}\right\|_{\infty}\left\|g^{\prime}\right\|_{\infty}\left[\int_{I} x^{2} h(x) d x \int_{I} h(x) d x-\left(\int_{I} x h(x) d x\right)^{2}\right] \tag{37}
\end{align*}
$$

provided that $h(x)>0$ and $f^{\prime}, g^{\prime}$ are differentiable and the first derivatives are bounded on $I$.

Theorem 3.2 For $k>0$, let $m, n, p, q>k$ and $r, s>-k$, then we have the following inequality for the $k$-beta function:

$$
\begin{align*}
\mid \beta_{k}(r & +k, s+k) \beta_{k}(m+p+r+k, n+q+s+k) \\
& -\beta_{k}(m+r+k, n+s+k) \beta_{k}(p+r+k, q+s+k) \mid \\
\leq & M_{\infty}^{\prime}(p, q) M_{\infty}^{\prime}(m, n)\left[\beta_{k}(r+3 k, s+k) \beta_{k}(r+k, s+k)-\beta_{k}^{2}(r+2 k, s+k)\right] \tag{38}
\end{align*}
$$

where

$$
M_{\infty}^{\prime}(p, q)=\frac{1}{k} \max \{p, q\} \frac{(p-k)^{\frac{p-k}{k}}(q-k)^{\frac{q-k}{k}}}{(p+q-2 k)^{\frac{(p+q-2 k)}{k}}}, \quad p, q>k>0 .
$$

Proof Consider the mappings

$$
f(x)=l_{m, n}=x^{\frac{m}{k}}(1-x)^{\frac{n}{k}}, \quad g(x)=l_{p, q}=x^{\frac{p}{k}}(1-x)^{\frac{q}{k}}, \quad h(x)=l_{r, s}=x^{\frac{r}{k}}(1-x)^{\frac{s}{k}},
$$

defined on the interval $[0,1]$. Using the generalized version of Lemma 3.1, i.e., (37), along with the mappings defined above, we get

$$
\begin{align*}
& \left|\int_{0}^{1} x^{\frac{r}{k}}(1-x)^{\frac{s}{k}} d x \int_{0}^{1} x^{\frac{r+m+p}{k}}(1-x)^{\frac{s+n+q}{k}} d x-\int_{0}^{1} x^{\frac{r+m}{k}}(1-x)^{\frac{s+n}{k}} d x \int_{0}^{1} x^{\frac{r+p}{k}}(1-x)^{\frac{s+q}{k}} d x\right| \\
& \quad \leq\left\|f^{\prime}\right\|_{\infty}\left\|g^{\prime}\right\|_{\infty}\left[\int_{0}^{1} x^{\frac{r}{k}+2}(1-x)^{\frac{s}{k}} d x \int_{0}^{1} x^{\frac{r}{k}}(1-x)^{\frac{s}{k}} d x\right. \\
& \left.\quad-\left(\int_{0}^{1} x^{\frac{r}{k}+1}(1-x)^{\frac{s}{k}} d x\right)^{2}\right] . \tag{39}
\end{align*}
$$

Applying (6), (39) gives

$$
\begin{aligned}
& \mid k^{2} \beta_{k}(r+k, s+k) \beta_{k}(m+p+r+k, n+q+s+k) \\
& \quad-k^{2} \beta_{k}(m+r+k, n+s+k) \beta_{k}(p+r+k, q+s+k) \mid \\
& \leq\left\|f^{\prime}\right\|_{\infty}\left\|g^{\prime}\right\|_{\infty}\left[k^{2} \beta_{k}(r+3 k, s+k) \beta_{k}(r+k, s+k)-\left(k \beta_{k}(r+2 k, s+k)\right)^{2}\right] .
\end{aligned}
$$

Now, taking into account the fact

$$
\left\|l_{m, n}^{\prime}\right\|_{\infty} \leq M_{\infty}^{\prime}(m, n), \quad\left\|l_{p, q}^{\prime}\right\|_{\infty} \leq M_{\infty}^{\prime}(p, q),
$$

for all $m, n, p, q>k$, we can deduce the desired inequality (38).

Corollary 3.3 For $k>0$ and $m, n, p, q>k$, we have the following inequality for the $k$-beta function:

$$
\left|\beta_{k}(m+p+k, n+q+k)-\beta_{k}(m+k, n+k) \beta_{k}(p+k, q+k)\right| \leq M_{\infty}^{\prime}(p, q) M_{\infty}^{\prime}(m, n)
$$

Proof Just use $r=s=0$ in Theorem 3.2 to get the required corollary.

Theorem 3.4 For $k>0, p, q>k$, and $r, s>-k$, we have the following inequality for the $k$-beta function:

$$
\begin{align*}
& \left|\beta_{k}(r+k, s+k) \beta_{k}(p+r+k, q+s+k)-\beta_{k}(p+r+k, s+k) \beta_{k}(r+k, q+s+k)\right| \\
& \quad \leq \frac{p q}{k^{2}}\left[\beta_{k}(r+3 k, s+k) \beta_{k}(r+k, s+k)-\beta_{k}^{2}(r+2 k, s+k)\right] . \tag{40}
\end{align*}
$$

Proof Consider the mappings

$$
f(x)=x^{\frac{p}{k}}, \quad g(x)=(1-x)^{\frac{q}{k}}, \quad h(x)=l_{r, s}(x)=x^{\frac{r}{k}}(1-x)^{\frac{s}{k}},
$$

defined on the interval $[0,1], k>0$. Now, we have

$$
\begin{aligned}
& f^{\prime}(x)=\frac{p}{k} x^{\frac{p}{k}-1}, \quad g^{\prime}(x)=-\frac{q}{k}(1-x)^{\frac{q}{k}-1}, \\
& \left\|f^{\prime}\right\|_{\infty}=\sup _{t \in(a, b)}\left|f^{\prime}(t)\right|=\frac{p}{k}, \quad\left\|g^{\prime}\right\|_{\infty}=\frac{q}{k} .
\end{aligned}
$$

Using the generalized version of Lemma 3.1, i.e., (37), along with the above results, we get

$$
\begin{aligned}
& \left|\int_{0}^{1} x^{\frac{r}{k}}(1-x)^{\frac{s}{k}} d x \int_{0}^{1} x^{\frac{r+p}{k}}(1-x)^{\frac{s+q}{k}}-\int_{0}^{1} x^{\frac{r+p}{k}}(1-x)^{\frac{s}{k}} \int_{0}^{1} x^{\frac{r}{k}}(1-x)^{\frac{s+q}{k}} d x\right| \\
& \quad \leq \frac{p q}{k^{2}}\left[\int_{0}^{1} x^{\frac{r}{k}+2}(1-x)^{\frac{s}{k}} d x \int_{0}^{1} x^{\frac{r}{k}}(1-x)^{\frac{s}{k}} d x-\left(\int_{0}^{1} x^{\frac{r}{k}+1}(1-x)^{\frac{s}{k}} d x\right)^{2}\right],
\end{aligned}
$$

which will be equivalent to Theorem 3.4 by applying (6) on both sides of the above inequality.

Corollary 3.5 If $r=s=0$ and $p, q>k>0$, then Theorem 3.4 takes the form

$$
\begin{equation*}
\left|\beta_{k}(p+k, q+k)-\frac{k}{(p+k)(q+k)}\right| \leq \frac{p q}{12 k^{3}} \tag{41}
\end{equation*}
$$

and inequality (41) is equivalent to

$$
\begin{align*}
& \max \left\{0, \frac{12 k^{4}-p^{2} q^{2}-p^{2} q k-p q^{2} k-p q k^{2}}{12 k^{3}(p+k)(q+k)}\right\} \\
& \quad \leq \beta_{k}(p+k, q+k) \leq \frac{12 k^{4}+p^{2} q^{2}+p^{2} q k+p q^{2} k+p q k^{2}}{12 k^{3}(p+k)(q+k)} . \tag{42}
\end{align*}
$$

Proof Taking $r=s=0$, in Theorem 3.4, we get

$$
\left|\beta_{k}(k, k) \beta_{k}(p+k, q+k)-\beta_{k}(p+k, k) \beta_{k}(k, q+k)\right| \leq \frac{p q}{k^{2}}\left[\beta_{k}(3 k, k) \beta_{k}(k, k)-\beta_{k}^{2}(2 k, k)\right] .
$$

Use of (5) and (7) implies

$$
\begin{aligned}
& \left|\frac{1}{k} \beta_{k}(p+k, q+k)-\frac{\Gamma_{k}(p+k) \Gamma_{k}(k)}{\Gamma_{k}(p+2 k)} \frac{\Gamma_{k}(k) \Gamma_{k}(q+k)}{\Gamma_{k}(q+2 k)}\right| \\
& \quad \leq \frac{p q}{k^{2}}\left[\frac{1}{k} \frac{\Gamma_{k}(3 k) \Gamma_{k}(k)}{\Gamma_{k}(4 k)}-\left(\frac{\Gamma_{k}(2 k) \Gamma_{k}(k)}{\Gamma_{k}(3 k)}\right)^{2}\right] .
\end{aligned}
$$

By (8) and (10), inequality (41) can be obtained and some algebraic calculations give the desired inequality (42).

## 4 Some other inequalities for $k$-beta mappings

In 1935, Grüss established an integral inequality which gives an estimation for the integral of a product in terms of the product of integrals [17]. We use the following lemma [21] to prove our next theorem which is based on the Grüss integral inequality.

Lemma 4.1 Iff and $g$ are two functions defined and integrable on ( $a, b$ ), then

$$
\begin{aligned}
& \left|\frac{1}{b-a} \int_{a}^{b} f(x) g(x) d x-\frac{1}{b-a} \int_{a}^{b} f(x) d x \cdot \frac{1}{b-a} \int_{a}^{b} g(x) d x\right| \\
& \quad \leq \begin{cases}\left\|f^{\prime}\right\|_{\infty}\|g\|_{1}, & \text { provided } g \in L_{1}[a, b], f^{\prime} \in L_{\infty}(a, b), \\
{\left[\frac{2}{c+1)(c+2)}\right]^{\frac{1}{c}}(b-a)^{\frac{1}{c}}\left\|f^{\prime}\right\|_{\infty}\|g\|_{d}} & \text { if } g \in L_{d}[a, b], f^{\prime} \in L_{\infty}(a, b), c>1, \frac{1}{c}+\frac{1}{d}=1, \\
\frac{b-a}{3}\left\|f^{\prime}\right\|_{\infty}\|g\|_{\infty}, & \text { provided } g, f^{\prime} \in L_{\infty}(a, b) .\end{cases}
\end{aligned}
$$

Theorem 4.2 Let $m, n>k$ and $p, q, k>0$, then we have the following inequality for the $k$-beta mapping:

$$
\begin{aligned}
& \left|\beta_{k}(m+p+k, n+q+k)-\beta_{k}(m+k, n+k) \beta_{k}(p+k, q+k)\right| \\
& \quad \leq\left\{\begin{array}{l}
M_{\infty}^{\prime}(m, n) k \beta_{k}(p+k, q+k), \\
{\left[\frac{2}{(c+1)(c+2)} \frac{1}{c} M_{\infty}^{\prime}(m, n)\left[k \beta_{k}(d p+k, d q+k)\right]^{\frac{1}{d}}, \quad \text { op } c>1, \frac{1}{c}+\frac{1}{d}=1,\right.} \\
\frac{1}{3} M_{\infty}^{\prime}(m, n) \frac{p k q}{(p k q)^{\frac{p+q}{k}}},
\end{array}\right.
\end{aligned}
$$

where

$$
M_{\infty}^{\prime}(m, n)=\frac{1}{k} \max \{m, n\} \frac{(m-k)^{\frac{m-k}{k}}(n-k)^{\frac{n-k}{k}}}{(m+n-2 k)^{\frac{(m+n-2 k}{k}}}, \quad m, n>k>0 .
$$

Proof Consider the mappings

$$
f(x)=l_{m, n}=x^{\frac{m}{k}}(1-x)^{\frac{n}{k}}, \quad g(x)=l_{p, q}=x^{\frac{p}{k}}(1-x)^{\frac{q}{k}},
$$

defined on the interval $[0,1]$. Using Lemma 4.1, along with the mappings defined above, we get

$$
\begin{aligned}
& \left|\int_{0}^{1} x^{\frac{m+p}{k}}(1-x)^{\frac{n+q}{k}} d x-\int_{0}^{1} x^{\frac{m}{k}}(1-x)^{\frac{n}{k}} d x \cdot \int_{0}^{1} x^{\frac{p}{k}}(1-x)^{\frac{q}{k}} d x\right| \\
& \quad \leq \begin{cases}\left\|l_{m, n}\right\|\left\|_{\infty}\right\| l_{p, q} \|_{1}, & m, n>k, p, q, k>0, \\
{\left[\frac{2}{(c+1)(c+2+2}\right]} \\
\frac{1}{3}\left\|l_{m, n}^{\frac{1}{c}}\right\| l_{m, n}\left\|_{\infty}\right\| l_{p, q, q} \|_{d} & \text { if } m, n>k, p, q, k>0, c>1, \frac{1}{c}+\frac{1}{d}=1, \\
m, n>k, p, q, k>0 .\end{cases}
\end{aligned}
$$

Applying (24) and (26) and using the fact $\left\|l_{m, n}^{\prime}\right\|_{\infty} \leq M_{\infty}^{\prime}(m, n)$ given in Section 2, we get

$$
\leq \begin{cases}M_{\infty}^{\prime}(m, n) k \beta_{k}(p+k, q+k), & m, n>1, p, q, k>0, \\ {\left[\frac{2}{(c+1)(c+2)}\right]^{\frac{1}{c}} M_{\infty}^{\prime}(m, n)\left[k \beta_{k}(d p+k, d q+k)\right]^{\frac{1}{d},},} & m, n, p, q>k>0, c>1, \frac{1}{c}+\frac{1}{d}=1, \\ \frac{1}{3} M_{\infty}^{\prime}(m, n) \frac{(p) \frac{q}{k}(q) \frac{q}{k},}{(p+q)^{p+q} \frac{1}{k}}, & m, n, p, q, k>0 .\end{cases}
$$

Theorem 4.3 Let $m, n>k$ and $p, q, k>0$, then the $k$-beta mapping satisfies the inequality

$$
\begin{aligned}
& \left|\beta_{k}(p+k, q+k)-\frac{k}{(p+k)(q+k)}\right| \\
& \quad \leq \begin{cases}\frac{p}{q+k}, & m, n>k, \text { if } p>k, q>-k, k>0, \\
{\left[\frac{2}{k(c+1)(c+2)}\right]^{\frac{1}{c}} \frac{p}{(d q+k)^{\frac{1}{d}}},} & p>k, q>0, c>1, \frac{1}{c}+\frac{1}{d}=1, \\
\frac{p}{3 k}, & p>k>0 .\end{cases}
\end{aligned}
$$

Proof Consider the mappings

$$
f(x)=x^{\frac{p}{k}}, \quad g(x)=(1-x)^{\frac{q}{k}},
$$

defined on the interval $[0,1]$. Here, we observe that

$$
\begin{aligned}
& f^{\prime}(x)=\frac{p}{k} x^{\frac{p}{k}-1}, \quad\left\|f^{\prime}\right\|_{\infty}=\frac{p}{k}, \quad p>k, \\
& \|g\|_{d}=\left(\int_{0}^{1}(1-x)^{\frac{d q}{k}} d x\right)^{\frac{1}{d}}=\left(\frac{k}{d q+k}\right)^{\frac{1}{d}},
\end{aligned}
$$

and

$$
\|g\|_{1}=\int_{0}^{1}(1-x)^{\frac{q}{k}} d x=\frac{k}{q+k}, \quad\|g\|_{\infty}=1 .
$$

Now, by Lemma 4.1, we get the required result.

The following inequality of Grüss type has been established in [22].

Lemma 4.4 Iff and $g$ are two functions defined and integrable on $(a, b)$, then

$$
\begin{aligned}
& \left|\frac{1}{b-a} \int_{a}^{b} f(x) g(x) d x-\frac{1}{b-a} \int_{a}^{b} f(x) d x \cdot \frac{1}{b-a} \int_{a}^{b} g(x) d x\right| \\
& \quad \leq \frac{1}{6}\left\|f^{\prime}\right\|_{c}\left\|g^{\prime}\right\|_{d}(b-a)
\end{aligned}
$$

provided

$$
f^{\prime} \in L_{c}(a, b), \quad g^{\prime} \in L_{d}(a, b), \quad c>1, \frac{1}{c}+\frac{1}{d}=1 .
$$

Theorem 4.5 Let m, $n, p, q, k>0$, then we have the following inequality for the $k$-beta mapping:

$$
\begin{aligned}
& \left|\beta_{k}(m+p+k, n+q+k)-\beta_{k}(m+k, n+k) \beta_{k}(p+k, q+k)\right| \\
& \leq \frac{k}{6} \max \{m, n\} \max \{p, q\}\left[\beta_{k}((m-k) c+k,(n-k) c+k)\right]^{\frac{1}{c}} \\
& \quad \times\left[\beta_{k}((p-k) d+k,(q-k) d+k)\right]^{\frac{1}{d}},
\end{aligned}
$$

where $c>1, \frac{1}{c}+\frac{1}{d}=1$.

Proof Consider the mappings

$$
f(x)=x^{\frac{m}{k}}(1-x)^{\frac{n}{k}}, \quad g(x)=x^{\frac{p}{k}}(1-x)^{\frac{q}{k}}, \quad m, n, p, q, k>0,
$$

defined on the interval $[0,1]$. Using Lemma 4.4, along with the mappings defined above, we get

$$
\left|\int_{0}^{1} x^{\frac{m+p}{k}}(1-x)^{\frac{n+q}{k}} d x-\int_{0}^{1} x^{\frac{m}{k}}(1-x)^{\frac{n}{k}} d x \cdot \int_{0}^{1} x^{\frac{p}{k}}(1-x)^{\frac{q}{k}} d x\right| \leq \frac{1}{6}\left\|f^{\prime}\right\|_{c}\left\|_{g^{\prime}}\right\|_{d^{\prime}}
$$

provided

$$
f^{\prime} \in L_{c}(a, b), \quad g^{\prime} \in L_{d}(a, b), \quad c>1, \frac{1}{c}+\frac{1}{d}=1 .
$$

Now, using the fact that

$$
\begin{equation*}
\left\|f^{\prime}\right\|_{c} \leq \max \{m, n\}\left[k \beta_{k}((m-k) c+k,(n-k) c+k)\right]^{\frac{1}{c}} \tag{43}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|g^{\prime}\right\|_{d} \leq \max \{p, q\}\left[k \beta_{k}((p-k) d+k,(q-k) d+k)\right]^{\frac{1}{d}} \tag{44}
\end{equation*}
$$

we have our required result.

Remarks If we use $k=1$, inequalities (43) and (44) are the results for the classical beta function proved in [22].

Lemma 4.6 Iff and $g$ are two functions defined and integrable on $(a, b)$, then we have the inequality

$$
\left|\frac{1}{b-a} \int_{a}^{b} f(x) g(x) d x-\frac{1}{b-a} \int_{a}^{b} f(x) d x \cdot \frac{1}{b-a} \int_{a}^{b} g(x) d x\right| \leq \frac{1}{6}\left\|f^{\prime}\right\|_{\infty}\left\|g^{\prime}\right\|_{1}(b-a) .
$$

Theorem 4.7 If $k>0$, the following inequalities for the $k$-beta mapping hold good:

$$
\begin{align*}
& \left|\beta_{k}(p+k, q+k)-\frac{k}{(p+k)(q+k)}\right| \\
& \quad \leq \frac{p q}{6 k^{2}}\left[\frac{1}{c(p-k)+k}\right]^{\frac{1}{c}}\left[\frac{1}{d(q-k)+k}\right]^{\frac{1}{d}}, \quad c>1, \frac{1}{c}+\frac{1}{d}=1, \tag{45}
\end{align*}
$$

and

$$
\begin{equation*}
\left|\beta_{k}(p+k, q+k)-\frac{k}{(p+k)(q+k)}\right| \leq \frac{p}{6 k^{2}}, \quad p>k \tag{46}
\end{equation*}
$$

Proof Consider the mappings

$$
f(x)=x^{\frac{p}{k}}, \quad g(x)=(1-x)^{\frac{q}{k}}, \quad p, q>k>0
$$

defined on the interval $[0,1]$. Here, we note that

$$
\left\|f^{\prime}\right\|_{c}=\frac{p}{k}\left[\frac{k}{c(p-k)+k}\right]^{\frac{1}{c}}, \quad\left\|g^{\prime}\right\|_{d}=\frac{q}{k}\left[\frac{k}{d(q-k)+k}\right]^{\frac{1}{d}} .
$$

Using Lemma 4.4 for the above results, we get

$$
\begin{aligned}
& \left|\int_{0}^{1} x^{\frac{p}{k}}(1-x)^{\frac{q}{k}} d x-\int_{0}^{1} x^{\frac{p}{k}} d x \cdot \int_{0}^{1}(1-x)^{\frac{q}{k}} d x\right| \\
& \quad \leq \frac{1}{6} \frac{p}{k}\left[\frac{k}{c(p-k)+k}\right]^{\frac{1}{c}} \frac{q}{k}\left[\frac{k}{d(q-k)+k}\right]^{\frac{1}{d}},
\end{aligned}
$$

provided

$$
f^{\prime} \in L_{c}(a, b), \quad g^{\prime} \in L_{d}(a, b), \quad c>1, \frac{1}{c}+\frac{1}{d}=1 .
$$

By (6) and the fact $\frac{1}{c}+\frac{1}{d}=1$, we have the inequality (45). Also, $\|f\|_{\infty}=\frac{p}{k}$ and $\|g\|_{1}=1$. Thus, using Lemma 4.6 we have the inequality (46).

## 5 Main results: via Ostrowski's inequality

In this section, we use the integral inequality which is known in the literature as Ostrowki's inequality [23]. The following lemma concerning Ostrowski's inequality for absolutely continuous mappings whose derivatives belong to $L_{p}$ spaces hold [24, 25]. Here, we give some lemmas which are helpful for the results involving $k$-beta mapping.

Lemma 5.1 Let $f:[a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping for which $f^{\prime} \in L_{p}[a, b]$, $p>1$. Then

$$
\begin{align*}
\left|f(x)-\frac{1}{b-a} \int_{a}^{b} f(t) d t\right| & \leq \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f^{\prime}\right\|_{p} \\
& \leq \frac{1}{(q+1)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}}\left\|f^{\prime}\right\|_{p} \tag{47}
\end{align*}
$$

for all $x \in[a, b]$, where

$$
\left\|f^{\prime}\right\|_{p}=\left(\int_{a}^{b}\left|f^{\prime}(t)\right|^{p} d t\right)^{\frac{1}{p}}
$$

and the best inequality for (47) is embodied in the form

$$
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a} \int_{a}^{b} f(t) d t\right| \leq \frac{1}{2} \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|f^{\prime}\right\|_{p} .
$$

For the application of the above inequalities to some numerical quadrature rules, we have the following lemma.

Lemma 5.2 Letf $:[a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping for which $f^{\prime} \in L_{p}[a, b]$, $p>1$. Then for any partition $I_{n}: a=x_{0}<x_{1}<\cdots<x_{n-1}<x_{n}=b$ of $[a, b]$ and any intermediate point vector $\xi=\left(\xi_{0}, \xi_{1}, \ldots, \xi_{n-1}\right)$ satisfying $\xi_{i} \in\left[x_{i}, x_{i+1}\right](i=0,1, \ldots, n-1)$, we have

$$
\int_{a}^{b} f(x) d x=A_{R}\left(f, I_{n}, \xi\right)+R_{R}\left(f, I_{n}, \xi\right)
$$

Here $A_{R}$ denotes the quadrature rule of the Riemann type defined by

$$
A_{R}\left(f, I_{n}, \xi\right)=\sum_{i=0}^{n-1} f\left(\xi_{i}\right) h_{i}, \quad h_{i}=x_{i+1}-x_{i}
$$

and the remainder satisfies the estimate

$$
\left|R_{R}\left(f, I_{n}, \xi\right)\right| \leq \frac{\left\|f^{\prime}\right\|_{p}}{(q+1)^{\frac{1}{q}}}\left(\sum_{i=0}^{n-1}\left[\left(\xi_{i}-x_{i}\right)^{q+1}+\left(x_{i+1}-\xi_{i}\right)^{q+1}\right]\right)^{\frac{1}{q}} \leq \frac{\left\|f^{\prime}\right\|_{p}}{(q+1)^{\frac{1}{q}}}\left(\sum_{i=0}^{n-1} h_{i}^{q+1}\right)^{\frac{1}{q}}
$$

where $h_{i}=x_{i+1}-x_{i}(i=0,1, \ldots, n-1)$. Lemmas 5.1 and 5.2 are proved in [19] and the best quadrature formula that can be obtained from the above result is one for which $\xi_{i}=\left(\frac{x_{i}+x_{i+1}}{2}\right)$, $i=0,1, \ldots, n-1$, and is given in the following corollary.

Corollary 5.3 Letf and $I_{n}$ be as in the Lemma 5.2, then

$$
\int_{a}^{b} f(x) d x=A_{M}\left(f, I_{n}\right)+R_{M}\left(f, I_{n}\right)
$$

where $A_{M}$ denotes the mid point quadrature rule i.e.,

$$
A_{M}\left(f, I_{n}\right)=\sum_{i=0}^{n-1} f\left(\frac{x_{i}+x_{i+1}}{2}\right) h_{i}
$$

and the remainder $R_{M}$ satisfies the estimation

$$
\left|R_{M}\left(f, I_{n}\right)\right| \leq \frac{1}{2} \frac{\left\|f^{\prime}\right\|_{p}}{(q+1)^{\frac{1}{q}}}\left(\sum_{i=0}^{n-1} h_{i}^{q+1}\right)^{\frac{1}{q}}
$$

We are now able to apply the above results for Euler's $k$-beta mapping.

Theorem 5.4 Let $s>1, p, q>2 k-\frac{1}{s}>k$, and $k>0$. Then we have the inequality for the $k$-beta function as

$$
\begin{aligned}
\left|k \beta_{k}(p, q)-x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}\right| \leq & \frac{1}{(l+1)^{\frac{1}{l}}}\left[x^{l+1}+(1-x)^{l+1}\right]^{\frac{1}{l}} \max \{p-k, q-k\} \\
& \times\left[\beta_{k}(s(p-k-1)+1, s(q-k-1)+1)\right]^{\frac{1}{s}}
\end{aligned}
$$

provided that $\frac{1}{s}+\frac{1}{l}=1$.

Proof Consider the mapping $f(t)=t^{\frac{p}{k}-1}(1-t)^{\frac{q}{k}-1}=l_{p-k, q-k}(t), t \in[0,1]$. From Lemma 5.1 along with this mapping, we get

$$
\begin{align*}
& \left|k \beta_{k}(p, q)-l_{p-k, q-k}(x)\right| \\
& \quad \leq \frac{1}{(l+1)^{\frac{1}{l}}}\left[x^{l+1}+(1-x)^{l+1}\right]^{\frac{1}{l}}\left\|l_{p-k, q-k}^{\prime}\right\|_{s^{\prime}} \quad x \in[0,1] \tag{48}
\end{align*}
$$

where $s>1$ and $\frac{1}{s}+\frac{1}{l}=1$. Now, taking the derivatives of the above mapping, we have

$$
l_{p-k, q-k}^{\prime}(t)=\frac{1}{k} l_{p-k-1, q-k-1}[p-k-(p+q-2 k) t], \quad t \in(0,1) .
$$

If $t \in\left(0, \frac{p-k}{p+q-2 k}\right), l_{p-k, q-k}^{\prime}>0$ and if $t \in\left(\frac{p-k}{p+q-2 k}, 1\right)$, then $l_{p-k, q-k}^{\prime}<0$, which shows that at $t_{0}=\frac{p-k}{p+q-2 k}$, we have a maximum for $l_{p-k, q-k}$ and

$$
\sup _{t \in(0,1)} l_{p-k, q-k}(t)=l_{p-k, q-k}\left(t_{0}\right)=\frac{(p-k)^{(p-k)}(q-k)^{(q-k)}}{(p+q-2 k)^{(p+q-2 k)}}, \quad p, q>k .
$$

Consequently, we have

$$
\begin{aligned}
\left|l_{p-k, q-k}^{\prime}(t)\right| & \leq \frac{1}{k}\left|l_{p-k-1, q-k-1}(t)\right| \max _{t \in(0,1)}|(p-k)-(p+q-2 k) t| \\
& \leq \frac{1}{k} \frac{(p-k)^{(p-k)}(q-k)^{(q-k)}}{(p+q-2 k)^{(p+q-2 k)}} \max \{p-k, q-k\}
\end{aligned}
$$

for all $t \in(0,1)$. Thus

$$
\begin{align*}
\left\|l_{p-k, q-k}^{\prime}(t)\right\|_{s} & =\left(\frac{1}{k} \int_{0}^{1} l_{p-k-1, q-k-1}^{s}(t)|p-k-(p+q-2 k)|^{s} d s\right)^{\frac{1}{s}} \\
& =\left(\frac{1}{k} \int_{0}^{1} t^{\frac{s(p-k-1)}{k}}(1-t)^{\frac{s(q-k-1)}{k}}|p-k-(p+q-2 k)|^{s} d s\right)^{\frac{1}{s}} \\
& =\max \{p-k, q-k\}\left[\beta_{k}(s(p-k-1)+1, s(q-k-1)+1)\right]^{\frac{1}{s}} . \tag{49}
\end{align*}
$$

Using (48) and (49), we get the desired Theorem 5.4.

Now, we have the result concerning the approximation of the $k$-beta function in terms of the Riemann sums.

Theorem 5.5 Let $s>1, p, q>2 k-\frac{1}{s}>k$ and $k>0$. If $I_{n}: 0=x_{0}<x_{1}<\cdots<x_{n-1}<x_{n}=1$ is a division of $[0,1], \xi_{i} \in\left[x_{i}, x_{i+1}\right], i=(0,1, \ldots, n-1)$ a sequence of intermediate points for $I_{n}$, then we have the formula for the $k$-beta function:

$$
\beta_{k}(p, q)=\sum_{i=0}^{n-1} \xi_{i}^{\frac{p}{k}-1}\left(1-\xi_{i}\right)^{\frac{q}{k}-1} h_{i}+T_{n}(p, q),
$$

where the remainder $T_{n}(p, q)$ satisfies the estimate

$$
\begin{aligned}
\left|T_{n}(p, q)\right| \leq & \frac{\max \{p-k, q-k\}}{(l+1)^{\frac{1}{l}}}\left[\beta_{k}(s(p-k-1)+1, s(q-k-1)+1)\right]^{\frac{1}{s}} \\
& \times\left(\sum_{i=0}^{n-1}\left[\left(\xi_{i}-x_{i}\right)^{l+1}+\left(x_{i+1}-\xi_{i}\right)^{l+1}\right]\right)^{\frac{1}{l}} \\
\leq & \frac{\max \{p-k, q-k\}}{(l+1)^{\frac{1}{l}}}\left[\beta_{k}(s(p-k-1)+1, s(q-k-1)+1)\right]^{\frac{1}{s}}\left(\sum_{i=0}^{n-1}\left(h_{i}\right)^{l+1}\right)^{\frac{1}{l}},
\end{aligned}
$$

where $h_{i}=x_{i+1}-x_{i}(i=0,1, \ldots, n-1)$ and $\frac{1}{s}+\frac{1}{l}=1$.
Proof Taking $f(t)=t^{\frac{p}{k}-1}(1-t)^{\frac{q}{k}-1}, t \in[0,1], k>0$ along with Lemma 5.2 we get Theorem 5.5. The proof of Lemma 5.2 is available in [11], so details are omitted.

## 6 Inequalities in probability theory and applications for $\boldsymbol{k}$-beta function

Here, we give some applications of the Ostrowski type inequality for the $k$-beta function and cumulative distribution functions. For this purpose, we need some basic concepts of random variable, distribution function, probability density function and expected values. A process which generates raw data is called an experiment and an experiment which gives different results under similar conditions, even though it is repeated a large number of times, is termed a random experiment. A variable whose values are determined by the outcomes of a random experiment is called a random variable or simply a variate. The random variables are usually denoted by capital letters, $X, Y$, and $Z$, while the values associated to them by corresponding small letters $x, y$, and $z$. The random variables are classified into two classes namely discrete and continuous random variables.
A random variable that can assume only a finite or countably infinite number of values is known as a discrete random variable, while a variable which can assume each and every value within some interval is called a continuous random variable. The distribution function of a random variable $X$, denoted by $F(x)$, is defined by $F(x)=\operatorname{Pr}(X \leq x)$ i.e., the distribution function gives the probability of the event that $X$ takes a value less than or equal to a specified value $x$.
A random variable $X$ may also be defined as continuous if its distribution function $F(x)$ is continuous and differentiable everywhere except at isolated points in the given range. Let the derivative of $F(x)$ be denoted by $f(x)$ i.e., $f(x)=\frac{d}{d x} F(x)$. Since $F(x)$ is a non-decreasing function of $x$,

$$
f(x) \geq 0 \quad \text { and } \quad F(x)=\int_{-\infty}^{x} f(x) d x, \quad \text { for all } x
$$

Here, the function $f(x)$ is called the probability density function, denoted by ( $p d f$ ) or simply a density function of the random variable $X$. A probability density function has the properties

$$
f(x) \geq 0, \quad \text { for all } x \text { and } \quad F(x)=\int_{-\infty}^{\infty} f(x) d x=1
$$

and the probability that the random variable $X$ takes on a value in the interval $[a, b]$ is given by

$$
P(a<x \leq b)=F(b)-F(a)=\int_{-\infty}^{b} f(x) d x-\int_{-\infty}^{a} f(x) d x=\int_{a}^{b} f(x) d x,
$$

which shows the area under the curve $y=f(x)$ between $X=a$ and $X=b$.
A moment designates the power to which the deviations are raised before averaging them. In statistics, we have three kinds of moments:
(i) Moments about any value $x=A$ is the $r$ th power of the deviation of variable from $A$ and is called the $r$ th moment of the distribution about $A$.
(ii) Moments about $x=0$ is the $r$ th power of the deviation of variable from 0 and is called the $r$ th moment of the distribution about 0 .
(iii) Moments about mean i.e., $x=\mu$ is the $r$ th power of the deviation of variable from mean and is called the $r$ th moment of the distribution about mean. If a random variable $X$ assumes all the values from $a$ to $b$, then for a continuous distribution, the $r$ th moments about the arbitrary number $A$ and 0 , respectively, are given by $\int_{a}^{b}(x-A)^{r} f(x) d x$ and $\int_{a}^{b}(x-0)^{r} f(x) d x$ (see [26-28]).

Definition 6.1 In a random experiment with $n$ outcomes, suppose a variable $X$ assumes the values $x_{1}, \ldots, x_{n}$ with corresponding probabilities $p_{1}, \ldots, p_{n}$, then the paring $\left(x_{i}, p\left(x_{i}\right)\right)$, $i=1,2, \ldots$, is called a probability distribution and $\Sigma p_{i}=1$ (in the case of discrete distributions). Also, if $f(x)$ is a continuous probability density function defined on an interval [ $a, b]$, then $\int_{a}^{b} f(x) d x=1$. The expected value of the variate is defined as the first moment of the probability distribution about $x=0$ i.e.,

$$
E(X)=\int_{a}^{b} x f(x) d x
$$

Definition 6.2 Let $X$ be a continuous random variable, then it is said to have a beta $k$ distribution of the first kind with two parameters $m$ and $n$, if its probability $k$-density function ( $p k d f$ ) is defined by [8]

$$
f_{k}(x)= \begin{cases}\frac{1}{k \beta_{k}(p, q)} x^{p} k^{p}-1(1-x)^{\frac{q}{k}-1}, & 0 \leq x \leq 1 ; p, q, k>0,  \tag{50}\\ 0, & \text { elsewhere. }\end{cases}
$$

In the above distribution, the $k$-beta variable of the first kind is referred to as $\beta_{1, k}(m, n)$ and its $k$-distribution function $F_{k}(x)$ is given by

$$
F_{k}(x)= \begin{cases}0, & x<0,  \tag{51}\\ \int_{0}^{1} \frac{1}{k \beta_{k}(p, q)} x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}, & 0 \leq x \leq 1 ; p, q, k>0, \\ 0, & x>1 .\end{cases}
$$

Remarks We can call the above function an incomplete $k$-beta function because, if $k=1$, it is an incomplete beta function, as tabulated in [29, 30].

Proposition 6.3 For the parameters $p, q, k>0$, the expected value of the $k$-beta random variable is given by

$$
E(X)=\frac{p}{p+q} .
$$

Proof For the $k$-beta random variable defined above, we observe that

$$
\begin{aligned}
E(X) & =\int_{0}^{x} x f_{k}(x) d x \\
& =\int_{0}^{x} \frac{1}{k \beta_{k}(p, q)} x \cdot x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1} d x, \quad 0 \leqq x \leqq 1 ; p, q>0 .
\end{aligned}
$$

Using (5), (6), and (8), we have

$$
\begin{aligned}
E(X) & =\int_{0}^{1} \frac{1}{k \beta_{k}(p, q)} x^{\frac{p}{k}}(1-x)^{\frac{q}{k}-1} d x=\frac{\beta_{k}(p+k, q)}{\beta_{k}(p, q)} \\
& =\frac{\Gamma_{k}(p+k) \Gamma_{k}(q) \Gamma_{k}(p+q)}{\Gamma_{k}(p) \Gamma_{k}(q) \Gamma_{k}(p+q+k)}=\frac{p}{p+q} .
\end{aligned}
$$

Lemma 6.4 Let $X$ be a random variable taking values in the finite interval $[a, b]$, with the cumulative distribution function $F(x)=\operatorname{Pr}(X \leq x)$. Then the following inequalities of Ostrowski type hold:

$$
\begin{aligned}
& \left|\operatorname{Pr}(X \leq x)-\frac{b-E(X)}{b-a}\right| \\
& \quad \leq \frac{1}{b-a}\left[[2 x-(a+b)] \operatorname{Pr}(X \leq x)+\int_{a}^{b} \operatorname{sgn}(t-x) F(t) d t\right] \\
& \quad \leq \frac{1}{b-a}[(b-x) \operatorname{Pr}(X \geq x)+(x-a) \operatorname{Pr}(X \leq x)] \\
& \quad \leq \frac{1}{2}+\frac{\left|x-\frac{(a+b)}{2}\right|}{b-a}
\end{aligned}
$$

for all $x \in[a, b]$. All the inequalities are sharp and the constant $\frac{1}{2}$ is the best possible. However, by the integration by parts formula for the Riemann-Stieltjes integral, we have

$$
\begin{aligned}
E(X) & =\int_{a}^{b} t d F(t)=\left.t F(t)\right|_{a} ^{b}-\int_{a}^{b} F(t) d t \\
& =b F(a)-a F(b)-\int_{a}^{b} F(t)=b-\int_{a}^{b} F(t)
\end{aligned}
$$

and

$$
\begin{equation*}
1-F(x)=\operatorname{Pr}(X \geq x) \tag{52}
\end{equation*}
$$

The proof of Lemma 6.4 is given in $[19,31]$. Now, we are able to give some applications for the $k$-beta random variable.

Theorem 6.5 Let $X$ be a $k$-beta random variable with parameters $p, q, k>0$, then we have the following inequalities:

$$
\left|\operatorname{Pr}(X \leq x)-\frac{q}{p+q}\right| \leq \frac{1}{2}+\left|x-\frac{1}{2}\right|
$$

and

$$
\left|\operatorname{Pr}(X \geq x)-\frac{p}{p+q}\right| \leq \frac{1}{2}+\left|x-\frac{1}{2}\right|
$$

for all $x \in[0,1]$ and, in particular,

$$
\left|\operatorname{Pr}\left(X \leq \frac{1}{2}\right)-\frac{q}{p+q}\right| \leq \frac{1}{2}
$$

and

$$
\left|\operatorname{Pr}\left(X \geq \frac{1}{2}\right)-\frac{p}{p+q}\right| \leq \frac{1}{2}
$$

Proof Using Lemma 6.4 along with the $k$-beta random variable and ( $p k d f$ ) defined in (50) and (51) and Proposition 6.3 for the expected values, we get

$$
\left|\operatorname{Pr}(X \leq x)-\frac{q}{p+q}\right| \leq \frac{1}{2}+\left|x-\frac{1}{2}\right|
$$

and, by (52), we have

$$
\left|\operatorname{Pr}(X \geq x)-\frac{p}{p+q}\right| \leq \frac{1}{2}+\left|x-\frac{1}{2}\right|
$$

for all $x \in[0,1]$. In particular, for the intermediate point of the interval $[0,1]$, i.e., at $x=\frac{1}{2}$, we have the remaining results of Theorem 6.5.

Lemma 6.6 Let $X$ be a random variable with probability density function $f:[a, b] \subset \mathbb{R} \rightarrow$ $\mathbb{R}_{+}$and with cumulative distribution function $F(x)=\operatorname{Pr}(X \leq x)$. Iff $\in L_{p}[a, b], p>1$, then the following inequality holds:

$$
\begin{aligned}
& \left|\operatorname{Pr}(X \leq x)-\frac{b-E(X)}{b-a}\right| \\
& \quad \leq \frac{s}{s+1}\|f\|_{r}(b-a)^{\frac{1}{s}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{1+s}{s}}+\left(\frac{b-x}{b-a}\right)^{\frac{1+s}{s}}\right] \\
& \quad \leq \frac{s}{s+1}\|f\|_{r}(b-a)^{\frac{1}{s}} \\
& \quad \leq \frac{1}{2}+\frac{\left|x-\frac{(a+b)}{2}\right|}{b-a}
\end{aligned}
$$

for all $x \in[a, b]$, where $\frac{1}{r}+\frac{1}{s}=1$.

Now, we have the application of the beta random variable $X$ in terms of the parameter $k>0$. A k-beta random variable $X$ with positive parameters $p, q$, and $k$ has the probability density function

$$
f_{k}(x ; p, q)=\frac{x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}}{k \beta_{k}(p, q)}, \quad 0 \leq x \leq 1,
$$

where $\beta_{k}(p, q)$ is the $k$-beta function. Here, we observe that

$$
\begin{aligned}
\left\|f_{k}(x ; p, q)\right\|_{r} & =\frac{1}{\beta_{k}(p, q)}\left(\frac{1}{k} \int_{0}^{1} x^{\frac{r p}{k}-r}(1-x)^{\frac{r q}{k}-r} d x\right)^{\frac{1}{r}} \\
& =\frac{1}{\beta_{k}(p, q)}\left(\frac{1}{k} \int_{0}^{1} x^{\frac{r p}{k}-r+1-1}(1-x)^{\frac{r q}{k}-r+1-1} d x\right)^{\frac{1}{r}} .
\end{aligned}
$$

Thus, we have

$$
\begin{equation*}
\left\|f_{k}(x ; p, q)\right\|_{r}=\frac{1}{\beta_{k}(p, q)}\left[\beta_{k}(r(p-k)+k, r(q-k)+k)\right]^{\frac{1}{r}}, \tag{53}
\end{equation*}
$$

provided

$$
(r(p-k)+k, r(q-k)+k)>0, \quad \text { i.e., } p>k-\frac{k}{r} \text { and } q>k-\frac{k}{r} .
$$

Theorem 6.7 Let $X$ be a $k$-beta random variable with parameters $p, q, k>0, p>k-\frac{k}{r}$ and $q>k-\frac{k}{r}$. Then we have the following inequalities:

$$
\left|\operatorname{Pr}(X \leq x)-\frac{q}{p+q}\right| \leq \frac{s}{s+k} \frac{\left[x^{\frac{s+k}{s}}+(1-x)^{\frac{s+k}{s}}\right]\left[\beta_{k}(r(p-k)+k, r(q-k)+k)\right]^{\frac{1}{r}}}{\beta_{k}(p, q)}
$$

for all $x \in[0,1]$ and, in particular

$$
\left|\operatorname{Pr}\left(X \leq \frac{1}{2}\right)-\frac{q}{p+q}\right| \leq \frac{s}{2^{k / s}(s+k)} \frac{\left[\beta_{k}(r(p-k)+k, r(q-k)+k)\right]^{\frac{1}{r}}}{\beta_{k}(p, q)} .
$$

Proof Using Lemma 6.6 along with the $k$-beta random variable and ( $p k d f$ ) defined in (50), (51), and (53) for the expected values, we get the required Theorem 6.7.

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

The authors AR and SM contributed and approved equally to the writing of this paper

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