FRACTIONAL CALCULUS OPERATOR AND ITS APPLICATIONS TO CERTAIN CLASSES OF ANALYTIC FUNCTIONS

A Study On Fractional Derivative Operator In Analytic And Multivalent Functions

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Abstract

The main object of this thesis is to obtain numerous applications of fractional derivative operator concerning analytic and p-valent (or multivalent) functions in the open unit disk by introducing new classes and deriving new properties. Our finding will provide interesting new results and indicate extensions of a number of known results. In this thesis we investigate a wide class of problems. First, by making use of certain fractional derivative operator, we define various new classes of p-valent functions with negative coefficients in the open unit disk such as classes of p-valent starlike functions involving results of (Owa, 1985a), classes of p-valent starlike and convex functions involving the Hadamard product (or convolution) and classes of k-uniformly p-valent starlike and convex functions, in obtaining, coefficient estimates, distortion properties, extreme points, closure theorems, modified Hadmard products and inclusion properties. Also, we obtain radii of convexity, starlikeness and close-to-convexity for functions belonging to those classes. Moreover, we derive several new sufficient conditions for starlikeness and convexity of the fractional derivative operator by using certain results of (Owa, 1985a), convolution, Jack's lemma and Nunokakawa' Lemma. In addition, we obtain coefficient bounds for the functional $\left|a_{p+2} - \theta a_{p+1}^2\right|$ of functions belonging to certain classes of p-valent functions of complex order which generalized the concepts of starlike, Bazilevič and non-Bazilevič functions. We use the method of differential subordination and superordination for analytic functions in the open unit disk in order to derive various new subordination, superordination and sandwich results involving the fractional derivative operator. Finally, we obtain some new strong differential subordination, superordination, sandwich results for p-valent functions associated with the fractional derivative operator by investigating appropriate classes of admissible functions. First order linear strong differential subordination properties are studied. Further results including strong differential subordination and superordination based on the fact that the coefficients of the functions associated with the fractional derivative operator are not constants but complex-valued functions are also studied.

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List of symbols

$\mathcal{A}$	Class of normalized analytic functions in the open unit disk $\mathcal{U}$
$\mathcal{A}(p)$	Class of normalized p-valent functions in $\mathcal{U}$
$\mathcal{A}_\zeta^*(p)$	Class of normalized p-valent functions in $\mathcal{U}\times\overline{\mathcal{U}}$
$\mathcal{B}(\alpha)$	Class of Bazilevič functions of type α
$\mathbb{C}$	Complex plane
$C(\alpha)$	Class of univalent convex functions of order α with negative coefficients
$C(p,\alpha)$	Class of p-valent convex functions of order α with negative coefficients
$C(b)$	Class of convex functions of complex order b
$\mathcal{C}$	Class of close-to-convex functions
$\mathcal{C}(\alpha)$	Class of close-to-convex functions of order α
$\mathcal{C}(p,\alpha)$	Class of p-valent close-to-convex functions of order α
D	Domain
$D_{0,z}^{\lambda}$	Fractional derivative operator of order λ
${}_2F_1(a,b;c;z)$	Gauss hypergeometric function
$\mathcal{H}=\mathcal{H}(\mathcal{U})$	Class of analytic functions in the unit disk $\mathcal{U}$
$\mathcal{H}(\mathcal{U}\times\overline{\mathcal{U}})$	Class of analytic functions in $\mathcal{U}\times\overline{\mathcal{U}}$
$\mathcal{H}[a,n]$	Class of analytic function in $\mathcal{U}$ of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots,(z\in\mathcal{U})$
$\mathcal{H}^*[a,n,\zeta]$	Class of analytic function in $\mathcal{U}\times\overline{\mathcal{U}}$ of the form $f(z,\zeta)=a+a_n(\zeta)z^n+a_{n+1}(\zeta)z^{n+1}+\cdots,(z\in\mathcal{U},\zeta\in\overline{\mathcal{U}})$

Im	Imaginary part of a complex number
$J_{0,z}^{\lambda,\mu,\eta}$	Generalized fractional derivative operator
K	Class of convex functions
$K(\alpha)$	Class of convex functions of order α
$K(z)$	Koebe function
$k-UCV(\alpha)$	Class of k-uniformly convex functions of order α
$k-UST(\alpha)$	Class of k-uniformly starlike functions of order α
$K(p)$	Class of p-valent convex functions
$K(p,\alpha)$	Class of p-valent convex functions of order α
K_ζ^*	Class of convex functions in $\mathcal{U}\times\overline{\mathcal{U}}$
$M_{0,z}^{\lambda,\mu,\eta}$	Modification of the fractional derivative operator
$\mathbb{N}$	Set of all positive integers
$\mathcal{N}(\alpha)$	Class of non-Bazilevič functions
$\mathcal{P}$	Class of functions with positive real part
$\mathcal{P}(A,B)$	Class of Janowski functions
$\mathbb{R}$	Set of all real numbers
Re	Real part of a complex number
$\mathcal{R}_\alpha$	Class of prestarlike functions of order α
$\mathcal{R}(\gamma,\alpha)$	Class of γ –prestarlike of order α
$\mathcal{R}_\gamma(\alpha,\beta)$	Class of γ –prestarlike functions of order α and type β
S	Class of normalized univalent functions
S_p	Class of uniformly starlike functions
S^*	Class of starlike functions
$S^*(\alpha)$	Class of starlike functions of order α

$S^*(b)$	Class of starlike functions of complex order b
$S^*(p)$	Class of p-valent starlike functions
$S^*(p,\alpha)$	Class of p-valent starlike functions of order α
S_ζ^*	Class of starlike functions in $\mathcal{U}\times\overline{\mathcal{U}}$
T	Class of univalent functions with negative coefficients
$T(p)$	Class of p-valent functions with negative coefficients
$T^*(\alpha)$	Class of univalent starlike functions of order α with negative coefficients
$T^*(p,\alpha)$	Class of p-valent starlike functions of order α with negative coefficients
UCV	Class of uniformly convex functions
$\mathcal{U}$	Open unit disk $\{z \in \mathbb{C}: \lvert z\rvert < 1\}$
$\overline{\mathcal{U}}$	Closed unit disk $\{z \in \mathbb{C}: \lvert z\rvert \leq 1\}$
$\prec$	Subordinate to
$\prec\prec$	Strong subordinate to
$f * g$	Hadamard product (or convolution) of f and g
$(\lambda)_n$	Pochhammer symbol
$\Psi_n[\Omega, q]$	Class of admissible functions

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Chapter 1

Introduction

The purpose of this chapter is to give introduction to primitive backgrounds and motivations for the remaining chapters. In section 1.1, we present the review of literature. In section 1.2, we state the basic notations and definitions of univalent and p-valent (or multivalent) functions in the open unit disk, and their related classes. The Hadamard products (or Convolutions) for analytic functions are also presented. Section 1.3 gives subordinate principle. In section 1.4, we study the class of functions with positive real part. In section 1.5, we consider some special classes, including, starlike, convex, close-to-convex, prestarlike, starlike of complex order, convex of complex order, uniformly starlike, uniformly convex, Bazilevic and non- Bazilevic functions. Section 1.6 presents some definitions of fractional derivative operators. Section 1.7 is devoted to the study of differential subordination and its corresponding problem, that is differential superordination. The notation of the strong differential subordination and strong differential superordination are given in section 1.8. The motivations and outlines of this study are given in section 1.9.

The thesis is organized with solutions to a number of problems. For example, we consider the following problems:

- To identity some classes of p-valent functions with negative coefficients associated with certain fractional derivative operator in the open unit disk $\mathcal{U}$ and find coefficient estimates, distortion properties, extreme points, closure theorems, modified Hadmard products and inclusion properties. Also, to obtain radii of starlikeness, and convexity and close-to-convexity for functions belonging to those classes.

- To find sufficient conditions for p-valent functions defined by certain fractional derivative operator to be starlike and convex by using some known results such as results of (Owa, 1985a), results involving the Hadamard product due to (Rusheweyh and Sheil-Small, 1973), Jack's Lemma (Jack, 1971) and Nunokakawa's Lemma (Nunokakawa, 1992).
- To define some classes of p-valent functions involving certain fractional derivative operator, and obtain bounds for the functional $|a_{p+2} - \theta a_{p+1}^2|$ and bounds for the coefficient a_{p+3} for functions belonging to those classes.
- By using the differential subordination and superordination techniques, to find the sufficient conditions for p-valent functions $f(z) \in \mathcal{A}(p)$ associated with a fractional derivative operator to satisfy $q_1(z) \prec p(z) \prec q_2(z)$ where $p(z)$ is analytic function in $\mathcal{U}$ and the functions q_1 and q_2 are given univalent in $\mathcal{U}$ with $q_1(0) = q_2(0) = 1$, so that, they become respectively, the best subordinant and best dominant.
- By using the notion of strong differential subordination and superordination techniques, to investigate appropriate classes of admissible functions involving fractional derivative operator and to obtain some strong differential subordination, superordination and

sandwich-type results. Also, to find the sufficient conditions for p-valent functions $f(z,\zeta) \in \mathcal{A}_{\zeta}^{*}(p)$ associated with a fractional derivative operator to satisfy, respectively $p(z,\zeta) \prec\prec q(z,\zeta)$ and $q(z,\zeta) \prec\prec p(z,\zeta)$ for $z \in \mathcal{U}$ and $\zeta \in \overline{\mathcal{U}}$ where $p(z,\zeta)$ is analytic function in $\mathcal{U} \times \overline{\mathcal{U}}$.

1.1 Review of literature

This section deals with the conceptual framework of the present research problem and primary matters regarding the research. A survey of related studies provides some insight regarding strong points and limitation of the previous studies

The studies reviewed focus on how interest introduce new classes of analytic and p-valent (or multivalent) functions and investigate their properties. Also, what effect of fractional derivative operator on functions belonging to these classes. The review of related literature studied by the researcher is divided in the following categories:

- Univalent and multivalent functions
- Fractional calculus operators
- Functions with negative coefficients and related classes
- Starlikeness and convexity conditions
- Coefficient bounds
- Differential subordination and superordination
- Strong Differential subordination and superordination
- conclusions

The studies have been analyzed by keeping objectives, methodology and findings of the study to draw the conclusion to strengthen the rationale of the present research.

1.1.1 Univalent and multivalent functions

The theory of univalent functions is a classical problem of complex analysis which belongs to one of the most beautiful subjects in geometric function theory. It deals with the geometric properties of analytic functions, found around the turn of the 20th century. In spite of the famous coefficient problem, the Biberbach conjecture which was solved by (Branges, 1985). The geometry theory of functions is mostly concerned with the study of properties of normalized univalent functions which belong to the class S and defined in the open unit disk $\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$. The image domain of $\mathcal{U}$ under univalent function is of interest if it has some nice geometry properties. A convex domain is outstanding example of a domain with nice properties. Another example such domain is starlike with respect to a point. Certain subclasses of those analytic univalent functions which map $\mathcal{U}$ onto these geometric domains, are introduced and their properties are widely investigated, for example, the classes K and S^* of convex and starlike functions, respectively, see (Goodman, 1983), (Duren, 1983). It was observed that both of these classes are related with each other through classical Alexander type relation $f(z) \in K \Leftrightarrow zf'(z) \in S^*$, see (Alexander, 1915) and (Goodman, 1983). The special subclasses of the classes K and S^* are the classes $K(\alpha)$ and $S^*(\alpha)$ of convex and starlike functions of order $\alpha,\ (0 \le \alpha < 1)$. If $\alpha = 0$, we obtain $K(0) = K$ and $S^*(0) = S^*$. These classes were first introduced by (Robertson, 1936) and were studied subsequently by (Schild, 1965), (Pinchuk, 1968), (Jack, 1971), and others. Moreover, the classes of convex and starlike functions are closely related with the class $\mathcal{P}$ of analytic functions with positive real part $p(z)$ which satisfies $p(0) = 1$ and $\mathrm{Re}\, p(z) > 0$, see (Pommerenke, 1975).

The natural generalization of univalent function is p-valent (or multivalent) function which belong to the class $\mathcal{A}(p)$, $(p \in \mathbb{N})$ and defined in the open unit disk $\mathcal{U}$. If $f(z)$ is p-valent function with $p = 1$, then $f(z)$ is univalent function. In addition, the classes K and S^* of convex and starlike functions were extended to the classes $K(p)$ and $S^*(p)$ of p-valent convex and starlike functions, respectively, by (Goodman, 1950). The special subclasses of the classes $K(p)$ and $S^*(p)$ are the classes $K(p, \alpha)$ and $S^*(p, \alpha)$ of p-valent convex and starlike functions of order α, $(0 \leq \alpha < p)$. If $\alpha = 0$, we obtain $K(p, 0) = K(p)$ and $S^*(p, 0) = S^*(p)$. The class $K(p, \alpha)$ was introduced by (Owa, 1985a) and the class $S^*(p, \alpha)$ was introduced by (Patil and Thakare, 1983).

1.1.2 Fractional calculus operators

The theory of fractional calculus (that is, derivatives and integrals of arbitrary real or complex order) has found interesting applications in the theory of analytic functions in recent years. The classical definitions of fractional derivative operators have been applied in introducing various classes of univalent and p-valent functions and obtaining several properties such as coefficient estimates, distortion theorems, extreme points, and radii of convexity and starlikeness. For numerous works on this subject, one may refer to the works by, (Altintas et al. 1995a), (Altintas et al. 1995b), (Khairnar and More, 2009), (Owa, 1978), (Owa and Shen, 1998), (Raina and Bolia, 1998), (Raina and Choi, 2002), (Raina and Nahar, 2002), (Raina and Srivastava, 1996), (Srivastava and Aouf, 1992), (Srivastava and Aouf, 1995), (Srivastava and Mishra, 2000), (Srivastava et al.,1988), (Srivastava and Owa, 1984), (Srivastava and Owa, 1987),(Srivastava and Owa, 1989), (Srivastava and Owa, 1991b), (Srivastava and Owa, 1992) and (Srivastava et al., 1998). Moreover, the fractional derivative operators were applied to obtain the sufficient conditions for starlikeness and convexity of univalent functions defined in the open unit disk by (Owa, 1985b), (Raina and Nahar, 2000) and (Irmak et al., 2002).

1.1.3 Functions with negative coefficients and related classes

In this subsection we present various classes of analytic univalent and p-valent functions with negative coefficients in the open unit disk. These functions are convex, starlike, prestarlike, uniformly convex and uniformly starlike which were introduced and their properties such as coefficient estimates, distortion theorems, extreme points, and radii of convexity and starlikeness were investigated by several authors. The problem of coefficient estimates is one of interesting problems which was studied by researchers for certain classes of starlike and convex (p-valent starlike and p-valent convex) functions with negative coefficient in the open unit disk. Closely related to this problem is to determine how large the modulus of a univalent or p-valent function together with its derivatives can be in particular subclass. Such results, referred to as distortion theorems which provide important information about the geometry of functions in that subclass. The result which is as inequality is called sharp (best possible or exact) in sense, that it is impossible to improve the inequality (decrease an upper bound, or increase a lower bound) under the conditions given and it can be seen by considering a function such that equality holds. This function is called extermal function. A function belong to the class of functions is called an extreme point if it cannot be written as a proper convex combination of two other members of this class. The radius of convexity (stalikeness) problem for the class of functions is to determine the largest disk $|z| < r$, i.e. the largest number of r $(0 < r \leq 1)$ such that each function $f(z)$ in the class is convex (starlike) in $|z| < r$. One may refer to the books by (Nehari, 1952), (Goodman, 1983) and (Duren, 1983). Those problems have attracted many mathematicians involved in geometry function theory, for example, (Silverman, 1975) introduced and studied the classes $T^*(\alpha)$ and $C(\alpha)$ of starlike and convex functions with negative coefficients of order α $(0 \leq \alpha < 1)$. These classes were generalized to the classes $T^*(p, \alpha)$ and $C(p, \alpha)$ of p-valent starlike and convex functions with negative coefficients of order α $(0 \leq \alpha < p)$, by (Owa, 1985a). (Srivastava and Owa, 1987) established some distortion theorems for fractional calculus operators of functions belonging to the classes which were introduced by (Owa, 1985a).

In order to derive the similar properties above, two subclasses $T^*(\alpha, \beta, \gamma)$ and $C(\alpha, \beta, \gamma)$ of univalent starlike functions with negative coefficients were introduced by (Srivastava and Owa, 1991a). In fact, these classes become the subclasses of the class which was introduced by (Gupta, 1984) when the function is univalent with negative coefficients. Using the results of

(Srivastava and Owa, 1991a), (Srivastava and Owa, 1991b) have obtained several distortion theorems involving fractional derivatives and fractional integrals of functions belonging to the these classes. Recently, (Aouf and Hossen, 2006) have generalized the classes of univalent starlike functions with negative coefficients due to (Srivastava and Owa, 1991a) to obtain coefficient estimates, distortion theorem and radius of convexity for certain classes $T^*(p,\alpha,\beta,\gamma)$ and $C(p,\alpha,\beta,\gamma)$ of p-valent starlike functions with negative coefficients.

Moreover, (Aouf ,1988) studied certain classes $T^*(p,\alpha,\beta)$ and $C(p,\alpha,\beta)$ of p-valent functions of order α and type β which are an extension of the familiar classes which were studied earlier by (Gupta and Jain, 1976). More recently, (Aouf and Silverman, 2007) introduced and studied some subclasses of p-valent γ-prestarlike functions of order α. Subsequently, (Aouf, 2007) introduced and studied the classes $R_\gamma^p[\alpha,\beta]$ and $C_\gamma^p[\alpha,\beta]$ of p-valent γ-prestarlike functions of order α and type β. There are many contributions on prestarlike function classes, for example (Ahuja and Silverman, 1983), (Owa and Uralegaddi, 1984), (Silverman and Silvia, 1984) and (Srivastava and Aouf, 1995)

In addition, many authors have turned attention to the so-called classes of uniformly convex (starlike) functions for various subclasses of univalent functions. Those classes were first introduced and studird by (Goodman,1991a) and (Goodman,1991b), and were studied subsequently by (Rønning 1991), (Rønning 1993a), (Minda and Ma, 1992), (Rønning 1993b), (Minda and Ma, 1993) and others. The classes of k-uniformly convex (starlike) functions were studied by (Kanas and Wisniowska, 1999) and

(Kanas and Wisniowska, 2000); where their geometric definitions and connections with the conic domains were considered. Encouraged by wide study of classes of univalent functions with negative coefficients, (Al-Kharsani and Al-Hajiry, 2006) introduced the classes of uniformly p-valent starlike and uniformly p-valent convex functions of order α. More recently, (Gurugusundaramoorthy and Themangani, 2009), presented a study for class of uniformly convex functions based on certain fractional derivative operator to obtain the similar properties above. There are many other researchers who studied the classes of uniformly starlike and uniformly convex functions including (AL-Refai and Darus, 2009), (Khairnar and More, 2009), (Sokôł and Wisniowska, 2011) and (Srivastava and Mishra, 2000).

1.1.4 Starlikeness and convexity conditions

There is a beautiful and simple sufficient condition for univalence due independently to (Noshiro, 1934-1935) and (Warschawski, 1935), and then onwards the result is known as Noshiro-Warschawski Theorem. This says, if a function $f(z)$ is analytic in a convex domain D and $\mathrm{Re}\, f'(z) > 0$, then $f(z)$ is univalent in D, see also (Duren, 1983) and (Goodman, 1983). The problem of sufficient conditions for starlikeness and convexity is concerning to find conditions under which function in certain class are starlike and convex, respectively. For example, (Owa and Shen, 1998) and (Raina and Nahar, 2000) introduced various sufficient conditions for starlikeness and convexity of class of univalent functions associated with certain fractional derivative operators by using known results for the classes of starlike and convex function due to (Silverman,1975) and by using results involving the

Hadamard product (or convolution) due to (Ruscheweyh and Sheil-Small, 1973).

In addition, two results of (Jack, 1971) and (Nunokawa, 1992) which popularly known as jack's Lemma and Nunokawa's Lemma in literature have applied to obtain many of sufficient conditions for starlikeness and convexity for analytic functions, see (Irmak and Cetin, 1999), (Irmak et al., 2002) and (Irmak and Piejko, 2005).

1.1.5 Coefficient bounds

The problem of estimating the functional $|a_3 - \mu a_2^2|$ where μ is real parameter for the class of univalent functions is intimately related with the coefficient problem which called Fekete and Szegö problem, see (Keogh and Merkes, 1969). The result is sharp in the sense that for each μ there is a function in the class under consideration for which the equality holds. Thus an attention to the so-called coefficient estimate problems for different subclasses of univalent and p-valent functions has been the main interest among authors. (Ma and Minda, 1994) discussed the similar coefficient problem for functions in the classes $C(\phi)$ and $S^*(\phi)$. There are now several results for this type in literature, each of them dealing with $|a_3 - \mu a_2^2|$ for various classes of functions. (Srivastava and Mishra, 2000) obtained Fekete-Szegö problem to parabolic starlike and uniformly convex functions defined by fractional calculus operator. Many of other researchers who successfully to obtain Fekete-Szegö problem for various classes of univalent and p-valent functions such as (Dixit and Pal, 1995), (Obradovič, 1998), (Ramachandran et al., 2007), (Ravichandran et al., 2004), (Ravichandran et al., 2005), (Rosy et al., 2009), (Tuneski and Darus, 2002), (Wang et al., 2005), and (Shanmugam et al., 2006a). On other hand, (Prokhorov and Szynal, 1981) obtained the estimate of the functional $|c_3 + \mu c_1 c_2 + v c_1^3|$ $(\mu, v \in \mathbb{R})$ within the class Ω of all analytic functions of the form $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ in the open unit disk and satisfying the condition $|w(z)| < 1$ $(z \in \mathcal{U})$. Very recently, (Ali et al., 2007) obtained the sharp coefficient inequalities for $\left|a_{p+2} - \mu a_{p+1}^2\right|$ and $\left|a_{p+3}\right|$ for various classes of p-valent analytic functions by using the results of (Ma and Minda, 1994) and (Prokhorov and Szynal, 1981).

1.1.6 Differential subordination and superordination

The study of differential subordinations, which is the generalization from the differential inequalities, began with the papers according to (Miller and Mocanu, 1981) and (Miller and Mocanu, 1985). In very simple terms, a differential subordination in the complex plane is the generalization of a differential inequality on the real line. Obtaining information about properties of a function from properties of its derivatives plays an important role in functions of real variable, for example, if $f'(x) > 0$, then f is an increasing function. Also, to characterizing the original function, a differential inequality can be used to find information about the range of the original function, a typical example is given by, if $f(0) = 1$ and $f'(x) + f(x) \leq 1$, then $f(x) \leq 1$.

In the theory of complex-valued functions there are several differential implications in which a characterization of a function is determined from a differential condition, for example, the Noshiro-Warschawski Theorem: if f is analytic in the unit disk $\mathcal{U}$, then $\operatorname{Re} f'(z) > 0$ implies f is univalent function in

$\mathcal{U}$, see (Noshiro, 1934-1935), (Warschawski, 1935), (Goodman, 1983) and (Duren, 1983). In addition, to obtain properties of the range of a function from the range of a combination of the derivatives of a the function, a typical example is given by, if α is real and $p(z)$ is analytic function in $\mathcal{U}$, then $\mathrm{Re}\left[p(z) + \alpha z\, p'(z)/p(z)\right] > 0$ implies $\mathrm{Re}\, p(z) > 0$, see (Miller and Mocanu, 2000).

The dual problem of differential subordination, that is differential superordination was introduced by (Miller and Mocanu, 2003) and studied by (Bulboaca, 2002a) and (Bulboaca, 2002b). The methods of differential subordination were used by (Ali et al., 2005), (Shanmugam et al., 2006b) for various classes of analytic functions.

1.1.7 Strong differential subordination and superordination

Some recent results in the theory of analytic functions were obtained by using a more strong form of the differential subordination and superordination introduced by (Antonino and Romaguera, 1994) and studied by (Antonino and Romaguera, 2006) called strong differential subordination and strong differential superordination, respectively. By using this notion, (G. Oros, 2007) and (G. Oros, 2009) introduced the dual notion of strong differential superordination following the theory of differential superordination introduced and developed by (Miller and Mocaun,1981) and (Miller and Mocaun,1985). Since then, many of interesting results have appeared in literature on this topic such as (G. Oros and Oros, 2007), (G. Oros and Oros, 2009), (Oros, 2010), (G. Oros, 2010) and (G. Oros, 2011).

1.1.8 conclusions

This research work provides the insight to have a concept regarding fractional derivative operators and analytic functions. Thus a perusal and scrutiny of the literature that though many studies on fractional derivative operators have been done for analytic functions with negative coefficients. Additional research is needed to introduce and study some classes of p-valent functions with negative coefficients based on certain fractional derivative operator which generalize the previous classes and investigate their properties. Sufficient conditions for stalikeness and convexity of fractional derivative operators and coefficient bounds of functions involving the fractional derivative operators are not up to the desired level. This is another area that will require additional research. The review of differential subordination and superordination, and strong differential subordination and superordination of analytic functions defined in the open unit disk on complex plane reveals the need for investigating properties associated with fractional derivative operator for p-valent functions. Thus it reveals the importance and need of the present study.

1.2 Univalent and multivalent functions

In this section we give the definitions of univalent and multivalent functions and their related classes S and $\mathcal{A}(p)$ in the unit disk $\mathcal{U}$. We also mention to the Hadamard product (or convolution) of any two functions in these classes. The classes T and $T(p)$ of analytic functions with negative coefficients are also defined.

A complex-valued function $f(z)$ of a complex-variable is differentiable at $z_0 \in \mathbb{C}$ ($\mathbb{C}$ is a complex plane), if it has a derivative (Duren, 1983)

$$f^{'}(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

at z_0. Such a function $f(z)$ is called analytic at z_0 If it is differentiable at every point in some neighbourhood of z_0. A function $f(z)$ defined on a domain D is called analytic in D if it has a derivative at each point of D.

A function $f\ (z)$ analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$ is said to be univalent in $\mathcal{U}$, if $w = f\ (z)$ assumes distinct values w for distinct z in $\mathcal{U}$. In this case the equation $f\ (z) = w$ has at most one root in $\mathcal{U}$. A function on D is called univalent if it provides one-to-one (injective) mapping onto its image. Various other terms are used for this concept such as simple, or schlicht (the German word for "simple"), see (Goodman, 1983).

The selection of open unit disk $\mathcal{U}$ above instead of an arbitrary domain D has the advantage of simplifying the computations and leading to short and elegant formulas.

We begin with the class $\mathcal{H}(\mathcal{U})$ of all analytic functions in $\mathcal{U}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(\mathcal{U})$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \tag{1.2.1}$$

with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$.

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}(\mathcal{U})$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n\ , \qquad (z \in \mathcal{U}). \tag{1.2.2}$$

which are analytic in $\mathcal{U}$ and normalized by $f(0)=0$ and $f'(0)=1$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by S. The well-known example in class S is the Koebe function, $K(z)$, defined by

$$K(z)=\frac{z}{(1-z)^2}=z+\sum_{n=2}^{\infty} nz^n, \qquad (z\in\mathcal{U}).$$

which is an extremal function for many subclasses of the class of univalent functions. It maps $\mathcal{U}$ one-to-one onto the domain D that consists of the entire complex plane except for a slit along the negative real axis from $w=-\infty$ to $w=-\frac{1}{4}$, see (Duren, 1983), (Goodman, 1983), (Pommerenke, 1975) and (Graham and Kohr, 2003).

A function $f(z)$ analytic in the open unit disk $\mathcal{U}$ is said to p-valent in $\mathcal{U}$, (or multivalent of order p) $(p=1,2,\dots)$ in $\mathcal{U}$ if the equation $f(z)=w$ has never more than p-solutions in $\mathcal{U}$ and there exists some w for which this equation has exactly p solutions. If $f(z)$ is p-valent with $p=1$, then $f(z)$ is univalent, see (Goodman, 1983) and (Hayman, 1958).

Let $\mathcal{A}(p)$ denote the subclass of $\mathcal{H}(\mathcal{U})$ consisting of all functions of the form

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n}z^{p+n}, \qquad (p\in\mathbb{N};\ z\in\mathcal{U}). \tag{1.2.3}$$

which are analytic and p-valent in the unit disk $\mathcal{U}$.

For functions $f(z)\in\mathcal{A}$ given by (1.2.2) and $g(z)\in\mathcal{A}$ given by

$$g(z)=z+\sum_{n=2}^{\infty} b_n z^n \qquad (z\in\mathcal{U}).$$

the Hadamard product (or convolution) of f and g is denoted by $(f*g)(z)$ and defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad (z \in \mathcal{U}).$$

For functions $f(z) \in \mathcal{A}(p)$ given by (1.2.3) and $g(z) \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \qquad (p \in \mathbb{N};\ z \in \mathcal{U}).$$

the Hadamard product (or convolution) of f and g is denoted by $(f * g)(z)$ and defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}, \quad (p \in \mathbb{N};\ z \in \mathcal{U}).$$

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0;\ z \in \mathcal{U}). \tag{1.2.4}$$

The class T is called the class of univalent functions with negative coefficients. Also, let $T(p)$ denote the subclass of $\mathcal{A}(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \qquad \left(a_{p+n} \geq 0;\ p \in \mathbb{N}\right). \tag{1.2.5}$$

The class $T(p)$ is called the class of p-valent functions with negative coefficients.

1.3 Subordinate principle

In this section we present the concept of subordination between analytic functions which was developed by (Littlewood, 1925, 1944) and (Rogosinski, 1939, 1943). Here, we start with the following classical result, which is known by the name of Schwarz's Lemma (Graham and Gabrela, 2003) as follows:

Let $w(z)$ be analytic function in $\mathcal{U}$ and let $w(0) = 0$. If $|w(z)| \leq 1$ ($z \in \mathcal{U}$) then $|w(z)| \leq |z|$ ($z \in \mathcal{U}$). The equality can hold only if $w(z) \equiv kz$ and $|k| = 1$. We denote by Ω the class of Schwarz functions; i.e. $w \in \Omega$ if and only if w is analytic function in $\mathcal{U}$ such that $w(0) = 0$ and $|w(z)| < 1$.
The formulation of Schwarz's Lemma seems to assign a special role to the origin of the two planes.

The subordinate principle says: Let the functions $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. The function f is said to be subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g\big(w(z)\big)$, $z \in \mathcal{U}$. We note that

$$f(z) \prec g(z) \Longrightarrow f(0) = g(0), \qquad f(\mathcal{U}) \subset g(\mathcal{U}).$$

Furthermore, if the function g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$ (Duren, 1983) and (Pommerenke, 1975).

1.4 Functions with positive real part

In this section we define class $\mathcal{P}$ of analytic functions with positive real part. These functions map the open unit disk $\mathcal{U}$ onto right half plane. Many problems are solved by using the properties of these functions. Some related classes are introduced and their basic properties are given in this section. These properties will be very useful in our later investigations.

Let $\mathcal{P}$ denotes the class of all functions $p(z) \in \mathcal{H}$, $p(0) = 1$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \qquad (z \in \mathcal{U}).$$

which satisfy the following inequality

$$\operatorname{Re} p(z) > 0 , \qquad (z \in \mathcal{U}).$$

The functions in the class $\mathcal{P}$ need not to be univalent. For example, the function

$$p(z) = 1 + z^n \in \mathcal{P}, \qquad (n \in \mathbb{N}).$$

but if n ≥ 2, this function is no longer to be univalent. The Möbius function

$$L_0(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n, \qquad (z \in \mathcal{U}).$$

plays a central role in the class $\mathcal{P}$. This function is in the class $\mathcal{P}$, it is analytic and univalent in $\mathcal{U}$, and it maps $\mathcal{U}$ onto the real half-plane (Goodman, 1983). By using the principle of subordination, any function in the class $\mathcal{P}$ is called a function with positive real part in $\mathcal{U}$ and satisfies

$$p(z) \in \mathcal{P} \Leftrightarrow p(z) \prec \frac{1+z}{1-z}.$$

Some special subclasses of $\mathcal{P}$ play an important role in geometric function theory because of their relations with subclasses of univalent functions. Many such classes have been introduced and studied; some became the well-known. For instance, for given arbitrary numbers A, B $(-1 \leq B < A \leq 1)$, we denote by $\mathcal{P}(A, B)$ the class of functions $p(z) \in \mathcal{H}$ which satisfy the following conditions $p(0) = 1$, $\operatorname{Re} p(z) > 0$ and

$$p(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}).$$

The class $\mathcal{P}(A, B)$ was first introduced by (Janowski, 1973), therefore we say that $f(z)$ is in the class $\mathcal{P}(A, B)$ of Janowski functions. We note that

(i) $\mathcal{P}(1, -1) = \mathcal{P}$,

(ii) $\mathcal{P}(1 - 2\alpha,\ -1) = \mathcal{P}(\alpha)$ $(0 \leq \alpha < 1)$ defined by $\operatorname{Re} p(z) > \alpha$.

1.5 Some special classes of analytic functions

In this section we consider some special classes of univalent and p-valent functions defined by simple geometric properties. They are closely connected with functions of positive real part and with subordination. These classes can be completely characterized by simple inequality.

1.5.1 Classes of Starlike and convex functions

Geometric function theory of a single-valued complex variable is mostly concerned with the study of the properties of univalent functions. Several special subsets in the complex plane $\mathbb{C}$ play an important role in univalent functions. The image domain of $\mathcal{U}$ under a univalent function is of interest if it has some nice geometric properties. Convex domain and starlike domain are outstanding examples of domains with interesting properties. In this subsection we introduce some classes of starlike and convex functions for univalent and p-valent functions in the open unit disk.

A domain D in $\mathbb{C}$ is said to be starlike with respect to a point w_o if the line segment connecting any point in D to w_o is contained in D. A function $f(z) \in S$ in $\mathcal{U}$ is said to be starlike with respect to w_o if $\mathcal{U}$ is mapped onto a domain starlike with respect to w_o. In the special case that $w_o = 0$, the function $f(z)$ is said to be starlike with respect to the origin (or starlike) (Goodman, 1983). Let S^* denotes the class of all starlike functions in S. An analytic description of the class S^* is given by

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad (z \in \mathcal{U}).$$

A special subclass of S^* is that the class of starlike functions of order α, with $0 \leq \alpha < 1$, is denoted $S^*(\alpha)$ and given by

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (0 \leq \alpha < 1;\ z \in \mathcal{U}).$$

A function $f(z) \in \mathcal{A}(p)$ is said to be p-valent starlike if $f(z)$ satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad (p \in \mathbb{N};\ z \in \mathcal{U}).$$

We denote by $S^*(p)$ the class of all p-valent starlike functions. A special subclass of $S^*(p)$ is that the class of p-valent starlike functions of order α, with $0 \leq \alpha < p,\ p \in \mathbb{N}$ which denoted by $S^*(p, \alpha)$ and consists of functions satisfy

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (z \in \mathcal{U}).$$

A domain D in $\mathbb{C}$ is said to be convex if the line segment joining any two points of D lies entirely in D. If a function $f(z) \in S$ maps $\mathcal{U}$ onto a convex domain, then $f(\mathrm{z})$ is called a convex function (Goodman, 1983). Let K denotes the class of all convex functions in S. An analytic description of the class K is given by

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \qquad (z \in \mathcal{U}).$$

A special subclass of K is the class of convex functions of order α, with $0 \leq \alpha < 1$, is denoted by $K(\alpha)$ and given by

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (0 \leq \alpha < 1; z \in \mathcal{U}).$$

A function $f(z) \in \mathcal{A}(p)$ is said to be p-valent convex if $f(z)$ satisfies the following inequality

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0, \qquad (p\in\mathbb{N};\ z\in\mathcal{U}).$$

We denote by $K(p)$ the class of all p-valent convex functions. A special subclass of $K(p)$ that is the class of p-valent convex functions of order α with $0\le\alpha<p,\ p\in\mathbb{N}$ which denoted by $K(p,\alpha)$ and consists of functions satisfy

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha, \qquad (z\in\mathcal{U}).$$

The class $S^*(p,\alpha)$ was introduced by (Patil and Thakare, 1983) and the class $K(p,\alpha)$ was introduced by (Owa, 1985a). For $\alpha=0$, we have $S^*(p,0)=S^*(p)$ and $K(p,0)=K(p)$ which were first studied by (Goodman, 1950). If $p=1$, we have $S^*(1,\alpha)=S^*(\alpha)$ and $K(1,\alpha)=K(\alpha)$ which were first introduced by (Robertson, 1936) and were studied subsequently by (Schild, 1965), (Pinchuk, 1968), (Jack, 1971), and others.

There is a closely analytic connection between convex and starlike functions that was first noticed by (Alexander, 1915), and then onwards the result is known as Alexander's Theorem. This says that, if $f(z)$ be analytic function in $\mathcal{U}$ with $f(0)=0$ and $f'(0)=1$, then $f(z)\in K$ if and only if $zf'(z)\in S^*$. Further we note that

$$f(z)\in K(\alpha)\Leftrightarrow zf'(z)\in S^*(\alpha),$$

and for $f(z)\in\mathcal{A}(p)$, we have

$$f(z)\in K(p,\alpha)\Leftrightarrow\frac{zf'(z)}{p}\in S^*(p,\alpha).$$

Furthermore, we denote by $T^*(p,\alpha)$ and $C(p,\alpha)$ the classes obtained by intersections, respectively, of the classes $S^*(p,\alpha)$ and $K(p,\alpha)$ with $T(p)$; that is

$$T^*(p,\alpha) = S^*(p,\alpha) \cap T(p),$$

and

$$C(p,\alpha) = K(p,\alpha) \cap T(p).$$

The classes $T^*(p,\alpha)$ and $C(p,\alpha)$ were introduced by (Owa, 1985a). In particular, the classes $T^*(1,\alpha) = T^*(\alpha)$ and $C(1,\alpha) = C(\alpha)$ when $p = 1$, were studied by (Silverman, 1975).

A function $f(z) \in \mathcal{A}(p)$ is called p-valent starlike of order α and type β if it satisfies

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\alpha} \right| < \beta, \qquad (z \in \mathcal{U}).$$

where $0 \leq \alpha < p,\ 0 < \beta \leq 1$ and $p \in \mathbb{N}$. We denote by $S^*(p,\alpha,\beta)$ the class of all p-valent starlike functions of order α and type β . A function $f(z) \in \mathcal{A}(p)$ is called p-valent convex of order α and type β if it satisfies

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} - p}{1 + \frac{zf''(z)}{f'(z)} + p - 2\alpha} \right| < \beta, \qquad (z \in \mathcal{U}).$$

where $0 \leq \alpha < p,\ 0 < \beta \leq 1$ and $p \in \mathbb{N}$. We denote by $K(p,\alpha,\beta)$ the class of all p-valent convex functions of order α and type β. We note that

$$f(z) \in K(p,\alpha,\beta) \Leftrightarrow \frac{zf'(z)}{p} \in S^*(p,\alpha,\beta).$$

The classes $S^*(p,\alpha,\beta)$ and $K(p,\alpha,\beta)$ were studied by (Aouf, 1988) and (Aouf, 2007) which are extensions of the familiar classes were studied earlier by (Gupta and Jain, 1976) when $p = 1$, we have $S^*(1,\alpha,\beta) = S^*(\alpha,\beta)$ and $K(1,\alpha,\beta) = K(\alpha,\beta)$. If $\beta = 1$, we have the classes $S^*(p,\alpha,1) = S^*(p,\alpha)$ and $K(p,\alpha,1) = K(p,\alpha)$ which were studied by (Patil and Thakare,1983) and

(Owa, 1985a), respectively. Also, we denote by $T^*(p,\alpha,\beta)$ and $C(p,\alpha,\beta)$ the classes obtained by taking intersections, respectively, of the classes $S^*(p,\alpha,\beta)$ and $K(p,\alpha,\beta)$ with $T(p)$. Thus we have

$$T^*(p,\alpha,\beta) = S^*(p,\alpha,\beta) \cap T(p),$$

and

$$C(p,\alpha,\beta) = K(p,\alpha,\beta) \cap T(p).$$

The classes $T^*(p,\alpha,\beta)$ and $C(p,\alpha,\beta)$ were studied by (Aouf, 1988). In particular, for $\beta = 1$, we have the classes $T^*(p,\alpha,1) = T^*(p,\alpha)$ and $C(p,\alpha,1) = C(p,\alpha)$ which were introduced by (Owa, 1985a) and the classes $T^*(1,\alpha,1) = T^*(\alpha)$ and $C(1,\alpha,1) = C(\alpha)$ when $p = 1$ and $\beta = 1$ were studied by (Silverman, 1975).

Let us next define certain classes of starlike and convex functions with respect to the analytic function $\phi(z)$ by using the principle of subordination, which will be very useful in our later investigations in chapter 3.

Let $\phi(z)$ be an analytic function with positive real part in the unit disk $\mathcal{U}$, with $\phi(0) = 1$ and $\phi'(0) > 0$ which maps the unit disk $\mathcal{U}$ onto a region starlike with respect to 1 which symmetric with respect to the real axis. A functions $f(z) \in \mathcal{A}(p)$ is said to be in the class $S_p^*(\phi)$ for which

$$\frac{1}{p}\frac{zf'(z)}{f(z)} \prec \phi(z), \qquad (p \in \mathbb{N};\ z \in \mathcal{U}).$$

A functions $f(z) \in \mathcal{A}(p)$ is said to be in the class $C_p(\phi)$ if it satisfies

$$\frac{1}{p}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \prec \phi(z), \qquad (p \in \mathbb{N};\ z \in \mathcal{U}).$$

The classes $S_p^*(\phi)$ and $C_p(\phi)$ were introduced and studied by (Ali, et al. 2007). For $p = 1$, we get the classes $S^*(\phi)$ and $C(\phi)$ which were first introduced and studied by (Ma and Minda, 1994). The classes $S^*(\phi)$ and

$C(\phi)$ can be reduced to the familiar class $S^*(\alpha)$ of starlike functions of order $\alpha\,(0 \le \alpha < 1)$ and the class $C(\alpha)$ of convex functions of order α, respectively, when

$$\phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \qquad (0 \le \alpha < 1).$$

Also, the classes $S^*(\phi)$ and $C(\phi)$ can be reduced to the classes $S^*(A, B)$ and $C(A, B)$ of Janowski starlike functions and Janowski convex functions, respectively, when

$$\phi(z) = \frac{1 + Az}{1 + Bz}, \qquad (-1 \le B < A \le 1).$$

1.5.2 Classes of close-to-convex functions

A function $f(z) \in \mathcal{A}$ is said to be close-to-convex of order $\alpha\ (0 \le \alpha < 1)$ if there is a convex function g such that

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha, \qquad (z \in \mathcal{U}).$$

An equivalent formulation would involve the existence of a starlike function $h(z)$ such that

$$\mathrm{Re}\left\{\frac{zf'(z)}{h(z)}\right\} > \alpha, \qquad (z \in \mathcal{U}).$$

We denote by $\mathcal{C}(\alpha)$ to the class of all close-to-convex functions of order α. For $\alpha = 0$, we have the class $\mathcal{C}$ of all close-to-convex function in $\mathcal{U}$.

A function $f(z) \in \mathcal{A}(p)$ is said to be p-valent close-to-convex of order $\alpha\ (0 \le \alpha < p)$ if there is a p-valent convex function $g(z)$ such that

$$\mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right) > \alpha, \qquad (z \in \mathcal{U}).$$

An equivalent formulation would involve the existence of p-valent starlike function $h(z)$ such that

$$\mathrm{Re}\left(\frac{zf^{'}(z)}{h(z)}\right) > \alpha, \qquad (z \in \mathcal{U}).$$

We denote by $\mathcal{C}(p,\alpha)$ to the class of all p-valent close-to-convex functions of order α. If $\alpha = 0$, we have $\mathcal{C}(p,\alpha) = \mathcal{C}(p)$, the class of all p-valent close-to-convex functions. For $p = 1$ and $\alpha = 0$, we have $\mathcal{C}(1,0) = \mathcal{C}$. If $p = 1$, we get $\mathcal{C}(1,\alpha) = \mathcal{C}(\alpha)$. See (Duren, 1983), (Goodman, 1983) and (Pommerenke, 1975).

1.5.3 Classes of prestarlike functions

The class of prestarlike functions of order α $(0 \leq \alpha < 1)$ was introduced by (Ruscheweyh, 1977). It is denoted by $\mathcal{R}_{\alpha}$. A function $f(z) \in S$ is called prestarlike of order α with $0 \leq \alpha < 1$, if

$$(f * S_{\alpha})(z) \in S^{*}(\alpha),$$

where

$$S_{\alpha}(z) = \frac{z}{(1-z)^{2(1-\alpha)}}.$$

Let $\mathcal{R}(\gamma,\alpha)$ be the class of all function $f(z) \in S$ which satisfy the following condition

$$\left(f * S_{\gamma}\right)(z) \in S^{*}(\alpha).$$

This class $\mathcal{R}(\gamma,\alpha)$ is called the class of γ-prestarlike functions of order α with $(0 \leq \gamma < 1,\ 0 \leq \alpha < 1)$. This class were studied by (Sheil-Small et al., 1982). For $\gamma = \alpha$, we have the class $\mathcal{R}(\alpha,\alpha) \equiv \mathcal{R}_{\alpha}$.

For a function $f(z) \in S$, the class $\mathcal{R}_\gamma(\alpha,\beta)$ is said to be the class of γ-prestarlike functions of order α and type β with $(0 \le \gamma < 1,\ 0 \le \alpha < 1,\ 0 < \beta \le 1)$ if

$$\left(f * S_\gamma\right)(z) \in S^*(\alpha,\beta).$$

This class was introduced by (Ahuja and Silverman, 1983).

A function $f(z) \in \mathcal{A}(p)$ is said to be p-valent α-prestarlike functions of order β $(0 \le \alpha < p,\ 0 < \beta \le 1,\ p \in \mathbb{N})$ if

$$\left(f * S_\alpha^p\right)(z) \in S^*(p,\beta),$$

where

$$S_\alpha^p(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}}.$$

We denote by $\mathcal{R}^p(\alpha,\beta)$ the class of all p-valent α-prestarlike functions of order β. Further let $C^p(\alpha,\beta)$ be the subclass of $\mathcal{A}(p)$ consisting of functions satisfying

$$f(z) \in C^p(\alpha,\beta) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{R}^p(\alpha,\beta)$$

The classes $\mathcal{R}^p(\alpha,\beta)$ and $C^p(\alpha,\beta)$ were introduced by (Aouf and Silverman, 2007). We note that, $\mathcal{R}^p(\alpha,\alpha) = \mathcal{R}^p(\alpha)(0 \le \alpha < p,\ p \in \mathbb{N})$, the class which was studied by (Kumar and Reddy, 1992). For $p = 1$, we have $\mathcal{R}^1(\alpha,\alpha) = \mathcal{R}_\alpha$.

1.5.4 Classes of starlike and convex functions of complex order

A function $f(z) \in \mathcal{A}(p)$ is said to be p-valent starlike functions of complex order $b \neq 0$, (b complex) if and only if $\frac{f(z)}{z} \neq 0,\ (z \in \mathcal{U})$, and

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{pf(z)} - 1\right)\right\} > 0, \qquad (z \in \mathcal{U}).$$

We denote by $S(p,b)$ the class of all such functions. A function $f(z) \in \mathcal{A}(p)$ is said to be p-valent convex function of complex order $b \neq 0$, (b complex) that is $f(z) \in K(p,b)$, if and only if $f'(z) \neq 0$ in $\mathcal{U}$ and

$$\mathrm{Re}\left\{1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0, \qquad (z \in \mathcal{U}).$$

We denote by $K(p,b)$ the class of all such functions.

For $p = 1$, we have $S(1,b) = S(b)$ the class of starlike functions of complex order b $(b \neq 0, b \text{ complex})$ which was introduced by (Nasr and Aouf, 1985) and, $K(1,b) = K(b)$ is the class of convex functions of complex order b $(b \neq 0, b \text{ complex})$ which was introduced earlier by (Wiatrowshi, 1970) and considered by (Nasr and Aouf, 1982). For $b = 1$, we have $S(1) = S^*$ and $K(1) = K$. If $b = 1-\alpha$, then we get $S(1-\alpha) = S^*(\alpha)$ and $K(1-\alpha) = K(\alpha)$ for $0 \leq \alpha < 1$. Notice that

$$f(z) \in K(b) \Leftrightarrow zf'(z) \in S(b).$$

1.5.5 Classes of Uniformly starlike and uniformly convex functions

A function $f(z) \in \mathcal{A}$ is called uniformly convex (uniformly starlike) if $f(z)$ maps every circular arc γ contained in $\mathcal{U}$ with centre $\zeta \in \mathcal{U}$ onto a convex (starlike) arc $f(\gamma)$ with respect to $f(\zeta)$. The classes of all uniformly convex and uniformly starlike functions were introduced by (Goodman, 1991a) and (Goodman, 1991b) which denoted by UCV and UST. (Ma and Minda, 1992) and (Rønning, 1993a) independently showed that a function $f(z)$ is uniformly convex if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, \qquad (z \in \mathcal{U}).$$

Thus, a function $f(z) \in UCV$ if the quantity

$$1+\frac{zf''(z)}{f'(z)},$$

lies in the parabolic region $\Omega = \{u+iv: v^2 < 2u-1\}$. A corresponding class S_p of uniformly starlike functions consisting of parabolic starlike functions $f(z)$, where $f(z) = zg'(z)$ for $g(z)$ in UCV, was introduced by (Rønning, 1993a) and studied by (Rønning,1993b). Clearly a function $f(z)$ is in the class S_p if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)}-1\right|, \qquad (z \in \mathcal{U}).$$

We note that,

$$f(z) \in UCV \Leftrightarrow zf'(z) \in S_p.$$

Furthermore, (Kanas and Wisniowska, 1999) and (Kanas and Wisniowska 2000) defined the functions $f(z) \in S$ to be k-uniformly convex (k-uniformly starlike) if for $0 \leq k < \infty$, the image of every circular arc γ contained in $\mathcal{U}$ with centre ζ where $\zeta \leq k$ is convex (starlike).

A function $f(z) \in \mathcal{A}$ is said to be k-uniformly convex of order α $(0 \leq \alpha < 1,\ k \geq 0)$, denoted by $k-UCV(\alpha)$, if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f(z)}-\alpha\right\} \geq k\left|\frac{zf''(z)}{f'(z)}\right|, \qquad (z \in \mathcal{U}).$$

A function $f(z) \in \mathcal{A}$ is said to be k-uniformly starlike of order α $(0 \leq \alpha < 1,$ $k \geq 0)$, denoted by $k-UST(\alpha)$, if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\} \geq k\left|\frac{zf'(z)}{f(z)}-1\right|, \qquad (z \in \mathcal{U}).$$

Notice that,

$$f(z) \in k-UCV(\alpha) \Leftrightarrow zf'(z) \in k-UST(\alpha).$$

The classes $k-UCV(\alpha)$ and $k-UST(\alpha)$ which were studied by various authors including (Ma and Minda,1993), (Kanas and Wisnionska, 1999, 2000), and (Rønning,1991). In particular, for $k=0$, we have $0-UCV(\alpha)=K(\alpha)$ and $0-UST(\alpha)=S^*(\alpha)$. If $k=1$, we have $1-UCV(\alpha)=UCV(\alpha)$ and $1-UST(\alpha)=UST(\alpha)$, the classes of uniformly convex and uniformly starlike functions of order α , respectively.

A function $f(z)\in\mathcal{A}(p)$ is said to be k-uniformly p-valent starlike of order $\alpha\ (-p<\alpha<p),\ k\geq 0$ and $z\in\mathcal{U}$, denoted by $k-UST(p,\alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}\geq k\left|\frac{zf'(z)}{f(z)}-p\right|,\qquad (z\in\mathcal{U}).$$

A function $f(z)\in\mathcal{A}(p)$ is said to be k-uniformly p-valent convex of order $\alpha\ (-p<\alpha<p),\ k\geq 0$ and $z\in\mathcal{U}$, denoted by $k-UCV(p,\alpha)$ if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f(z)}-\alpha\right\}\geq k\left|1+\frac{zf''(z)}{f(z)}-p\right|,\qquad (z\in\mathcal{U}).$$

We note that,

$$1-UST(p,\alpha)=UST(p,\alpha),$$

and

$$1-UCV(p,\alpha)=UCV(p,\alpha).$$

where $UST(p,\alpha)$ and $UCV(p,\alpha)$ are the classes of uniformly p-valent starlike and uniformly p-valent convex functions of order $\alpha\ (-p<\alpha<p)$ which were introduced by (AL-Kharsani and AL-Hjiry, 2006). The classes $0-UST(p,\alpha)=S^*(p,\alpha)$ and $0-UCV(p,\alpha)=K(p,\alpha)$ of p-valent starlike and convex functions of order α. Furthermore, $k-UST(1,\alpha)=k-UST(\alpha)$ and $k-UCV(1,\alpha)=k-UCV(\alpha)$ are the classes of k-uniformly starlike and k-uniformly convex functions of order α.

1.5.6 Classes of Bazilevič and non- Bazilevič functions

A functions $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{B}(\alpha)$ if it satisfies the following condition

$$\operatorname{Re}\left\{\frac{zf^{'}(z)(f(z))^{\alpha-1}}{(g(z))^{\alpha}}\right\} > 0, \qquad (z \in \mathcal{U}). \tag{1.5.6.1}$$

for some $\alpha > 0$ where $g(z) \in S^{*}$. Furthermore, we denote by $\mathcal{B}_1(\alpha)$ the subclass of $\mathcal{B}(\alpha)$ for which $g(z) = z$ in (1.5.6.1), for functions satisfying

$$\operatorname{Re}\left\{\frac{zf^{'}(z)(f(z))^{\alpha-1}}{z^{\alpha}}\right\} > 0, \qquad (z \in \mathcal{U}). \tag{1.5.6.2}$$

Note that $\mathcal{B}(0) = \mathcal{B}_1(0) = S^{*}$. The class $\mathcal{B}(\alpha)$ is called the class of Bazilevič functions of type α and was studied by (Singh, 1973).

On the other hand, the class of non-Bazilevič functions was introduced by (Obradović, 1998). This class of functions is said to be non-Bazilevič type and denoted by $\mathcal{N}(\alpha)$ for $0 < \alpha < 1$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{N}(\alpha)$ if and only if

$$\operatorname{Re}\left\{f^{'}(z)\left(\frac{z}{f(z)}\right)^{1+\alpha}\right\} > 0 \qquad (z \in \mathcal{U}) \tag{1.5.6.3}$$

1.6 Fractional derivative operators

The study of operators plays an important role in geometric function theory. A large number of classes of analytic univalent and p-valent functions are defined by means of fractional derivative operators. For numerous references on the subject, one may refer to (Srivastava and Owa, 1989) and (Srivastava and Owa, 1992). In this section, we recall some definitions of the fractional derivative operators which are helpful in our later investigations.

Let us begin with the operator $D_{0,z}^{\lambda}$ which was studied by (Owa, 1978), (Owa, 1985b), (Srivastava and Owa, 1984) and (Srivastava and Owa, 1989).

The fractional derivative operator of order λ is denoted by $D_{0,z}^{\lambda}$ and defined by

$$D_{0,z}^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi, \quad (0 \le \lambda < 1). \tag{1.6.1}$$

where $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ involved in (1.6.1) is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Next we define the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ which was given by (Srivastava, et al. 1988) and (Srivastava and Owa, 1989) in terms of the Gauss's hypergeometric function ${}_2F_1(a,b,c;z)$, for $z \in \mathcal{U}$, see (Srivastava and Karlsson, 1985)

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n \, (b)_n \, z^n}{(c)_n \, n!}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & , \; n = 0 \\ \lambda(\lambda+1)(\lambda+2) \ldots . (\lambda+n-1) & , n \in \mathbb{N} \end{cases} \tag{1.6.2}$$

$$(\lambda \ne 0, -1, -2, \ldots)$$

The generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is defined by

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi) \, {}_2F_1\left(\mu-\lambda, 1-\eta, 1-\lambda; 1-\frac{\xi}{z}\right) d\xi \right). \tag{1.6.3}$$

for $0 \le \lambda < 1$ and $\mu, \ \eta \in \mathbb{R}$ where $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin with the order $f(z) =$

$O(|z|^{\varepsilon}),\ z \to 0,$ where $\varepsilon > \max\{0, \mu - \eta\} - 1,$ and the multiplicity of $(z-\xi)^{-\lambda}$ in (1.6.3) is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

Under the hypothesis of the definition (1.6.3), the fractional derivative operator $J_{0,z}^{\lambda+m,\mu+m,\eta+m}$ of a function $f(z)$ is defined by

$$J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z), \qquad (1.6.4)$$

$$(0 \le \lambda < 1;\ m = 0,1,2, \dots).$$

Notice that

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_{0,z}^{\lambda} f(z), \qquad (0 \le \lambda < 1). \qquad (1.6.5)$$

By means of the above definition (1.6.3), (Raina and Nahar, 2002) obtained

$$J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(1+k)\Gamma(1+k-\mu+\eta)}{\Gamma(1+k-\mu)\Gamma(1+k-\lambda+\eta)} z^{k-\mu}, \qquad (1.6.6)$$

where $\lambda, \mu, \eta \in \mathbb{R}$ such that $\lambda \ge 0$ and $k > \max(0, \mu - \eta) - 1$.

For $f(z) \in \mathcal{A}$, the fractional derivative operator $\Omega^{\lambda} f(z)$ is defined by

$$\Omega^{\lambda} f(z) = \Gamma(2-\lambda)\, z^{\lambda}\, D_{0,z}^{\lambda} f(z), \quad (\lambda \in \mathbb{R},\ \lambda \ne 2,3, \dots), \qquad (1.6.7)$$

$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n\, z^n .$$

We note that

$$\Omega^0 f(z) = f(z), \qquad \Omega^1 f(z) = z f'(z)$$

The operator $\Omega^{\lambda} f(z)$ was introduced by (Owa and Srivastava, 1987) and studied by (Owa and Shen, 1998) and (Srivastava et al., 1998).

For $f(z) \in \mathcal{A}$, the fractional derivative operator $P_{0,Z}^{\lambda,\mu,\eta} f(z)$ is defined by

$$P_{0,Z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(2-\mu)\Gamma(2-\lambda+\eta)}{\Gamma(2-\mu+\eta)}\, z^{\mu}\, J_{0,z}^{\lambda,\mu,\eta} f(z), \qquad (1.6.8)$$

where $\lambda \ge 0,\ \mu < 2$ and $\eta > \max\{\lambda, \mu\} - 2$.

The operator $P_{0,Z}^{\lambda,\mu,\eta}f(z)$ was introduced by (Raina and Nahar, 2000). Notice that, for $\mu=\lambda$ we have $P_{0,Z}^{\lambda,\lambda,\eta}f(z)=\Omega^{\lambda}f(z)$.

1.7 Differential subordinations and superordinations

In the theory of differential equations of real-valued functions there are many examples of differential inequalities that have important applications in the general theory. In those cases bounds on a function f are often determined from an inequality involving several of the derivatives of f. In two articles (Miller and Mocanu, 1981) and (Miller and Mocanu, 1985), the authors extended these ideas involving differential inequalities for real-valued functions to complex-valued functions. In this section we present the concepts of differential subordination and differential superordination for analytic functions which will be helpful for our investigations in chapter 4.

Let us begin with the differential subordination for analytic functions in the open unit disk, which was introduced by (Miller and Mocanu, 1981).

Let $\psi(r,s,t;\,z):\mathbb{C}^3\times\mathcal{U}\rightarrow\mathbb{C}$ and let $h(z)$ be univalent in $\mathcal{U}$. If $p(z)$ is analytic in $\mathcal{U}$ and satisfies the (second-order) differential subordination

$$\psi\left(p(z),zp'(z),z^2\,p''(z);z\right)\prec h(z), \tag{1.7.1}$$

then $p(z)$ is said to be a solution of the differential subordination (1.7.1). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p(z)\prec q(z)$ for all $p(z)$ satisfies (1.7.1). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z)\prec q(z)$ for all dominants $q(z)$ of (1.7.1) is said to be the best dominant of (1.7.1).

Let Ω be a subset of $\mathbb{C}$ and suppose (1.7.1) be replaced by

$$\psi\big(p(z), zp'(z), z^2\, p''(z); z\big) \in \Omega, \qquad (z \in \mathcal{U}). \tag{1.7.2}$$

the condition in (1.7.2) will also be referred as a (second-order) differential subordination (Miller and Mocanu, 2000).

The first order linear differential subordination was defined by (Miller and Mocanu, 1985) in the following subordination condition

$$A(z)zp'(z) + B(z)p(z) \prec h(z),$$

or

$$zp'(z) + P(z)p(z) \prec h(z),$$

and the second order linear differential subordination is defined by

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z),$$

where $A,\ B,\ C,\ D$ and h are complex functions.

Next let us present the dual concept of differential subordination, that is, differential superordination which was recently investigated by (Miller and Mocanu, 2003).

Let $\psi(r, s, t;\ z) :\ \mathbb{C}^3\ \times \mathcal{U} \to \mathbb{C}$ and let $h(z)$ be analytic in $\mathcal{U}$. If $p(z)$ and $\psi(p(z), zp'(z),\ z^2\, p''(z);\ z)$ are univalent functions in $\mathcal{U}$, and satisfies the (second-order) differential superordination

$$h(z) \prec \psi\big(p(z), zp'(z), z^2\, p''(z); z\big). \tag{1.7.3}$$

then $p(z)$ is called a solution of the differential superordination (1.7.3). The analytic function q is called a subordinant of the differential superordination, or more simply a subordinant if $q(z)\ \prec\ p(z)$ for all $p(z)$ satisfies (1.7.3). An univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.7.3) is said to be the best subordinant.

Let Ω be a subset of $\mathbb{C}$ and suppose (1.7.3) be replaced by

$$\Omega \subset \big\{\psi\big(p(z), zp'(z), z^2\, p''(z); z\big):\ z \in \mathcal{U}\big\}. \tag{1.7.4}$$

the condition in (1.7.4) will also be referred as a (second-order) differential superordination, see (Miller and Mocanu, 2000).

1.8 Strong differential subordinations and superordinations

Some recent results in the theory of analytic functions were obtained by using a more strong form of the differential subordination and superordination introduced by (Antonino and Romaguera, 1994) and studied by (Antonino and Romaguera, 2006) called strong differential subordination and strong differential superordination, respectively, which were developed by (G. Oros, 2007) and (G. Oros, 2009). In this section we present the concepts of strong differential subordination and strong differential superordination for analytic functions which will be helpful for our investigations in chapter 5.

Let us begin with some notations of strong differential subordination of analytic functions.

Let $H(z,\zeta)$ analytic functions in $\mathcal{U}\times\overline{\mathcal{U}}$, where $\overline{\mathcal{U}}=\{z\in\mathbb{C}:|z|\leq 1\}$ is the closed unit disk of the complex plane. Let $f(z)$ be analytic and univalent in $\mathcal{U}$. The function $H(z,\zeta)$ is said to be strongly suborordinate to $f(z)$ written

$$H(z,\zeta)\prec\prec f(z),$$

if for $\zeta\in\overline{\mathcal{U}}$, the function of z, $H(z,\zeta)$ is subordinate to $f(z)$. (Antonino and Romaguera, 1994) and (G. Oros, 2011). Since $f(z)$ is analytic and univalent, then $H(0,\zeta)=f(0)$ and $H(\mathcal{U}\times\overline{\mathcal{U}})\subset f(\mathcal{U})$. If $H(z,\zeta)\equiv H(z)$, then the strong differential subordinations becomes the usual differential subordinations.

Let $\psi: \mathbb{C}^3 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$, and let $h(z)$ be univalent in $\mathcal{U}$. If $p(z)$ is analytic in $\mathcal{U}$ and satisfies the following (second-order) strong differential subordination

$$\psi\left(p(z), zp'(z), z^2p''(z); z, \zeta\right) \prec\prec h(z), \qquad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}), \tag{1.8.1}$$

then $p(z)$ is called a solution of the strong differential subordination. The univalent function $q(z)$ is called a domaint of the solution of the strong differential subordination or, more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.8.1). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.8.1) is said to be the best dominant.

Let Ω be a set in $\mathbb{C}$ and suppose (1.8.1) is replaced by

$$\psi\left(p(z), zp'(z), z^2p''(z); z, \zeta\right) \in \Omega, \qquad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}). \tag{1.8.2}$$

the condition in (1.8.2) will also be referred as a (second-order) strong differential subordination (G. Oros, 2011).

A strong differential subordination of the form (G. Oros, 2011)

$$A(z,\zeta)\, zp'(z) + B(z,\zeta)\, p(z) \prec\prec h(z), \quad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}), \tag{1.8.3}$$

where $A(z,\zeta)zp'(z) + B(z,\zeta)p(z)$ is analytic in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$ and $h(z)$ is an analytic and univalent function in $\mathcal{U}$ is called first order linear strong differential subordination.

Now let us present the dual concept of strong differential subordination, that is, strong differential superordination which was introduced recently by (G. Oros, 2009).

Let $f(z)$ be analytic in $\mathcal{U}$ and let $H(z,\zeta)$ be analytic functions in $\mathcal{U} \times \overline{\mathcal{U}}$ and univalent in $\mathcal{U}$. The function $f(z)$ is said to be strongly subordinate to $H(z,\zeta)$ written

$$f(z) \prec\prec H(z,\zeta),$$

if there exists a function $w(z)$ analytic in $\mathcal{U}$ with $w(0)=0$ and $|w(z)|<1$ and such that $f(z)=H(w(z),\zeta)$. If $H(z,\zeta)$ is univalent in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$, then $f(z) \prec\prec H(z,\zeta)$ if $f(0)=H(0,\zeta),\ \zeta \in \overline{\mathcal{U}}$ and $f(\mathcal{U}) \subset H(\mathcal{U}\times\overline{\mathcal{U}})$.

Let $\psi\colon \mathbb{C}^3 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$, and let $h(z)$ be univalent in $\mathcal{U}$. If $p(z)$ and $\psi\big(p(z), zp'(z), z^2p''(z); z,\zeta\big)$ are univalent in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$ and satisfy the following (second-order) strong differential superordination

$$h(z) \prec\prec \psi\big(p(z), zp'(z), z^2p''(z); z,\zeta\big), \qquad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}), \tag{1.8.4}$$

then $p(z)$ is called a solution of the strong differential superordination. The univalent function $q(z)$ is called a subordinant of the solution of the strong differential superordination or, more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.8.4). A univalent dominant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.8.4) is said to be the best subordinant.

Let Ω be a set in $\mathbb{C}$ and suppose (1.8.4) is replaced by

$$\Omega \subset \big\{\psi\big(p(z), zp'(z), z^2p''(z); z,\zeta\big)\colon\ z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\big\}, \tag{1.8.5}$$

the condition in (1.8.5) will also be referred as a (second-order) strong differential superordination.

A strong differential superordination which was defined by (G. Oros, 2007) in the form

$$h(z) \prec\prec A(z,\zeta)zp'(z) + B(z,\zeta)p(z), \qquad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}),$$

where $h(z)$ is analytic in $\mathcal{U}$ and $A(z,\zeta)zp'(z) + B(z,\zeta)p(z)$ is univalent in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$, is called first order linear strong differential superordination.

The next classes consist in the fact that the coefficients of the functions in those classes are not constants but complex-valued functions. Using those classes, a new approach in studying the strong differential subordinations can be developed (G. Oros, 2011).

Let $\mathcal{H}(\mathcal{U}\times\overline{\mathcal{U}})$ denote the class of analytic functions in $\mathcal{U}\times\overline{\mathcal{U}}$ and let

$\mathcal{H}^*[a,n,\zeta] = \{ f \in \mathcal{H}(\mathcal{U}\times\overline{\mathcal{U}})$:

$$f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots, z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\},$$

where $a_k(\zeta)$ are analytic functions in $\overline{\mathcal{U}}$, $k \geq n$, $n \in \mathbb{N}$ and $a \in \mathbb{C}$, and

$$\mathcal{H}_u(\mathcal{U}) = \{f \in \mathcal{H}^*[a,n,\zeta]\colon\ f(z,\zeta) \text{ univalent in } \mathcal{U} \text{ for all } \zeta \in \overline{\mathcal{U}}\}.$$

Let

$$S_\zeta^* = \left\{f \in \mathcal{H}^*[a,n,\zeta]\colon\ \mathrm{Re}\left\{\frac{zf'(z,\zeta)}{f(z,\zeta)}\right\} > 0, z \in \mathcal{U}, \forall\zeta \in \overline{\mathcal{U}}\right\},$$

be the class of starlike functions in $\mathcal{U}\times\overline{\mathcal{U}}$, and

$$K_\zeta^* = \left\{f \in \mathcal{H}^*[a,n,\zeta]\colon\ \mathrm{Re}\left\{1 + \frac{zf''(z,\zeta)}{f'(z,\zeta)}\right\} > 0, z \in \mathcal{U}, \forall\zeta \in \overline{\mathcal{U}}\right\},$$

be the class of convex functions in $\mathcal{U}\times\overline{\mathcal{U}}$.

Let $f(z,\zeta)$ and $H(z,\zeta)$ analytic functions in $\mathcal{U}\times\overline{\mathcal{U}}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ or $H(z,\zeta)$ is said to be strongly superordinate to $f(z,\zeta)$ if there exists a function w analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{\mathcal{U}}$. In such a case we write

$$f(z,\zeta) \prec\prec H(z,\zeta), \qquad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}).$$

If $f(z,\zeta)$ is analytic functions in $\mathcal{U}\times\overline{\mathcal{U}}$, and univalent in $\mathcal{U}$, for all $\zeta \in \overline{\mathcal{U}}$, then $f(0,\zeta) = H(0,\zeta)$, for all $\zeta \in \overline{\mathcal{U}}$ and $f(\mathcal{U}\times\overline{\mathcal{U}}) \subset H(\mathcal{U}\times\overline{\mathcal{U}})$. If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, then the strong subordination becomes the usual notation of subordination.

1.9 Motivations and outlines

The attention to the so-called coefficient estimate problems for different subclasses of univalent and p-valent functions has been the main interest among authors. Hence there are many new subclasses and new properties of univalent and p-valent functions have been introduced. The study of operators plays a vital role in mathematics. To apply the definitions of fractional calculus operators (that are derivatives and integrals) for univalent and p-valent functions and then study its properties, is one of the hot areas of current ongoing research in the geometric function theory.

In this thesis, motivated by wide applications of fractional calculus operators in the study of univalent and p-valent functions including (Altintas et al. 1995a), (Altintas et al. 1995b), (Khairnar and More, 2009), (Irmak et al., 2002), (Owa, 1978), (Owa, 1985b), (Owa and Shen, 1998), (Raina and Bolia, 1998), (Raina and Nahar, 2000), (Raina and Choi, 2002), (Raina and Nahar, 2002), (Raina and Srivastava, 1996), (Srivastava and Aouf, 1992), (Srivastava and Aouf, 1995), (Srivastava and Mishra, 2000), (Srivastava et al.,1988), (Srivastava and Owa, 1984), (Srivastava and Owa, 1987),(Srivastava and Owa, 1989), (Srivastava and Owa, 1991b), (Srivastava and Owa, 1992) and (Srivastava et al., 1998) we present a study based on fractional derivative operator and its applications to certain classes of p-valent (or multivalent) functions in the open unit disk regarding various properties of some classes of functions with negative coefficients, sufficient conditions for starlikeness and convexity, sharp coefficient bounds, differential subordination and superordination, and strong differential subordination and superordination. Our finding will provide interesting new results and extensions of an number known results.

1.9.1 Functions with negative coefficients and related classes

Several classes of univalent functions have been extended to the case of p-valent functions in obtaining some properties such as coefficient estimates, distortion theorem, extreme points, inclusion properties, modified Hadamard product and radius of convexity and starlikeness. (Aouf and Hossen, 2006) have generalized certain classes of univalent starlike functions with negative coefficients due to (Srivastava and Owa, 1991a) to obtain coefficient estimates, distortion theorem and radius of convexity for certain class of p-valent starlike functions with negative coefficients. More recently, (Aouf and Silverman, 2007) studied certain classes of p-valent γ-prestarlike functions of order α. Subsequently, (Aouf, 2007) extended the classes of (Aouf and Silverman, 2007) to case p-valent γ-prestarlike functions of order α and type β. Moreover, (Gurugusundaramoorthy and Themangani, 2009) introduced class of uniformly convex functions based on certain fractional derivative operator.

The above observations motivate us to define some new classes of p-valent functions with negative coefficients $f(z) \in T(p)$ in the open unit disk by using certain fractional derivative operator. This leads to the results presented in Chapter 2. Some of the results established in this chapter provide extensions of those given in earlier works.

An outline of chapter 2 is as follows:

- Section 2.1 is an introductory section.

- Section 2.2 consists the definitions of the modification of fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ and the classes $T_{\lambda,\mu,\eta}^{*}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ of $T(p)$ as follows:

A function $f(z) \in T(p)$ is said to be in $T_{\lambda,\mu,\eta}^{*}(p,\alpha,\beta,\gamma)$ if it satisfies the following inequality

$$\left| \frac{\frac{z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{g(z)} - p}{\frac{z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{g(z)} + p - 2\beta} \right| < \gamma, \qquad (z \in \mathcal{U}), \qquad (1.9.1.1)$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N};$$

$$0 \leq \alpha < p;\ 0 \leq \beta < p;\ 0 < \gamma \leq 1),$$

for the function

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n}\, z^{p+n}, \qquad \left(b_{p+n} \geq 0;\ p \in \mathbb{N}\right),$$

belonging to $T^{*}(p,\alpha)$, where

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = \phi_p(\lambda,\mu,\eta)\, z^{\mu}\, J_{0,Z}^{\lambda,\mu,\eta} f(z), \qquad (1.9.1.2)$$

and

$$\phi_p(\lambda,\mu,\eta) = \frac{\Gamma(1-\mu+p)\Gamma(1+\eta-\lambda+p)}{\Gamma(1+p)\Gamma(1+\eta-\mu+p)}\,.$$

Further, if $f(z) \in T(p)$ satisfies the condition (1.9.1.1) for $g(z) \in C(p,\alpha)$, we say that $f(z) \in C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$.

Also, We obtain coefficient inequalities, distortion properties and convexity of functions in these classes.

- Section 2.3 gives the definition of the classes $S_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$ and $C_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$ of $T(p)$ by using the Hadamard product (or convolution) involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ as follows:

  A function $f(z) \in T(p)$ is said to be in the class $S_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$ if and only if

$$\left| \frac{\dfrac{z\left(\Omega_p^{\lambda,\mu,\eta}f(z)\right)'}{\Omega_p^{\lambda,\mu,\eta}f(z)} - p}{\dfrac{z\left(\Omega_p^{\lambda,\mu,\eta}f(z)\right)'}{\Omega_p^{\lambda,\mu,\eta}f(z)} + p - 2\alpha} \right| < \beta, \qquad (z \in \mathcal{U}),$$

with

$$\Omega_p^{\lambda,\mu,\eta}f(z) = \varphi_p(a,c;z) * M_{0,z}^{\lambda,\mu,\eta}f(z),$$

where $\varphi_p(a,c;z)$ is given by

$$\varphi_p(a,c;z) = z^p + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{p+n},$$

and $M_{0,z}^{\lambda,\mu,\eta}f(z)$ is given by (1.9.1.2), for $a \in \mathbb{R}$; $c \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$; $0 \le \alpha < p$; $0 < \beta \le 1$; $\lambda \ge 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu) - p - 1$ and $p \in \mathbb{N}$. Further, a function $f(z) \in T(p)$ is said to be in the class $C_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$ if and only if

$$\frac{zf'(z)}{p} \in S_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta).$$

Here, we study coefficient estimates, distortion properties, extreme points, modified Hadmard products, inclusion properties, radii of close-to-convex, starlikeness, and convexity for functions belonging to these classes.

- Section 2.4 presents the definition of the classes $k-UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(p,\alpha)$ and $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(p,\alpha)$ of k-uniformly p-valent starlike and convex functions in the open unit disk as follows:

  The function $f(z) \in \mathcal{A}(p)$ is said to be in the class $k-UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(p,\alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{pM_{0,z}^{\lambda,\mu,\eta}f(z)}{M_{0,z}^{\beta,\gamma,\xi}f(z)}-\alpha\right\} \geq k\left|\frac{pM_{0,z}^{\lambda,\mu,\eta}f(z)}{M_{0,z}^{\beta,\gamma,\xi}f(z)}-p\right|, \qquad (z \in \mathcal{U}),$$

$$(k \geq 0;\ 0 \leq \alpha < p; \lambda \geq 0;\ 0 \leq \mu < 1+p; \beta \geq 0;\ 0 \leq \gamma < 1+p;$$

$$\eta > \max(\lambda,\mu)-p-1;\ \xi > \max(\beta,\gamma)-p-1),$$

  where $M_{0,z}^{\lambda,\mu,\eta}f(z)$ and $M_{0,z}^{\beta,\gamma,\xi}f(z)$ as given in (1.9.1.2). We let

$$k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(p,\alpha) = k-UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha) \cap T(p)$$

  Also, we derive some properties for these classes including coefficient estimates, distortion theorems, extreme points, closure theorems and radii of k-uniform starlikeness, convexity and close-to-convexity.

1.9.2 Starlikeness, convexity and coefficient bounds

The problem of sufficient conditions for starlikeness and convexity is concerning to find conditions under which function in certain class are starlike and convex, respectively. (Owa and Shen, 1998) and (Raina and Nahar, 2000) introduced various sufficient conditions for starlikeness and convexity of some classes of univalent functions associated with certain fractional derivative operators. Also, the results of (Jack, 1971) and (Nunokawa, 1992) which popularly known as jack's Lemma and Nunokawa's Lemma in literature have applied to obtain many of sufficient conditions for starlikeness and convexity for analytic functions and were studied by (Irmak and Cetin, 1999), (Irmak and Piejko, 2005) and (Irmak et al., 2002).

There are now several results for Fekete and Szegö problem in literature, each of them dealing with $|a_3 - \mu a_2^2|$ for various classes of functions. The unified treatment of various subclasses of starlike and convex functions (Ma and Minda, 1994) and the coefficient bounds for various classes (Ali et al., 2007), (Ramachandran et al., 2007), (Rosy et al., 2009) and (Shanmugam et al., 2006a) motivate one to consider similar classes defined by subordination.

The above contributions on sufficient conditions for starlikeness and convexity of univalent functions and sharp coefficient bounds for some classes of univalent and p-valent functions encourage us to obtain conditions for starlikeness and convexity to case p-valent functions associated with certain fractional derivative operator and also, to obtain coefficient bounds for $\left|a_{p+2} - \mu a_{p+1}^2\right|$ and $\left|a_{p+3}\right|$ for certain classes of p-valent analytic function associated with fractional derivative operator. This leads to the results presented in Chapter 3. Some of our results in this chapter generalize previously known results. This chapter contains of three sections:

An outline of chapter 3 is as follows:

- Section 3.1 is an introductory section and contains some preliminary results which are absolutely essential for completing the results used in subsequent sections.

- Section 3.2 gives some sufficient conditions for starlikeness and convexity and divided into three subsections.
- Subsection 3.2.1 gives some sufficient conditions for starlikeness and convexity by using the results of the classes $S^*(p,\alpha)$ and $K(p,\alpha)$ due to (Owa, 1985a).
- subsection 3.2.2 contains some sufficient conditions for starlikeness and convexity involving the Hadamard product (or convolution).
- Subsection 3.2.3 is concerned to apply Jack's Lemma and Nunokakawa's Lemma for p-valent functions involving the operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ to obtain some sufficient conditions for starlikeness and convexity.
- Section 3.3 gives coefficient bounds for p-valent functions associated with the operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ belonging to certain classes and is divided into three subsections.
- Subsection 3.3.1 gives the definition of the classes $S^*_{p,\lambda,\mu,\eta}(\phi)$, $S^*_{b,p,\lambda,\mu,\eta}(\phi)$ of $\mathcal{A}(p)$ as follows:

  A function $f(z) \in \mathcal{A}(p)$ is in the class $S^*_{b,p,\lambda,\mu,\eta}(\phi)$ if

$$1+\frac{1}{b}\left\{\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-1\right\} \prec \phi(z), \quad (z \in \mathcal{U}, b \in \mathbb{C}\backslash\{0\}).$$

  Also, we let $S^*_{1,p,\lambda,\mu,\eta}(\phi) = S^*_{p,\lambda,\mu,\eta}(\phi)$ **.**

  Here, we obtain some coefficient bounds for functions belonging to the classes $S^*_{p,\lambda,\mu,\eta}(\phi)$ and $S^*_{b,p,\lambda,\mu,\eta}(\phi)$.

- Subsection 3.3.2 gives the definitions of some classes of p-valent Bazilevič functions such as the classes $R_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ and $R_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ of $\mathcal{A}(p)$ as follows:

  A function $f(z) \in \mathcal{A}(p)$ is in the class $R_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ if

$$1+\frac{1}{b}\left\{(1-\beta)\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\alpha}+\beta\,\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\alpha}-1\right\} \prec \phi(z),$$

  where

$$(\alpha \geq 0;\ 0 \leq \beta \leq 1;\ \lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu)-p-1;$$
$$p \in \mathbb{N};\ b \in \mathbb{C}\backslash\{0\};\ z \in \mathcal{U}).$$

  Also, we let $R_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi) = R_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$.

  Here, we obtain some coefficient bounds for functions belonging to the classes $R_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ and $R_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$.

  Moerover, we define the classes $M_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ and $M_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, of $\mathcal{A}(p)$ as follows:

  A function $f(z) \in \mathcal{A}(p)$ is in the class $M_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ if

$$1+\frac{1}{b}\left\{\Psi_{\lambda,\mu,\eta}(\alpha,\beta,p)-1\right\} \prec \phi(z),$$

  where

$$\Psi_{\lambda,\mu,\eta}(\alpha,\beta,p) =$$
$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\alpha}+\beta\left[1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}\right.$$
$$\left.-(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}+\alpha\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-1\right)\right],$$

$$(\alpha \geq 0;\ \beta \geq 0;\ \lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;$$

$$p \in \mathbb{N};\ b \in \mathbb{C}\backslash\{0\};\ z \in \mathcal{U}).$$

Also, we let $M_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi) = M_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$.

We obtain some coefficient bounds for functions belonging to the classes $M_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ and $M_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$.

- Subsection 3.3.3 contains the definitions of the classes $N_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ and $N_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ of p-valent non-Bazilevič functions as follows:

  A function $f(z) \in \mathcal{A}(p)$ is in the class $N_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ if

$$1 + \frac{1}{b}\left\{(1+\beta)\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta} f(z)}\right)^{\alpha} - \beta\,\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta} f(z)}\right)^{\alpha} - 1\right\} \prec \phi(z),$$

where

$$(0 < \alpha < 1;\ \beta \in \mathbb{C};\ \lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}$$

$$b \in \mathbb{C}\backslash\{0\};\ z \in \mathcal{U}).$$

Also, we let $N_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi) = N_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$.

Here, we obtain some coefficient bounds for functions belonging to the classes $N_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ and $N_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$.

1.9.3 Differential subordination and superordination

By using the differential superordination, (Miller and Mocann, 2003) obtained conditions on $h(z)$, $q(z)$ and ψ for which the following implication holds

$$h(z) \prec \psi\big(p(z), zp'(z), z^2p''(z);\ z\big) \Rightarrow q(z) \prec p(z).$$

With the results of (Miller and Mocann, 2003), (Bulboaca, 2002a) investigated certain classes of first order differential superordinations as well as superordination-preserving integral operators (Bulboaca, 2002b). (Ali, et al., 2004) used the results obtained by (Bulboaca, 2002b) and gave the sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$. (Shanmugam et al., 2006b) obtained sufficient conditions for normalized analytic functions to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z),$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\left(f(z)\right)^2} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$.

Motivated by the above results, we investigate some results concerning an application of first order differential subordination, superordination for p-valent functions involving certain fractional derivative operators. This leads to the results presented in Chapter 4.

An outline of Chapter 4 is as follows:

- Section 4.1 is an introductory section.
- Section 4.2 contains some new differential subordination results for analytic functions associated with the operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$.

- Section 4.3 contains some new differential superordination results for analytic functions associated with the operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$.
- Section 4.4 contains some sandwich results for analytic functions associated with the operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ by combining the results of sections 4.2 and 4.3.

1.9.4 Strong Differential subordination and superordination

As a motivation of some works on strong differential subordination and superordination due to (G. Oros and Oros, 2007), (G. Oros, 2007), (G. Oros and Oros, 2009) and (G. Oros, 2009), we study strong differential subordination and superordination for p-valent functions involving certain fractional derivative operator in the open unit disk. This leads to the results presented in Chapter 5.

An outline of Chapter 5 is as follows:

- Section 5.1 is introductory section.
- Section 5.2 gives new results for strong differential subordination and superordination for analytic functions involving the operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained.
- Section 5.3 discusses some results of first order linear strong differential subordination involving the operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$.
- Section 5.4 discusses some results of strong differential subordination and superordination involving the operator $M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)$ based on the

fact that the coefficients of the functions are not constants but complex-valued functions.

Chapter 2

Properties for certain classes of p-valent functions with negative coefficients

This chapter is devoted to the study of certain classes of $T(p)$ of p-valent functions whose non-zero coefficients, from the second on, are negative defined by a fractional derivative operator with an aim to obtain coefficient conditions for functions to be in some subclasses of $T(p)$ and distortion theorems. Further results given extermal properties, closure theorems, modified Hadamard product, inclusion properties, and the radii of close-to-convexity, starlikeness, and convexity for functions belonging to those subclasses are also considered. Moreover, relevant connections of the results which are presented in this chapter with various known results are also discussed. In section 2.1, we give preliminary details which are require to prove our results. In section 2.2, we give the definition of fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ and introduce two new classes $T_{\lambda,\mu,\eta}^{*}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ of p-valent functions by using results of (Owa, 1985a). We obtain coefficient inequalities, distortion properties, and the radii of convexity for functions belonging to those classes . In section 2.3, we define the classes $S_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$ and $C_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$ of p-valent functions by using the Hadamard product in order to obtain coefficient estimates and distortion properties. Results including extreme points, modified Hadamard products, inclusion properties, and the radii of convexity, starlikeness, and close-to-convexity for functions belonging to those classes are also discussed. Section 2.4 is mainly concerned with the classes $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(p,\alpha)$ of k-uniformly p-valent functions. The results presented include coefficient estimates, distortion properties, extreme points and closure theorems. The radii of convexity, starlikenesss and close-to-convexity for functions belonging to those classes are also determined.

The results of sections 2.2 and 2.3 are, respectively, from the published papers in Sutra: Int. J. Math. Sci. Education. (Amsheri and V. Zharkova, 2011a) and Int. J. Contemp. Math. Sciences (Amsheri and V. Zharkova, 2011b), while the results of section 2.4 are from British Journal of Mathematics & Computer Science (Amsheri and V. Zharkova, 2012j) and from Int. J. Mathematics and statistics (Amsheri and V. Zharkova, 2012a).

2.1 Introduction and preliminaries

We refer to Chapter 1 for related definitions and notations used in this chapter. First, to introduce our main results in section 2.2, we consider the classes $T^*(p,\alpha,\beta,\gamma)$ and $C(p,\alpha,\beta,\gamma)$, of p-valent starlike functions with negative coefficients in $\mathcal{U}$ which were introduced by (Aouf and Hossen, 2006) and defined as follows:

A function $f(z) \in T(p)$ is said to be in the class $T^*(p,\alpha,\beta,\gamma)$ if it satisfies the condition

$$\left|\frac{\frac{zf'(z)}{g(z)}-p}{\frac{zf'(z)}{g(z)}+p-2\beta}\right|<\gamma, \qquad (z\in\mathcal{U}). \tag{2.1.1}$$

for $g(z)\in T^*(p,\alpha)(0\le\alpha<p)$ defined by

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}z^{p+n}\ , \qquad \left(b_{p+n}\ge 0;\ p\in\mathbb{N}\right). \tag{2.1.2}$$

where $0\le\beta<p$ and $0<\gamma\le 1$. If $f(z)\in T(p)$ satisfies the condition (2.1.1) for $g(z)\in C(p,\alpha)(0\le\alpha<p), 0\le\beta<p$ and $0<\gamma\le 1$, we say that the function $f(z)$ is in the class $C(p,\alpha,\beta,\gamma)$.

For these classes, results concerning coefficient estimates, distortion theorems and the radii of convexity are obtained by authors. In fact, these classes are extensions of the classes which introduced and studied by (Srivastava and Owa, 1991a) and (Srivastava and Owa, 1991b) when $p=1$.

Next, to introduce our main results in section 2.3, we consider the classes $S^*(p,\alpha,\beta)$ and $K(p,\alpha,\beta)$ of $\mathcal{A}(p)$ consisting, respectively, of functions which are p-valent starlike functions of order α and type β and p-valent convex of order α and type β which were studied by (Aouf, 1988) and (Aouf, 2007). These classes are extensions of the familiar classes were studied earlier by (Gupta and Jain, 1976) when $p=1$. For $\beta=1$, the classes $S^*(p,\alpha,1)=S^*(p,\alpha)$ and $K(p,\alpha,1)=K(p,\alpha)$ were studied by (Patil and Thakare,1983) and (Owa, 1985a), respectively. We denote by $T^*(p,\alpha,\beta)$ and $C(p,\alpha,\beta)$ the classes obtained by taking intersections, respectively, of the classes $S^*(p,\alpha,\beta)$ and $K(p,\alpha,\beta)$ with $T(p)$. The classes $T^*(p,\alpha,\beta)$ and $C(p,\alpha,\beta)$ were studied by (Aouf, 1988). In particular, for $\beta=1$, we have the classes $T^*(p,\alpha,1)=T^*(p,\alpha)$ and $C(p,\alpha,1)=C(p,\alpha)$ which were introduced by (Owa, 1985a) and the classes $T^*(1,\alpha,1) = T^*(\alpha)$ and $C(1,\alpha,1) = C(\alpha)$ when $p = 1$ and $\beta = 1$ were studied by (Silverman, 1975). Furthermore, we define the class $\mathcal{R}_\gamma^p[\alpha,\beta]$ of $T(p)$ which was studied by (Aouf, 2007) by means of the Hadamard product (or convolution) as follows:

A function $f(z) \in T(p)$ is said to be in the class $\mathcal{R}_\gamma^p[\alpha,\beta]$ if it satisfies the condition

$$\left(f * S_\gamma^p\right)(z) \in T^*(p,\alpha,\beta), \tag{2.1.3}$$

where

$$S_\gamma^p(z) = \frac{z^p}{(1-z)^{2(p-\gamma)}}, \qquad (0 \le \gamma < p;\ p \in \mathbb{N}). \tag{2.1.4}$$

The class $\mathcal{R}_\gamma^p[\alpha,\beta]$ is called the class of p-valent γ-prestarlike functions of order α and type β where $0 \le \gamma < p,\ 0 \le \alpha < p,\ 0 < \beta \le 1$ and $p \in \mathbb{N}$. The class $C_\gamma^p[\alpha,\beta]$ for functions satisfy

$$f(z) \in C_\gamma^p[\alpha,\beta] \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{R}_\gamma^p[\alpha,\beta],$$

is also studied. (Aouf, 2007) obtained several results for functions with negative coefficients belonging the classes $\mathcal{R}_\gamma^p[\alpha,\beta]$ and $C_\gamma^p[\alpha,\beta]$ such as coefficient estimates, distortion theorems, extreme points and radii of starlikeness and convexity. Further results concerning the modified Hadamard product are also established. The classes of functions $\mathcal{R}_\gamma^p[\alpha,\beta]$ and $C_\gamma^p[\alpha,\beta]$ include, as its special cases various other classes were studied in many earlier works, for example, (Ahuja and Silverman,1983), (Aouf and Silverman, 2007), (Owa and Uralegaddi, 1984), (Silverman, 1975) and (Srivastava and Aouf, 1995).

Finally, to introduce our main results in section 2.4, we consider the classes of uniformly convex functions and uniformly starlike functions which were first introduced and studied by (Goodman, 1991a) and (Goodman, 1991b), and were studied subsequently by (Rønning 1991), (Rønning 1993a), (Rønning 1993b), (Minda and Ma, 1992), (Minda and Ma, 1993) and others. More recently, (Murugusundaramoorthy and Themangani, 2009) introduced and studied certain class of uniformly convex functions based on fractional calculus operator and defined as:

A function $f(z) \in \mathcal{A}$ is said to be in the class $UCV(\alpha,\beta,\gamma)$ if it satisfies

$$\operatorname{Re}\left\{\frac{\Omega^{\alpha} f(z)}{\Omega^{\beta} f(z)} - \gamma\right\} \geq \left|\frac{\Omega^{\alpha} f(z)}{\Omega^{\beta} f(z)} - 1\right|, \tag{2.1.5}$$

where $0 \leq \gamma < 1,\ 0 \leq \alpha < 2,\ 0 \leq \beta < 2,\ z \in \mathcal{U}$ and

$$\Omega^{\delta} f(z) = \Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z).$$

We let $TUCV(\alpha,\beta,\gamma) = UCV(\alpha,\beta,\gamma) \cap T$. Here, the authors investigated some results such as coefficient estimates, extreme points and distortion bounds.

In this chapter, motivated by the above discussion we introduce new classes of p-valent functions with negative coefficients associated with certain fractional derivative operator. These classes generalize the concepts of starlike and convex, prestarlike, and uniformly starlike and uniformly convex functions. We obtain coefficient estimates and distortion theorems. Further results given extermal properties, closure theorems, modified Hadamard product, inclusion properties, and the radii of close-to-convexity, starlikeness, and convexity for functions belonging to those classes are also considered. Moreover, relevant connections of the results which are presented in this chapter with various known results are also discussed.

Let us now give the following lemmas 2.1.1 and 2.1.2 for the classes $T^*(p,\alpha)$ and $C(p,\alpha)$ following the methodology by (Owa, 1985a) which will be required in the investigation presented in the next section.

Lemma 2.1.1. Let the function $g(z)$ defined by

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \qquad \left(b_{p+n} \geq 0;\ p \in \mathbb{N}\right). \tag{2.1.6}$$

Then $g(z)$ is in the class $T^*(p,\alpha)$ if and only if

$$\sum_{n=1}^{\infty} (p+n-\alpha) b_{p+n} \leq (p-\alpha)\ . \tag{2.1.7}$$

Lemma 2.1.2. Let the function $g(z)$ defined by (2.1.6). Then $g(z)$ is in the class $C(p,\alpha)$ if and only if

$$\sum_{n=1}^{\infty} (p+n)(p+n-\alpha) b_{p+n} \leq p(p-\alpha). \tag{2.1.8}$$

2.2 Classes of p-valent starlike functions involving results of Owa

In this section we first give the definition of the modification of fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}$ (Amsheri and Zharkova, 2011a) for $f(z) \in \mathcal{A}(p)$ by

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = \phi_p(\lambda,\mu,\eta)\, z^{\mu}\, J_{0,Z}^{\lambda,\mu,\eta} f(z), \tag{2.2.1}$$

where

$$\phi_p(\lambda,\mu,\eta) = \frac{\Gamma(1-\mu+p)\Gamma(1+\eta-\lambda+p)}{\Gamma(1+p)\Gamma(1+\eta-\mu+p)}. \tag{2.2.2}$$

for $\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}$. By using (1.2.3), we can write $M_{0,z}^{\lambda,\mu,\eta} f(z)$ in the form

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \delta_n(\lambda,\mu,\eta,p)\, a_{p+n} z^{p+n}\ , \tag{2.2.3}$$

If $f(z) \in T(p)$, we can write $M_{0,z}^{\lambda,\mu,\eta} f(z)$ in the form

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p - \sum_{n=1}^{\infty} \delta_n(\lambda,\mu,\eta,p)\, a_{p+n} z^{p+n}\,, \tag{2.2.4}$$

where

$$\delta_n(\lambda,\mu,\eta,p) = \frac{\phi_p(\lambda,\mu,\eta)}{\phi_{p+n}(\lambda,\mu,\eta)} = \frac{(1+p)_n(1+\eta-\mu+p)_n}{(1-\mu+p)_n(1+\eta-\lambda+p)_n}. \tag{2.2.5}$$

It is easily verified from (2.2.3) that (Amsheri and Zharkova, 2011d)

$$z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)' = (p-\mu)\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z) + \mu\, M_{0,z}^{\lambda,\mu,\eta} f(z). \tag{2.2.6}$$

This identity plays a critical role in obtaining the information about functions defined by use of the fractional derivative operator. We note that

$$M_{0,z}^{0,0,\eta} f(z) = f(z), \qquad M_{0,z}^{1,1,\eta} f(z) = \frac{zf'(z)}{p}$$

Now, let us give the following definition of the classes $T_{\lambda,\mu,\eta}^{*}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ of p-valent starlike functions based on the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ (Amsheri and Zharkova, 2011a).

Definition 2.2.1. The function $f(z) \in T(p)$ is said to be in the class $T_{\lambda,\mu,\eta}^{*}(p,\alpha,\beta,\gamma)$ if

$$\left| \frac{\dfrac{z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{g(z)} - p}{\dfrac{z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{g(z)} + p - 2\beta} \right| < \gamma, \qquad (z \in \mathcal{U}), \tag{2.2.7}$$

$$(\lambda \geq 0;\ \mu < p+1\,;\ \eta > \max(\lambda,\mu) - p - 1\,;$$

$$0 \leq \alpha < p;\ 0 \leq \beta < p;\ 0 < \gamma \leq 1;\ p \in \mathbb{N}).$$

for the function $g(z)$ defined by (2.1.6) belonging to the class $T^*(p,\alpha)$ and $M_{0,z}^{\lambda,\mu,\eta} f(z)$ is given by (2.2.4). Further, if $f(z) \in T(p)$ satisfies the condition (2.2.7) for $g(z) \in C(p,\alpha)$, we say that $f(z) \in C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$.

The above-defined classes $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ contain many well-known classes of analytic functions. In particular, for $\lambda = \mu = 0$, we have

$$T^*_{0,0,\eta}(p,\alpha,\beta,\gamma) = T^*(p,\alpha,\beta,\gamma)$$

and

$$C_{0,0,\eta}(p,\alpha,\beta,\gamma) = C(p,\alpha,\beta,\gamma)$$

where $T^*(p,\alpha,\beta,\gamma)$ and $C(p,\alpha,\beta,\gamma)$ are precisely the classes of p-valent starlike functions which were studied by (Aouf and Hossen, 2006). Furthermore, for $\lambda = \mu = 0$ and $p = 1$, we obtain

$$T^*_{0,0,\eta}(1,\alpha,\beta,\gamma) = T^*(\alpha,\beta,\gamma)$$

and

$$C_{0,0,\eta}(1,\alpha,\beta,\gamma) = C(\alpha,\beta,\gamma)$$

where $T^*(\alpha,\beta,\gamma)$ and $C(\alpha,\beta,\gamma)$ are the classes of starlike functions which were studied by (Srivastava and Owa, 1991a) and (Srivastava and Owa, 1991b).

In next subsections let us obtain some properties for functions belonging to the classes $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$.

2.2.1 Coefficient estimates

In this subsection, we first state and prove the sufficient condition for the functions $f(z) \in T(p)$ in the form (1.2.5) to be in the class $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ according (Amsheri and Zharkova, 2011a).

Theorem 2.2.1.1. Let the function $f(z)$ defined by (1.2.5). If $f(z)$ belongs to the class $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$, then

$$\sum_{n=1}^{\infty} \delta_n(\lambda,\mu,\eta,p)(p+n)\,(1+\gamma)a_{p+n} - \frac{(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{p+n-\alpha} \le 2\gamma(p-\beta). \tag{2.2.1.1}$$

where $\delta_n(\lambda,\mu,\eta,p)$ is given by (2.2.5).

Proof. We have from (2.2.4) that

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p - \sum_{n=1}^{\infty} \frac{(1+p)_n(1+\eta-\mu+p)_n}{(1-\mu+p)_n(1+\eta-\lambda+p)_n}\, a_{p+n}\, z^{p+n}.$$

Since $f(z) \in T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$, there exist a function $g(z)$ belonging to the class $T^*(p,\alpha)$ such that

$$\left| \frac{z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)' - pg(z)}{z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)' + (p-2\beta)g(z)} \right| < \gamma, \qquad (z \in \mathcal{U}). \tag{2.2.1.2}$$

It follows from (2.2.1.2) that

$$\mathrm{Re}\left\{ \frac{\sum_{n=1}^{\infty}[\delta_n(\lambda,\mu,\eta,p)(p+n)\,a_{p+n} - p\,b_{p+n}]\ z^n}{2(p-\beta) - \sum_{n=1}^{\infty}[\delta_n(\lambda,\mu,\eta,p)(p+n)a_{p+n} + (p-2\beta)b_{p+n}]z^n} \right\} < \gamma. \tag{2.2.1.3}$$

Choosing values of z on the real axis so that $z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)'/g(z)$ is real, and letting $z \to 1^-$ through real axis, we have

$$\sum_{n=1}^{\infty}[\delta_n(\lambda,\mu,\eta,p)(p+n)\,a_{p+n} - p\,b_{p+n}] \le$$

$$\gamma\left\{2(p-\beta) - \sum_{n=1}^{\infty}[\delta_n(\lambda,\mu,\eta,p)(p+n)a_{p+n} + (p-2\beta)b_{p+n}]\right\},$$

or, equivalently,

$$\sum_{n=1}^{\infty}\{\delta_n(\lambda,\mu,\eta,p)(p+n)\,(1+\gamma)a_{p+n} - [p(1-\gamma)+2\gamma\beta]\,b_{p+n}\} \leq 2\gamma(p-\beta).$$

$$(2.2.1.4)$$

Note that, by using Lemma 2.1.1, $g(z) \in T^*(p,\alpha)$ implies

$$b_{p+n} \leq \frac{p-\alpha}{p+n-\alpha}\,. \qquad (2.2.1.5)$$

Making substituting (2.2.1.5) in (2.2.1.4), we complete the proof of Theorem 2.2.1.1.

Now we can obtain the following corollary from Theorem 2.2.1.1 (Amsheri and Zharkova, 2011a).

Corollary 2.2.1.2. Let the function $f(z)$ defined by (1.2.5) be in the class $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then

$$a_{p+n} \leq \frac{2\gamma(p-\beta)(p+n-\alpha)+(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{\delta_n(\lambda,\mu,\eta,p)(p+n)(1+\gamma)(p+n-\alpha)}. \qquad (2.2.1.6)$$

where $\delta_n(\lambda,\mu,\eta,p)$ is given by (2.2.5). The result (2.2.1.6) is sharp for a function of the form:

$$f(z)=z^p-\frac{2\gamma(p-\beta)(p+n-\alpha)+(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{\delta_n(\lambda,\mu,\eta,p)(p+n)(1+\gamma)(p+n-\alpha)}z^{p+n}, \qquad (2.2.1.7)$$

with respect to

$$g(z)=z^p-\frac{p-\alpha}{p+n-\alpha}z^{p+n}, \qquad (n \geq 1). \qquad (2.2.1.8)$$

Remark 1. By letting $p=1$, $\lambda=\mu=0$ and $\alpha=0$ in Corollary 2.2.1.2, we obtain the result which was proven by [(Gupta, 1984), Theorem 3].

In the similar manner, Lemma 2.1.2 can be used to prove the following theorem for coefficient estimates of the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ (Amsheri and Zharkova, 2011a).

Theorem 2.2.1.3. Let the function $f(z)$ defined by (1.2.5) be in the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then

$$\sum_{n=1}^{\infty}\delta_n(\lambda,\mu,\eta,p)(p+n)\,(1+\gamma)a_{p+n}-\frac{p(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{(p+n)(p+n-\alpha)}\leq 2\gamma(p-\beta). \tag{2.2.1.9}$$

where $\delta_n(\lambda,\mu,\eta,p)$ is given by (2.2.5).

Now we can obtain the following corollary from Theorem 2.2.1.3 (Amsheri and Zharkova, 2011a).

Corollary 2.2.1.4. Let the function $f(z)$ defined by (1.2.5) be in the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then

$$a_{p+n}\leq\frac{2\gamma(p-\beta)(p+n)(p+n-\alpha)+p(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{\delta_n(\lambda,\mu,\eta,p)(p+n)^2(1+\gamma)(p+n-\alpha)}. \tag{2.2.1.10}$$

where $\delta_n(\lambda,\mu,\eta,p)$ is given by (2.2.5). The result (2.2.1.10) is sharp for a function of the form:

$$f(z)=z^p-\frac{2\gamma(p-\beta)(p+n)(p+n-\alpha)+p(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{\delta_n(\lambda,\mu,\eta,p)(p+n)^2(1+\gamma)(p+n-\alpha)}z^{p+n}, \tag{2.2.1.11}$$

with respect to

$$g(z)=z^p-\frac{p(p-\alpha)}{(p+n)(p+n-\alpha)}z^{p+n},\quad (n\geq 1). \tag{2.2.1.12}$$

2.2.2 Distortion Properties

Let us investigate the modulus of the function $f(z)$ and its derivative for the class $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ (Amsheri and Zharkova, 2011a).

Theorem 2.2.2.1. Let $\lambda,\ \mu,\ \eta\in\mathbb{R}$ such that

$$\lambda\geq 0;\ \mu<p+1;\quad \eta\geq\lambda\left(1-\frac{p+2}{\mu}\right);\quad p\in\mathbb{N} \tag{2.2.2.1}$$

Also, let $f(z)$ defined by (1.2.5) be in the class $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then

$$|f(z)| \geq |z|^p - A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)\,|z|^{p+1}, \tag{2.2.2.2}$$

$$|f(z)| \leq |z|^p + A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^{p+1}, \tag{2.2.2.3}$$

$$|f'(z)| \geq p|z|^{p-1} - (\mathrm{p}+1)A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^p, \tag{2.2.2.4}$$

and

$$|f'(z)| \leq p|z|^{p-1} + (\mathrm{p}+1)A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^p. \tag{2.2.2.5}$$

for $z \in \mathcal{U}$, provided that $0 \leq \alpha < p,\ 0 \leq \beta < p$ and $0 < \gamma \leq 1$ where

$$A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma) = \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)\}}{(1+p+\eta-\mu)(p+1)^2(1+\gamma)(p+1-\alpha)}. \tag{2.2.2.6}$$

The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

Proof. We observe that the function $\delta_n(\lambda,\mu,\eta,p)$ defined by (2.2.5) satisfy the inequality

$$\delta_n(\lambda,\mu,\eta,p) \leq \delta_{n+1}(\lambda,\mu,\eta,p), \qquad (\forall n \in \mathbb{N}).$$

provided that $\eta \geq \lambda\left(1-\frac{p+2}{\mu}\right)$. Thereby, showing that $\delta_n(\lambda,\mu,\eta,p)$ is non-decreasing. Thus under conditions stated in (2.2.2.1) we have for all $n \in \mathbb{N}$

$$0 < \frac{(1+p)(1+p+\eta-\mu)}{(1+p-\mu)(1+p+\eta-\lambda)} = \delta_1(\lambda,\mu,\eta,p) \leq \delta_n(\lambda,\mu,\eta,p). \tag{2.2.2.7}$$

For $f(z) \in T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$, (2.2.1.4) implies

$$\delta_1(\lambda,\mu,\eta,p)(p+1)(1+\gamma)\sum_{n=1}^{\infty} a_{p+n} - [p(1-\gamma)+2\gamma\beta]\sum_{n=1}^{\infty} b_{p+n} \leq 2\gamma(p-\beta). \tag{2.2.2.8}$$

For $g(z) \in T^*(p,\alpha)$, Lemma 2.2.1 yields

$$\sum_{n=1}^{\infty} b_{p+n} \leq \frac{p-\alpha}{p+1-\alpha}, \tag{2.2.2.9}$$

so that (2.2.2.8) reduces to

$$\sum_{n=1}^{\infty} a_{p+n} \le \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)\}}{(1+p+\eta-\mu)(p+1)^2(1+\gamma)(p+1-\alpha)}$$

$$= A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma). \tag{2.2.2.10}$$

Consequently,

$$|f(z)| \ge |z|^p - |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n}, \tag{2.2.2.11}$$

and

$$|f(z)| \le |z|^p + |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n}. \tag{2.2.2.12}$$

On using (2.2.2.11), (2.2.2.12) and (2.2.2.10), we easily arrive at the desired results (2.2.2.2) and (2.2.2.3).

Furthermore, we note from (2.2.1.4) that

$$\delta_1(\lambda,\mu,\eta,p)(1+\gamma)\sum_{n=1}^{\infty}(p+n)a_{p+n} - [p(1-\gamma)+2\gamma\beta]\sum_{n=1}^{\infty} b_{p+n} \le 2\gamma(p-\beta), \tag{2.2.2.13}$$

which in view of (2.2.2.9), becomes

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \le \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)\}}{(1+p+\eta-\mu)(p+1)(1+\gamma)(p+1-\alpha)}$$

$$= (p+1)A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma). \tag{2.2.2.14}$$

Thus, we have

$$|f'(z)| \ge p|z|^{p-1} - |z|^p\sum_{n=1}^{\infty}(p+n)a_{p+n}, \tag{2.2.2.15}$$

and

$$|f'(z)| \le p|z|^{p-1} + |z|^p\sum_{n=1}^{\infty}(p+n)\,a_{p+n}. \tag{2.2.2.16}$$

On using (2.2.2.15), (2.2.2.16) and (2.2.2.14), we arrive at the desired results (2.2.2.4) and (2.2.2.5).

Finally, we can prove that the estimates for $|f(z)|$ and $|f'(z)|$ are sharp by taking the function

$$f(z) = z^p - \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)\}}{(1+p+\eta-\mu)(p+1)^2(1+\gamma)(p+1-\alpha)} z^{p+1}, \tag{2.2.2.17}$$

with respect to

$$g(z) = z^p - \frac{p-\alpha}{p+1-\alpha} z^{p+1} . \tag{2.2.2.18}$$

This completes the proof of Theorem 2.2.2.1.

Remark 2. By letting $p = 1,\ \lambda = \mu = 0$ and $\alpha = 0$ in Theorem 2.2.2.1, we obtain the result which was proven by [(Gupta, 1984), Theorem 4].

Let us now investigate the modulus of the function $f(z)$ and its derivative for the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ (Amsheri and Zharkova, 2011a).

Theorem 2.2.2.2. Under the conditions stated in (2.2.2.1), let the function $f(z)$ defined by (1.2.5) be in the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then

$$|f(z)| \geq |z|^p - B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^{p+1}, \tag{2.2.2.19}$$

$$|f(z)| \leq |z|^p + B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^{p+1}, \tag{2.2.2.20}$$

$$|f'(z)| \geq p|z|^{p-1} - (p+1)B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^p, \tag{2.2.2.21}$$

and

$$|f'(z)| \leq p|z|^{p-1} + (p+1)B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^p, \tag{2.2.2.22}$$

for $z \in \mathcal{U}$, provided that $0 \leq \alpha < p,\ 0 \leq \beta < p$ and $0 < \gamma \leq 1$, where

$$B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma) = \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)(p+1)(p+1-\alpha)+p(p-\alpha)[p(1-\gamma)+2\gamma\beta]\}}{(1+p+\eta-\mu)(p+1)^3(1+\gamma)(p+1-\alpha)}. \tag{2.2.2.23}$$

The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

Proof. By using Lemma 2.1.2, we have

$$\sum_{n=1}^{\infty} b_{p+n} \leq \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}. \tag{2.2.2.24}$$

Since $g(z) \in C(p,\alpha)$, the assertions (2.2.2.19), (2.2.2.20), (2.2.2.21) and (2.2.2.22) of Theorem 2.2.2.2 follow if we apply (2.2.2.24) to (2.2.1.4). The estimates for $|f(z)|$ and $|f'(z)|$ are attained by the function

$$f(z) = z^p - \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)(\mathrm{p}+1)(p+1-\alpha)+p(p-\alpha)[\mathrm{p}(1-\gamma)+2\gamma\beta]\}}{(1+p+\eta-\mu)(\mathrm{p}+1)^3(1+\gamma)(p+1-\alpha)} z^{p+1}, \tag{2.2.2.25}$$

with respect to

$$g(z) = z^p - \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)} z^{p+1}. \tag{2.2.2.26}$$

This completes the proof of Theorem 2.2.2.2.

Next let us investigate further distortion properties for the class $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ involving generalized fractional derivative operator $J_{0,Z}^{\lambda,\mu,\eta}$ (Amsheri and Zharkova, 2011a).

Theorem 2.2.2.3. Let $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu) - p - 1$ and $p \in \mathbb{N}$. Also, let the function $f(z)$ defined by (1.2.5) be in the class $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then

$$\left|J_{0,Z}^{\lambda,\mu,\eta} f(z)\right| \geq \frac{|z|^{p-\mu}}{\phi_p(\lambda,\mu,\eta)} \left\{1 - \frac{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)}{(1+\gamma)(p+1-\alpha)}|\mathrm{z}|\right\}, \tag{2.2.2.27}$$

and

$$\left|J_{0,Z}^{\lambda,\mu,\eta} f(z)\right| \leq \frac{|z|^{p-\mu}}{\phi_p(\lambda,\mu,\eta)} \left\{1 + \frac{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)}{(1+\gamma)(p+1-\alpha)}|\mathrm{z}|\right\}. \tag{2.2.2.28}$$

for $z \in \mathcal{U}$ and $\phi_p(\lambda,\mu,\eta)$ is given by (2.2.2).

Proof. Consider the function $M_{0,z}^{\lambda,\mu,\eta}f(z)$ defined by (2.2.4). With the aid of (2.2.2.7) and (2.2.2.14), we find that

$$\left|\mathrm{M}_{0,z}^{\lambda,\mu,\eta}f(z)\right| \geq |z|^p - \delta_1(\lambda,\mu,\eta,p)|z|^{p+1}\sum_{n=1}^{\infty}(p+n)a_{p+n}$$

$$\geq |z|^p - \frac{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)}{(1+\gamma)(p+1-\alpha)}|z|^{p+1}, \qquad (2.2.2.29)$$

and

$$\left|\mathrm{M}_{0,z}^{\lambda,\mu,\eta}f(z)\right| \leq |z|^p + \delta_1(\lambda,\mu,\eta,p)|z|^{p+1}\sum_{n=1}^{\infty}(p+n)a_{p+n}$$

$$\leq |z|^p + \frac{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)}{(1+\gamma)(p+1-\alpha)}|z|^{p+1}. \qquad (2.2.2.30)$$

which yields the inequalities (2.2.2.27) and (2.2.2.28) of Theorem 2.2.2.3.

In the similar manner, we can establish the distortion property for the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ (Amsheri and Zharkova, 2011a).

Theorem 2.2.2.4. Let $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu)-p-1$ and $p \in \mathbb{N}$. let the function $f(z)$ defined by (1.2.5) be in the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then

$$\left|J_{0,Z}^{\lambda,\mu,\eta}f(z)\right| \geq$$

$$\frac{|z|^{p-\mu}}{\phi_p(\lambda,\mu,\eta)}\left\{1-\frac{2\gamma(p-\beta)(\mathrm{p}+1)(p+1-\alpha)+p(p-\alpha)[\mathrm{p}(1-\gamma)+2\gamma\beta]}{(1+\mathrm{p})(1+\gamma)(p+1-\alpha)}|\mathrm{z}|\right\}, \qquad (2.2.2.31)$$

and

$$\left|J_{0,Z}^{\lambda,\mu,\eta}f(z)\right| \leq$$

$$\frac{|z|^{p-\mu}}{\phi_p(\lambda,\mu,\eta)}\left\{1+\frac{2\gamma(p-\beta)(\mathrm{p}+1)(p+1-\alpha)+p(p-\alpha)[\mathrm{p}(1-\gamma)+2\gamma\beta]}{(1+\mathrm{p})(1+\gamma)(p+1-\alpha)}|\mathrm{z}|\right\}. \qquad (2.2.2.32)$$

for $z \in \mathcal{U}$ and $\phi_p(\lambda,\mu,\eta)$ is given by (2.2.2).

Remark 3. By letting $p = 1$, $\mu = \lambda$ and using the relationship (1.6.5) in Theorem 2.2.2.3, Theorem 2.2.2.4, we obtain the results, which were proven by [(Srivastava and Owa, 1991b), Theorem 5 and Theorem 6, respectively].

2.2.3 Radii of Convexity

Let us solve the radius of convexity problem that is to determine the largest disk $|z| < r\ (0 < r \leq 1)$ such that each function $f(z)$ in the class $T^*_{\lambda,\mu,\eta}(p, \alpha, \beta, \gamma)$ is p-valent convex in $|z| < r$ (Amsheri and Zharkova, 2011a).

Theorem 2.2.3.1. Let the function $f(z)$ defined by (1.2.5) be in the class $T^*_{\lambda,\mu,\eta}(p, \alpha, \beta, \gamma)$. Then $f(z)$ is p-valent convex in the disk $|z| < r_1$, where

$$r_1 = \inf_{n\in\mathbb{N}} \left\{ \frac{p^2}{(p+1)(p+n)\, A_{\lambda,\mu,\eta}(p, \alpha, \beta, \gamma)} \right\}^{1/n}. \qquad (2.2.3.1)$$

and $A_{\lambda,\mu,\eta}(p, \alpha, \beta, \gamma)$ is given by (2.2.2.6).

Proof. It suffices to prove

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p\,, \qquad (|z| < r_1). \qquad (2.2.3.2)$$

Indeed we have

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| = \left| \frac{-\sum_{n=1}^{\infty} n(p+n) a_{p+n} z^n}{p - \sum_{n=1}^{\infty} (p+\mathrm{n}) a_{p+n} z^n} \right|$$

$$\leq \frac{\sum_{n=1}^{\infty} n(p+n)\, a_{p+n} |z|^n}{p - \sum_{n=1}^{\infty} (p+n) a_{p+n} |z|^n}. \qquad (2.2.3.3)$$

Hence (2.2.3.2) is true if

$$\sum_{n=1}^{\infty} n(p+n)\, a_{p+n} |z|^n \leq p^2 - \sum_{n=1}^{\infty} p(p+n) a_{p+n} |z|^n, \qquad (2.2.3.4)$$

that is, if

$$\sum_{n=1}^{\infty}(p+n)^2\, a_{p+n}|z|^n \leq p^2. \tag{2.2.3.5}$$

with the aid of (2.2.2.14), (2.2.3.5) is true if

$$(p+n)|z|^n \leq \frac{p^2}{(p+1)A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)}\,, \tag{2.2.3.6}$$

Solving (2.2.3.6) for $|z|$, we get

$$|z| \leq \left\{\frac{p^2}{(p+1)(p+n)\, A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)}\right\}^{1/n}, \qquad (n \geq 1). \tag{2.2.3.7}$$

This completes the proof of Theorem 2.2.3.1.

In the similar manner, we can find the radius of convexity for functions in the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$.

Theorem 2.2.3.2. Let the function $f(z)$ defined by (1.2.5) be in the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then $f(z)$ is p-valent convex in the disk $|z| < r_2$, where

$$r_2 = \inf_{n\in\mathbb{N}}\left\{\frac{p^2}{(p+1)(p+n)\, B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)}\right\}^{1/n}, \tag{2.2.3.8}$$

and $B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ is given by (2.2.2.23).

2.3 Classes of p-valent starlike and convex functions involving the Hadamard product

In this section we introduce new certain classes of p-valent starlike and convex functions with negative coefficients by using the Hadamard product (or convolution) involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ given by (2.2.4) and investigate some properties for functions belonging to these classes. Let us begin with the following definition according to (Amsheri and V. Zharkova, 2011b).

Definition 2.3.1. A function $f(z) \in T(p)$ is said to be in the class $S_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$ if and only if

$$\left| \frac{\dfrac{z\left(\Omega_p^{\lambda,\mu,\eta} f(z)\right)'}{\Omega_p^{\lambda,\mu,\eta} f(z)} - p}{\dfrac{z\left(\Omega_p^{\lambda,\mu,\eta} f(z)\right)'}{\Omega_p^{\lambda,\mu,\eta} f(z)} + p - 2\alpha} \right| < \beta, \qquad (z \in \mathcal{U}). \qquad (2.3.1)$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}$$

$$0 \leq \alpha < p;\ 0 < \beta \leq 1;\ a,c \in \mathbb{R}\backslash\{0,-1,-2,\dots\}).$$

with

$$\Omega_p^{\lambda,\mu,\eta} f(z) = \varphi_p(a,c;z) * M_{0,z}^{\lambda,\mu,\eta} f(z), \qquad (2.3.2)$$

where

$$\varphi_p(a,c;z) = z^p + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{p+n}.$$

and $M_{0,z}^{\lambda,\mu,\eta} f(z)$ is given by (2.2.4). Further, a function $f(z) \in T(p)$ is said to be in the class $C_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$ if and only if

$$\frac{zf'(z)}{p} \in S_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta). \qquad (2.3.3)$$

We note that, by specifying the parameters $a, c, \alpha, \beta, \lambda, \mu$ and p for those generalized classes, we obtain the most of the subclasses which were studied by various authors:

1. For $a = c$ and $\lambda = \mu = 0$, we get $S_{0,0,\eta}^{p}(a,a,\alpha,\beta) = T^*(p,\alpha,\beta)$, that is the class of p-valent starlike functions of order α and type β, which was studied by (Aouf, 1988).

2. For $a = c$, $\lambda = \mu = 0$ and $p = 1$, we have $S_{0,0,\eta}^{1}(a, a, \alpha, \beta) = S^*(\alpha, \beta)$, that is the class of starlike functions of order α and type β, which was studied by (Gupta and Jain, 1976).
3. For $a = c$, $\lambda = \mu = 0$ and $\beta = 1$, we obtain the class $S_{0,0,\eta}^{p}(a, a, \alpha, 1) = T^*(p, \alpha)$, which was introduced by (Owa, 1985a).
4. For $a = c$, $\lambda = \mu = 0$, $p = 1$ and $\beta = 1$, we have $S_{0,0,\eta}^{1}(a, a, \alpha, 1) = T^*(\alpha)$, which was studied by (Silverman, 1975).
5. For $a = 2(p - \gamma)$ $(0 \leq \gamma < p)$, $c = 1$ and $\lambda = \mu = 0$, we obtain $S_{0,0,\eta}^{p}(2(p - \gamma), 1, \alpha, \beta) = \mathcal{R}_{\gamma}^{p}[\alpha, \beta]$, that is the class of p-valent γ-prestarlike functions of order α and type β, which was studied by (Aouf, 2007).
6. For $a = 2(p - \gamma)$ $(0 \leq \gamma < p)$, $c = 1$, $\lambda = \mu = 0$ and $\beta = 1$, we have $S_{0,0,\eta}^{p}(2(p - \gamma), 1, \alpha, 1) = \mathcal{R}^{p}[\gamma, \alpha]$, that is the class of p-valent γ-prestarlike functions of order α, which was studied by (Aouf and Silverman, 2007).
7. For $a = c$, $\lambda = \mu = 1$ and $p = 1$, we have the class $C_{1,1,\eta}^{1}(a, a, \alpha, \beta) = C^*(\alpha, \beta)$, which was studied by (Gupta and Jain, 1976).
8. For $a = c$ and $\lambda = \mu = 1$, we have the class $C_{1,1,\eta}^{p}(a, a, \alpha, \beta) = C(p, \alpha, \beta)$, that is the class of p-valent convex functions of order α and type β, which was studied by (Aouf, 1988).
9. For $a = c$, $\lambda = \mu = 1$ and $\beta = 1$, we have the class $C_{1,1,\eta}^{p}(a, a, \alpha, 1) = C(p, \alpha)$, which was studied by (Owa, 1985a).
10. For $a = c$, $\lambda = \mu = 1$, $p = 1$ and $\beta = 1$, we obtain the class $C_{1,1,\eta}^{1}(a, a, \alpha, 1) = C(\alpha)$, which was studied by (Silverman, 1975).

11. For $a = 2(p-\gamma)$ $(0 \le \gamma < p)$, $c = 1$ and $\lambda = \mu = 1$, we obtain the class $C^p_{1,1,\eta}(2(p-\gamma), 1, \alpha, \beta) = C^p_\gamma[\alpha, \beta]$, which was studied by (Aouf, 2007).

12. For $a = 2(p-\gamma)$ $(0 \le \gamma < p)$, $c = 1$, $\lambda = \mu = 1$ and $\beta = 1$, we have $C^p_{1,1,\eta}(2(p-\gamma), 1, \alpha, 1) = C^p[\gamma, \alpha]$, which was studied by (Aouf and Silverman, 2007).

Thus, the generalization classes $S^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$ and $C^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$ defined in this section is proven to account for most available classes discussed in the previous papers and generalize the concept of prestarlike functions.

In the next subsections let us obtain some properties for functions belonging to the classes $S^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$ and $C^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$.

2.3.1 Coefficient estimates

In this subsection we state and prove the necessary and sufficient conditions for functions to be in the classes $S^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$according to (Amsheri and V. Zharkova, 2011b).

Theorem 2.3.1.1. Let the function $f(z)$ to be defined by (1.2.5). Then $f(z)$ belongs to the class $S^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty} [n + \beta(n + 2p - 2\alpha)]\, \Delta^p_n(a, c, \lambda, \mu, \eta) a_{p+n} \le 2\beta(p-\alpha), \qquad (2.3.1.1)$$

where

$$\Delta^p_n(a, c, \lambda, \mu, \eta) = \frac{(a)_n(1+p)_n(1+\eta-\mu+p)_n}{(c)_n(1-\mu+p)_n(1+\eta-\lambda+p)_n}\,. \qquad (2.3.1.2)$$

Proof. We have from (2.3.2) that

$$\Omega^{\lambda,\mu,\eta}_p f(z) = z^p - \sum_{n=1}^{\infty} \frac{(a)_n(1+p)_n(1+\eta-\mu+p)_n}{(c)_n(1-\mu+p)_n(1+\eta-\lambda+p)_n}\, a_{p+n} z^{p+n}. \qquad (2.3.1.3)$$

Let the function $f(z)$ be in the class $S^{p}_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$. Then in view of (2.3.1), we have

$$\left|\frac{\dfrac{z\left(\Omega_p^{\lambda,\mu,\eta}f(z)\right)'}{\Omega_p^{\lambda,\mu,\eta}f(z)}-p}{\dfrac{z\left(\Omega_p^{\lambda,\mu,\eta}f(z)\right)'}{\Omega_p^{\lambda,\mu,\eta}f(z)}+p-2\alpha}\right| =$$

$$\left|\frac{\sum_{n=1}^{\infty} n\,\Delta_n^p(a,c,\lambda,\mu,\eta)a_{p+n}\, z^n}{2(p-\alpha)-\sum_{n=1}^{\infty}(n+2p-2\alpha)\Delta_n^p(a,c,\lambda,\mu,\eta)a_{p+n}\, z^n}\right| < \beta. \qquad (2.3.1.4)$$

Since $|\mathrm{Re}(z)| \leq |z|$ for all z we have

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty} n\,\Delta_n^p(a,c,\lambda,\mu,\eta)a_{p+n}\, z^n}{2(p-\alpha)-\sum_{n=1}^{\infty}(n+2p-2\alpha)\Delta_n^p(a,c,\lambda,\mu,\eta)a_{p+n}\, z^n}\right\} < \beta. \qquad (2.3.1.5)$$

Choosing values of z on the real axis so that $z\left(\Omega_p^{\lambda,\mu,\eta}f(z)\right)'/\,\Omega_p^{\lambda,\mu,\eta}f(z)$ is real, and letting $z \to 1^-$ through real axis, we get

$$\sum_{n=1}^{\infty} n\,\Delta_n^p(a,c,\lambda,\mu,\eta)a_{p+n} \leq$$

$$\beta\left\{2(p-\alpha)-\sum_{n=1}^{\infty}(n+2p-2\alpha)\Delta_n^p(a,c,\lambda,\mu,\eta)a_{p+n}\right\}.$$

which implies that the assertion (2.3.1.1).

Conversely, let the inequality (2.3.1.1) holds true, then

$$\left|z\left(\Omega_p^{\lambda,\mu,\eta}f(z)\right)' - p\,\Omega_p^{\lambda,\mu,\eta}f(z)\right| - \beta\left|z\left(\Omega_p^{\lambda,\mu,\eta}f(z)\right)' + (p-2\alpha)\,\Omega_p^{\lambda,\mu,\eta}f(z)\right| \leq$$

$$\sum_{n=1}^{\infty}[n+\beta(n+2p-2\alpha)]\,\Delta_n^p(a,c,\lambda,\mu,\eta)a_{p+n} - 2\beta(p-\alpha) \leq 0\,. \qquad (2.3.1.6)$$

by the assumption. This implies that $f(z) \in S^{p}_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$.

Now we can obtain the following corollary from Theorem 2.3.1.1 according (Amsheri and Zharkova, 2011b).

Corollary 2.3.1.2. If the function $f(z)$ is in the class $S^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$, then

$$a_{p+n} \leq \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]\Delta^p_n(a,c,\lambda,\mu,\eta)}, \quad (p,n \in \mathbb{N}). \qquad (2.3.1.7)$$

where $\Delta^p_n(a,c,\lambda,\mu,\eta)$ is given by (2.3.1.2). The result (2.3.1.7) is sharp for the function $f(z)$ of the form

$$f(z) = z^p - \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]\Delta^p_n(a,c,\lambda,\mu,\eta)} z^{p+n}, \quad (p,n \in \mathbb{N}). \quad (2.3.1.8)$$

In the similar manner, we can establish the necessary and sufficient conditions for functions to be in the classes $C^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ according (Amsheri and V. Zharkova, 2011b).

Theorem 2.3.1.3. The function $f(z)$ belongs to the class $C^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ if and only if

$$\sum_{n=1}^{\infty}(p+n)[n+\beta(n+2p-2\alpha)]\,\Delta^p_n(a,c,\lambda,\mu,\eta)a_{p+n} \leq 2\beta p(p-\alpha). \quad (2.3.1.9)$$

where $\Delta^p_n(a,c,\lambda,\mu,\eta)$ is given by (2.3.1.2).

Now we can obtain the following corollary from Theorem 2.3.1.3 (Amsheri and V. Zharkova, 2011b).

Corollary 2.3.1.4. If the function $f(z)$ is in the class $C^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$, then

$$a_{p+n} \leq \frac{2\beta p(p-\alpha)}{(p+n)[n+\beta(n+2p-2\alpha)]\Delta^p_n(a,c,\lambda,\mu,\eta)}, \quad (p,n \in \mathbb{N}). \quad (2.3.1.10)$$

where $\Delta^p_n(a,c,\lambda,\mu,\eta)$ is given by (2.3.1.2). The result (2.3.1.10) is sharp for the function $f(z)$ of the form

$$f(z) = z^p - \frac{2\beta p(p-\alpha)}{(p+n)[n+\beta(n+2p-2\alpha)]\Delta^p_n(a,c,\lambda,\mu,\eta)} z^{p+n}, \quad (p,n \in \mathbb{N}).$$

(2.3.1.11)

2.3.2 Distortion Properties

Let us find the modulus of $f(z)$ and its derivative for the class $S^{p}_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ according to (Amsheri and V. Zharkova, 2011b).

Theorem 2.3.2.1. Let $\lambda,\ \mu,\ \eta \in \mathbb{R}$ such that $a, c \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$; $\lambda \geq 0$; $\mu < p+1$; $\eta \leq \lambda\left(1-\frac{p+2}{\mu}\right)$; $0 \leq \alpha < p$; $0 < \beta \leq 1$. If $f(z)$ belongs to the class $S^{p}_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$, then

$$|f(z)| \geq |z|^{p} - \frac{2\beta c(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{a[1+\beta(1+2p-2\alpha)](1+p)(1+p+\eta-\mu)}|z|^{p+1},$$

$$(2.3.2.1)$$

$$|f(z)| \leq |z|^{p} + \frac{2\beta c(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{a[1+\beta(1+2p-2\alpha)](1+p)(1+p+\eta-\mu)}|z|^{p+1},$$

$$(2.3.2.2)$$

$$|f'(z)| \geq p|z|^{p-1} - \frac{2\beta c(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{a[1+\beta(1+2p-2\alpha)](1+p+\eta-\mu)}|z|^{p},$$

$$(2.3.2.3)$$

and

$$|f'(z)| \leq p|z|^{p-1} + \frac{2\beta c(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{a[1+\beta(1+2p-2\alpha)](1+p+\eta-\mu)}|z|^{p}.$$

$$(2.3.2.4)$$

for $z \in \mathcal{U}$ and $p \in \mathbb{N}$. The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

Proof. Under the hypothesis of the theorem, we observe that the function $\Delta^{p}_{n}(a,c,\lambda,\mu,\eta)$is a decreasing function for $n \geq 1$, that is

$$0 < \Delta^{p}_{n+1}(a,c,\lambda,\mu,\eta) \leq \Delta^{p}_{n}(a,c,\lambda,\mu,\eta),$$

for all $n \in \mathbb{N}$, thus

$$0 < \Delta^{p}_{n+1}(a,c,\lambda,\mu,\eta) \leq \Delta^{p}_{1}(a,c,\lambda,\mu,\eta)$$

$$= \frac{a(1+p)(1+\eta-\mu+p)}{c(1-\mu+p)(1+\eta-\lambda+p)}. \qquad (2.3.2.5)$$

Therefore from (2.3.1.1) we have

$$\sum_{n=1}^{\infty} a_{p+n} \leq \frac{2\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)]\Delta_1^p(a,c,\lambda,\mu,\eta)}$$

$$= \frac{2\beta c(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{a[1+\beta(1+2p-2\alpha)](1+p)(1+p+\eta-\mu)}. \qquad (2.3.2.6)$$

since

$$|f(z)| \geq |z|^p - |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n}, \qquad (2.3.2.7)$$

and

$$|f(z)| \leq |z|^p + |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n}. \qquad (2.3.2.8)$$

On using (2.3.2.6) to (2.3.2.7) and (2.3.2.8), we easily arrive at the desired results (2.3.2.1) and (2.3.2.2). Furthermore, we observe that

$$|f'(z)| \geq p|z|^{p-1} - (p+1)|z|^p \sum_{n=1}^{\infty} a_{p+n}, \qquad (2.3.2.9)$$

and

$$|f'(z)| \leq p|z|^{p-1} + (p+1)|z|^p \sum_{n=1}^{\infty} a_{p+n}. \qquad (2.3.2.10)$$

On using (2.3.2.6) to (2.3.2.9) and (2.3.2.10), we easily arrive at the desired results (2.3.2.3) and (2.3.2.4).

Finally, we can see that the estimates for $|f(z)|$ and $|f'(z)|$ are sharp for the function

$$f(z) = z^p - \frac{2\beta c(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{a[1+\beta(1+2p-2\alpha)](1+p)(1+p+\eta-\mu)} z^{p+1}. \qquad (2.3.2.11)$$

The proof is complete.

In the similar manner, we can establish the following distortion properties for functions in the class $C^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ (Amsheri and Zharkova, 2011b).

Theorem 2.3.2.2. Let $\lambda,\ \mu,\ \eta \in \mathbb{R}$ such that $a,c \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$; $\lambda \geq 0$; $\mu < p+1$; $\eta \leq \lambda\left(1-\frac{p+2}{\mu}\right)$; $0 \leq \alpha < p$; $0 < \beta \leq 1$. If $f(z)$ belongs to the class $C^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$, then

$$|f(z)| \geq |z|^p - \frac{2\beta pc(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{a[1+\beta(1+2p-2\alpha)](1+p)^2(1+p+\eta-\mu)}|z|^{p+1}, \tag{2.3.2.12}$$

$$|f(z)| \leq |z|^p + \frac{2\beta pc(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{a[1+\beta(1+2p-2\alpha)](1+p)^2(1+p+\eta-\mu)}|z|^{p+1}, \tag{2.3.2.13}$$

$$|f'(z)| \geq p|z|^{p-1} - \frac{2\beta pc(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{a[1+\beta(1+2p-2\alpha)](1+p)(1+p+\eta-\mu)}|z|^p, \tag{2.3.2.14}$$

and

$$|f'(z)| \leq p|z|^{p-1} + \frac{2\beta pc(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{a[1+\beta(1+2p-2\alpha)](1+p)(1+p+\eta-\mu)}|z|^p. \tag{2.3.2.15}$$

for $z \in \mathcal{U}$ and $p \in \mathbb{N}$. The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

In the similar manner, we can establish further distortion properties for the class $S^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ involving the operator $\Omega_p^{\lambda,\mu,\eta}$ defined by (2.3.2) (Amsheri and Zharkova, 2011b).

Theorem 2.3.2.3. Let $\lambda \geq 0$; $\mu < p+1$; $\eta > \max\{\lambda,\mu\} - p - 1$; $0 \leq \alpha < p$; $0 < \beta \leq 1$; $a,c \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$ and $p \in \mathbb{N}$. Also, let the function $f(z)$ be in the class $S^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$. Then

$$\left| \Omega_p^{\lambda,\mu,\eta} f(z) \right| \geq |z|^p - \frac{2\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)]} |z|^{p+1},$$

$$\left| \Omega_p^{\lambda,\mu,\eta} f(z) \right| \leq |z|^p + \frac{2\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)]} |z|^{p+1} ,$$

$$\left| \left(\Omega_p^{\lambda,\mu,\eta} f(z) \right)' \right| \geq p|z|^{p-1} - \frac{2\beta(p-\alpha)(1+p)}{[1+\beta(1+2p-2\alpha)]} |z|^p,$$

and

$$\left| \left(\Omega_p^{\lambda,\mu,\eta} f(z) \right)' \right| \leq p|z|^{p-1} + \frac{2\beta(p-\alpha)(1+p)}{[1+\beta(1+2p-2\alpha)]} |z|^p.$$

for $z \in \mathcal{U}$ and $\Omega_p^{\lambda,\mu,\eta} f(z)$ is defined by (2.3.2).

Also, we can establish further distortion properties for the class $C_{\lambda,\mu,\eta}^p(a,c,\alpha,\beta)$ involving the operator $\Omega_p^{\lambda,\mu,\eta}$ defined by (2.3.2) (Amsheri and Zharkova, 2011b)

Theorem 2.3.2.4. Let $\lambda \geq 0$; $\mu < p+1$; $\eta > \max\{\lambda,\mu\} - \mathrm{p} - 1$; $0 \leq \alpha < p$; $0 < \beta \leq 1$; $a, c \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$ and $p \in \mathbb{N}$. Also, let the function $f(z)$ be in the class $C_{\lambda,\mu,\eta}^p(a,c,\alpha,\beta)$. Then

$$\left| \Omega_p^{\lambda,\mu,\eta} f(z) \right| \geq |z|^p - \frac{2\beta p(p-\alpha)}{(1+p)[1+\beta(1+2p-2\alpha)]} |z|^{p+1},$$

$$\left| \Omega_p^{\lambda,\mu,\eta} f(z) \right| \leq |z|^p + \frac{2\beta p(p-\alpha)}{(1+p)[1+\beta(1+2p-2\alpha)]} |z|^{p+1} ,$$

$$\left| \left(\Omega_p^{\lambda,\mu,\eta} f(z) \right)' \right| \geq p|z|^{p-1} - \frac{2\beta p(p-\alpha)}{[1+\beta(1+2p-2\alpha)]} |z|^p,$$

and

$$\left| \left(\Omega_p^{\lambda,\mu,\eta} f(z) \right)' \right| \leq p|z|^{p-1} + \frac{2\beta p(p-\alpha)}{[1+\beta(1+2p-2\alpha)]} |z|^p.$$

for $z \in \mathcal{U}$ and $\Omega_p^{\lambda,\mu,\eta} f(z)$ is defined by (2.3.2).

2.3.3 Extreme points

Let us investigate the extreme points which are functions belonging to the class $S^{p}_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ following (Amsheri and Zharkova, 2011b).

Theorem 2.3.3.1. Let

$$f_p(z) = z^p, \qquad (p \in \mathbb{N}), \tag{2.3.3.1}$$

and

$$f_{p+n}(z) = z^p - \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]\Delta^p_n(a,c,\lambda,\mu,\eta)} z^{p+n}, \qquad (p,n \in \mathbb{N}). \tag{2.3.3.2}$$

Then $f(z) \in S^{p}_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \varepsilon_{p+n} f_{p+n}(z), \tag{2.3.3.3}$$

where

$$\varepsilon_{p+n} \geq 0, \quad \sum_{n=0}^{\infty} \varepsilon_{p+n} = 1\,. \tag{2.3.3.4}$$

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} \varepsilon_{p+n} f_{p+n}(z)$$

$$= z^p - \sum_{n=1}^{\infty} \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]\Delta^p_n(a,c,\lambda,\mu,\eta)} \varepsilon_{p+n} z^{p+n}\,. \tag{2.3.3.5}$$

Then, in view of (2.3.3.4), it follows that

$$\sum_{n=1}^{\infty} \frac{[n+\beta(n+2p-2\alpha)]\Delta^p_n(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)} \left\{ \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]\Delta^p_n(a,c,\lambda,\mu,\eta)} \varepsilon_{p+n} \right\}$$

$$= \sum_{n=1}^{\infty} \varepsilon_{p+n} = 1 - \varepsilon_p \leq 1\,. \tag{2.3.3.6}$$

So, by Theorem 2.3.1.1, $f(z)$ belongs to the class $S^{p}_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$.

Conversely, let the function $f(z)$ belongs to the class $S^{p}_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$. Then

$$a_{p+n} \leq \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]\Delta^{p}_{n}(a,c,\lambda,\mu,\eta)}, \qquad (p,n \in \mathbb{N}). \qquad (2.3.3.7)$$

Setting

$$\varepsilon_{p+n} = \frac{[n+\beta(n+2p-2\alpha)]\Delta^{p}_{n}(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)} a_{p+n}, \quad (p,n \in \mathbb{N}). \qquad (2.3.3.8)$$

and

$$\varepsilon_p = 1 - \sum_{n=1}^{\infty} \varepsilon_{p+n}. \qquad (2.3.3.9)$$

we see that $f(z)$ can be expressed in the form (2.3.3.3). This completes the proof of the Theorem 2.3.3.1.

Now we can obtain the following corollary from Theorem 2.3.3.1 according to (Amsheri and Zharkova, 2011b).

Corollary 2.3.3.2. The extreme points of the class $S^{p}_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ are the functions $f_p(z)$ and $f_{p+n}(z)$, given by (2.3.3.1) and (2.3.3.2), respectively.

In the similar manner, we can obtain the extreme points for the class $C^{p}_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$.

Theorem 2.3.3.3. Let

$$f_p(z) = z^p, \qquad (p \in \mathbb{N}). \qquad (2.3.3.10)$$

and

$$f_{p+n}(z) = z^p - \frac{2\beta p(p-\alpha)}{(p+n)[n+\beta(n+2p-2\alpha)]\Delta^{p}_{n}(a,c,\lambda,\mu,\eta)} z^{p+n}, \quad (p,n \in \mathbb{N}).$$

(2.3.3.11)

Then $f(z) \in C^{p}_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \varepsilon_{p+n} f_{p+n}(z), \qquad (2.3.3.12)$$

where

$$\varepsilon_{p+n} \geq 0, \quad \sum_{n=0}^{\infty} \varepsilon_{p+n} = 1 . \qquad (2.3.3.13)$$

Now we can obtain the following corollary from Theorem 2.3.3.3 according to (Amsheri and Zharkova, 2011b).

Corollary 2.3.3.4. The extreme points of the class $C_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$ are the functions $f_p(z)$ and $f_{p+n}(z)$ given by (2.3.3.10) and (2.3.3.11), respectively.

2.3.4 Modified Hadmard Products

Let us obtain the Hadamard product of any two functions in the class $S_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$ following (Amsheri and Zharkova, 2011b).

Theorem 2.3.4.1. Let the functions $f_i(z)$ $(i = 1, 2)$ defined by

$$f_i(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,i}\, z^{p+n}, \qquad (p \in \mathbb{N}). \qquad (2.3.4.1)$$

be in the class $S_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$. Then $(f_1 * f_2)(z) \in S_{\lambda,\mu,\eta}^{p}(a,c,\delta,\beta)$, where

$$\delta = p - \frac{2\beta c(1+\beta)(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}{a(1+p)(1+p+\eta-\mu)[1+\beta(1+2p-2\alpha)]^2 - 4\beta^2 c(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}. \qquad (2.3.4.2)$$

Proof. To prove the theorem, we need to find the largest δ such that

$$\sum_{n=1}^{\infty} \frac{[n+\beta(n+2p-2\delta)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\delta)} a_{p+n,1} a_{p+n,2} \leq 1 . \qquad (2.3.4.3)$$

since

$$\sum_{n=1}^{\infty} \frac{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)} a_{p+n,1} \leq 1, \qquad (2.3.4.4)$$

and

$$\sum_{n=1}^{\infty} \frac{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)} a_{p+n,2} \le 1\,. \qquad (2.3.4.5)$$

we have

$$\sum_{n=1}^{\infty} \frac{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)} \sqrt{a_{p+n,1}a_{p+n,2}} \le 1. \qquad (2.3.4.6)$$

Thus, it is sufficient to show that

$$\frac{[n+\beta(n+2p-2\delta)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{(p-\delta)} a_{p+n,1}a_{p+n,2} \le$$

$$\frac{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{(p-\alpha)} \sqrt{a_{p+n,1}a_{p+n,2}}\,, \qquad (2.3.4.7)$$

That is, that

$$\sqrt{a_{p+n,1}a_{p+n,2}} \le \frac{[n+\beta(n+2p-2\alpha)](p-\delta)}{[n+\beta(n+2p-2\delta)](p-\alpha)}\,. \qquad (2.3.4.8)$$

Note that

$$\sqrt{a_{p+n,1}a_{p+n,2}} \le \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}\,, \qquad (n \in \mathbb{N}). \quad (2.3.4.9)$$

Consequently, we need only to prove that

$$\frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)} \le \frac{[n+\beta(n+2p-2\alpha)](p-\delta)}{[n+\beta(n+2p-2\delta)](p-\alpha)},$$

$$(2.3.4.10)$$

or, equivalently that

$$\delta \le p - \frac{2\beta(1+\beta)n(p-\alpha)^2}{[n+\beta(n+2p-2\alpha)]^2\Delta_n^p(a,c,\lambda,\mu,\eta) - 4\beta^2(p-\alpha)^2}\,. \qquad (2.3.4.11)$$

Let

$$A(n) = p - \frac{2\beta(1+\beta)n(p-\alpha)^2}{[n+\beta(n+2p-2\alpha)]^2\Delta_n^p(a,c,\lambda,\mu,\eta) - 4\beta^2(p-\alpha)^2}\,.$$

$$(2.3.4.12)$$

Letting $n = 1$ in (2.3.4.12), we obtain

$$\delta = p - \frac{2\beta c(1+\beta)(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}{a(1+p)(1+p+\eta-\mu)[1+\beta(1+2p-2\alpha)]^2-4\beta^2 c(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}. \qquad (2.3.4.13)$$

which completes the proof of Theorem 2.3.4.1.

In the similar manner, we can obtain the Hadamard product of any two functions in the class $C^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ according to (Amsheri and Zharkova, 2011b).

Theorem 2.3.4.2. Let the functions $f_i(z)$ $(i = 1,2)$ defined by (2.3.4.1) be in the class $C^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$. Then $(f_1 * f_2)(z) \in C^p_{\lambda,\mu,\eta}(a,c,\sigma,\beta)$, where

$$\sigma = p - \frac{2\beta pc(1+\beta)(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}{a(1+p)^2(1+p+\eta-\mu)[1+\beta(1+2p-2\alpha)]^2-4\beta^2 pc(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}.$$

2.3.5 Inclusion properties

In this subsection let us investigate inclusion property for any two functions in the classes $S^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ according to (Amsheri and Zharkova, 2011b).

Theorem 2.3.5.1 Let the functions $f_i(z)$ $(i = 1,2)$ defined by (2.3.4.1) be in the class $S^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$. Then the function

$$h(z) = z^p - \sum_{n=1}^{\infty} \left(a^2_{p+n,1} + a^2_{p+n,2}\right) z^{p+n}. \qquad (2.3.5.1)$$

belongs to the class $S^p_{\lambda,\mu,\eta}(a,c,\delta,\beta)$, where

$$\delta = p - \frac{4\beta c(1+\beta)(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}{a(1+p)(1+p+\eta-\mu)[1+\beta(1+2p-2\alpha)]^2-8\beta^2 c(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}. \qquad (2.3.5.2)$$

Proof. By virtue of Theorem 2.3.1.1, we obtain

$$\sum_{n=1}^{\infty} \left\{\frac{[n+\beta(n+2p-2\alpha)]\Delta^p_n(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)}\right\}^2 a^2_{p+n,1} \leq$$

$$\left\{\sum_{n=1}^{\infty}\frac{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)}\ a_{p+n,1}\right\}^2 \leq 1, \tag{2.3.5.3}$$

and

$$\sum_{n=1}^{\infty}\left\{\frac{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)}\right\}^2 a_{p+n,2}^2 \leq$$

$$\left\{\sum_{n=1}^{\infty}\frac{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)}\ a_{p+n,2}\right\}^2 \leq 1. \tag{2.3.5.4}$$

It follows from (2.3.5.3) and (2.3.5.4) that

$$\sum_{n=1}^{\infty}\frac{1}{2}\left\{\frac{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)}\right\}^2\left(a_{p+n,1}^2+a_{p+n,2}^2\right)\leq 1. \tag{2.3.5.5}$$

Therefore we need to find the largest δ such that

$$\frac{[n+\beta(n+2p-2\delta)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\delta)} \leq$$

$$\frac{1}{2}\left\{\frac{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)}\right\}^2, \tag{2.3.5.6}$$

that is

$$\delta \leq p-\frac{4\beta(1+\beta)n(p-\alpha)^2}{[n+\beta(n+2p-2\alpha)]^2\Delta_n^p(a,c,\lambda,\mu,\eta)-8\beta^2(p-\alpha)^2}. \tag{2.3.5.7}$$

Let

$$B(n)=p-\frac{4\beta(1+\beta)n(p-\alpha)^2}{[n+\beta(n+2p-2\alpha)]^2\Delta_n^p(a,c,\lambda,\mu,\eta)-8\beta^2(p-\alpha)^2}. \tag{2.3.5.8}$$

Letting $n=1$ in (2.3.5.8), we obtain

$$\delta=p-\frac{4\beta c(1+\beta)(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}{a(1+p)(1+p+\eta-\mu)[1+\beta(1+2p-2\alpha)]^2-8\beta^2c(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}. \tag{2.3.5.9}$$

which completes the proof of this theorem.

In the similar manner, we can establish the inclusion property for the class $C_{\lambda,\mu,\eta}^p(a,c,\alpha,\beta)$.

Theorem 2.3.5.2. Let the functions $f_i(z)$ $(i = 1, 2)$ defined by (2.3.4.1) be in the class $C^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$. Then the function $h(z)$ defined by (2.3.5.1) belongs to the class $C^p_{\lambda,\mu,\eta}(a, c, \sigma, \beta)$, where

$$\sigma = p - \frac{4\beta pc(1+\beta)(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}{a(1+p)^2(1+p+\eta-\mu)[1+\beta(1+2p-2\alpha)]^2-8\beta^2 pc(p-\alpha)^2(1+p-\mu)(1+p+\eta-\lambda)}.$$

Next let us investigate further inclusion property for functions in the class $S^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$ according to (Amsheri and Zharkova, 2011b).

Theorem 2.3.5.3. Let the functions $f_i(z)$ $(i = 1, 2, \ldots m)$ be in the class $S^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$. Then the function

$$h(z) = z^p - \frac{1}{m}\sum_{n=1}^{\infty}\sum_{i=1}^{m} a_{p+n,i}\, z^{p+n}\,. \tag{2.3.5.10}$$

belongs to the class $S^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$.

Proof. Since $f_i(z) \in S^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$, by Theorem 2.3.1.1, we have

$$\sum_{n=1}^{\infty} \frac{[n + \beta(n + 2p - 2\alpha)]\Delta^p_n(a, c, \lambda, \mu, \eta)}{2\beta(p-\alpha)}\ a_{p+n,i} \le 1, \quad (i = 1,2, \ldots m).$$

so

$$\sum_{n=1}^{\infty} \frac{[n + \beta(n + 2p - 2\alpha)]\Delta^p_n(a, c, \lambda, \mu, \eta)}{2\beta(p-\alpha)} \left(\frac{1}{m}\sum_{i=1}^{m} a_{p+n,i}\right) =$$

$$\frac{1}{m}\sum_{i=1}^{m}\left\{\sum_{n=1}^{\infty} \frac{[n + \beta(n + 2p - 2\alpha)]\Delta^p_n(a, c, \lambda, \mu, \eta)}{2\beta(p-\alpha)}\right\} a_{p+n,i} \le 1.$$

which shows that $f(z) \in S^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$.

In the similar manner, we can establish further inclusion property for functions in the class $C^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$ according to (Amsheri and Zharkova, 2011b).

Theorem 2.3.5.4. Let the functions $f_i(z)\ (i = 1, 2, \ldots, m)$ be in the class $C^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$. Then the function $h(z)$ defined by (2.3.5.10) belongs to the class $C^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$.

2.3.6 Radii of close-to-convexity, starlikeness, and convexity

Let us obtain the largest disk for functions in the class $S^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$ to be p-valent close-to-convex according to (Amsheri and Zharkova, 2011b).

Theorem 2.3.6.1. Let the function $f(z)$ be in the class $S^p_{\lambda,\mu,\eta}(a, c, \alpha, \beta)$. Then $f(z)$ is p-valent close-to-convex of order $\delta\ (0 \le \delta < p)$ in $|z| < r_1$, where

$$r_1 = \inf_{n\in\mathbb{N}} \left\{ \frac{(p-\delta)[n+\beta(n+2p-2\alpha)]\Delta^p_n(a, c, \lambda, \mu, \eta)}{2\beta(p-\alpha)(p+n)} \right\}^{1/n} \tag{2.3.6.1}$$

and $\Delta^p_n(a, c, \lambda, \mu, \eta)$ is given by (2.3.1.2). The result is sharp with the extremal function $f(z)$ given by (2.3.1.8).

Proof. It suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \le p - \delta \qquad (|z| < r_1) \tag{2.3.6.2}$$

Indeed, we have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \le \sum_{n=1}^{\infty} (p+n) a_{p+n} |z|^n \tag{2.3.6.3}$$

Hence (2.3.6.3) is true if

$$\sum_{n=1}^{\infty} (p+n) a_{p+n} |z|^n \le p - \delta$$

or

$$\sum_{n=1}^{\infty} \frac{(p+n)}{(p-\delta)} a_{p+n} |z|^n \le 1 \tag{2.3.6.4}$$

By Theorem 2.3.1.1, (2.3.6.4) is true if

$$\frac{(p+n)|z|^n}{(p-\delta)} \leq \frac{[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)} \quad (n \geq 1) \quad (2.3.6.5)$$

Solving (2.3.6.5) for $|z|$, we get the desired result (2.3.6.1).

In the similar manner, we can obtain the radii of starlikeness for functions in the class $S_{\lambda,\mu,\eta}^p(a,c,\alpha,\beta)$ according to (Amsheri and Zharkova, 2011b).

Theorem 2.3.6.2. Let the function $f(z)$ be in the class $S_{\lambda,\mu,\eta}^p(a,c,\alpha,\beta)$. Then $f(z)$ is p-valently starlike of order δ $(0 \leq \delta < p)$ in $|z| < r_2$, where

$$r_2 = \inf_{n\in\mathbb{N}} \left\{\frac{(p-\delta)[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)(n+p-\delta)}\right\}^{1/n}$$

and $\Delta_n^p(a,c,\lambda,\mu,\eta)$ is given by (2.3.1.2). The result is sharp with the extremal function $f(z)$ given by (2.3.1.8).

Also, we can obtain the radii of convexity for functions in the class $S_{\lambda,\mu,\eta}^p(a,c,\alpha,\beta)$ according to (Amsheri and Zharkova, 2011b).

Theorem 2.3.6.3. Let the function $f(z)$ be in the class $S_{\lambda,\mu,\eta}^p(a,c,\alpha,\beta)$. Then $f(z)$ is p-valently convex of order δ $(0 \leq \delta < p)$ in $|z| < r_3$, where

$$r_3 = \inf_{n\in\mathbb{N}} \left\{\frac{p(p-\delta)[n+\beta(n+2p-2\alpha)]\Delta_n^p(a,c,\lambda,\mu,\eta)}{2\beta(p-\alpha)(p+n)(n+p-\delta)}\right\}^{1/n}$$

and $\Delta_n^p(a,c,\lambda,\mu,\eta)$ is given by (2.3.1.2). The result is sharp with the extremal function $f(z)$ given by (2.3.1.8).

2.4 Classes of k-uniformly p-valent starlike and convex functions

In this section we introduce new certain classes of k-uniformly p-valent starlike and convex functions defined by the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ given by (2.2.1) and investigate some properties for functions belonging to these classes. Let us begin with the following definition according to (Amsheri and Zharkova, 2012j).

Definition 2.4.1. The function $f(z) \in \mathcal{A}(p)$ is said to be in the class $k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ if and only if

$$\mathrm{Re}\left\{\frac{pM_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\beta,\gamma,\xi} f(z)} - \alpha\right\} \geq k\left|\frac{pM_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\beta,\gamma,\xi} f(z)} - p\right|, \qquad (z \in \mathcal{U}). \tag{2.4.1}$$

for

$$(k \geq 0;\ 0 \leq \alpha < p;\ \lambda \geq 0;\ 0 \leq \mu < 1+p;\ \beta \geq 0;\ 0 \leq \gamma < 1+p;$$

$$\eta > \max(\lambda,\mu) - p - 1;\ \xi > \max(\beta,\gamma) - p - 1).$$

where $M_{0,z}^{\lambda,\mu,\eta} f(z)$ and $M_{0,z}^{\beta,\gamma,\xi} f(z)$ are given by (2.2.1). We let

$$k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha) = k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha) \cap T(p). \tag{2.4.2}$$

The above-defined class $k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ contain subclass $k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(\alpha)$ of k-uniformly starlike and convex functions when $p = 1$ for $f(z) \in \mathcal{A}$ which satisfies the condition (Amsheri and Zharkova, 2012a)

$$\mathrm{Re}\left\{\frac{P_{0,z}^{\lambda,\mu,\eta} f(z)}{P_{0,z}^{\beta,\gamma,\xi} f(z)} - \alpha\right\} \geq k\left|\frac{P_{0,z}^{\lambda,\mu,\eta} f(z)}{P_{0,z}^{\beta,\gamma,\xi} f(z)} - 1\right|, \qquad (z \in \mathcal{U}), \tag{2.4.3}$$

where $P_{0,z}^{\lambda,\mu,\eta} f(z)$ is defined by (1.6.8). We let

$$k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(\alpha) = k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha) \cap T. \tag{2.4.4}$$

Also, for $\mu = \lambda,\ \gamma = \beta,\ k = 1$ and $p = 1$, we have

$$1 - UCV_{\beta,\beta,\xi}^{\lambda,\lambda,\eta}(1,\alpha) = UCV(\lambda,\beta,\alpha),$$

and

$$1 - TUCV_{\beta,\beta,\xi}^{\lambda,\lambda,\eta}(1,\alpha) = TUCV(\lambda,\beta,\alpha).$$

where $UCV(\lambda,\beta,\alpha)$ and $TUCV(\lambda,\beta,\alpha)$ are precisely the subclasses of uniformly convex functions which were studied by (Gurugusundaramoorthy and Themangani, 2009). Furthermore, by specifying the parameters $\lambda,\ \mu,\ \beta,\ \gamma,\ k,\ \alpha$ and p, we obtain the most of subclasses which were studied by various other authors:

1. For $\mu = \lambda = 1,\ \beta = \gamma = 0$ and $k = 1$, the class $k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ can be reduced to $UST(p,\alpha)$, the class of uniformly p-valent starlike functions of order α, see (Al-Kharsani and AL-Hajiry, 2006).
2. For $\mu = \lambda = 1,\ \beta = \gamma = 0,\ p = 1$ and $k = 1$, we obtain $UST(\alpha)$, the class of uniformly starlike functions of order α, see (Owa, 1998) and (Rønning, 1991).
3. For $\mu = \lambda = 1,\ \beta = \gamma = 0,\ p = 1,\ \alpha = 0$ and $k = 1$, we obtain UST, the class of uniformly starlike functions, see (Goodman, 1991b).
4. For $\mu = \lambda = 1,\ \beta = \gamma = 0$ and $k = 0$, we obtain $S^*(p,\alpha)$, the class of all p-valent starlike functions of order α, see (Partil and Thakare, 1983).
5. For $\mu = \lambda = 1,\ \beta = \gamma = 0,\ p = 1$ and $k = 0$, we have $S^*(\alpha)$, the class of starlike functions of order α, see (Duren, 1983), (Jack, 1971), (Robertson, 1936), (Pinchuk, 1968) and (Schild, 1965).
6. For $\mu = \lambda = 1,\ \beta = \gamma = 0,\ p = 1,\ \alpha = 0$ and $k = 0$, we have S^*, the class of starlike functions , see (Duren, 1983).

Thus, the generalization class $k-UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ defined in this section is proven to account for most available subclasses discussed in the previous papers and generalize the concept of uniformy starlike and uniformly convex functions.

In the next subsections let us obtain some properties of functions belonging to the classes $k-UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ and $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$.

2.4.1 Coefficient estimates

In this subsection we start with the coefficient estimates for the class $k-UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ following (Amsheri and Zharkova, 2012j).

Theorem 2.4.1.1. The function $f(z)$ defined by (1.2.3) is in the class $k-UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ if

$$\sum_{n=1}^{\infty}[p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]\left|a_{p+n}\right|\leq p-\alpha, \qquad (2.4.1.1)$$

where

$$\delta_n(\lambda,\mu,\eta,p)=\frac{\phi_p(\lambda,\mu,\eta)}{\phi_{p+n}(\lambda,\mu,\eta)}=\frac{(1+p)_n(1+\eta-\mu+p)_n}{(1-\mu+p)_n(1+\eta-\lambda+p)_n}, \qquad (2.4.1.2)$$

and

$$\delta_n(\beta,\gamma,\xi,p)=\frac{\phi_p(\beta,\gamma,\xi)}{\phi_{p+n}(\beta,\gamma,\xi)}=\frac{(1+p)_n(1+\xi-\gamma+p)_n}{(1-\gamma+p)_n(1+\xi-\beta+p)_n}. \qquad (2.4.1.3)$$

with $\phi_p(\lambda,\mu,\eta)$ and $\phi_p(\beta,\gamma,\xi)$ are given by (2.2.2).

Proof. We have from (2.2.3) that

$$M_{0,z}^{\lambda,\mu,\eta}f(z)=z^p+\sum_{n=1}^{\infty}\delta_n(\lambda,\mu,\eta,p)\,a_{p+n}z^{p+n},$$

and

$$M_{0,z}^{\beta,\gamma,\xi} f(z) = z^p + \sum_{n=1}^{\infty} \delta_n(\beta,\gamma,\xi,p)\, a_{p+n} z^{p+n}.$$

Since $f(z) \in k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$, it suffices to show that

$$k \left| \frac{pM_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\beta,\gamma,\xi} f(z)} - p \right| - \operatorname{Re}\left\{ \frac{pM_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\beta,\gamma,\xi} f(z)} - p \right\} \leq p - \alpha,$$

Notice that

$$k \left| \frac{pM_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\beta,\gamma,\xi} f(z)} - p \right| - \operatorname{Re}\left\{ \frac{pM_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\beta,\gamma,\xi} f(z)} - p \right\} \leq (1+k) \left| \frac{pM_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\beta,\gamma,\xi} f(z)} - p \right|$$

$$\leq (1+k) \left| \frac{\sum_{n=1}^{\infty} p[\delta_n(\lambda,\mu,\eta,p) - \delta_n(\beta,\gamma,\xi,p)] a_{p+n}\, z^{p+n}}{z^p + \sum_{n=1}^{\infty} \delta_n(\beta,\gamma,\xi,p)\, a_{p+n} z^{p+n}} \right|$$

$$\leq \frac{(1+k)\sum_{n=1}^{\infty} p[\delta_n(\lambda,\mu,\eta,p) - \delta_n(\beta,\gamma,\xi,p)] |a_{p+n}|}{1 - \sum_{n=1}^{\infty} \delta_n(\beta,\gamma,\xi,p)\, |a_{p+n}|}.$$

The last inequality above is bounded by $(p - \alpha)$ if

$$\sum_{n=1}^{\infty} [p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)] |a_{p+n}| \leq p - \alpha.$$

This completes the proof.

Now by letting $p = 1$ in Theorem 2.4.1.1 we obtain the coefficient estimates for the class $k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$ following (Amsheri and Zharkova, 2012a).

Theorem 2.4.1.2. The function $f(z)$ defined by (1.2.2) is in the class $k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(\alpha)$ if

$$\sum_{n=2}^{\infty} [(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha)\delta_n(\beta,\gamma,\xi)] |a_n| \leq 1 - \alpha\,, \tag{2.4.1.4}$$

where

$$\delta_n(\lambda,\mu,\eta) = \frac{(2)_{n-1}(2+\eta-\mu)_{n-1}}{(2-\mu)_{n-1}(2+\eta-\lambda)_{n-1}}, \tag{2.4.1.5}$$

and

$$\delta_n(\beta,\gamma,\xi) = \frac{(2)_{n-1}(2+\xi-\gamma)_{n-1}}{(2-\gamma)_{n-1}(2+\xi-\beta)_{n-1}}\,. \qquad (2.4.1.6)$$

Proof. We have from (1.6.8) that

$$P_{0,z}^{\lambda,\mu,\eta} f(z) = z + \sum_{n=2}^{\infty} \delta_n(\lambda,\mu,\eta)\, a_n z^n,$$

and

$$P_{0,z}^{\beta,\gamma,\xi} f(z) = z + \sum_{n=2}^{\infty} \delta_n(\beta,\gamma,\xi)\, a_n z^n.$$

Since $f(z) \in k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$, it suffices to show that

$$k\left|\frac{P_{0,z}^{\lambda,\mu,\eta} f(z)}{P_{0,z}^{\beta,\gamma,\xi} f(z)} - 1\right| - \mathrm{Re}\left\{\frac{P_{0,z}^{\lambda,\mu,\eta} f(z)}{P_{0,z}^{\beta,\gamma,\xi} f(z)} - 1\right\} \le 1-\alpha.$$

Notice that

$$k\left|\frac{P_{0,z}^{\lambda,\mu,\eta} f(z)}{P_{0,z}^{\beta,\gamma,\xi} f(z)} - 1\right| - \mathrm{Re}\left\{\frac{P_{0,z}^{\lambda,\mu,\eta} f(z)}{P_{0,z}^{\beta,\gamma,\xi} f(z)} - 1\right\} \le (1+k)\left|\frac{P_{0,z}^{\lambda,\mu,\eta} f(z)}{P_{0,z}^{\beta,\gamma,\xi} f(z)} - 1\right|$$

$$\le (1+k)\left|\frac{\sum_{n=2}^{\infty}[\delta_n(\lambda,\mu,\eta) - \delta_n(\beta,\gamma,\xi)]a_n\, z^n}{z + \sum_{n=2}^{\infty}\, \delta_n(\beta,\gamma,\xi)\, a_n z^n}\right|$$

$$\le \frac{(1+k)\sum_{n=2}^{\infty}[\delta_n(\lambda,\mu,\eta) - \delta_n(\beta,\gamma,\xi)]|a_n\,|}{1 - \sum_{n=2}^{\infty}\delta_n(\beta,\gamma,\xi)\,|a_n\,|}.$$

The last inequality above is bounded by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty}[(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha)\delta_n(\beta,\gamma,\xi)]|a_n| \le 1-\alpha.$$

This completes the proof.

Next, let us obtain the necessary and sufficient conditions for $f(z)$ to be in the classes $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ following (Amsheri and Zharkova, 2012j).

Theorem 2.4.1.3. The function $f(z)$ defined by (1.2.5) is in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(p,\alpha)$ if and only if

$$\sum_{n=1}^{\infty}[p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]a_{p+n} \leq p-\alpha. \qquad (2.4.1.7)$$

where $\delta_n(\lambda,\mu,\eta,p)$ and $\delta_n(\beta,\gamma,\xi,p)$ are given by (2.4.1.2) and (2.4.1.3) respectively.

Proof. In view of Theorem 2.4.1.1, we need to prove the sufficient part. Let $f(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ and z be real, then by the inequality (2.4.1)

$$\mathrm{Re}\left\{\frac{pM_{0,z}^{\lambda,\mu,\eta}f(z)}{M_{0,z}^{\beta,\gamma,\xi}f(z)}-\alpha\right\} \geq k\left|\frac{pM_{0,z}^{\lambda,\mu,\eta}f(z)}{M_{0,z}^{\beta,\gamma,\xi}f(z)}-p\right|,$$

or

$$\frac{p-\sum_{n=1}^{\infty}p\,\delta_n(\lambda,\mu,\eta,p)\,a_{p+n}\,z^n}{1-\sum_{n=1}^{\infty}\delta_n(\beta,\gamma,\xi,p)a_{p+n}\,z^n}-\alpha \geq$$

$$k\left|\frac{\sum_{n=1}^{\infty}p[\delta_n(\lambda,\mu,\eta,p)-\delta_n(\beta,\gamma,\xi,p)]a_{p+n}\,z^n}{1-\sum_{n=1}^{\infty}\delta_n(\beta,\gamma,\xi,p)a_{p+n}\,z^{p+n}}\right|.$$

Letting $z \to 1$ along the real axis, we obtain

$$\frac{(p-\alpha)-\sum_{n=1}^{\infty}[p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]a_{p+n}}{1-\sum_{n=1}^{\infty}\delta_n(\beta,\gamma,\xi,p)a_{p+n}} \geq 0.$$

This is only possible if (2.4.1.7) holds. Therefore we obtain the desired result and the proof is complete.

Next, let us obtain the necessary and sufficient condition for $f(z)$ to be in the classes $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$, following (Amsheri and Zharkova, 2012a).

Theorem 4.2.1.4. The function $f(z)$ defined by (1.2.4) is in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(\alpha)$ if and only if

$$\sum_{n=2}^{\infty}[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)]a_n \le 1-\alpha\,. \qquad (2.4.1.8)$$

where $\delta_n(\lambda,\mu,\eta)$ and $\delta_n(\beta,\gamma,\xi)$ are given by (2.4.1.5) and (2.4.1.6) respectively.

Proof. In view of Theorem 2.4.1.2, we need to prove the sufficient part. Let $f(z) \in k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(\alpha)$ and z be real, then by the inequality (2.4.3)

$$\mathrm{Re}\left\{\frac{P_{0,z}^{\lambda,\mu,\eta}f(z)}{P_{0,z}^{\beta,\gamma,\xi}f(z)}-\alpha\right\} \ge k\left|\frac{P_{0,z}^{\lambda,\mu,\eta}f(z)}{P_{0,z}^{\beta,\gamma,\xi}f(z)}-1\right|,$$

or

$$\frac{1-\sum_{n=2}^{\infty}\delta_n(\lambda,\mu,\eta)\,a_{p+n}\,z^{n-1}}{1-\sum_{n=2}^{\infty}\delta_n(\beta,\gamma,\xi)a_n\,z^{n-1}}-\alpha \ge k\left|\frac{\sum_{n=2}^{\infty}[\delta_n(\lambda,\mu,\eta)-\delta_n(\beta,\gamma,\xi)]a_n\,z^{n-1}}{1-\sum_{n=2}^{\infty}\delta_n(\beta,\gamma,\xi)a_n\,z^{n-1}}\right|.$$

Letting $z \to 1$ along the real axis, we obtain

$$\frac{(1-\alpha)-\sum_{n=2}^{\infty}[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)]a_n}{1-\sum_{n=2}^{\infty}\delta_n(\beta,\gamma,\xi)a_n} \ge 0.$$

This is only possible if (2.4.1.8) holds. Therefore we obtain the desired results and he proof is complete.

Now we can obtain the following corollary from Theorem 2.4.1.3 according to (Amsheri and Zharkova, 2012j).

Corollary 2.4.1.5. Let the function $f(z)$ defined by (1.2.5) be in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$, then

$$a_{p+n} \le \frac{p-\alpha}{[p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}, \quad (p,n \in \mathbb{N}).$$

with equality for the function $f(z)$ given by

$$f(z) = z^p - \frac{p-\alpha}{[p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}z^{p+n}, \quad (p,n \in \mathbb{N}).$$

(2.4.1.9)

Also we can obtain the following corollary from Theorem 2.4.1.4 according to (Amsheri and Zharkova, 2012a).

Corollary 2.4.1.6. Let the function $f(z)$ defined by (1.2.4) be in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(\alpha)$, then

$$a_n \leq \frac{1-\alpha}{[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)]}, \quad (n \geq 2).$$

with equality for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)]} z^n, \quad (n \geq 2). \quad (2.4.1.10)$$

2.4.2 Distortion properties

Next let us obtain the modulus for functions $f(z)$ belonging to the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ according to (Amsheri and Zharkova, 2012j).

Theorem 2.4.2.1. Let the function $f(z)$ defined by (1.2.5) be in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ such that $k \geq 0,\ 0 \leq \alpha < p,\ \lambda \geq 0,\ 0 \leq \mu < 1+p,\ \beta \geq 0,$ $0 \leq \gamma < 1+p,\ \gamma \leq \mu,\ \eta \geq \lambda\left(1-\frac{2+p}{\mu}\right)$ and $\xi \geq \beta\left(1-\frac{2+p}{\gamma}\right)$. Then

$$|z|^p - A_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)|z|^{p+1} \leq |f(z)| \leq |z|^p + A_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)|z|^{p+1}, \quad (2.4.2.1)$$

where

$$A_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha) = \frac{p-\alpha}{[p(1+k)\delta_1(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_1(\beta,\gamma,\xi,p)]}. \quad (2.4.2.2)$$

The estimates for $|f(z)|$ are sharp.

Proof. We observe that the functions $\delta_n(\lambda,\mu,\eta,p)$ and $\delta_n(\beta,\gamma,\xi,p)$ defined by (2.4.1.2) and (2.4.1.3), respectively, satisfy the inequalities $\delta_n(\lambda,\mu,\eta,p) \leq \delta_{n+1}(\lambda,\mu,\eta,p)$ and $\delta_n(\beta,\gamma,\xi,p) \leq \delta_{n+1}(\beta,\gamma,\xi,p),\ \forall n \in \mathbb{N}$ provided that

$\eta \geq \lambda\left(1-\frac{2+p}{\mu}\right)$ and $\xi \geq \beta\left(1-\frac{2+p}{\gamma}\right)$. So $\delta_n(\lambda,\mu,\eta,p)$ and $\delta_n(\beta,\gamma,\xi,p)$ are non-decreasing functions for all $n \in \mathbb{N}$

$$0<\frac{(1+p)(1+\eta-\mu+p)}{(1-\mu+p)(1+\eta-\lambda+p)}=\delta_1(\lambda,\mu,\eta,p) \leq \delta_n(\lambda,\mu,\eta,p), \qquad (2.4.2.3)$$

and

$$0<\frac{(1+p)(1+\xi-\gamma+p)}{(1-\gamma+p)(1+\xi-\beta+p)}=\delta_1(\beta,\gamma,\xi,p) \leq \delta_n(\beta,\gamma,\xi,p). \qquad (2.4.2.4)$$

Since $f(z) \in k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$, then

$$[p(1+k)\delta_1(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_1(\beta,\gamma,\xi,p)]\sum_{n=1}^{\infty}a_{p+n} \leq$$

$$\sum_{n=1}^{\infty}[p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]a_{p+n} \leq p-\alpha\,. \qquad (2.4.2.5)$$

So that (2.4.2.5) reduces to

$$\sum_{n=1}^{\infty}a_{p+n} \leq \frac{p-\alpha}{[p(1+k)\delta_1(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_1(\beta,\gamma,\xi,p)]}=A_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha).$$

(2.4.2.6)

From (1.2.5), we obtain

$$|f(z)| \leq |z|^p+|z|^{p+1}\sum_{n=1}^{\infty}a_{p+n}, \qquad (2.4.2.7)$$

and

$$|f(z)| \geq |z|^p-|z|^{p+1}\sum_{n=1}^{\infty}a_{p+n}\,. \qquad (2.4.2.8)$$

on using (2.4.2.6) to (2.4.2.7) and (2.4.2.8), we arrive at the desired result (2.4.2.1).

Finally, we can see that the estimates for $|f(z)|$ are sharp by taking the function

$$f(z) = z^p - \frac{p-\alpha}{[p(1+k)\delta_1(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_1(\beta,\gamma,\xi,p)]} z^{p+1}. \quad (2.4.2.9)$$

This completes the proof of Theorem 2.4.2.1.

Now by letting $p = 1$ in Theorem 2.4.2.1, we can obtain the modulus for functions $f(z)$ belonging to the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$ according to (Amsheri and Zharkova, 2012a).

Theorem 2.4.2.2. Let the function $f(z)$ defined by (1.2.4) be in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(\alpha)$ such that $k \geq 0, 0 \leq \alpha < 1, \lambda \geq 0,\ 0 \leq \mu < 2,\ \beta \geq 0,\ 0 \leq \gamma < 2,\ \gamma \leq \mu,\ \eta \geq \lambda\left(\frac{\mu-3}{\mu}\right)$ and $\xi \geq \beta\left(\frac{\gamma-3}{\gamma}\right)$. Then

$$|f(z)| \leq |z| + A_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)|z|^2, \quad (2.4.2.10)$$

and

$$|f(z)| \geq |z| - A_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)|z|^2. \quad (2.4.2.11)$$

where

$$A_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha) = \frac{1-\alpha}{[(1+k)\delta_2(\lambda,\mu,\eta) - (k+\alpha)\delta_2(\beta,\gamma,\xi)]}. \quad (2.4.2.12)$$

The estimates for $|f(z)|$ are sharp.

Proof. We observe that the functions $\delta_n(\lambda,\mu,\eta)$ and $\delta_n(\beta,\gamma,\xi)$ defined by (2.4.1.5) and (2.4.1.6), respectively, satisfy the inequalities $\delta_n(\lambda,\mu,\eta) \leq \delta_{n+1}(\lambda,\mu,\eta)$ and $\delta_n(\beta,\gamma,\xi) \leq \delta_{n+1}(\beta,\gamma,\xi)$, $\forall n \geq 2$ provided that $\eta \geq \lambda\left(\frac{\mu-3}{\mu}\right)$ and $\xi \geq \beta\left(\frac{\gamma-3}{\gamma}\right)$. So $\delta_n(\lambda,\mu,\eta)$ and $\delta_n(\beta,\gamma,\xi)$ are non-decreasing functions for all $n \geq 2$

$$0 < \frac{2(2+\eta-\mu)}{(2-\mu)(2+\eta-\lambda)} = \delta_2(\lambda,\mu,\eta) \leq \delta_n(\lambda,\mu,\eta), \quad (2.4.2.13)$$

and

$$0 < \frac{2(2+\xi-\gamma)}{(2-\gamma)(2+\xi-\beta)} = \delta_2(\beta,\gamma,\xi) \leq \delta_n(\beta,\gamma,\xi). \qquad (2.4.2.14)$$

Since $f(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$, then

$$[(1+k)\delta_2(\lambda,\mu,\eta) - (k+\alpha)\delta_2(\beta,\gamma,\xi)] \sum_{n=2}^{\infty} a_n \leq$$

$$\sum_{n=2}^{\infty} [(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha)\delta_n(\beta,\gamma,\xi)] a_n \leq 1-\alpha\,. \qquad (2.4.2.15)$$

So that (2.4.2.15) reduces to

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\alpha}{[(1+k)\delta_2(\lambda,\mu,\eta) - (k+\alpha)\delta_2(\beta,\gamma,\xi)]} = A_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)\,. \qquad (2.4.2.16)$$

From (1.2.4), we obtain

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n, \qquad (2.4.2.17)$$

and

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n\,. \qquad (2.4.2.18)$$

On using (2.4.2.16) to (2.4.2.17) and (2.4.2.18), we arrive at the desired results (2.4.2.10) and (2.4.2.11). Finally, we can prove that the estimate for $|f(z)|$ are sharp by taking the function

$$f(z) = z - \frac{1-\alpha}{[(1+k)\delta_2(\lambda,\mu,\eta) - (k+\alpha)\delta_2(\beta,\gamma,\xi)]} z^2\,. \qquad (2.4.2.19)$$

This completes the proof of Theorem 2.4.2.2.

2.4.3 Extreme points

Let us obtain the extreme points for the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$, following (Amsheri and Zharkova, 2012j).

Theorem 2.4.3.1. Let $f_p(z) = z^p$ $(p \in \mathbb{N})$ and

$$f_{p+n}(z) = z^p - \frac{p-\alpha}{[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]} z^{p+n}, \quad (p,n \in \mathbb{N}). \tag{2.4.3.1}$$

Then $f(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \theta_{p+n} f_{p+n}(z). \tag{2.4.3.2}$$

where $\theta_{p+n} \geq 0$ and $\sum_{n=0}^{\infty} \theta_{p+n} = 1$.

Proof. Let $f(z)$ be expressible in the form

$$f(z) = \sum_{n=0}^{\infty} \theta_{p+n} f_{p+n}(z).$$

Then

$$f(z) = z^p - \sum_{n=1}^{\infty} \frac{p-\alpha}{[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]} \theta_{p+n} z^{p+n}.$$

Now

$$\sum_{n=1}^{\infty} \Bigg(\frac{[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{p-\alpha}$$

$$\left\{ \frac{(p-\alpha)\theta_{p+n}}{[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]} \theta_{p+n} \right\} \Bigg)$$

$$= \sum_{n=1}^{\infty} \theta_{p+n} = 1 - \theta_p \leq 1.$$

Therefore, $f(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$.

Conversely, suppose that $f(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$. Thus

$$a_{p+n} \leq \frac{p-\alpha}{[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}, \qquad (p,n \in \mathbb{N}).$$

Setting

$$\theta_{p+n} = \frac{[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{p-\alpha} a_{p+n},$$

and

$$\theta_p = 1 - \sum_{n=1}^{\infty} \theta_{p+n}.$$

we see that $f(z)$ can be expressed in the form (2.4.3.2).The proof is complete.

Now by letting $p = 1$ in Theorem 2.4.3.1, we can obtain the extreme points for the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$, following (Amsheri and Zharkova, 2012a).

Theorem 2.4.3.2. Let $f_1(z) = z$ and

$$f_n(z) = z - \frac{1-\alpha}{[(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha)\delta_n(\beta,\gamma,\xi)]} z^n, \quad (n \geq 2). \qquad (2.4.3.3)$$

Then $f(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \theta_n f_n(z). \qquad (2.4.3.4)$$

where $\theta_n \geq 0$ and $\sum_{n=0}^{\infty} \theta_n = 1$.

Proof. Let $f(z)$ be expressible in the form

$$f(z) = \sum_{n=1}^{\infty} \theta_n f_n(z).$$

Then

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1-\alpha}{[(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha)\delta_n(\beta,\gamma,\xi)]} \theta_n z^n.$$

Now

$$\sum_{n=2}^{\infty} \left(\frac{[(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha)\delta_n(\beta,\gamma,\xi)]}{1-\alpha} \right.$$

$$\left\{\frac{(1-\alpha)\theta_n}{[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)]}\theta_n\right\}\Bigg)$$

$$=\sum_{n=2}^{\infty}\theta_n = 1-\theta_1 \le 1.$$

Therefore, $f(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(\alpha)$.

Conversely, suppose that $f(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(\alpha)$. Thus

$$a_n \le \frac{1-\alpha}{[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)]}, \qquad (n \ge 2).$$

Setting

$$\theta_n = \frac{[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)]}{1-\alpha}a_n,$$

and

$$\theta_1 = 1-\sum_{n=2}^{\infty}\theta_n.$$

we see that $f(z)$ can be expressed in the form (2.4.3.4). The proof is complete.

Now from Theorem 2.4.3.1 we have the following corollary for functions in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$, following (Amsheri and Zharkova, 2012j).

Corollary 2.4.3.3. The extreme points of the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ are

$$f_p(z) = z^p,$$

and

$$f_{p+n}(z) = z^p - \frac{p-\alpha}{[p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}z^{p+n}, (p,n \in \mathbb{N}).$$

Also from Theorem 2.4.3.2 we have the following corollary for functions in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$, following (Amsheri and Zharkova, 2012a).

Corollary 2.4.3.4. The extreme points of the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(\alpha)$ are

$$f_1(z) = z,$$

and

$$f_n(z) = z - \frac{1-\alpha}{[(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha)\delta_n(\beta,\gamma,\xi)]} z^n, \quad (n \geq 2).$$

2.4.4 Closure properties

Let the function $f(z) \in T(p)$ defined by (1.2.5) and the function $g(z)$ be in the class $T(p)$ defined by (2.1.6), the class $T(p)$ is said to be convex if

$$\rho f(z) + (1-\rho)g(z) \in T(p)$$

where $0 \leq \rho \leq 1$.

Now let us prove that the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ is convex according to (Amsheri an Zharkova, 2012j).

Theorem 2.4.4.1. The class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ is convex.

Proof. Let $f(z)$ defined by (1.2.5) and $g(z)$ defined by (2.1.6) be in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$, then

$$\rho f(z) + (1-\rho)g(z) = z^p - \sum_{n=1}^{\infty} \left[\rho a_{p+n} + (1-\rho)\, b_{p+n}\right] z^{p+n}.$$

Applying Theorem 2.4.1.2 for the functions $f(z)$ and $g(z)$, we get

$$\sum_{n=1}^{\infty} [p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]\left[\rho a_{p+n} + (1-\rho)\, b_{p+n}\right] \leq$$

$$\rho(p-\alpha) + (1-\rho)(p-\alpha) = (p-\alpha).$$

This completes the proof of the Theorem 2.4.4.1.

Next by letting $p = 1$ in Theorem 2.4.4.1 we can prove that the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$ is convex according to (Amsheri and (Zharkova, 2012a).

Theorem 2.4.4.2. The class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$ is convex.

Proof. Let $f(z)$ defined by (1.2.4) and $g(z)$ defined by

$$g(z)=z-\sum_{n=2}^{\infty}b_n z^n, \tag{2.4.4.1}$$

be in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$, then

$$\rho f(z)+(1-\rho)g(z)=z-\sum_{n=2}^{\infty}[\rho a_n+(1-\rho)\,b_n]z^n.$$

Applying Theorem 2.4.1.4 for the functions $f(z)$ and $g(z)$, we get

$$\sum_{n=2}^{\infty}[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)][\rho a_n+(1-\rho)\,b_n]\leq$$

$$\rho(1-\alpha)+(1-\rho)(1-\alpha)=1-\alpha.$$

This completes the proof of the Theorem 2.4.4.2.

Let us now prove further theorem for functions $f_i(z)$ in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha_i)$ following (Amsheri and Zharkova, 2012j), where $f_i(z)\in T(p)(i=1,2,\dots,m)$ defined by

$$f_i(z)=z^p-\sum_{n=1}^{\infty}a_{i,p+n}z^{p+n},\quad \left(a_{i,p+n}\geq 0;\ p\in\mathbb{N}\right). \tag{2.4.4.2}$$

Theorem 2.4.4.3. Let the function $f_i(z)$ defined by (2.4.4.2) be in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha_i)$ for each $(i=1,2,\dots,m)$. Then the function $h(z)$ defined by

$$h(z)=z^p-\frac{1}{m}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m}a_{i,p+n}\right)z^{p+n}\ , \tag{2.4.4.3}$$

is in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ where $\alpha=\min_{1\leq i\leq m}\{\alpha_i\}$ with $0\leq\alpha_i<p$.

Proof. Since

$$f_i(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha_i), \qquad (i = 1,2,\dots,m).$$

By applying Theorem 2.4.1.2, we observe that

$$\sum_{n=1}^{\infty} [p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha_i)\delta_n(\beta,\gamma,\xi,p)]\, a_{i,p+n} \leq p - \alpha_i.$$

Hence

$$\sum_{n=1}^{\infty} [p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha_i)\delta_n(\beta,\gamma,\xi,p)] \left(\frac{1}{m}\sum_{i=1}^{m} a_{i,p+n}\right)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\left(\sum_{n=1}^{\infty} [p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha_i)\delta_n(\beta,\gamma,\xi,p)] a_{i,p+n}\right)$$

$$\leq \frac{1}{m}\sum_{i=1}^{m}(p - \alpha_i) \leq p - \alpha.$$

which in view of Theorem 2.4.1.2, again implies that

$$h(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\zeta}(p,\alpha).$$

The proof is complete.

Next by letting $p = 1$ in Theorem 2.4.4.3 we can prove further theorem for functions $f_i(z)$ in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha_i)$ following (Amsheri and Zharkova, 2012a), where $f_i(z) \in T,\ (i = 1,2,\dots,m)$ defined by

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{i,n} z^n\ , \quad \left(a_{i,n} \geq 0\right). \tag{2.4.4.4}$$

Theorem 2.4.4.4. Let the function $f_i(z)$ defined by (2.4.4.4) be in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha_i)$ for each $(i = 1,2,\dots,m)$. Then the function $h(z)$ defined by

$$h(z) = z - \frac{1}{m}\sum_{n=2}^{\infty}\left(\sum_{i=1}^{m} a_{i,n}\right) z^n. \tag{2.4.4.5}$$

is in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$ where $\alpha = min_{1 \leq i \leq m}\{\alpha_i\}$ with $0 \leq \alpha_i < 1$.

Proof. Since $f_i(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha_i)$ $(i = 1,2,\dots,m)$, by applying Theorem 2.4.1.4, we observe that

$$\sum_{n=2}^{\infty}[(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha_i)\delta_n(\beta,\gamma,\xi)]\, a_{i,n} \le 1-\alpha_i.$$

Hence

$$\sum_{n=2}^{\infty}[(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha_i)\delta_n(\beta,\gamma,\xi)]\left(\frac{1}{m}\sum_{i=1}^{m} a_{i,n}\right) =$$

$$\frac{1}{m}\sum_{i=1}^{m}\left(\sum_{n=2}^{\infty}[(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha_i)\delta_n(\beta,\gamma,\xi)]a_{i,n}\right) \le$$

$$\frac{1}{m}\sum_{i=1}^{m}(1-\alpha_i) \le 1-\alpha.$$

which in view of Theorem 2.4.1.4, again implies that

$$h(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha).$$

The proof is complete.

2.4.5 Radii of starlikeness, convexity, and close-to-convexity

Let us obtain the radii of starlikeness for functions in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ according to (Amsheri and Zharkova, 2012j).

Theorem 2.4.5.1. Let the function $f(z)$ defined by (1.2.5) be in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$. Then $f(z)$ is p-valent starlike of order σ $(0 \le \sigma < p)$ in the disk $|z| < r_1$, where

$$r_1 = \inf_{n\in\mathbb{N}}\left\{\frac{(p-\sigma)[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{(p-\alpha)(n+p-\sigma)}\right\}^{\frac{1}{n}}. \tag{2.4.5.1}$$

The result is sharp with the extremal function given by (2.4.1.9).

Proof. It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \sigma\,, \qquad (|z| < r_1). \tag{2.4.5.2}$$

Indeed we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{-\sum_{n=1}^{\infty} n\, a_{p+n}\, z^n}{1 - \sum_{n=1}^{\infty}\, a_{p+n}\, z^n}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty} n\, a_{p+n}\, |z|^n}{1 - \sum_{n=1}^{\infty}\, a_{p+n}\, |z|^n}\ . \tag{2.4.5.3}$$

Hence (2.4.5.3) is true if

$$\sum_{n=1}^{\infty} n\, a_{p+n}\, |z|^n \leq (p-\sigma) - \sum_{n=1}^{\infty}\, (p-\sigma) a_{p+n}\, |z|^n.$$

That is, if

$$\sum_{n=1}^{\infty} (n + p - \sigma)\, a_{p+n}\, |z|^n \leq p - \sigma.$$

or

$$\sum_{n=1}^{\infty} \left(\frac{n+p-\sigma}{p-\sigma}\right) a_{p+n}\, |z|^n \leq 1, \tag{2.4.5.4}$$

By Theorem 2.4.1.2, (2.4.5.3) is true if

$$\frac{n+p-\sigma}{p-\sigma}\, |z|^n \leq \frac{[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{(p-\alpha)}. \tag{2.4.5.5}$$

Solving (2.4.5.5) for $|z|$, we get

$$|z| \leq \left\{\frac{(p-\sigma)[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{(p-\alpha)(n+p-\sigma)}\right\}^{1/n},$$

or

$$r_1 = \inf_{n\in\mathbb{N}} \left\{\frac{(p-\sigma)[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{(p-\alpha)(n+p-\sigma)}\right\}^{\frac{1}{n}}\ . \tag{2.4.5.6}$$

The proof is complete.

Now by letting $p = 1$ in Theorem 2.4.5.1 we can obtain the radii of starlikeness for functions in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$ according to (Amsheri and Zharkova, 2012a).

Theorem 2.4.5.2. Let the function $f(z)$ defined by (1.2.4) be in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$. Then $f(z)$ is starlike of order $\delta\,(0 \leq \delta < 1)$ in the disk $|z| < r_2$ where

$$r_2 = \inf_{n\geq 2}\left\{\frac{(1-\delta)[(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha)\delta_n(\beta,\gamma,\xi)]}{(1-\alpha)(n+1-\delta)}\right\}^{1/(n-1)} \qquad (2.4.5.7)$$

The result is sharp with the extremal function given by (2.4.1.10).

Proof. It suffices to prove

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\delta\,, \qquad (|z| < r_2). \qquad (2.4.5.8)$$

Indeed we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{-\sum_{n=2}^{\infty} na_n\, z^{n-1}}{1-\sum_{n=2}^{\infty} a_n\, z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty} n\, a_n\, |z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n\, |z|^{n-1}}\,. \qquad (2.4.5.9)$$

Hence (2.4.5.9) is true if

$$\sum_{n=2}^{\infty} n\, a_n\, |z|^{n-1} \leq (1-\delta) - \sum_{n=2}^{\infty} (1-\delta)a_n\, |z|^{n-1}.$$

That is, if

$$\sum_{n=2}^{\infty} (n+1-\delta)\, a_n\, |z|^{n-1} \leq 1-\delta,$$

or

$$\sum_{n=2}^{\infty} \left(\frac{n+1-\delta}{1-\delta}\right) a_n\, |z|^{n-1} \leq 1\,. \qquad (2.4.5.10)$$

By Theorem 2.4.1.4, (2.4.5.9) is true if

$$\frac{n+1-\delta}{1-\delta}|z|^{n-1} \leq \frac{[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)]}{(1-\alpha)}\ , \qquad (2.4.5.11)$$

Solving (2.4.5.11) for $|z|$, we get

$$|z| \leq \left\{\frac{(1-\delta)[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)]}{(1-\alpha)(n+1-\delta)}\right\}^{1/(n-1)},$$

or

$$r_2 = \inf_{n\geq 2}\left\{\frac{(1-\delta)[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)]}{(1-\alpha)(n+1-\delta)}\right\}^{1/(n-1)}. \quad (2.4.5.12)$$

The proof is complete.

In the similar manner, we can obtain the radii of convexity for the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ following (Amsheri and Zharkova, 2012j).

Theorem 2.4.5.3. Let the function $f(z)$ defined by (1.2.5) be in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$. Then $f(z)$ is p-valent convex of order σ $(0 \leq \sigma < p)$ in the disk $|z| < r_3$, where

$$r_3 = \inf_{n\in\mathbb{N}}\left\{\frac{p(p-\sigma)[p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{(p+n)(p-\alpha)(n+p-\sigma)}\right\}^{\frac{1}{n}}.$$

The result is sharp with the extremal function given by (2.4.1.9).

By letting $p=1$ in Theorem 2.4.5.3 we can obtain the radii of convexity for the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$ following (Amsheri and Zharkova, 2012a).

Theorem 2.4.5.4. Let the function $f(z)$ defined by (1.2.4) be in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$. Then $f(z)$ is convex of order δ $(0 \leq \delta < 1)$ in the disk $|z| < r_4$ where

$$r_4 = \inf_{n\geq 2}\left\{\frac{(1-\delta)[(1+k)\delta_n(\lambda,\mu,\eta)-(k+\alpha)\delta_n(\beta,\gamma,\xi)]}{(1+n)(1-\alpha)(n+1-\delta)}\right\}^{1/(n-1)}.$$

The result is sharp with the extremal function given by (2.4.1.10).

Also, we can obtain the radii of close-to-convexity for the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ following (Amsheri and Zharkova, 2012j).

Theorem 2.4.5.5. Let the function $f(z)$ defined by (1.2.5) be in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$. Then $f(z)$ is p-valent close-to-convex of order σ $(0 \leq \sigma < p)$ in the disk $|z| < r_5$, where

$$r_5 = \inf_{n\in\mathbb{N}} \left\{\frac{(p-\sigma)[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{(p+n)(p-\alpha)}\right\}^{\frac{1}{n}}.$$

The result is sharp with the extremal function given by (2.4.1.9).

By letting $p = 1$ in Theorem 2.4.5.5 we can obtain the radii of close-to-convexity for the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$ following (Amsheri and Zharkova, 2012a).

Theorem 2.4.5.6. Let the function $f(z)$ defined by (1.2.4) be in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(\alpha)$. Then $f(z)$ is close-to-convex of order δ $(0 \leq \delta < 1)$ in the disk $|z| < r_6$ where

$$r_6 = \inf_{n\geq 2} \left\{\frac{(1-\delta)[(1+k)\delta_n(\lambda,\mu,\eta) - (k+\alpha)\delta_n(\beta,\gamma,\xi)]}{(1+n)(1-\alpha)}\right\}^{1/(n-1)}.$$

The result is sharp with the extremal function given by (2.4.1.10).

Chapter 3

Properties of certain classes and inequalities involving p-valent functions

This chapter is composed of two types of problems. The first type is concerned with the sufficient conditions for starlikeness and convexity of p-valent functions associated with fractional derivative operator, while the second type is concerned with the coefficient bounds for some classes of p-valent functions by making use of certain fractional derivative operator. This chapter is organized as follows: Section 3.1 is introductory in nature and contains some lemmas those are require to prove our results. In section 3.2, we present some sufficient conditions for starlikeness and convexity by using the results of (Owa, 1985a). Further results involving the Hadamard product (or convolution) are obtained. Sufficient conditions for starlikeness and convexity by using Jack's Lemma and Nunokakawa's Lemma are also studied. In section 3.3 we obtain the coefficient bound for the functional $\left|a_{p+2}-\theta a_{p+1}^{2}\right|$ and bounds for the coefficient a_{p+3} of the function belonging to some classes of p-valent functions in the open unit disk involving certain fractional derivative operator. We obtain the coefficient bounds for the function $f(z)$ belonging to the classes $S_{p,\lambda,\mu,\eta}^{*}(\phi)$, $S_{b,p,\lambda,\mu,\eta}^{*}(\phi)$ of starlike functions. In addition, we study the similar problem to the classes $R_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$**,** $R_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, $M_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ and $M_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ of Bazilevič functions and to the classes

$N_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ and $N_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ of non-Bazilevič functions. Relevant connections of some results obtained in this chapter with those in earlier works are considered.

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3.1 Introduction and Preliminaries

We refer to Chapter 1 for related definitions and notations used in this chapter. First, to obtain the coefficient conditions for starlikeness and convexity in subsections 3.2.1 and 3.2.2 by using the results of (Owa, 1985a) and the Hadamard product, we consider the fractional derivative operator $P_{0,Z}^{\lambda,\mu,\eta}f(z)$ defined by (1.6.8), which was studied by (Raina and Nahar, 2000) in order to obtain many of sufficient conditions for starlikenesss and convexity, that are extensions of the results by (Owa and Shen, 1998) when $\mu=\lambda$. Moreover, to introduce our main results in the subsection 3.2.3, we consider Jack's Lemma (Jack, 1971) or (Miller and Mocanue, 2000) and Nunokakawa's Lemma (Nunokakawa, 1992) which have been applied in obtaining various sufficient conditions of starlikeness and convexity by many authors, including (Imark and Cetin, 1999), (Imark and Piejko, 2005) and (Imark, et al., 2002).

In addition, to investigate our main results in section 2.3 concerning the coefficient bounds for some classes of p-valent functions in the open unit disk defined by the fractional derivative $M_{0,z}^{\lambda,\mu,\eta} f(z)$ given as in (2.2.1), we consider the class $\mathcal{P}$ which defined in Chapter 1, section 1.4, for all analytic functions with positive real part in the open unit disk $\mathcal{U}$ defined by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ $(|z| < 1)$. It is well known (C. Pommerenke, 1975) that $|c_n| \leq 2$ $(n = 1,2, \dots)$. (Livingston, 1969) proved that $|c_1^2 - c_2| \leq 2$ and (Ma and Minda, 1993) obtained that $\left|c_2 - \frac{1}{2}c_1^2\right| \leq 2 - \frac{1}{2}|c_1|^2$. (Ma and Minda, 1994) introduced the classes $S^*(\phi)$ and $C(\phi)$ of the analytic function ϕ with positive real part in the unit disk $\mathcal{U}$, such that $\phi(0) = 1$, $\phi'(0) > 0$, where ϕ maps $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. They also determined bounds for the associated Fekete-Szegö functional. (Ali et al., 2007) defined and studied the class $S_{b,p}^*(\phi)$ of functions $f(z) \in \mathcal{A}(p)$ for which

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z), \qquad (z \in \mathcal{U}; b \in \mathbb{C} \setminus \{0\}).$$

and the class $C_{b,p}(\phi)$ of functions for which

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z), \qquad (z \in \mathcal{U}; b \in \mathbb{C} \setminus \{0\}).$$

Also, (Ali et al., 2007) defined and studied the class $R_{b,p}(\phi)$ to be the class of all functions $f(z) \in \mathcal{A}(p)$ for which

$$1+\frac{1}{b}\left(\frac{f^{'}(z)}{pz^{p-1}}-1\right) \prec \phi(z), \qquad (z \in \mathcal{U}\,; b \in \mathbb{C} \setminus \{0\}).$$

Note that, $S_{1,1}^{*}(\phi) = S^{*}(\phi)$ and $C_{1,1}(\phi) = C(\phi)$. The familiar class $S^{*}(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order $\alpha,$ $(0 \le \alpha < 1)$ are the special case of $S_{1,1}^{*}(\phi)$ and $C_{1,1}(\phi)$, respectively, when

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z}.$$

To present our main results in the subsection 3.3.2 concerning the coefficient bounds for some classes of Bazilevič functions, we consider the class of Bazilevič functions $H_p(A,B,\alpha,\beta)$ which was introduced by (Owa, 2000) for all functions $f(z) \in \mathcal{A}(p)$ satisfying

$$(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta\frac{zf^{'}(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec \frac{1+Az}{1+Bz}.$$

where $z \in \mathcal{U}$, $-1 \le \mathrm{B} < A \le 1,$ $0 \le \beta \le 1,$ $\alpha \ge 0$. Following the classes $H_p(A,B,\alpha,\beta)$ and $R_{b,p}(\phi)$ which were studied, respectively, by (Owa, 2000) and (Ali et al., 2007), (Ramachandran et al., 2007) obtained the coefficient bounds for the class $R_{b,p,\alpha,\beta}(\phi)$, defined by

$$1+\frac{1}{b}\left\{(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta\frac{zf^{'}(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} - 1\right\} \prec \phi(z).$$

where $0 \le \beta \le 1,$ $\alpha \ge 0$.

Moreover, (Guo and Liu, 2007) introduced and studied the class of Bazilevič functions $M(\alpha,\beta,\rho)$ for all functions $f(z) \in S$ satisfying

$$\mathrm{Re}\left\{\frac{zf^{'}(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \beta\left[1+\frac{zf^{''}(z)}{f'(z)} - \frac{zf^{'}(z)}{f(z)} + \alpha\left(\frac{zf^{'}(z)}{f(z)}-1\right)\right]\right\} > \rho.$$

where $\alpha \geq 0;\ \beta \geq 0; 0 \leq \rho < 1$. Following the class $M(\alpha, \beta, \rho)$, (Rosy et al., 2009) obtained coefficient bounds for the class $M_{\alpha,\beta}(\phi)$, defined by

$$\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \beta\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} \prec \phi(z).$$

where $\alpha \geq 0;\ \beta \geq 0$.

On the other hand, to present our main results in the subsection 3.3.3 concerning the coefficient bounds for some classes of non-Bazilevič functions, we consider the class of non-Bazilevič functions which was introduced by (Obradović, 1998) for all functions $f(z) \in S$ such that

$$\mathrm{Re}\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\alpha}\right\} > 0.$$

where $0 < \alpha < 1$ and $z \in \mathcal{U}$. (Tuneski and Darus, 2002) obtained the Fekete-Szegö inequality for this non-Bazilevič class of functions. Using this non-Bazilevič class, (Wang et al., 2005) studied many subordination results for the class $N(\alpha, \beta, A, B)$ of functions $f(z) \in S$ such that

$$(1+\beta)\left(\frac{z}{f(z)}\right)^{\alpha} - \beta f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha} \prec \frac{1+Az}{1+Bz}.$$

for $-1 \leq \mathrm{B} < A \leq 1,\ \beta \in \mathbb{C},\ 0 < \alpha < 1$. Following this class, (Shanmugam et al., 2006a) obtained the Fekete-Szegö inequality for the class $N_{\alpha,\beta}(\phi)$, defined by

$$(1+\beta)\left(\frac{z}{f(z)}\right)^{\alpha} - \beta f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha} \prec \phi(z).$$

where $\beta \in \mathbb{C},\ 0 < \alpha < 1$.

Now, in order to prove our results in the subsection 3.2.1 for starlikeness and convexity, we need the following coefficient conditions that are sufficient for the functions to be in the classes $S^*(p,\alpha)$ and $K(p,\alpha)$ according to (Owa, 1985a).

Lemma 3.1.1 Let the function $f(z) \in \mathcal{A}(p)$. If $f(z)$ satisfies

$$\sum_{n=1}^{\infty}(p+n-\alpha)\left|a_{p+n}\right| \leq (p-\alpha).$$

Then $f(z)$ is in the class $S^*(p,\alpha)$.

Lemma 3.1.2 Let the function $f(z) \in \mathcal{A}(p)$. If $f(z)$ satisfies

$$\sum_{n=1}^{\infty}(p+n)(p+n-\alpha)\left|a_{p+n}\right| \leq p(p-\alpha).$$

Then $f(z)$ is in the class $K(p,\alpha)$.

Next, in order to prove our results in the subsection 3.2.2 for starlikeness and convexity by using the Hadamard product, we need the following result due to (Ruscheweyh and Sheil-Small, 1973).

Lemma 3.1.3. Let $\varphi(z)$ and $g(z)$ be analytic in $|z|<1$ and satisfy $\varphi(0)=g(0)=0$, $\varphi'(0)\neq 0, g'(0)\neq 0$. Suppose also that

$$\varphi(z) * \left\{\frac{1+abz}{1-bz}g(z)\right\} \neq 0, \qquad (0<|z|<1).$$

for a and b on the unit circle. Then, for a function $F(z)$ analytic in $|z|<1$ such that

$$\mathrm{Re}\,\{F(z)\} > 0,$$

satisfies the inequality:

$$\mathrm{Re}\left\{\frac{(\varphi * Fg)(z)}{(\varphi * g)(z)}\right\} > 0, \qquad (|z|<1).$$

Next to prove our results in the subsection 3.2.3 for starlikeness and convexity by using Jack's Lemma and Nunokawa's Lemma, we need to the following results of Jack and Nunokawa (Lemma 3.1.4 and Lemma 3.1.5) which are popularly known as Jack's Lemma (Jack, 1971) or (Miller and Mocaun, 2000) and Nunokawa's Lemma (Nunokawa, 1992), respectively.

Lemma 3.1.4. Let $w(z)$ be non-constant and analytic function in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r,\ (0 < r < 1)$ at the point z_0, then $z_0 w^{'}(z_0) = cw(z_0)$, where $c \geq 1$.

Lemma 3.1.5. Let $p(z)$ be an analytic function in $\mathcal{U}$ with $p(0) = 1$. If there exists a point $z_0 \in \mathcal{U}$ such that

$$\mathrm{Re}\{p(z)\} > 0 \quad (|z| < |z_0|), \quad \mathrm{Re}\{p(z_0)\} = 0,\ \ p(z_0) \neq 0.$$

then

$$p(z_0) = i\,a\,, \qquad \frac{z_0 p'(z_0)}{p(z_0)} = i\frac{c}{2}\left(a + \frac{1}{a}\right).$$

where $a \neq 0$ and $c \geq 1$.

Now, to prove our main results in section 3.3, we mention to the following lemma 3.1.6 for functions $p(z)$ in the class $\mathcal{P}$ according to (Ma and Minda, 1994) to obtain the sharp bound on coefficient functional $|c_2 - vc_1^2|$.

Lemma 3.1.6. Let $p(z) \in \mathcal{P}$. Then

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2 & \text{if} \quad v \leq 0 \\ 2 & \text{if} \quad 0 \leq v \leq 1 \\ 4v - 2 & \text{if} \quad v \geq 1 \end{cases} ,$$

when $v < 0$ or $v > 1$, the equality holds if and only if

$$p(z) = \frac{1+z}{1-z},$$

or one of its rotations. If $0 < v < 1$, then equality holds if and only if

$$p(z) = \frac{1+z^2}{1-z^2},$$

or one of its rotations. Inequality becomes equality when $v = 0$ if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z}, \qquad (0 \leq \lambda \leq 1).$$

or one of its rotations, while for $v = 1$, the equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that equality in the case of $v = 0$. Although the above upper bound is sharp, it can be improved as follows when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2, \qquad \left(0 < v \leq \frac{1}{2}\right),$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2, \qquad \left(\frac{1}{2} \leq v < 1\right).$$

Also, to prove our main results in section 3.3, we need to the following lemmas regarding the coefficients of analytic functions of the form $w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots$ in the class Ω in the open unit disk $\mathcal{U}$ satisfying $|w(z)| < 1$. Lemma 3.1.7 is formulated according to (Ali et al., 2007) which is a reformulation of the corresponding result Lemma 3.1.6 for functions with positive real part.

Lemma 3.1.7. If $w \in \Omega$, then

$$|w_2 - tw_1^2| \leq \begin{cases} -t, & t < -1 \\ 1, & -1 \leq t \leq 1, \\ t, & t > 1 \end{cases}$$

when $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if

$$w(z) = z\frac{\lambda + z}{1 + \lambda z}, \qquad (0 \leq \lambda \leq 1),$$

or one of its rotations, while for $t = 1$, the equality holds if and only if

$$w(z) = -z\frac{\lambda + z}{1 + \lambda z}, \qquad (0 \le \lambda \le 1),$$

or one of its rotations. Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$:

$$|w_2 - tw_1^2| + (t+1)|w_1|^2 \le 1\,, \qquad (-1 < t \le 0),$$

and

$$|w_2 - tw_1^2| + (1-t)|w_1|^2 \le 1, \qquad (0 < t < 1).$$

Also, for functions in the class Ω, we need to the following result to prove our main results in section 3.3, which is according to [(Keogh and Merkes, 1969), Inequality 7, p.10].

Lemma 3.1.8. If $w \in \Omega$, then for any complex number t,

$$|w_2 - tw_1^2| \le \max(1, |t|)$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

We also need to the following result which is due to (Prokhorov and Szynal, 1981), see also (Ali et al., 2007)

Lemma 3.1.9. If $w \in \Omega$, then for any real numbers q_1 and q_2, the following sharp estimate holds

$$|w_3 + q_1 w_1 w_2 + q_2 w_1^3| \le H(q_1, q_2),$$

where

$$H(q_1,q_2)$$

$$= \begin{cases} 1 & \text{for } (q_1,q_2) \in D_1 \cup D_2, \\ |q_2| & \text{for } (q_1,q_2) \in \cup_{k=3}^{7} D_k\,, \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{1/2} & \text{for } (q_1,q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{1/2} & \text{for } (q_1,q_2) \in D_{10} \cup D_{11} \setminus \{\pm 2,1\}, \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{1/2} & \text{for } (q_1,q_2) \in D_{12}. \end{cases}$$

The sets $D_k,\ k = 1,2,\dots,12$, are defined as follows:

$$D_1 = \left\{(q_1, q_2): |q_1| \le \frac{1}{2}, |q_2| \le 1 \right\},$$

$$D_2 = \left\{(q_1, q_2): \frac{1}{2} \le |q_1| \le 2, \frac{4}{27}(|q_1| + 1)^3 - (|q_1| + 1) \le q_2 \le 1 \right\},$$

$$D_3 = \left\{(q_1, q_2): |q_1| \le \frac{1}{2}, q_2 \le -1 \right\},$$

$$D_4 = \left\{(q_1, q_2): |q_1| \ge \frac{1}{2}, q_2 \le -\frac{2}{3}\ (|q_1| + 1)\right\},$$

$$D_5 = \{(q_1, q_2): |q_1| \le 2, q_2 \ge 1 \},$$

$$D_6 = \left\{(q_1, q_2): 2 \le |q_1| \le 4, q_2 \ge \frac{1}{12}(q_1^2 + 8)\right\},$$

$$D_7 = \left\{(q_1, q_2): |q_1| \ge 4, q_2 \ge \tfrac{2}{3}(|q_1| - 1)\right\},$$

$$D_8 = \left\{(q_1, q_2): \frac{1}{2} \le |q_1| \le 2, -\frac{2}{3}\ (|q_1| + 1) \le q_2 \le \frac{4}{27}(|q_1| + 1)^3 - (|q_1| + 1) \right\},$$

$$D_9 = \left\{(q_1, q_2): |q_1| \ge 2, -\frac{2}{3}\ (|q_1| + 1) \le q_2 \le \frac{2|q_1|(|q_1| + 1)}{q_1^2 + 2|q_1| + 4}\right\},$$

$$D_{10} = \left\{(q_1, q_2): 2 \le |q_1| \le 4, \frac{2|q_1|(|q_1| + 1)}{q_1^2 + 2|q_1| + 4} \le q_2 \le \frac{1}{12}(q_1^2 + 8)\right\},$$

$$D_{11} = \left\{(q_1, q_2): |q_1| \ge 4, \frac{2|q_1|(|q_1| + 1)}{q_1^2 + 2|q_1| + 4} \le q_2 \le \frac{2|q_1|(|q_1| - 1)}{q_1^2 - 2|q_1| + 4}\right\},$$

$$D_{12} = \left\{(q_1, q_2): |q_1| \ge 4, \frac{2|q_1|(|q_1| - 1)}{q_1^2 - 2|q_1| + 4} \le q_2 \le \frac{2}{3}\ (|q_1| - 1)\right\}.$$

3.2 Sufficient conditions for starlikeness and convexity of p-valent functions

In this section we mainly concentrate in obtaining the sufficient conditions for starlikeness and convexity of p-valent functions defined by fractional derivative operator.

3.2.1 Sufficient conditions involving results of Owa

Let us first obtain the sufficient conditions for starlikeness of $M_{0,z}^{\lambda,\mu,\eta} f(z)$ as given in (2.2.1) by using Lemmas 3.1.1 following the results by (Amsheri and Zharkova, 2010).

Theorem 3.2.1.1. Let $\lambda,\ \mu,\ \eta \in \mathbb{R}$ such that

$$\lambda \geq 0;\ \mu < p+1;\ \max(\lambda,\mu) - p - 1 < \ \eta \leq \lambda\left(1 - \frac{p+2}{\mu}\right);\ p \in \mathbb{N}. \qquad (3.2.1.1)$$

Also, let the function $f(z) \in \mathcal{A}(p)$ satisfies

$$\sum_{n=1}^{\infty} \frac{(p+n-\alpha)}{(p-\alpha)} |a_{p+n}| \leq \frac{(1+p-\mu)(1+p+\eta-\lambda)}{(1+p)(1+p+\eta-\mu)}. \qquad (3.2.1.2)$$

for $0 \leq \alpha < p$. Then $M_{0,z}^{\lambda,\mu,\eta} f(z) \in S^*(p,\alpha)$.

Proof. We have from (2.2.3)

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \delta_n(\lambda,\mu,\eta,p)\, a_{p+n} z^{p+n},$$

where $\delta_n(\lambda,\mu,\eta,p)$ is given by (2.2.5). We observe that the function $\delta_n(\lambda,\mu,\eta,p)$ satisfies the inequality

$$\delta_{n+1}(\lambda,\mu,\eta,p) \leq \delta_n(\lambda,\mu,\eta,p), \qquad (\ \forall n \in \mathbb{N}).$$

provided that $\eta \leq \lambda\left(1 - \frac{p+2}{\mu}\right)$. Thereby, showing that $\delta_n(\lambda,\mu,\eta,p)$ is non-increasing. Thus under conditions stated in (3.2.1.1), we have

$$0 < \delta_n(\lambda,\mu,\eta,p) \leq \delta_1(\lambda,\mu,\eta,p) = \frac{(1+p)(1+p+\eta-\mu)}{(1+p-\mu)(1+p+\eta-\lambda)}. \qquad (3.2.1.3)$$

Therefore, (3.2.1.2) and (3.2.1.3) yield

$$\sum_{n=1}^{\infty} \frac{(p+n-\alpha)}{(p-\alpha)} \delta_n(\lambda,\mu,\eta,p) |a_{p+n}| \leq$$

$$\delta_1(\lambda,\mu,\eta,p) \sum_{n=1}^{\infty} \frac{(p+n-\alpha)}{(p-\alpha)} |a_{p+n}| \leq 1. \qquad (3.2.1.4)$$

Hence, by Lemma 3.1.1, we conclude that

$$M_{0,z}^{\lambda,\mu,\eta} f(z) \in S^*(p,\alpha).$$

and the proof is complete.

Remark 1. The equality in (3.2.1.2) is attained for the function $f(z)$ defined by

$$f(z) = z^p + \frac{(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{(p+1-\alpha)(1+p)(1+p+\eta-\mu)} z^{p+1}\,. \tag{3.2.1.5}$$

In the similar manner, we can prove with the help of Lemma 3.1.2 the sufficient conditions for convexity of $M_{0,z}^{\lambda,\mu,\eta} f(z)$ according to (Amsheri and Zharkova, 2010).

Theorem 3.2.1.2. Under the conditions stated in (3.2.1.1), let the function $f(z) \in \mathcal{A}(p)$ satisfies

$$\sum_{n=1}^{\infty} \frac{(p+n)(p+n-\alpha)}{p(p-\alpha)} |a_{p+n}| \le \frac{(1+p-\mu)(1+p+\eta-\lambda)}{(1+p)(1+p+\eta-\mu)}. \tag{3.2.1.6}$$

for $0 \le \alpha < p$. Then $M_{0,z}^{\lambda,\mu,\eta} f(z) \in K(p,\alpha)$.

Remark 2. The equality in (3.2.1.6) is attained for the function $f(z)$ defined by

$$f(z) = z^p + \frac{p(p-\alpha)(1+p-\mu)(1+p+\eta-\lambda)}{(p+1)^2(p+1-\alpha)(1+p+\eta-\mu)} z^{p+1}\,. \tag{3.2.1.7}$$

3.2.2 Sufficient conditions involving the Hadamard product

Let us obtain the sufficient conditions for starlikeness of $M_{0,z}^{\lambda,\mu,\eta} f(z)$ as given in (2.2.1) by using Lemmas 3.1.3 following the results by (Amsheri and Zharkova, 2010).

Theorem 3.2.2.1. Let the conditions stated in (3.2.1.1) hold true, and let the function $f(z) \in \mathcal{A}(p)$ be in the class $S^*(p,\alpha)$, and satisfies:

$$\psi(z) * \left\{ \frac{1+abz}{1-bz} f(z) \right\} \neq 0, \qquad (z \in \mathcal{U} - \{0\}). \tag{3.2.2.1}$$

for a and b on the unit circle, where

$$\psi(z) = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1+\eta-\mu+p)_n}{(1-\mu+p)_n (1+\eta-\lambda+p)_n} z^{p+n} \ . \tag{3.2.2.2}$$

then $M_{0,z}^{\lambda,\mu,\eta} f(z) \in S^*(p,\alpha)$.

Proof. Using (2.2.3) and (3.2.2.2), we have

$$\begin{aligned} M_{0,Z}^{\lambda,\mu,\eta} f(z) &= z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1+\eta-\mu+p)_n}{(1-\mu+p)_n (1+\eta-\lambda+p)_n} a_{p+n} z^{p+n} \\ &= (\psi * f)(z). \end{aligned} \tag{3.2.2.3}$$

By setting $\varphi(z) = \psi(z)$, $g(z) = f(z)$ and $F(z) = \frac{zf'(z)}{f(z)} - \alpha$, in Lemma 3.1.3, we find with the help of (3.2.2.3) that

$$\operatorname{Re}\left\{ \frac{(\varphi * Fg)(z)}{(\varphi * g)(z)} \right\} > 0$$

$$\Rightarrow \operatorname{Re}\left\{ \frac{(\psi * zf')(z)}{(\psi * f)(z)} \right\} - \alpha > 0$$

$$\Rightarrow \operatorname{Re}\left\{ \frac{z(\psi * f)'(z)}{(\psi * f)(z)} \right\} - \alpha > 0$$

$$\Rightarrow \operatorname{Re}\left\{ \frac{z\left(\mathrm{M}_{0,Z}^{\lambda,\mu,\eta} f(z)\right)'}{\mathrm{M}_{0,Z}^{\lambda,\mu,\eta} f(z)} \right\} - \alpha > 0$$

$$\Rightarrow M_{0,z}^{\lambda,\mu,\eta} f(z) \in S^*(p,\alpha).$$

and the proof is complete.

Next let us obtain the sufficient conditions for convexity of $M_{0,z}^{\lambda,\mu,\eta} f(z)$ as given in (2.2.1) by using Lemmas 3.1.3 following the results by (Amsheri and Zharkova, 2010).

Theorem 3.2.2.2. Let the conditions stated in (3.2.1.1) hold true, and let the function $f(z) \in \mathcal{A}(p)$ be in the class $K(p,\alpha)$, and satisfies:

$$\psi(z) * \left\{ \frac{1+abz}{1-bz} z f^{'}(z) \right\} \neq 0, \qquad (z \in \mathcal{U} - \{0\}). \tag{3.2.2.4}$$

for a and b on the unit circle, where $\psi(z)$ is given by (3.2.2.2). Then $M_{0,z}^{\lambda,\mu,\eta} f(z)$ is also in the class $K(p,\alpha)$.

Proof. Using (2.2.3) and Theorem 3.2.2.1, we observe that

$$\begin{aligned}
f(z) \in K(p,\alpha) &\Leftrightarrow \frac{z f^{'}(z)}{p} \in S^*(p,\alpha) \\
&\Rightarrow M_{0,Z}^{\lambda,\mu,\eta} \left(\frac{z f^{'}(z)}{p} \right) \in S^*(p,\alpha) \\
&\Leftrightarrow \left(\psi * \frac{z f^{'}}{p} \right)(z) \in S^*(p,\alpha) \\
&\Leftrightarrow \frac{z(\psi * f)^{'}(z)}{p} \in S^*(p,\alpha) \\
&\Leftrightarrow (\psi * f)(z) \in K(p,\alpha) \\
&\Leftrightarrow M_{0,Z}^{\lambda,\mu,\eta} f(z) \in K(p,\alpha).
\end{aligned}$$

which completes the proof of Theorem 3.2.2.2.

Remark 3. The results in subsections 3.2.1 and 3.2.2 can be reduced to the well known results, which were proven by (Raina and Nahar, 2000) when $p = 1$, and to the results which were proven by (Owa and Shen, 1998) when $p = 1$ and $\mu = \lambda$.

3.2.3 Sufficient conditions involving Jack's and Nunokawa's Lemmas

Let us obtain the sufficient conditions for starlikeness of $M_{0,z}^{\lambda,\mu,\eta}f(z)$ as given in (2.2.1) by using Jack's lemma 3.1.4 and Nunokawa's lemma 3.1.5 following the results by (Amsheri and Zharkova, 2013b).

Theorem 3.2.3.1. Let $z \in \mathcal{U};\ \lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1$ and $f(z) \in \mathcal{A}(p)$.

1. If

$$\mathrm{Re}\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} - (p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right\} > \frac{-(3+\alpha)}{2(1+\alpha)}. \tag{3.2.3.1}$$

then

$$\mathrm{Re}\left\{\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right\} > \frac{1+\alpha}{2}, \qquad (0 \leq \alpha < 1)\,. \tag{3.2.3.2}$$

2. If

$$\mathrm{Re}\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} - (p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right\} > -1\,. \tag{3.2.3.3}$$

then

$$\mathrm{Re}\left\{\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right\} > \alpha, \qquad (0 \leq \alpha < 1). \tag{3.2.3.4}$$

Proof. First, we prove (1). Since

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} = 1 + d_1 z + d_2 z^2 + \cdots. \qquad (z \in \mathcal{U})$$

Define the function $w(z)$ by

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}=\frac{1+\alpha w(z)}{1+w(z)}\,, \qquad (z\in\mathcal{U};\;\; 0\le\alpha<1). \tag{3.2.3.5}$$

It is clear that $w(z)$ is analytic in $\mathcal{U}$ with $w(0)=0$. Also, we can find from (3.2.3.5) that

$$\frac{z\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-\frac{z\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{M_{0,z}^{\lambda,\mu,\eta}f(z)}=\frac{\alpha zw'(z)}{1+\alpha w(z)}-\frac{zw'(z)}{1+w(z)}. \tag{3.2.3.6}$$

by using (2.2.6) to (3.2.3.6), we have

$$(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}$$

$$=\frac{\alpha zw'(z)}{1+\alpha w(z)}-\frac{zw'(z)}{1+w(z)}-1. \tag{3.2.3.7}$$

If there exists a point $z_0\in\mathcal{U}$ such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1.$$

then by Lemma 3.1.4, we have

$$z_0w'(z_0)=c\,w(z_0), \qquad\qquad (c\ge 1).$$

Therefore, since $w(z_0)=e^{i\theta}$, we obtain

$$\mathrm{Re}\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z_0)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z_0)}-(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z_0)}{M_{0,z}^{\lambda,\mu,\eta}f(z_0)}\right\}$$

$$=\mathrm{Re}\left\{\frac{\alpha z_0w'(z_0)}{1+\alpha w(z_0)}-\frac{z_0w'(z_0)}{1+w(z_0)}-1\right\}$$

$$=\mathrm{Re}\left\{\frac{\alpha ce^{i\theta}}{1+\alpha e^{i\theta}}-\frac{ce^{i\theta}}{1+e^{i\theta}}-1\right\}\le\frac{-(3+\alpha)}{2(1+\alpha)}.$$

which is a contradiction to the condition (3.2.3.1). Therefore, $|w(z)|<1$ for all $\in\mathcal{U}$. Hence (3.2.3.5) yields

$$\left|\frac{1-\dfrac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}}{\dfrac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-\alpha}\right| = |w(z)| < 1, \quad (0 \le \alpha < 1;\ z \in \mathcal{U}).$$

which implies the inequality (3.2.3.2). This completes the proof of (1) in the Theorem 3.2.3.1.

For the proof of (2), we define a new function $p(z)$ by

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} = \alpha + (1-\alpha)p(z), \qquad (0 \le \alpha < 1;\ z \in \mathcal{U}). \tag{3.2.3.8}$$

where $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = 1$. Then we find from (3.2.3.8) that

$$\frac{z\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} - \frac{z\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{M_{0,z}^{\lambda,\mu,\eta}f(z)} = \frac{(1-\alpha)zp'(z)}{\alpha+(1-\alpha)p(z)}\,. \tag{3.2.3.9}$$

by using (2.2.6) to (3.2.3.9), we have

$$(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} - (p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} + 1$$

$$= \frac{(1-\alpha)zp'(z)}{\alpha+(1-\alpha)p(z)}\,. \tag{3.2.3.10}$$

If there exists a point $z_0 \in \mathcal{U}$ such that

$$\mathrm{Re}\{p(z)\} > 0 \ \ (|z| < |z_0|); \quad \mathrm{Re}\{p(z_0)\} = 0;\ \ p(z_0) \neq 0;\ z \in \mathcal{U}.$$

Then by using Lemma 3.1.5, we have

$$p(z_0) = i\,a\,, \quad \frac{z_0 p'(z_0)}{p(z_0)} = i\frac{c}{2}\left(a+\frac{1}{a}\right) \qquad (a \neq 0, c \ge 1)$$

Thus from (2.2.6) and (3.2.3.10), we have

$$\mathrm{Re}\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z_0)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z_0)} - (p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z_0)}{M_{0,z}^{\lambda,\mu,\eta}f(z_0)} + 1\right\} =$$

$$\mathrm{Re}\left\{\frac{(1-\alpha)z_0p'(z_0)}{p(z_0)}\frac{p(z_0)}{\alpha+(1-\alpha)p(z_0)}\right\} = \frac{-c\alpha(1-\alpha)(1+a^2)}{2[\alpha^2+a^2(1-\alpha)^2a^2]} \le 0\,.$$

which contradicts the condition (3.2.3.3). Hence, $\mathrm{Re}\{p(z)\} > 0$ for all $z \in \mathcal{U}$ and the equality (3.2.3.8) implies the condition (3.2.3.4). Therefore, the proof of the Theorem 3.2.3.1 is complete.

Now, to obtain the sufficient conditions for convexity of $M_{0,z}^{\lambda,\mu,\eta} f(z)$ as given in (2.2.1) we put $zf'(z)/p$ instead of $f(z)$ in the Theorem 3.2.3.1, then we have the following theorem according to (Amsheri and Zharkova, 2013b).

Theorem 3.2.3.2. Let $z \in \mathcal{U}$; $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu) - p - 1$ and $f(z) \in \mathcal{A}(p)$.

1. If

$$\mathrm{Re}\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}\left(\frac{zf'(z)}{p}\right)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left(\frac{zf'(z)}{p}\right)} - (p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left(\frac{zf'(z)}{p}\right)}{M_{0,z}^{\lambda,\mu,\eta}\left(\frac{zf'(z)}{p}\right)}\right\} > \frac{-(3+\alpha)}{2(1+\alpha)}, \tag{3.2.3.11}$$

then

$$\mathrm{Re}\left\{\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left(\frac{zf'(z)}{p}\right)}{M_{0,z}^{\lambda,\mu,\eta}\left(\frac{zf'(z)}{p}\right)}\right\} > \frac{1+\alpha}{2}, \qquad (0 \leq \alpha < 1). \tag{3.2.3.12}$$

2. If

$$\mathrm{Re}\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}\left(\frac{zf'(z)}{p}\right)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left(\frac{zf'(z)}{p}\right)} - (p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left(\frac{zf'(z)}{p}\right)}{M_{0,z}^{\lambda,\mu,\eta}\left(\frac{zf'(z)}{p}\right)}\right\} > -1, \tag{3.2.3.13}$$

then

$$\mathrm{Re}\left\{\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left(\frac{zf'(z)}{p}\right)}{M_{0,z}^{\lambda,\mu,\eta}\left(\frac{zf'(z)}{p}\right)}\right\} > \alpha, \qquad (0 \leq \alpha < 1). \tag{3.2.3.14}$$

Now by setting $\lambda = \mu = 0$ in Theorem 3.2.3.1, we obtain the sufficient conditions for starlikeness of p-valent functions in $\mathcal{U}$ following (Amsheri and Zharkova, 2013b).

Corollary 3.2.3.3. Let $f(z) \in \mathcal{A}(p),\ z \in \mathcal{U},\ 0 \leq \alpha < 1.$

1. If

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > \frac{-(3+\alpha)}{2(1+\alpha)},$$

then

$$\operatorname{Re}\left\{\frac{zf'(z)}{pf(z)}\right\} > \frac{1+\alpha}{2}, \qquad (0 \leq \alpha < 1).$$

2. If

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > -1,$$

then

$$\operatorname{Re}\left\{\frac{zf'(z)}{pf(z)}\right\} > \alpha, \qquad (0 \leq \alpha < 1).$$

Remark 4. By setting $p = 1$ in Corollary 3.2.3.3, we get the corresponding result obtained by (Irmak and Piejko, 2005, Corollary 2.3).

Corollary 3.2.3.4. Let $f(z) \in \mathcal{A},\ z \in \mathcal{U},\ 0 \leq \alpha < 1.$

1. If

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > \frac{-(3+\alpha)}{2(1+\alpha)},$$

then

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1+\alpha}{2}, \qquad (0 \leq \alpha < 1).$$

2. If

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\}>-1,$$

then

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha, \qquad (0\le\alpha<1).$$

Next by setting $\lambda=\mu=0$ in Theorem 3.2.3.2, we obtain the sufficient conditions for convexity of p-valent functions in $\mathcal{U}$ following (Amsheri and Zharkova, 2013b).

Corollary 3.2.3.5. Let $f(z)\in\mathcal{A}(p),\ z\in\mathcal{U},\ 0\le\alpha<1$.

1. If

$$\operatorname{Re}\left\{\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}\right\}>\frac{\alpha-1}{2(1+\alpha)},$$

then

$$\operatorname{Re}\left\{\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\frac{1+\alpha}{2}. \qquad (0\le\alpha<1).$$

2. If

$$\operatorname{Re}\left\{\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}\right\}>0,$$

then

$$\operatorname{Re}\left\{\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha, \qquad (0\le\alpha<1).$$

Remark 5. By setting $p=1$ in Corollary 3.2.3.5, we get the corresponding result obtained by (Irmak and Piejko, 2005, Corollary 2.4).

Corollary 3.2.3.6. Let $f(z)\in\mathcal{A},\ z\in\mathcal{U},\ 0\le\alpha<1$.

1. If

$$\operatorname{Re}\left\{\frac{z^2 f'''(z) + 2zf''(z)}{zf''(z) + f'(z)} - \frac{zf''(z)}{f'(z)}\right\} > \frac{\alpha - 1}{2(1+\alpha)},$$

then

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1+\alpha}{2}, \qquad (0 \leq \alpha < 1).$$

2. If

$$\operatorname{Re}\left\{\frac{z^2 f'''(z) + 2zf''(z)}{zf''(z) + f'(z)} - \frac{zf''(z)}{f'(z)}\right\} > 0,$$

then

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (0 \leq \alpha < 1).$$

3.3 Coefficient bounds for some classes of generalized starlike and related functions

In this section we introduce various new classes of complex order of p-valent functions associated with the fractional derivative $M_{0,z}^{\lambda,\mu,\eta} f(z)$ as given in (2.2.1), in order to obtain the coefficient bounds of $\left|a_{p+2} - \theta a_{p+1}^2\right|$ and bounds for the coefficient a_{p+3} of the function belonging to those classes. Relevant connections of the results obtained in this section with those in earlier works are also considered. We set $\delta_n(\lambda, \mu, \eta, p) \equiv \delta_n$ which defined as in (2.2.5).

3.3.1 Coefficient bounds for classes of p-valent starlike functions

Motivated by the class $S_{b,p}^*(\phi)$ which was studied by (Ali et al., 2007), we now define a more general class of complex order $S_{b,p,\lambda,\mu,\eta}^*(\phi)$ of p-valent starlike functions associated with fractional derivative operator following the results by (Amsheri and Zharkova, 2012b).

Definition 3.3.1.1. Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis, $\phi(0)=1$ and $\phi'(0)>0$. A function $f(z)\in\mathcal{A}(p)$ is in the class $S^{*}_{b,p,\lambda,\mu,\eta}(\phi)$ if

$$1+\frac{1}{b}\left\{\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-1\right\}\prec\phi(z),\qquad (z\in\mathcal{U},b\in\mathbb{C}\backslash\{0\}).$$

Also, we let $S^{*}_{1,p,\lambda,\mu,\eta}(\phi)=S^{*}_{p,\lambda,\mu,\eta}(\phi)$.

The above class $S^{*}_{b,p,\lambda,\mu,\eta}(\phi)$ contains many well-known subclasses of analytic functions. In particular, for $\lambda=\mu=0$, we have

$$S^{*}_{b,p,0,0,\eta}(\phi)=S^{*}_{b,p}(\phi)$$

where $S^{*}_{b,p}(\phi)$ is precisely the class which was studied by (Ali et al., 2007). Furthermore, by specifying the parameters b,p,λ and μ we obtain the most of subclasses which were studied by other authors:

1. For $b=1$, $p=1$ and $\lambda=\mu=0$, we get the class $S^{*}_{1,1,0,0,\eta}(\phi)=S^{*}(\phi)$ which studied by (Ma and Minda, 1994).
2. For $p=1$ and $\lambda=\mu=0$, we have the class $S^{*}_{b,1,0,0,\eta}(\phi)=S^{*}_{b}(\phi)$ which studied by (Ravichandran et al., 2005).
3. For $b=1$ and $\lambda=\mu=0$, we have the class $S^{*}_{1,p,0,0,\eta}(\phi)=S^{*}_{p}(\phi)$ which studied by (Ali et al., 2007).

Thus, the generalization class $S^*_{b,p,\lambda,\mu,\eta}(\phi)$ defined in this subsection is proven to account for most available classes discussed in the previous papers and generalize the concept of starlike functions.

Now, to obtain the coefficient bounds of functions belonging to the class $S^*_{1,p,\lambda,\mu,\eta}(\phi)$, we use lemmas 3.1.7- 3.1.9 following (Amsheri and Zharkova, 2012b).

Theorem 3.3.1.2. Let $0 \leq \theta \leq 1$; $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda, \mu) - p - 1$ and $p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ where B_n are real with $B_1 > 0$, $B_2 \geq 0$, and

$$\sigma_1 = \frac{(B_2 - B_1)\delta_1^2 + (p-\mu)\delta_1^2 B_1^2}{2\delta_2 (p-\mu) B_1^2}\,, \tag{3.3.1.1}$$

$$\sigma_2 = \frac{(B_2 + B_1)\delta_1^2 + (p-\mu)\delta_1^2 B_1^2}{2\delta_2 (p-\mu) B_1^2}\,, \tag{3.3.1.2}$$

$$\sigma_3 = \frac{B_2\delta_1^2 + (p-\mu)\delta_1^2 B_1^2}{2\delta_2 (p-\mu) B_1^2}\,. \tag{3.3.1.3}$$

If $f(z) \in \mathcal{A}(p)$ belongs to $S^*_{1,p,\lambda,\mu,\eta}(\phi)$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \leq$$

$$\begin{cases} \dfrac{(p-\mu)}{2\delta_2}\left(B_2 - \dfrac{(p-\mu)(2\delta_2\theta - \delta_1^2)}{\delta_1^2} B_1^2\right), & \theta \leq \sigma_1, \\ \dfrac{(p-\mu)B_1}{2\delta_2}, & \sigma_1 \leq \theta \leq \sigma_2, \\ -\dfrac{(p-\mu)}{2\delta_2}\left(B_2 - \dfrac{(p-\mu)(2\delta_2\theta - \delta_1^2)}{\delta_1^2} B_1^2\right), & \theta \geq \sigma_2. \end{cases} \tag{3.3.1.4}$$

Further, if $\sigma_1 \leq \theta \leq \sigma_3$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| + \frac{\delta_1^2}{2\delta_2 (p-\mu) B_1}\left\{1 - \frac{B_2}{B_1} + \frac{(p-\mu)(2\delta_2\theta - \delta_1^2)}{\delta_1^2} B_1\right\}\left|a_{p+1}\right|^2$$

$$\leq \frac{(p-\mu)B_1}{2\delta_2}\,. \tag{3.3.1.5}$$

If $\sigma_3 \leq \theta \leq \sigma_2$, then

$$\left|a_{p+2}-\theta a_{p+1}^2\right|+\frac{\delta_1^2}{2\delta_2(p-\mu)B_1}\left\{1+\frac{B_2}{B_1}-\frac{(p-\mu)(2\delta_2\theta-\delta_1^2)}{\delta_1^2}B_1\right\}\left|a_{p+1}\right|^2$$

$$\leq \frac{(p-\mu)B_1}{2\delta_2}. \tag{3.3.1.6}$$

For any complex number θ,

$$\left|a_{p+2}-\theta a_{p+1}^2\right| \leq \frac{(p-\mu)B_1}{2\delta_2}\max\left\{1,\left|\frac{(p-\mu)(2\delta_2\theta-\delta_1^2)}{\delta_1^2}B_1-\frac{B_2}{B_1}\right|\right\}. \tag{3.3.1.7}$$

Further,

$$\left|a_{p+3}\right| \leq \frac{(p-\mu)B_1}{3\delta_3}H(q_1,q_2). \tag{3.3.1.8}$$

where $H(q_1,q_2)$ is as defined in Lemma 3.1.9,

$$q_1=\frac{4B_2+3(p-\mu){B_1}^2}{2B_1}, \tag{3.3.1.9}$$

and

$$q_2=\frac{2B_3+3(p-\mu)B_1B_2+(p-\mu)^2{B_1}^3}{2B_1}. \tag{3.3.1.10}$$

Proof. If $f(z) \in S^*_{1,p,\lambda,\mu,\eta}(\phi)$, then there is a Schwarz function

$$w(z)=w_1z+w_2z^2+\cdots \in \Omega,$$

such that

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}=\phi\big(w(z)\big). \tag{3.3.1.11}$$

since

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}=1+\frac{\delta_1}{p-\mu}a_{p+1}z+\frac{1}{p-\mu}\big(2\delta_2a_{p+2}-\delta_1^2a_{p+1}^2\big)z^2+$$

$$\frac{1}{p-\mu}\big(3\delta_3a_{p+3}-3\delta_1\delta_2a_{p+1}a_{p+2}+\delta_1^3a_{p+1}^3\big)z^3+\cdots$$

we have from (3.3.1.11),

$$a_{p+1} = \frac{(p-\mu)w_1}{\delta_1} B_1, \tag{3.3.1.12}$$

$$a_{p+2} = \frac{(p-\mu)}{2\delta_2} \{B_1 w_2 + (B_2 + (p-\mu)B_1^2)w_1^2\}, \tag{3.3.1.13}$$

and

$$a_{p+3} = \frac{(p-\mu)B_1}{3\delta_3} \left\{ w_3 + \frac{4B_2 + 3(p-\mu){B_1}^2}{2B_1} w_1 w_2 + \right.$$

$$\left. \frac{2B_3 + 3(p-\mu)B_1 B_2 + (p-\mu)^2 {B_1}^3}{2B_1} w_1^3 \right\}. \tag{3.3.1.14}$$

Therefore, we have

$$a_{p+2} - \theta a_{p+1}^2 = \frac{(p-\mu)B_1}{2\delta_2} \{w_2 - v w_1^2\}, \tag{3.3.1.15}$$

where

$$v := \frac{(p-\mu)B_1(2\delta_2\theta - \delta_1^2)}{\delta_1^2} - \frac{B_2}{B_1}. \tag{3.3.1.16}$$

Making use of (3.3.1.12)-(3.3.1.16), the results (3.3.1.4) - (3.3.1.7) are established by an application of Lemma 3.1.7, inequality (3.3.1.7) by Lemma 3.1.8, and (3.3.1.8) follows from Lemma 3.1.9. To show that the bounds in (3.3.1.4) - (3.3.1.7) are sharp, we define the functions $K_{\phi n}$ $(n = 2,3,\ldots)$ by

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} K_{\phi n}(z)}{M_{0,z}^{\lambda,\mu,\eta} K_{\phi n}(z)} = \phi(z^{n-1}), \qquad \left(K_{\phi n}(0) = \left(K_{\phi n}\right)'(0) - 1 = 0\right).$$

and the functions $F_r,\ G_r$ $(0 \le r \le 1)$ defined by

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} F_r(z)}{M_{0,z}^{\lambda,\mu,\eta} F_r(z)} = \phi\left(\frac{z(z+r)}{1+rz}\right), \qquad (F_r(0) = {F_r}'(0) - 1 = 0).$$

and

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}G_r(z)}{M_{0,z}^{\lambda,\mu,\eta}G_r(z)} = \phi\left(-\frac{z(z+r)}{1+rz}\right), \quad (G_r(0) = G_r{}'(0) - 1 = 0).$$

respectively. It is clear that the functions $K_{\phi n}$, F_r and G_r belong to the class $S^*_{1,p,\lambda,\mu,\eta}(\phi)$. If $\theta < \sigma_1$ or $\theta > \sigma_2$, then the equality holds if and only if f is $K_{\phi 2}$ or one of its rotations. When $\sigma_1 < \theta < \sigma_2$, the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\theta = \sigma_1$, then the equality holds if and only if f is F_r or one of its rotations. If $\theta = \sigma_2$, then the equality holds if and only if f is G_r or one of its rotations. The proof is complete.

In the similar manner, we can obtain the coefficient bound for $\left|a_{p+2} - \theta a_{p+1}^2\right|$ of functions in the class $S^*_{b,p,\lambda,\mu,\eta}(\phi)$ according to (Amsheri and Zharkova, 2012b).

Theorem 3.3.1.3. Let $0 \le \theta \le 1;\ \lambda \ge 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1; b \in \mathbb{C}\backslash\{0\}$ and $p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ where B_n are real with $B_1 > 0$ and $B_2 \ge 0$. If $f(z) \in \mathcal{A}(p)$ belongs to $S^*_{b,p,\lambda,\mu,\eta}(\phi)$, then for any complex number θ, we have

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \le \frac{(p-\mu)|b|B_1}{2\delta_2} \max\left\{1, \left|\frac{(p-\mu)b(2\delta_2\theta - \delta_1^2)}{\delta_1^2} B_1 - \frac{B_2}{B_1}\right|\right\} \tag{3.3.1.17}$$

3.3.2 Coefficient bounds for classes of p-valent Bazilevič functions

Motivated by the class $R_{b,p}(\phi)$ which was studied by (Ali et al., 2007) and the class $R_{b,p,\alpha,\beta}(\phi)$ of p-valent Bazilevič functions which was studied by (Ramachandran et al., 2007), we define a new general class of complex order $R^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$ of p-valent Bazilevič functions associated with the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ as given in (2.2.1) following the results by (Amsheri and Zharkova, 2012c).

Definition 3.3.2.1. Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f(z) \in \mathcal{A}(p)$ is in the class $R_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ if

$$1 + \frac{1}{b}\left\{(1-\beta)\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\alpha} + \beta \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\alpha} - 1\right\} \prec \phi(z) \tag{3.3.2.1}$$

where $\alpha \geq 0;\ 0 \leq \beta \leq 1;\ \lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N};\ b \in \mathbb{C}\backslash\{0\}$ and $z \in \mathcal{U}$. Also, we let $R_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi) = R_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$.

The above class $R_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ contains many well-known subclasses of analytic functions. In particular; for $\lambda = \mu = 0$, we have

$$R_{p,b,\alpha,\beta}^{0,0,\eta}(\phi) = R_{p,b,\alpha,\beta}(\phi)$$

where $R_{p,b,\alpha,\beta}(\phi)$ is precisely the class which was studied by (Ramachandran et al., 2007). Furthermore, when $b = 1,\ \lambda = \mu = 0$ and $\phi(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, we have

$$R_{p,1,\alpha,\beta}^{0,0,\eta}(\phi) = H_p(A, B, \alpha, \beta)$$

where $H_p(A, B, \alpha, \beta)$ is the class which introduced by (Owa, 2000).

Now, to obtain the coefficient bounds of functions belonging to the class $R_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, we use lemmas 3.1.7- 3.1.9 according (Amsheri and Zharkova, 2012c).

Theorem 3.3.2.2. Let $0 \le \theta \le 1$; $\lambda \ge 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu) - p - 1$ and $p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ where B_n are real with $B_1 > 0$, $B_2 \ge 0$, and

$$\sigma_1 = \frac{\delta_1^2[\alpha(p-\mu)+\beta]^2}{2\delta_2 B_1^2 (p-\mu)[\alpha(p-\mu)+2\beta]} \left\{ 2(B_2 - B_1) - \frac{B_1^2(p-\mu)(\alpha-1)[\alpha(p-\mu)+2\beta]}{[\alpha(p-\mu)+\beta]^2} \right\}, \tag{3.3.2.2}$$

$$\sigma_2 = \frac{\delta_1^2[\alpha(p-\mu)+\beta]^2}{2\delta_2 B_1^2 (p-\mu)[\alpha(p-\mu)+2\beta]} \left\{ 2(B_2 + B_1) - \frac{B_1^2(p-\mu)(\alpha-1)[\alpha(p-\mu)+2\beta]}{[\alpha(p-\mu)+\beta]^2} \right\}, \tag{3.3.2.3}$$

$$\sigma_3 = \frac{\delta_1^2[\alpha(p-\mu)+\beta]^2}{2\delta_2 B_1^2 (p-\mu)[\alpha(p-\mu)+2\beta]} \left\{ 2B_2 - \frac{B_1^2(p-\mu)(\alpha-1)[\alpha(p-\mu)+2\beta]}{[\alpha(p-\mu)+\beta]^2} \right\}, \tag{3.3.2.4}$$

and

$$A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta) = \frac{[\alpha(p-\mu)+2\beta][(\alpha-1)\delta_1^2 + 2\theta\delta_2]}{2\delta_1^2[\alpha(p-\mu)+\beta]^2}. \tag{3.3.2.5}$$

If $f(z) \in \mathcal{A}(p)$ belongs to $R_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \le$$

$$\begin{cases} \dfrac{(p-\mu)}{\delta_2[\alpha(p-\mu)+2\beta]}\left(B_2 - (p-\mu)B_1^2 A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)\right), & \theta < \sigma_1, \\ \dfrac{(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]}, & \sigma_1 \le \theta \le \sigma_2, \\ -\dfrac{(p-\mu)}{\delta_2[\alpha(p-\mu)+2\beta]}\left(B_2 - (p-\mu)B_1^2 A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)\right), & \theta > \sigma_2. \end{cases} \tag{3.3.2.6}$$

Further, if $\sigma_1 \le \theta \le \sigma_3$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| + \frac{\delta_1^2}{2\delta_2(p-\mu)B_1}$$

$$\times\left\{2\left(1-\frac{B_2}{B_1}\right)\frac{[\alpha(p-\mu)+\beta]^2}{[\alpha(p-\mu)+2\beta]}+(p-\mu)B_1[(\alpha-1)\delta_1^2+2\theta\delta_2]\right\}\left|a_{p+1}\right|^2$$

$$\leq\frac{(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]}. \tag{3.3.2.7}$$

If $\sigma_3\leq\theta\leq\sigma_2$, then

$$\left|a_{p+2}-\theta a_{p+1}^2\right|+\frac{\delta_1^2}{2\delta_2(p-\mu)B_1}$$

$$\times\left\{2\left(1+\frac{B_2}{B_1}\right)\frac{[\alpha(p-\mu)+\beta]^2}{[\alpha(p-\mu)+2\beta]}-(p-\mu)B_1[(\alpha-1)\delta_1^2+2\theta\delta_2]\right\}\left|a_{p+1}\right|^2$$

$$\leq\frac{(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]}. \tag{3.3.2.8}$$

For any complex number θ,

$$\left|a_{p+2}-\theta a_{p+1}^2\right|\leq$$

$$\frac{(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]}\max\left\{1,\left|(p-\mu)B_1A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)-\frac{B_2}{B_1}\right|\right\}. \tag{3.3.2.9}$$

Further,

$$\left|a_{p+3}\right|\leq\frac{(p-\mu)B_1}{\delta_3[\alpha(p-\mu)+3\beta]}H(q_1,q_2). \tag{3.3.2.10}$$

where $H(q_1,q_2)$ is as defined in Lemma 3.1.9,

$$q_1=\frac{2B_2}{B_1}-\frac{(p-\mu)B_1(\alpha-1)[\alpha(p-\mu)+3\beta]}{[\alpha(p-\mu)+\beta][\alpha(p-\mu)+2\beta]}, \tag{3.3.2.11}$$

and

$$q_2=\frac{B_3}{B_1}+\frac{(p-\mu)^2B_1^2(\alpha-1)(2\alpha-1)[\alpha(p-\mu)+3\beta]}{6[\alpha(p-\mu)+\beta]^3}$$

$$-\frac{(p-\mu)B_2(\alpha-1)[\alpha(p-\mu)+3\beta]}{[\alpha(p-\mu)+\beta][\alpha(p-\mu)+2\beta]}. \tag{3.3.2.12}$$

Proof. If $f(z)\in R_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, then there is a Schwarz function

$$w(z)=w_1z+w_2z^2+\cdots\in\Omega.$$

such that

$$(1-\beta)\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\alpha}+\beta\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\alpha}=\phi\big(w(z)\big).$$

(3.3.2.13)

since

$$(1-\beta)\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\alpha}+\beta\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\alpha}$$

$$=1+\frac{[\alpha(p-\mu)+\beta]}{(p-\mu)}\delta_1 a_{p+1}z+\left(\frac{[\alpha(p-\mu)+2\beta]}{(p-\mu)}\delta_2 a_{p+2}\right.$$

$$\left.+\frac{(\alpha-1)[\alpha(p-\mu)+2\beta]}{2(p-\mu)}\delta_1^2 a_{p+1}^2\right)z^2+\left(\frac{[\alpha(p-\mu)+3\beta]}{(p-\mu)}\delta_3 a_{p+3}\right.$$

$$+\frac{(\alpha-1)[\alpha(p-\mu)+3\beta]}{(p-\mu)}\delta_1\delta_2 a_{p+1}a_{p+2}$$

$$\left.+\frac{(\alpha-1)(\alpha-2)[\alpha(p-\mu)+3\beta]}{6(p-\mu)}\delta_1^3 a_{p+1}^3\right)z^3+\cdots. \qquad (3.3.2.14)$$

we have from (3.3.2.13),

$$a_{p+1}=\frac{(p-\mu)B_1 w_1}{\delta_1[\alpha(p-\mu)+\beta]}, \qquad (3.3.2.15)$$

$$a_{p+2}=\frac{(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]}\left\{w_2\right.$$

$$\left.-\left(\frac{(p-\mu)B_1(\alpha-1)[\alpha(p-\mu)+2\beta]}{2[\alpha(p-\mu)+\beta]^2}-\frac{B_2}{B_1}\right)w_1^2\right\}. \qquad (3.3.2.16)$$

and

$$a_{p+3}=\frac{(p-\mu)B_1}{\delta_3[\alpha(p-\mu)+3\beta]}\{w_3+q_1w_1w_2+q_2w_1^3\}. \qquad (3.3.2.17)$$

where q_1 and q_2 as defined (3.3.2.11) and (3.3.2.12), respectively. Therefore, we have

$$a_{p+2}-\theta a_{p+1}^2=\frac{(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]}\{w_2-vw_1^2\}, \qquad (3.3.2.18)$$

where

$$v := (p-\mu)B_1 A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta) - \frac{B_2}{B_1}. \tag{3.3.2.19}$$

By making use of (3.3.2.15)-(33.2.19), the results (3.3.2.6) - (3.3.2.9) are established by an application of Lemma 3.1.7, inequality (3.3.2.9) by Lemma 3.1.8, and (3.3.2.10) follows from Lemma 3.1.9. To show that the bounds in (3.3.2.6) - (3.3.2.9) are sharp, we define the functions $K_{\phi n}$ $(n = 2,3,\dots)$ by

$$(1-\beta)\left(\frac{M_{0,z}^{\lambda,\mu,\eta}K_{\phi n}(z)}{z^p}\right)^{\alpha} + \beta\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}K_{\phi n}(z)}{M_{0,z}^{\lambda,\mu,\eta}K_{\phi n}(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}K_{\phi n}(z)}{z^p}\right)^{\alpha} = \phi(z^{n-1}),$$

$$K_{\phi n}(0) = \left(K_{\phi n}\right)'(0) - 1 = 0.$$

and the functions $F_r,\ G_r$ $(0 \le r \le 1)$ defined by

$$(1-\beta)\left(\frac{M_{0,z}^{\lambda,\mu,\eta}F_r(z)}{z^p}\right)^{\alpha} + \beta\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}F_r(z)}{M_{0,z}^{\lambda,\mu,\eta}F_r(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}F_r(z)}{z^p}\right)^{\alpha} = \phi\left(\frac{z(z+r)}{1+rz}\right),$$

$$F_r(0) = F_r{}'(0) - 1 = 0.$$

and

$$(1-\beta)\left(\frac{M_{0,z}^{\lambda,\mu,\eta}G_r(z)}{z^p}\right)^{\alpha} + \beta\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}G_r(z)}{M_{0,z}^{\lambda,\mu,\eta}G_r(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}G_r(z)}{z^p}\right)^{\alpha}$$

$$= \phi\left(-\frac{z(z+r)}{1+rz}\right),$$

$$G_r(0) = G_r{}'(0) - 1 = 0.$$

respectively. It is clear that the functions $K_{\phi n}$, F_r and G_r belong to the class $R_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$. If $\theta < \sigma_1$ or $\theta > \sigma_2$, then the equality holds if and only if f is $K_{\phi 2}$ or one of its rotations. When $\sigma_1 < \theta < \sigma_2$, the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\theta = \sigma_1$, then the equality holds if and only if f

is F_r or one of its rotations. If $\theta = \sigma_2$, then the equality holds if and only if f is G_r or one of its rotations. The proof is complete.

Remark 1. By specifying the parameters $p, \alpha, \beta, \lambda$ and μ in Theorem 3.3.2.2, we have the most the coefficient bound results which were obtained by other authors:

1. Letting $p = 1, \alpha = 0, \beta = 1, B_1 = \frac{8}{\pi^2}, B_2 = \frac{16}{3\pi^2}$ and $\mu = \lambda$, we get the corresponding result due to (Srivastava and Mishra, 2000).
2. Letting $p = 1,\ \alpha = 0,\ \beta = 1$ and $\mu = \lambda = 0$, we obtain the corresponding result due to (Ma and Minda, 1994) for the class $S^*(\phi)$.
3. Letting $\alpha = 0,\ \beta = 1$ and $\mu = \lambda = 0$, we obtain the result which was proven by (Ali et al., 2007) for the class $S_p^*(\phi)$.
4. Letting $p = 1, \beta = 1$ and $\mu = \lambda = 0$, we obtain the result which was proven by (Ravichandran et al., 2004) for the class $B^{\alpha}(\phi)$.
5. Letting $\mu = \lambda = 0$, we obtain the result which was proven by (Ramachandran et al., 2007) for the class $R_{p,1,\alpha,\beta}(\phi)$.

Thus, the generalization of classes $R_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ defined in this subsection is proven to account for most available classes discussed in the previous papers generalize the concept of starlike and Bazilevič functions.

In the similar manner, we can obtain the coefficient bound for the functional $\left|a_{p+2} - \theta a_{p+1}^2\right|$ of functions belonging to the class $R_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ according to (Amsheri and Zharkova, 2012c).

Theorem 3.3.2.3. Let $0 \le \theta \le 1;\ \lambda \ge 0;\ \mu < p + 1;\ \eta > \max(\lambda, \mu) - p - 1$ and $p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ where B_n are real with $B_1 > 0$, $B_2 \geq 0$. If $f(z) \in \mathcal{A}(p)$ belongs to $R^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$, then for any complex number θ,

$$\left|a_{p+2} - \theta a^2_{p+1}\right| \leq \frac{(p-\mu)|b|B_1}{\delta_2[\alpha(p-\mu)+2\beta]} \max\left\{1, \left|(p-\mu)bB_1 A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta) - \frac{B_2}{B_1}\right|\right\} \tag{3.3.2.20}$$

where $A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)$ is given by (3.3.2.5).

Remark 2. By specifying the parameters $p, \alpha, \beta, \lambda$ and μ in Theorem 3.3.2.3, we the most coefficient bound results which were obtained by other authors.

1. Letting $p = 1$, $\alpha = 0$, $\beta = 1$ and $\mu = \lambda = 0$, we obtain the corresponding result due to (Ravichandran et al., 2005) for the class $S^*_b(\phi)$.
2. Letting $p = 1, \alpha = 1$, $\beta = 1$, $\mu = \lambda = 0$ and $\phi(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, we obtain the results which were proven by (Dixit and Pal., 1995) for the class $R^b(A, B)$.
3. Letting $\alpha = 1$, $\beta = 1$ and $\mu = \lambda = 0$, we obtain the result which was proven by (Ali et al., 2007) for the class $R_{b,p}(\phi)$.

Thus, the generalization of classes $R^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$ defined in this subsection is proven to account for most available classes discussed in the previous papers.

Next, motivated by the class $M_{\alpha,\beta}(\phi)$ which introduced by (Rosy et al., 2009), we introduce a more general class of complex order $M^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$ of Bazilevič functions by using the fractional derivative operator $M^{\lambda,\mu,\eta}_{0,z}f(z)$ as given in (2.2.1) following the results by (Amsheri and Zharkova, 2013a).

Definition 3.3.2.4. Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f(z) \in \mathcal{A}(p)$ is in the class $M_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ if

$$1 + \frac{1}{b}\left\{\Psi_{\lambda,\mu,\eta}(\alpha,\beta,p)f(z) - 1\right\} \prec \phi(z), \tag{3.3.2.21}$$

where

$$\Psi_{\lambda,\mu,\eta}(\alpha,\beta,p)f(z) =$$

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\alpha} + \beta\left[1 + (p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}\right.$$

$$\left.-(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} + \alpha\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} - 1\right)\right]. \tag{3.3.2.22}$$

where $\alpha \geq 0$; $\beta \geq 0$; $\lambda \geq 0$; $\mu < p + 1$; $\eta > \max(\lambda,\mu) - p - 1$; $p \in \mathbb{N}$; $b \in \mathbb{C}\backslash\{0\}$ and $z \in \mathcal{U}$. Also, we let $M_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi) = M_{p,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$.

The above class $M_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ contains many well-known subclasses of analytic functions. In particular; for $p = 1$, $b = 1$ and $\lambda = \mu = 0$, we have

$$M_{1,1,\alpha,\beta}^{0,0,\eta}(\phi) = M_{\alpha,\beta}(\phi)$$

where $M_{\alpha,\beta}(\phi)$ is precisely the class which was studied by (Rosy et al., 2009).

Now, to obtain the coefficient bounds of functions belonging to the class $M_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, we use lemmas 3.1.7- 3.1.9 following (Amsheri and Zharkova, 2013a).

Theorem 3.3.2.5. Let $0 \leq \theta \leq 1;\ \alpha \geq 0;\ \beta \geq 0;\ \lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1$ and $p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ where B_n are real with $B_1 > 0,\ B_2 \geq 0$, and

$$\sigma_1 = \frac{\delta_1^2\,\tau^2}{2\delta_2(p-\mu)\xi B_1^2}\left\{2(B_2 - B_1) - \frac{(\rho-\gamma)B_1^2}{\tau^2}\right\}, \tag{3.3.2.23}$$

$$\sigma_2 = \frac{\delta_1^2\,\tau^2}{2\delta_2(p-\mu)\xi B_1^2}\left\{2(B_2 + B_1) - \frac{(\rho-\gamma)B_1^2}{\tau^2}\right\}, \tag{3.3.2.24}$$

$$\sigma_3 = \frac{\delta_1^2\,\tau^2}{2\delta_2(p-\mu)\xi B_1^2}\left\{2B_2 - \frac{(\rho-\gamma)B_1^2}{\tau^2}\right\}, \tag{3.3.2.25}$$

$$\tau = 1 + \alpha(p-\mu) + \beta(1+\alpha), \tag{3.3.2.26}$$

$$\xi = 2 + \alpha(p-\mu) + 2\beta(\alpha+2), \tag{3.3.2.27}$$

$$\rho = (\alpha-1)(p-\mu)[\alpha(p-\mu)+2], \tag{3.3.2.28}$$

$$\gamma = 2\beta[1 + (p-\mu)(\alpha+2)], \tag{3.3.2.29}$$

$$\epsilon_1 = 3 + \alpha(p-\mu) + 3\beta(\alpha+3), \tag{3.3.2.30}$$

$$\epsilon_2 = (\alpha-1)(p-\mu)[\alpha(p-\mu)+3], \tag{3.3.2.31}$$

$$\epsilon_3 = 3\beta[2 + (p-\mu)(\alpha+3)], \tag{3.3.2.32}$$

$$\epsilon_4 = 6\beta[1 + (p-\mu)(\alpha+3)], \tag{3.3.2.33}$$

and

$$A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta) = \frac{\delta_1^2(\rho-\gamma) + 2\theta\delta_2(p-\mu)\xi}{2\delta_1^2\tau^2}\,. \tag{3.3.2.34}$$

If $f(z) \in \mathcal{A}(p)$ belongs to $M_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, then

$\left|a_{p+2} - \theta a_{p+1}^2\right| \leq$

$$\leq \begin{cases} \dfrac{(p-\mu)}{\delta_2\xi}\left(B_2 - A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)B_1^2\right), & \theta < \sigma_1, \\ \dfrac{(p-\mu)B_1}{\delta_2\xi}, & \sigma_1 \leq \theta \leq \sigma_2, \\ -\dfrac{(p-\mu)}{\delta_2\xi}\left(B_2 - A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)B_1^2\right), & \theta > \sigma_2. \end{cases} \tag{3.3.2.35}$$

Further, if $\sigma_1 \leq \theta \leq \sigma_3$, then

$$\left|a_{p+2}-\theta a_{p+1}^2\right|+\frac{\delta_1^2\,\tau^2}{\delta_2(p-\mu)\xi B_1}\left\{1-\frac{B_2}{B_1}+A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)B_1\right\}\left|a_{p+1}\right|^2$$

$$\leq \frac{(p-\mu)B_1}{\delta_2\xi}. \tag{3.3.2.36}$$

If $\sigma_3 \leq \theta \leq \sigma_2$, then

$$\left|a_{p+2}-\theta a_{p+1}^2\right|+\frac{\delta_1^2\,\tau^2}{\delta_2(p-\mu)\xi B_1}\left\{1+\frac{B_2}{B_1}-A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)B_1\right\}\left|a_{p+1}\right|^2$$

$$\leq \frac{(p-\mu)B_1}{\delta_2\xi}. \tag{3.3.2.37}$$

For any complex number θ,

$$\left|a_{p+2}-\theta a_{p+1}^2\right| \leq \frac{(p-\mu)B_1}{\delta_2\xi}\max\left\{1,\left|A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)B_1-\frac{B_2}{B_1}\right|\right\}. \tag{3.3.2.38}$$

Further,

$$\left|a_{p+3}\right| \leq \frac{(p-\mu)B_1}{\delta_3\epsilon_1}H(q_1,q_2). \tag{3.3.2.39}$$

where $H(q_1,q_2)$ is as defined in Lemma 3.1.9,

$$q_1=\frac{2B_2}{B_1}-\frac{(\epsilon_2-\epsilon_3)B_1}{\tau\xi}, \tag{3.3.2.40}$$

and

$$q_2=\frac{B_3}{B_1}-\frac{(\epsilon_2-\epsilon_3)B_2}{\tau\xi}+\frac{(3(\epsilon_2-\epsilon_3)(\rho-\gamma)\xi^2-\tau^3[(p-\mu)(\alpha-2)\epsilon_2+\epsilon_4])B_1^2}{6\tau^3\xi^3}. \tag{3.3.2.41}$$

Proof. If $f(z) \in M_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, then there is a Schwarz function

$$w(z)=w_1z+w_2z^2+\cdots \in \Omega.$$

such that

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\alpha}+\beta\left[1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-\right.$$

$$(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}+\alpha\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-1\right)\Bigg]=\phi\big(w(z)\big). \quad (3.3.2.42)$$

since

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\alpha}+\beta\Bigg[1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-$$

$$(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}+\alpha\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-1\right)\Bigg]=1+$$

$$\frac{[1+\alpha(p-\mu)+\beta(\alpha+1)]}{(p-\mu)}\delta_1 a_{p+1}z+\Bigg(\frac{[2+\alpha(p-\mu)+2\beta(\alpha+2)]}{(p-\mu)}\delta_2 a_{p+2}-$$

$$\frac{(1-\alpha)(p-\mu)[\alpha(p-\mu)+2]+2\beta[1+(p-\mu)(\alpha+2)]}{2(p-\mu)^2}\delta_1^2 a_{p+1}^2\Bigg)z^2+$$

$$\Bigg(\frac{[3+\alpha(p-\mu)+3\beta(\alpha+3)]}{(p-\mu)}\delta_3 a_{p+3}+\Bigg\{\frac{(\alpha-1)[\alpha(p-\mu)+3]}{(p-\mu)}-$$

$$\frac{3\beta[2+(p-\mu)(\alpha+3)]}{(p-\mu)^2}\Bigg\}\delta_1\delta_2 a_{p+1}a_{p+2}+\Bigg\{\frac{(\alpha-1)(\alpha-2)[\alpha(p-\mu)+3]}{6(p-\mu)}+$$

$$\frac{\beta[1+(p-\mu)(\alpha+3)]}{(p-\mu)^3}\Bigg\}\delta_1^3 a_{p+1}^3\Bigg)z^3+\cdots$$

we have from (3.3.2.42),

$$a_{p+1}=\frac{(p-\mu)B_1 w_1}{\delta_1\tau}, \quad (3.3.2.43)$$

$$a_{p+2}=\frac{(p-\mu)B_1}{\delta_2\xi}\left\{w_2-\left(\frac{(\rho-\gamma)B_1}{2\tau^2}-\frac{B_2}{B_1}\right)w_1^2\right\}, \quad (3.3.2.44)$$

$$a_{p+3}=\frac{(p-\mu)B_1}{\delta_3\epsilon_1}\{w_3+q_1w_1w_2+q_2w_1^3\}, \quad (3.3.2.45)$$

where q_1 and q_2 as defined in (3.3.2.40) and (3.3.2.41), respectively. Therefore, we have

$$a_{p+2}-\theta a_{p+1}^2=\frac{(p-\mu)B_1}{\delta_2\xi}\{w_2-vw_1^2\}, \quad (3.3.2.46)$$

where

$$v := A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)B_1 - \frac{B_2}{B_1}\,. \tag{3.3.2.47}$$

By making use of (3.3.2.43)-(3.3.2.47), the results (3.3.2.35)-(3.3.2.38) are established by an application of Lemma 3.1.7, inequality (3.3.2.38) by Lemma 3.1.8, and (3.3.2.39) follows from Lemma 3.1.9. To show that the bounds in (3.3.2.35)-(3.3.2.38) are sharp, we define the functions $K_{\phi n}$ $(n = 2,3,\dots)$ by

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}K_{\phi n}(z)}{M_{0,z}^{\lambda,\mu,\eta}K_{\phi n}(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}K_{\phi n}(z)}{z^p}\right)^{\alpha} +$$

$$\beta\left[1 + (p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}K_{\phi n}(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}K_{\phi n}(z)} - \right.$$

$$\left.(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}K_{\phi n}(z)}{M_{0,z}^{\lambda,\mu,\eta}K_{\phi n}(z)} + \alpha\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}K_{\phi n}(z)}{M_{0,z}^{\lambda,\mu,\eta}K_{\phi n}(z)} - 1\right)\right] = \phi(z^{n-1}),$$

$$K_{\phi n}(0) = \left(K_{\phi n}\right)'(0) - 1 = 0.$$

and the functions $F_r,\ G_r\ (0 \le r \le 1)$ defined by

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}F_r(z)}{M_{0,z}^{\lambda,\mu,\eta}F_r(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}F_r(z)}{z^p}\right)^{\alpha} + \beta\left[1 + (p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}F_r(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}F_r(z)} - \right.$$

$$\left.(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}F_r(z)}{M_{0,z}^{\lambda,\mu,\eta}F_r(z)} + \alpha\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}F_r(z)}{M_{0,z}^{\lambda,\mu,\eta}F_r(z)} - 1\right)\right] = \phi\left(\frac{z(z+r)}{1+rz}\right),$$

$$F_r(0) = F_r{}'(0) - 1 = 0.$$

and

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}G_r(z)}{M_{0,z}^{\lambda,\mu,\eta}G_r(z)}\left(\frac{M_{0,z}^{\lambda,\mu,\eta}G_r(z)}{z^p}\right)^{\alpha} + \beta\left[1 + (p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}G_r(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}G_r(z)} - \right.$$

$$\left.(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}G_r(z)}{M_{0,z}^{\lambda,\mu,\eta}G_r(z)} + \alpha\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}G_r(z)}{M_{0,z}^{\lambda,\mu,\eta}G_r(z)} - 1\right)\right] = \phi\left(-\frac{z(z+r)}{1+rz}\right),$$

$$G_r(0) = G_r{}'(0) - 1 = 0.$$

respectively. It is clear that the functions $K_{\phi n},\ F_r$ and G_r belong to the class $M_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$. If $\theta < \sigma_1$ or $\theta > \sigma_2$, then the equality holds if and only if f is $K_{\phi 2}$ or one of its rotations. When $\sigma_1 < \theta < \sigma_2$, the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\theta = \sigma_1$, then the equality holds if and only if f is F_r or one of its rotations. If $\theta = \sigma_2$, then the equality holds if and only if f is G_r or one of its rotations. The proof is complete.

Remark 3. By specifying the parameters $p, \alpha, \beta, \lambda$ and μ in Theorem 3.3.2.5, we have the most the coefficient bound results which were obtained by other authors:

1. By letting $p = 1, \alpha = 0, \beta = 0, B_1 = \frac{8}{\pi^2}, B_2 = \frac{16}{3\pi^2}$ and $\mu = \lambda$, we get the corresponding result due to (Srivastava and Mishra, 2000).
2. By letting $p = 1, \alpha = 0, \beta = 0$ and $\mu = \lambda = 0$, we obtain the corresponding result due to (Ma and Minda, 1994) for the class $S^*(\phi)$.
3. By letting $\alpha = 0$ and $\beta = 0$, we obtain the result which was proven by (Amsheri and Zahrkova, 2012b) for the class $S_{p,\lambda,\mu,\eta}^*(\phi)$.
4. By letting $\alpha = 0,\ \beta = 0$ and $\mu = \lambda = 0$, we obtain the result which was provenby (Ali et al., 2007) for the class $S_p^*(\phi)$.
5. By letting $p = 1, \alpha = 0,\ \beta = 1$ and $\mu = \lambda = 0$, we obtain the result according to (Ma and Minda, 1994) for the class $C(\phi)$.
6. By letting $p = 1, \beta = 0$ and $\mu = \lambda = 0$, we obtain the corresponding result due to (Ravichandran et al., 2004) for the class $B^\alpha(\phi)$.
7. By letting $p = 1$ and $\mu = \lambda = 0$, we obtain the corresponding result due to (Rosy et al., 2009) for the class $M_{\alpha,\beta}(\phi)$.

Thus, the generalization of classes $M^{\lambda,\mu,\eta}_{p,1,\alpha,\beta}(\phi)$ defined in this subsection is proven to account for most available classes discussed in the previous papers and generalize the concept of starlike and Bazilevič functions.

In the similar manner, we can obtain the coefficient bound for the functional $\left|a_{p+2}-\theta a_{p+1}^2\right|$ of functions belonging to the class $M^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$ following (Amsheri and Zharkova, 2013a).

Theorem 3.3.2.6. Let $0 \le \theta \le 1;\ \alpha \ge 0;\ \beta \ge 0;\ \lambda \ge 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ b \in \mathbb{C} \setminus \{0\}$ and $p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ where B_n are real with $B_1 > 0$ and $B_2 \ge 0$. If $f(z) \in \mathcal{A}(p)$ belongs to $M^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$, then for any complex number θ, we have

$$\left|a_{p+2}-\theta a_{p+1}^2\right| \le \frac{(p-\mu)|b|B_1}{\delta_2 \xi} \max\left\{1, \left|A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta) b B_1 - \frac{B_2}{B_1}\right|\right\}. \tag{3.3.2.48}$$

where ξ and $A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)$ are given by (3.3.2.27) and (3.3.2.34) respectively.

Remark 4. By specializing the parameters $p, \alpha, \beta, \lambda$ and μ in Theorem 3.3.2.6, we have the most the coefficient bound results which were obtained by other authors:

1. Letting $p = 1, \alpha = 0, \beta = 0$ and $\mu = \lambda = 0$, we obtain the corresponding result due to (Ravichandran et al., 2005) for the class $S_b^*(\phi)$.
2. Letting $p = 1,\ \alpha = 1, \beta = 0,\ \lambda = \mu = 0$ and $\phi(z) = \frac{1+Az}{1+Bz}, -1 \le B < A \le 1$, we obtain the results which were proven by (Dixit and Pal, 1995) for the class $R^b(A,B)$.

3. Letting $\alpha = 1$, $\beta = 0$ and $\mu = \lambda = 0$, we obtain the result which was proven by (Ali et al. 2007) for the class $R_{b,p}(\phi)$.
4. Letting $\alpha = 0$ and $\beta = 0$, we obtain the corresponding result due to (Amsheri and Zaharkova, 2012b) for the class $S^*_{p,b,\lambda,\mu,\eta}(\phi)$.

Thus, the generalization of classes $M^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$ defined in this subsection is proven to account for most available classes discussed in the previous papers.

3.3.3 Coefficient bounds for classes of p-valent non-Bazilevič functions

Motivated by the class $N_{\alpha,\beta}(\phi)$ which was introduced by (Shanmugam et al., 2006a), we introduce a more general class of complex order $N^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$ of p-valent non-Bazilevič functions by using the fractional derivative operator $M^{\lambda,\mu,\eta}_{0,z} f(z)$ as given in (2.2.1) following the results by (Amsheri and Zharkova, 2012d).

Definition 3.3.3.1. Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f(z) \in \mathcal{A}(p)$ is in the class $N^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$ if

$$1 + \frac{1}{b}\left\{(1+\beta)\left(\frac{z^p}{M^{\lambda,\mu,\eta}_{0,z} f(z)}\right)^{\alpha} - \beta \frac{M^{\lambda+1,\mu+1,\eta+1}_{0,z} f(z)}{M^{\lambda,\mu,\eta}_{0,z} f(z)}\left(\frac{z^p}{M^{\lambda,\mu,\eta}_{0,z} f(z)}\right)^{\alpha} - 1\right\} \prec \phi(z). \tag{3.3.3.1}$$

where $0 < \alpha < 1$; $\beta \in \mathbb{C}$; $\lambda \geq 0$; $\mu < p + 1$; $\eta > \max(\lambda, \mu) - p - 1$; $p \in \mathbb{N}$; $b \in \mathbb{C} \backslash \{0\}$ and $z \in \mathcal{U}$. Also, we let $N^{\lambda,\mu,\eta}_{p,1,\alpha,\beta}(\phi) = N^{\lambda,\mu,\eta}_{p,\alpha,\beta}(\phi)$.

The above class $N_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ contains many well-known classes of analytic functions. In particular; for $\lambda = \mu = 0,\ p = 1$, and $b = 1$ we have

$$N_{1,1,\alpha,\beta}^{0,0,\eta}(\phi) = N_{\alpha,\beta}(\phi)$$

where $N_{\alpha,\beta}(\phi)$ is precisely the class which was studied by (Shanmugam et al., 2006a).

Now, to obtain the coefficient bounds of functions belonging to the class $N_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, we use lemmas 3.1.7- 3.1.9 following (Amsheri and Zharkova, 2012d).

Theorem 3.3.3.2. Let $0 \leq \theta \leq 1;\ \lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1, 0 < \alpha < 1, \beta \in \mathbb{C}$ and $p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ where B_n are real with $B_1 > 0,\ B_2 \geq 0$, and

$$\sigma_1 = \frac{\delta_1^2[\alpha(p-\mu)+\beta]^2}{2\delta_2 B_1^2(p-\mu)[\alpha(p-\mu)+2\beta]}\left\{\frac{B_1^2(p-\mu)(\alpha+1)[\alpha(p-\mu)+2\beta]}{[\alpha(p-\mu)+\beta]^2} - 2(B_2 - B_1)\right\}, \tag{3.3.3.2}$$

$$\sigma_2 = \frac{\delta_1^2[\alpha(p-\mu)+\beta]^2}{2\delta_2 B_1^2(p-\mu)[\alpha(p-\mu)+2\beta]}\left\{\frac{B_1^2(p-\mu)(\alpha+1)[\alpha(p-\mu)+2\beta]}{[\alpha(p-\mu)+\beta]^2} - 2(B_2 + B_1)\right\}, \tag{3.3.3.3}$$

$$\sigma_3 = \frac{\delta_1^2[\alpha(p-\mu)+\beta]^2}{2\delta_2 B_1^2(p-\mu)[\alpha(p-\mu)+2\beta]}\left\{\frac{B_1^2(p-\mu)(\alpha+1)[\alpha(p-\mu)+2\beta]}{[\alpha(p-\mu)+\beta]^2} - 2B_2\right\}, \tag{3.3.3.4}$$

and

$$A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta) = \frac{[\alpha(p-\mu)+2\beta][(\alpha+1)\delta_1^2 - 2\theta\delta_2]}{2\delta_1^2[\alpha(p-\mu)+\beta]^2}. \tag{3.3.3.5}$$

If $f(z) \in \mathcal{A}(p)$ belongs to $N_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \leq$$

$$\begin{cases} \dfrac{(p-\mu)}{\delta_2[\alpha(p-\mu)+2\beta]}\left(-B_2 + (p-\mu)B_1^2 A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)\right), & \theta < \sigma_1, \\ \dfrac{-(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]}, & \sigma_1 \leq \theta \leq \sigma_2, \\ \dfrac{(p-\mu)}{\delta_2[\alpha(p-\mu)+2\beta]}\left(B_2 - (p-\mu)B_1^2 A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)\right), & \theta > \sigma_2. \end{cases} \quad (3.3.3.6)$$

Further, if $\sigma_1 \leq \theta \leq \sigma_3$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| -$$

$$\frac{\delta_1^2[\alpha(p-\mu)+\beta]^2}{\delta_2(p-\mu)B_1^2[\alpha(p-\mu)+2\beta]}\left\{B_1 - B_2 + (p-\mu)B_1^2 A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)\right\}\left|a_{p+1}\right|^2$$

$$\leq \frac{-(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]}. \quad (3.3.3.7)$$

If $\sigma_3 \leq \theta \leq \sigma_2$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| -$$

$$\frac{\delta_1^2[\alpha(p-\mu)+\beta]^2}{\delta_2(p-\mu)B_1^2[\alpha(p-\mu)+2\beta]}\left\{B_1 + B_2 - (p-\mu)B_1^2 A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)\right\}\left|a_{p+1}\right|^2$$

$$\leq \frac{-(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]}. \quad (3.3.3.8)$$

For any complex number θ,

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \leq \frac{-(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]} \max\left\{1, \left|(p-\mu)B_1 A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta) - \frac{B_2}{B_1}\right|\right\}.$$

(3.3.3.9)

Further,

$$\left|a_{p+3}\right| \leq \frac{(p-\mu)B_1}{\delta_3[\alpha(p-\mu)+3\beta]} H(q_1, q_2), \quad (3.3.3.10)$$

where $H(q_1, q_2)$ is as defined in Lemma 3.1.9,

$$q_1 = \frac{2B_2}{B_1} - \frac{(p-\mu)B_1(\alpha+1)[\alpha(p-\mu)+3\beta]}{[\alpha(p-\mu)+\beta][\alpha(p-\mu)+2\beta]}, \tag{3.3.3.11}$$

$$q_2 = \frac{B_3}{B_1} + \frac{(p-\mu)^2 B_1^2(\alpha+1)(2\alpha+1)[\alpha(p-\mu)+3\beta]}{6[\alpha(p-\mu)+\beta]^3}$$

$$- \frac{(p-\mu)B_2(\alpha+1)[\alpha(p-\mu)+3\beta]}{[\alpha(p-\mu)+\beta][\alpha(p-\mu)+2\beta]}. \tag{3.3.3.12}$$

Proof. If $f(z) \in N_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, then there is a Schwarz function

$$w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega.$$

such that

$$(1+\beta)\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^{\alpha} - \beta \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^{\alpha} = \phi(w(z)). \tag{3.3.3.13}$$

since

$$(1+\beta)\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^{\alpha} - \beta \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^{\alpha} =$$

$$1 - \frac{[\alpha(p-\mu)+\beta]}{(p-\mu)}\delta_1 a_{p+1} z +$$

$$\frac{[\alpha(p-\mu)+2\beta]}{(p-\mu)}\left(\frac{(\alpha+1)}{2}\delta_1^2 a_{p+1}^2 - \delta_2 a_{p+2}\right) z^2 +$$

$$\frac{[\alpha(p-\mu)+3\beta]}{(p-\mu)}\left(-\delta_3 a_{p+3} + (\alpha+1)\delta_1\delta_2 a_{p+1}a_{p+2} - \frac{(\alpha+1)(\alpha+2)}{6}\delta_1^3 a_{p+1}^3\right) z^3$$

$$+ \cdots \tag{3.3.3.14}$$

we have from (3.3.3.13),

$$a_{p+1} = \frac{-(p-\mu)B_1 w_1}{\delta_1[\alpha(p-\mu)+\beta]}, \tag{3.3.3.15}$$

$$a_{p+2} = \frac{-(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]}\left\{w_2 - \left(\frac{(p-\mu)B_1(\alpha+1)[\alpha(p-\mu)+2\beta]}{2[\alpha(p-\mu)+\beta]^2} - \frac{B_2}{B_1}\right) w_1^2\right\} \tag{3.3.3.16}$$

and

$$a_{p+3} = \frac{-(p-\mu)B_1}{\delta_3[\alpha(p-\mu)+3\beta]}\{w_3 + q_1 w_1 w_2 + q_2 w_1^3\}. \qquad (3.3.3.17)$$

where q_1 and q_2 as defined (3.3.3.11) and (3.3.3.12), respectively. Therefore, we have

$$a_{p+2} - \theta a_{p+1}^2 = \frac{-(p-\mu)B_1}{\delta_2[\alpha(p-\mu)+2\beta]}\{w_2 - v w_1^2\}\,, \qquad (3.3.3.18)$$

where

$$v \coloneqq (p-\mu)B_1 A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta) - \frac{B_2}{B_1}\,. \qquad (3.3.3.19)$$

By making use of (3.3.3.15)-(3.3.3.19), the results (3.3.3.6) - (3.3.3.9) are established by an application of Lemma 3.1.7, inequality (3.3.3.9) by Lemma 3.1.8, and (3.3.3.10) follows from Lemma 3.1.9. To show that the bounds in (3.3.3.6) - (3.3.3.9) are sharp, we define the functions $K_{\phi n}$ $(n = 2,3,\dots)$ by

$$(1+\beta)\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta}K_{\phi n}(z)}\right)^{\alpha} - \beta\,\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}K_{\phi n}(z)}{M_{0,z}^{\lambda,\mu,\eta}K_{\phi n}(z)}\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta}K_{\phi n}(z)}\right)^{\alpha} = \phi(z^{n-1}),$$

$$K_{\phi n}(0) = \left(K_{\phi n}\right)'(0) - 1 = 0.$$

and the functions $F_r,\ G_r$ $(0 \le r \le 1)$ defined by

$$(1+\beta)\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta}F_r(z)}\right)^{\alpha} - \beta\,\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}F_r(z)}{M_{0,z}^{\lambda,\mu,\eta}F_r(z)}\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta}F_r(z)}\right)^{\alpha} = \phi\left(\frac{z(z+r)}{1+rz}\right),$$

$$F_r(0) = F_r{}'(0) - 1 = 0.$$

and

$$(1+\beta)\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta}G_r(z)}\right)^{\alpha} - \beta\,\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}G_r(z)}{M_{0,z}^{\lambda,\mu,\eta}G_r(z)}\left(\frac{z^p}{M_{0,z}^{\lambda,\mu,\eta}G_r(z)}\right)^{\alpha}$$

$$= \phi\left(-\frac{z(z+r)}{1+rz}\right),$$

$$G_r(0) = G_r{}'(0) - 1 = 0.$$

respectively. It is clear that the functions $K_{\phi n}$, F_r and G_r belong to the class $N_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$. If $\theta < \sigma_1$ or $\theta > \sigma_2$, then the equality holds if and only if f is $K_{\phi 2}$ or one of its rotations. When $\sigma_1 < \theta < \sigma_2$, the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\theta = \sigma_1$, then the equality holds if and only if f is F_r or one of its rotations. If $\theta = \sigma_2$, then the equality holds if and only if f is G_r or one of its rotations. The proof is complete.

Remark 1. By specifying the parameters $p, \alpha, \beta, \lambda$ and μ in Theorem 3.3.3.2, we have the most the coefficient bound results which were obtained by other authors:

1. Letting $p = 1$ and $\mu = \lambda = 0$, we obtain the results which were proven by [(Shanmugam et al., 2006a), Theorem 2.1, Remark 2.4] for the class $N_{\alpha,\beta}(\phi)$.
2. Letting $p = 1$ and $\mu = \lambda$, we obtain the result which was proven by [(Shanmugam et al., 2006a), Theorem 3.1] for the class $N_{\alpha,\beta}(\phi)$.

Thus, the generalization of class $N_{p,1,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ defined in this subsection is proven to some classes discussed in the previous papers and generalize the concept of non-Bazilevič functions.

In the similar manner, we can obtain the coefficient bound for the functional $\left|a_{p+2} - \theta a_{p+1}^2\right|$ of functions belonging to the class $N_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ following (Amsheri and Zharkova, 2012d).

Theorem 3.3.3.3. Let $0 \leq \theta \leq 1;\ \lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1, 0 < \alpha < 1, \beta \in \mathbb{C}$ and $p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ where B_n are real with $B_1 > 0,\ B_2 \geq 0$. If $f(z) \in \mathcal{A}(p)$ belongs to $N_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, then for any complex number θ,

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \leq \frac{-(p-\mu)|b|B_1}{\delta_2[\alpha(p-\mu)+2\beta]} \max\left\{1, \left|(p-\mu)bB_1 A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta) - \frac{B_2}{B_1}\right|\right\} \tag{3.3.3.20}$$

where $A_{\lambda,\mu,\eta}(p,\alpha,\beta,\theta)$ is given by (3.3.3.5).

Chapter 4

Differential subordination, superordination and sandwich results for p-valent functions

The main objective of this chapter is to apply a method based upon the first order differential subordination and superordination, in order to derive some new differential subordination and superordination results for p-valent functions in the open unit disk described in the previous chapters involving certain fractional derivative operator. Section 4.1 consists of introduction and some lemmas required to prove our results. In section 4.2, we obtain differential subordination results. In section 4.3, the corresponding differential superordination problems are investigated. section 4.4, discusses various differential sandwich results.

The results of sections 4.2, 4.3 and 4.4 are published in Kargujevac journal of mathematics (Amsheri and Zharkova, 2011d) and Global Journal of pure and applied mathematics (Amsheri and Zharkova, 2011c).

4.1 Introduction and preliminaries

In this chapter we will use the related definitions and notations described in Chapter 1, section 1.7. Let $\phi(r,s;\,z):\mathbb{C}^2\times\mathcal{U}\to\mathbb{C}$ and let $h(z)$ be univalent in $\mathcal{U}$. If $p(z)$ is analytic in $\mathcal{U}$ and satisfies the (first-order) differential subordination

$$\phi\big(p(z), zp'(z); z\big) \prec h(z), \tag{4.1.1}$$

then $p(z)$ is said to be a solution of the differential subordination (4.1.1) The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (4.1.1), or more simply a dominant, if $p(z) \prec q(z)$ for all $p(z)$ satisfies (4.1.1). The univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (4.1.1) is called the best dominant. If $p(z)$ and $\phi\big(p(z), zp'(z); z\big)$ are univalent functions in $\mathcal{U}$ and if $p(z)$ satisfies the (first-order) differential superordination

$$h(z) \prec \phi\big(p(z), zp'(z); z\big), \tag{4.1.2}$$

then $p(z)$ is said to be a solution of the differential superordination (4.1.2). The univalent function $q(z)$ is called a subordinant of the solutions of the differential superordination (4.1.2), or more simply a subordinant, if $q(z) \prec p(z)$ for all $p(z)$ satisfies (4.1.2). The univalent subordinant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (4.1.2) is called the best subordinant, see (Miller and Mocanu, 2002).

To introduce our main results concerning differential subordination, differential superordination and sandwich type results, we consider the differential superordination which was given by (Miller and Mocanu, 2003) to obtain the conditions on $h(z)$, $q(z)$ and ϕ for which the following implication holds true:

$$h(z) \prec \phi(p(z), zp'(z); z) \Rightarrow q(z) \prec p(z)$$

With the results of (Miller and Mocanu, 2003), (Bulboaca, 2002a) investigated certain classes of first order differential superordinations as well as superordination-preserving integral operators (Bulboaca, 2002b). (Ali et al., 2005) have used the results of (Bulboaca, 2002b) to obtain sufficient conditions for normalized analytic functions $f(z) \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$. Recently, (Shanmugam et al., 2006b) obtained sufficient conditions for a normalized analytic functions $f(z) \in \mathcal{A}$ to satisfy the conditions

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z),$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\left(f(z)\right)^2} \prec q_2(z).$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$.

In this chapter, we will derive several subordination, superordination and sandwich results involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ as defined in (2.2.1) for p-valent functions $f(z) \in \mathcal{A}(p)$.

Let us first mention the following known definition according to (Miller and Mocanu, 2003) for a class $\mathcal{Q}$ of univalent functions defined on the unit disk.

Definition 4.1.1. Denoted by $\mathcal{Q}$ the set of all functions q that are analytic and injective in $\overline{\mathcal{U}} - E(q)$ where

$$E(q) = \left\{\xi \in \partial\mathcal{U} : \lim_{z \to \xi} q(z) = \infty\right\}.$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial\mathcal{U} - E(q)$. Further let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

In order to prove our results, we need to the following result according to (Shanmugam et al., 2006b), which deals with finding the best dominant from the differential subordination.

Lemma 4.1.2. Let q be univalent in the open unit disk $\mathcal{U}$ with $q(0) = 1$ and $\alpha,\ \gamma \in \mathbb{C}$. Further assume that

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\mathrm{Re}\left(\frac{\alpha}{\gamma}\right)\right\}.$$

If $p(z)$ is analytic in $\mathcal{U}$, and

$$\alpha p(z) + \gamma z p'(z) \prec \alpha q(z) + \gamma z q'(z).$$

then

$$p(z) \prec q(z).$$

and q is the best dominant.

We also need to the following result according to (Shanmugam et al., 2006b), which deals with finding the best subordinant from the differential superordination.

Lemma 4.1.3. Let q be univalent in the open unit disk $\mathcal{U}$ with $q(0) = 1$. Let $\alpha,\ \gamma \in \mathbb{C}$ and $\mathrm{Re}\left(\frac{\alpha}{\gamma}\right) > 0$. If $p(z) \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$, $\alpha p(z) + \gamma z p'(z)$ is univalent in $\mathcal{U}$, and

$$\alpha q(z) + \gamma z q'(z) \prec \alpha p(z) + \gamma z p'(z).$$

then

$$q(z) \prec p(z).$$

and q is the best subordinant.

4.2 Differential subordination results

Let us begin with establishing some new differential subordination results between analytic functions involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$, by making use of lemma 4.1.2. Theorem 4.2.1 deals with finding the best dominant from the differential subordination according to (Amsheri and Zharkova, 2011d).

Theorem 4.2.1. Let $q(z)$ be univalent in $\mathcal{U}$ with $q(0) = 1$, and suppose that

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\operatorname{Re}\left(\frac{1}{\gamma}\right)\right\}. \tag{4.2.1}$$

If $f(z) \in \mathcal{A}(p)$, and

$$\Phi_{\lambda,\mu,\eta}(\gamma, f)(z) = \gamma\left[(p-\mu) - (p-\mu-1)\frac{M_{0,z}^{\lambda,\mu,\eta} f(z) M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)\right)^2}\right]$$

$$+(1-\gamma)\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}, \tag{4.2.2}$$

If q satisfies the following subordination:

$$\Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q(z) + \gamma z q'(z), \tag{4.2.3}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N};\ \gamma \in \mathbb{C}).$$

then

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \prec q(z). \tag{4.2.4}$$

and q is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}.$$

So that, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)} = \frac{z\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{M_{0,z}^{\lambda,\mu,\eta}f(z)} - \frac{z\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} \quad . \tag{4.2.5}$$

By using the identity (2.2.6), we obtain

$$\gamma\left[(p-\mu)-(p-\mu-1)\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)^2}\right]$$

$$+(1-\gamma)\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} = p(z)+\gamma zp'(z),$$

The assertion (4.2.4) of Theorem 4.2.1 now follows by an application of Lemma 4.1.2, with $\alpha = 1$.

Remark 1. For the choice $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, in Theorem 4.2.1, we get the following corollary according to (Amsheri and Zharkova, 2011d).

Corollary 4.2.2. Let $-1 \le B < A \le 1$, and suppose that

$$\mathrm{Re}\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\mathrm{Re}\left(\frac{1}{\gamma}\right)\right\}. \tag{4.2.6}$$

If $f(z) \in \mathcal{A}(p)$, and

$$\Phi_{\lambda,\mu,\eta}(\gamma,f)(z) \prec \frac{1+Az}{1+Bz} + \frac{\gamma(A-B)z}{(1+Bz)^2},$$

$$(\lambda \ge 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu)-p-1;\ p \in \mathbb{N};\ \gamma \in \mathbb{C}).$$

where $\Phi_{\lambda,\mu,\eta}(\gamma,f)(z)$ is as defined in (4.2.2), then

$$\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} \prec \frac{1+Az}{1+Bz}.$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Next, let us investigate further differential subordination results for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$, which deal with finding the best dominant from the differential subordination according to (Amsheri and Zharkova, 2011d).

Theorem 4.2.3. Let $q(z)$ be univalent in $\mathcal{U}$ with $q(0)=1$, and assume that (4.2.1) holds. Let $f(z) \in \mathcal{A}(p)$, and

$$\Psi_{\lambda,\mu,\eta}(\gamma,f)(z) = [1+\gamma(\mu-p-1)]\frac{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} +$$

$$2\,\gamma(p-\mu)\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p} - \gamma(p-\mu-1)\frac{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2 M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{z^p\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)^2}, \quad (4.2.7)$$

If q satisfies the following subordination:

$$\Psi_{\lambda,\mu,\eta}(\gamma,f)(z) \prec q(z) + \gamma z q'(z),$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N};\ \gamma \in \mathbb{C}).$$

then

$$\frac{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} \prec q(z). \quad (4.2.8)$$

and q is the best dominant

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}.$$

So that, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)} = \frac{2z\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{M_{0,z}^{\lambda,\mu,\eta}f(z)} - p - \frac{z\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}\ . \quad (4.2.9)$$

By using the identity (2.2.6), we obtain

$$[1+\gamma(\mu-p-1)]\frac{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} + 2\,\gamma(p-\mu)\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}$$

$$-\gamma(p-\mu-1)\frac{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2 M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{z^p\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)^2}=p(z)+\gamma zp'(z),$$

The assertion (4.2.8) of Theorem 4.2.3 now follows by an application of Lemma 4.1.2, with $\alpha=1$.

Remark 2. For the choice $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, in Theorem 4.2.3, we get the following result according to (Amsheri and Zharkova, 2011d).

Corollary 4.2.4. Let $-1\leq B<A\leq 1$, and assume that (4.2.6) holds. If $f(z)\in\mathcal{A}(p)$, and

$$\Psi_{\lambda,\mu,\eta}(\gamma,f)(z)\prec\frac{1+Az}{1+Bz}+\frac{\gamma(A-B)z}{(1+Bz)^2},$$

$$(\lambda\geq 0;\ \ \mu<p+1;\ \eta>\max(\lambda,\mu)-p-1;\ p\in\mathbb{N};\ \gamma\in\mathbb{C}).$$

where $\Psi_{\lambda,\mu,\eta}(\gamma,f)(z)$ is as defined in (4.2.7), then

$$\frac{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}{z^p\,M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}\prec\frac{1+Az}{1+Bz}.$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Next, let us investigate further differential subordination results for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$, which deal with finding the best dominant from the differential subordination according to (Amsheri and Zharkova, 2011c).

Theorem 4.2.5. Let q be univalent in $\mathcal{U}$ with $q(0)=1$, and assume that (4.2.1) holds. If $f(z)\in\mathcal{A}(p)$, and

$$F_{\lambda,\mu,\eta}(\gamma,f)(z)=[1+\gamma(p-\mu+1)]\frac{z^p\,M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}+$$

$$\gamma(p-\mu-1)\frac{z^p\, M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}-2\gamma(p-\mu)\frac{z^p\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)^2}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^3}. \quad (4.2.10)$$

If q satisfies the following subordination:

$$F_{\lambda,\mu,\eta}(\gamma,f)(z) \prec q(z)+\gamma z q'(z), \quad (4.2.11)$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu)-p-1;\ p \in \mathbb{N};\ \gamma \in \mathbb{C}).$$

then

$$\frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2} \prec q(z). \quad (4.2.12)$$

and q is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z)=\frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}.$$

So that, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)}=p+\frac{z\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-\frac{2z\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{M_{0,z}^{\lambda,\mu,\eta}f(z)}. \quad (4.2.13)$$

By using the identity (2.2.6), a simple computation shows that

$$[1+\gamma(p-\mu+1)]\frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}+\gamma(p-\mu-1)\frac{z^p\, M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}-$$

$$2\gamma(p-\mu)\frac{z^p\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)^2}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^3}=p(z)+\gamma z p'(z)$$

The assertion (4.2.12) of Theorem 4.2.5 now follows by an application of Lemma 4.1.2, with $\alpha = 1$.

Remark 3. For the choice $q(z)=\frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 4.2.5, we get the following corollary according to (Amsheri and Zharkova, 2011c).

Corollary 4.2.6. Let $-1 \leq \mathrm{B} < \mathrm{A} \leq 1$, and assume that (4.2.6) holds. If $f(z) \in \mathcal{A}(p)$, and

$$F_{\lambda,\mu,\eta}(\gamma, f)(z) \prec \frac{1+Az}{1+Bz} + \frac{\gamma(A-B)z}{(1+Bz)^2},$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N};\ \gamma \in \mathbb{C}).$$

where $F_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (4.2.10), then

$$\frac{z^p\ M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} \prec \frac{1+Az}{1+Bz}.$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Now, let us prove further differential subordination result for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ following the results by (Amsheri and Zharkova, 2011c).

Theorem 4.2.7. Let q be univalent in $\mathcal{U}$ with $q(0) = 1$, and assume that (4.2.1) holds. If $f(z) \in \mathcal{A}(p)$, and

$$G_{\lambda,\mu,\eta}(\gamma, f)(z) = (1+\gamma)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,Z}^{\lambda,\mu,\eta} f(z)} + \gamma(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}$$

$$-\gamma(p-\mu)\frac{\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)\right)^2}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2}. \tag{4.2.14}$$

If q satisfies the following subordination:

$$G_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q(z) + \gamma z q'(z),$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N};\ \gamma \in \mathbb{C}).$$

then

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec q(z). \tag{4.2.15}$$

and q is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}.$$

So that, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)} = \frac{z\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)\right)'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} - \frac{z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{M_{0,z}^{\lambda,\mu,\eta} f(z)}. \qquad (4.2.16)$$

By using the identity (2.2.6), a simple computation shows that

$$(1+\gamma)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} + \gamma(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} -$$

$$\gamma(p-\mu)\frac{\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)\right)^2}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} = p(z) + \gamma z p'(z).$$

The assertion (4.2.15) of Theorem 4.2.7 now follows by an application of Lemma 4.1.2, with $\alpha = 1$.

Remark 3. For the choice $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, in Theorem 4.2.7, we get the following result according to (Amsheri and Zharkova, 2011c).

Corollary 4.2.8. Let $-1 \le B < A \le 1$, and assume that (4.2.6) holds. If $f(z) \in \mathcal{A}(p)$, and

$$G_{\lambda,\mu,\eta}(\gamma, f)(z) \prec \frac{1+Az}{1+Bz} + \frac{\gamma(A-B)z}{(1+Bz)^2},$$

$$(\lambda \ge 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N};\ \gamma \in \mathbb{C}).$$

where $G_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (4.2.14), then

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+\mathrm{Az}}{1+\mathrm{Bz}}$ is the best dominant.

4.3 Differential superordination results

In this section Let us investigate some new differential superordination results between analytic functions involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$, by making use of lemma 4.1.3. The following Theorems 4.3.1 and 4.3.2 deal with finding the best subordinant from the differential superordination according to (Amsheri and Zharkova, 2011d).

Theorem 4.3.1. Let $q(z)$ be convex in $\mathcal{U}$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If $f(z) \in \mathcal{A}(p)$,

$$0 \neq \frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}\,,$$

and $\Phi_{\lambda,\mu,\eta}(\gamma,f)(z)$ is univalent in $\mathcal{U}$, then

$$q(z) + \gamma z q'(z) \prec \Phi_{\lambda,\mu,\eta}(\gamma,f)(z), \tag{4.3.1}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}).$$

implies

$$q(z) \prec \frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}\,. \tag{4.3.2}$$

and q is the best subordinant where $\Phi_{\lambda,\mu,\eta}(\gamma,f)(z)$ is as defined in (4.2.2).

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}.$$

Then from the assumption of Theorem 4.3.1, the function $p(z)$ is analytic in $\mathcal{U}$ and (4.2.5) holds. Hence, (4.3.1) is equivalent to

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z).$$

The assertion (4.3.2) of Theorem 4.3.1 now follows by an application of Lemma 4.1.3.

Theorem 4.3.2. Let $q(z)$ be convex in $\mathcal{U}$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If $f(z) \in \mathcal{A}(p)$,

$$0 \neq \frac{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2}{z^p \, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}\,,$$

and $\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)$ is univalent in $\mathcal{U}$, then

$$q(z) + \gamma z q'(z) \prec \Psi_{\lambda,\mu,\eta}(\gamma, f)(z), \tag{4.3.3}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}).$$

implies

$$q(z) \prec \frac{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2}{z^p \, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}\,. \tag{4.3.4}$$

and q is the best subordinant where $\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (4.2.7).

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2}{z^p \, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}.$$

Then from the assumption of Theorem 4.3.2, the function $p(z)$ is analytic in $\mathcal{U}$ and (4.2.9) holds. Hence, (4.3.3) is equivalent to

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z).$$

The assertion (4.3.4) of Theorem 4.3.2 now follows by an application of Lemma 4.1.3.

Next, by making use of lemma 4.1.3, we prove the following Theorems 4.3.3 and 4.3.4, which deal with finding the best subordinant from differential superordination according to (Amsheri and Zharkova, 2011c).

Theorem 4.3.3. Let q be convex in $\mathcal{U}$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If $f(z) \in \mathcal{A}(p)$,

$$0 \neq \frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}\,.$$

and $F_{\lambda,\mu,\eta}(\gamma, f)(z)$ is univalent in $\mathcal{U}$, then

$$q(z) + \gamma z q'(z) \prec F_{\lambda,\mu,\eta}(\gamma, f)(z), \tag{4.3.5}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}).$$

implies

$$q(z) \prec \frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2}. \tag{4.3.6}$$

and q is the best subordinant where $F_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (4.2.10).

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2}.$$

Then from the assumption of Theorem 4.3.3, the function $p(z)$ is analytic in $\mathcal{U}$ and (4.2.13) holds. Hence, (4.3.5) is equivalent to

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z).$$

The assertion (4.3.6) of Theorem 4.3.3 now follows by an application of Lemma 4.1.3.

Theorem 4.3.4. Let q be convex in $\mathcal{U}$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re} \gamma > 0$. If $f(z) \in \mathcal{A}(p)$,

$$0 \neq \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}\,.$$

and $G_{\lambda,\mu,\eta}(\gamma, f)(z)$ is univalent in $\mathcal{U}$, then

$$q(z) + \gamma z q'(z) \prec G_{\lambda,\mu,\eta}(\gamma, f)(z), \tag{4.3.7}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}).$$

implies

$$q(z) \prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} . \tag{4.3.8}$$

and q is the best subordinant where $G_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (4.2.14).

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} .$$

Then from the assumption of Theorem 4.3.4, the function $p(z)$ is analytic in $\mathcal{U}$ and (4.2.16) holds. Hence, (4.3.7) is equivalent to

$$q(z) + \gamma z q^{'}(z) \prec p(z) + \gamma z p^{'}(z)$$

The assertion (4.3.8) of Theorem 4.3.4 now follows by an application of Lemma 4.1.3.

4.4 Differential sandwich results

In this section we obtain the differential sandwich type results by combining the differential subordination results from section 4.2 and the differential superordination results from section 4.3. Let us begin by combining Theorem 4.2.1 and Theorem 4.3.1 to get the following sandwich theorem for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ according to (Amsheri and Zharkova, 2011d).

Theorem 4.4.1. Let q_1 and q_2 be univalent functions in $\mathcal{U}$ such that $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\operatorname{Re} \gamma > 0$. If $f(z) \in \mathcal{A}(p)$ such that

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \in \mathcal{H}[1, 1] \cap \mathcal{Q} ,$$

and $\Phi_{\lambda,\mu,\eta}(\gamma, f)(z)$ is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec \Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z),$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}).$$

implies

$$q_1(z) \prec \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \prec q_2(z)\,.$$

and q_1 and q_2 are respectively the best subordinant and the best dominant where $\Phi_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (4.2.2).

Remark 1. For $\lambda = \mu = 0$ in Theorem 4.4.1, we get differential sandwich result for p-valent function $f(z) \in \mathcal{A}(p)$ in the open unit disk according to (Amsheri and Zharkova, 2011d).

Corollary 4.4.2. Let q_1 and q_2 be convex functions in $\mathcal{U}$ with $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If $f(z) \in \mathcal{A}(p)$ such that

$$\frac{pf(z)}{zf'(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}\,,$$

and let

$$\Phi_1(\gamma, f)(z) = \gamma p\left[1 - \frac{f''(z)f(z)}{(f'(z))^2}\right] + p(1-\gamma)\frac{f(z)}{zf'(z)}.$$

is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec \Phi_1(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z),$$

implies

$$q_1(z) \prec \frac{pf(z)}{zf'(z)} \prec q_2(z).$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

Now, by combining Theorem 4.2.4 and Theorem 4.3.2, we get the sandwich theorem for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ according to (Amsheri and Zharkova, 2011d).

Theorem 4.4.3. Let q_1 and q_2 be univalent functions in $\mathcal{U}$ such that $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If $f(z) \in \mathcal{A}(p)$ such that

$$\frac{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q},$$

and $\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)$ is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec \Psi_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z),$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}).$$

implies

$$q_1(z) \prec \frac{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \prec q_2(z).$$

and q_1 and q_2 are respectively the best subordinant and the best dominant where $\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (4.2.7).

Remark 2. For $\lambda = \mu = 0$ in Theorem 4.4.3, we get differential sandwich result for p-valent function $f(z) \in \mathcal{A}(p)$ in the open unit disk according to (Amsheri and Zharkova, 2011d).

Corollary 4.4.4. Let q_1 and q_2 be convex functions in $\mathcal{U}$ with $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If $f(z) \in \mathcal{A}(p)$ such that

$$\frac{p\big(f(z)\big)^2}{z^{p+1} f'(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q},$$

and let

$$\Psi_1(\gamma, f)(z) = [1 - \gamma(p+1)]\frac{p\big(f(z)\big)^2}{z^{p+1} f'(z)} + 2\gamma p\,\frac{f(z)}{z^p} - \gamma p\,\frac{f''(z)\big(f(z)\big)^2}{z^p\big(f'(z)\big)^2}.$$

is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec \Phi_1(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z),$$

implies

$$q_1(z) \prec \frac{p\big(f(z)\big)^2}{z^{p+1}f'(z)} \prec q_2(z)\,.$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

Next, by combining Theorem 4.2.5 and Theorem 4.3.3, we get the following sandwich theorem for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ according to (Amsheri and Zharkova, 2011c).

Theorem 4.4.4. Let q_1 and q_2 be convex functions in $\mathcal{U}$ with $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If $f(z) \in \mathcal{A}(p)$ such that

$$\frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}\,,$$

and $F_{\lambda,\mu,\eta}(\gamma, f)(z)$ is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec F_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z),$$

$$(\lambda \geq 0;\ \ \mu < p + 1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in \mathbb{N};\ \gamma \in \mathbb{C}).$$

implies

$$q_1(z) \prec \frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} \prec q_2(z).$$

and q_1 and q_2 are respectively the best subordinant and the best dominant where $F_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (4.2.10).

Remark 3. For $\lambda = \mu = 0$ in Theorem 4.4.4, we get the following differential sandwich result for p-valent function $f(z) \in \mathcal{A}(p)$ in the open unit disk according to (Amsheri and Zharkova, 2011c).

Corollary 4.4.5. Let q_1 and q_2 be convex functions in $\mathcal{U}$ with $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If $f(z) \in \mathcal{A}(p)$ such that

$$\frac{z^{p+1}f'(z)}{p\big(f(z)\big)^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}\,,$$

and let

$$F_1(\gamma, f)(z) = [1 + \gamma(p+1)]\frac{z^{p+1} f'(z)}{p\big(f(z)\big)^2} + \gamma\frac{z^{p+2} f''(z)}{p\big(f(z)\big)^2} - 2\gamma\frac{z^{p+2}\big(f'(z)\big)^2}{p\big(f(z)\big)^3},$$

is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec F_1(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z).$$

implies

$$q_1(z) \prec \frac{z^{p+1}\, f'(z)}{p\big(f(z)\big)^2} \prec q_2(z).$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

Next, by combining Theorem 4.2.7 and Theorem 4.3.4, we get the following sandwich theorem for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ according to (Amsheri and Zharkova, 2011c).

Theorem 4.4.6. Let q_1 and q_2 be convex functions in $\mathcal{U}$ with $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If $f(z) \in \mathcal{A}(p)$ such that

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}\,,$$

and $G_{\lambda,\mu,\eta}(\gamma, f)(z)$ is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec G_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z),$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N};\ \gamma \in \mathbb{C}).$$

implies

$$q_1(z) \prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec q_2(z)\,.$$

and q_1 and q_2 are respectively the best subordinant and the best dominant where $G_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (4.2.14).

Remark 4. For $\lambda = \mu = 0$ in Theorem 4.4.6, we get differential sandwich result for p-valent function $f(z) \in \mathcal{A}(p)$ in the open unit disk according to (Amsheri and Zharkova, 2011c).

Theorem 4.4.7. Let q_1 and q_2 be convex functions in $\mathcal{U}$ with $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\operatorname{Re} \gamma > 0$. If $f(z) \in \mathcal{A}(p)$ such that

$$\frac{zf'(z)}{pf(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}\,,$$

and let

$$G_1(\gamma, f)(z) = \frac{(1+\gamma)}{p} \frac{zf'(z)}{f(z)} + \frac{\gamma}{p} \left\{ \frac{z^2 f''(z)}{f(z)} - \frac{z^2 \left(f'(z)\right)^2}{\left(f(z)\right)^2} \right\},$$

is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec G_1(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z)\,.$$

implies

$$q_1(z) \prec \frac{zf'(z)}{pf(z)} \prec q_2(z).$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

Chapter 5

Strong differential subordination and superordination for p-valent functions

In this chapter we derive several results for strong differential subordination and superordination of p-valent functions involving certain fractional derivative operator. Section 5.1 consists of introduction and some lemmas those are required to prove our results. In section 5.2, strong differential subordination and superordination properties are determined for some families of p-valent functions with certain fractional derivative operator by investigating appropriate classes of admissible functions. In addition, new strong differential sandwich-type results are also obtained. In section 5.3, we derive first order linear strong differential subordination results for certain fractional derivative operator of p-valent functions. In section 5.4, we obtain some new first order strong differential subordination and superordination results based on the fact that the coefficients of functions defined by the operator are not constants but complex-valued functions.

The results of section 5.2 are published in Pioneer Journal of Mathematics and Mathematical Sciences (Amsheri and V. Zharkova, 2012f). The results of section 5.3 are published in Far East J. Math. Sci. (FJMS) (Amsheri and V. Zharkova, 2012g). The results of section 5.4 are published in International journal of Mathematical Analysis (Amsheri and V. Zharkova, 2012h) and in Journal of Mathematical Sciences: Advances and Applications (Amsheri and V Zharkova, 2012i).

5.1 Introduction and preliminaries

Some recent results in the theory of analytic functions were obtained by using a more strong form of the differential subordination and superordination introduced by (Antonino and Romaguera, 1994) and studied by (Antonino and Romaguera, 2006) called strong differential subordination and strong differential superordination, respectively. By using this notion, (G. Oros and Oros, 2007), (G. Oros, 2007), (G. Oros and Oros, 2009) and (G. Oros, 2009) introduced the notions of strong differential superordination and strong differential subordination following the theory of differential subordination introduced by (Miller and Mocaun,1981) and was developed by (Miller and Mocaun,2000) and the dual problem differential superordination which was introduced by (Miller and Mocanu, 2003).

To introduce our main results concerning strong differential subordination, and strong differential superordination, we consider the strong differential superordination which was given by (G. Oros, 2009). Let $\psi: \mathbb{C}^2 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$, and let $h(z)$ be univalent in $\mathcal{U}$. If $p(z)$ is analytic in $\mathcal{U}$ and satisfies the following (first-order) strong differential subordination

$$\psi\left(p(z), zp'(z); z, \zeta\right) \prec\prec h(z) \qquad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}). \qquad (5.1.1)$$

then $p(z)$ is called a solution of the strong differential subordination. The univalent function $q(z)$ is called a domainant of the solution of the strong differential subordination or, more simply, a dominant if $p(z) \prec q(z)$ for all

$p(z)$ satisfying (5.1.1). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (5.1.1) is said to be the best dominant. If $p(z)$ and $\psi(p(z), zp'(z); z)$ are univalent functions in $\mathcal{U}$ and if $p(z)$ satisfies the (first-order) strong differential superordination

$$h(z) \prec\prec \psi(p(z), zp'(z); z, \zeta), \qquad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}). \qquad (5.1.2)$$

then $p(z)$ is said to be a solution of the strong differential superordination (5.1.2). The univalent function $q(z)$ is called a subordinant of the solutions of the strong differential superordination (5.1.2), or more simply a subordinant, if $q(z) \prec p(z)$ for all $p(z)$ satisfies (5.1.2). The univalent subordinant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (5.1.2) is called the best subordinant, see (G. Oros, 2011).

In this chapter we investigate appropriate classes of admissible functions involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ which is as defined in (2.2.1) for p-valent functions by using the related definitions and notations defined in section 1.8, in order to obtain some new strong differential subordination, superordination, and sandwich type results. In addition, we obtain some new first order strong differential subordination and superordination results by considering that the coefficients of functions defined by the operator are not constants but complex-valued functions.

We refer to Chapter 4 of related Definition 4.1.1 for the class $\mathcal{Q}$. In order to prove our main results let us define the class of admissible functions $\Psi_n[\Omega, q]$ following (G. Oros and Oros, 2009) **.**

Definition 5.1.1. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$, consists of those functions $\psi: \mathbb{C}^3 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\psi(r,s,t;z,\zeta) \notin \Omega, \tag{5.1.3}$$

whenever $r = q(\xi),\ s = k\xi q'(\xi)$, and

$$\mathrm{Re}\left\{\frac{t}{s}+1\right\} \geq k\ \mathrm{Re}\left\{1+\frac{\xi q''(\xi)}{q'(\xi)}\right\},$$

where $z \in \mathcal{U},\ \xi \in \partial\mathcal{U}\backslash E(q),\ \zeta \in \overline{\mathcal{U}}$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In the special case when Ω is a simply connected domain, $\Omega \neq \mathbb{C}$, and h is a conformal mapping of $\mathcal{U}$ onto Ω, we denote this class by $\Psi_n[h, q]$.

If $\psi: \mathbb{C}^2 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$, then the admissibility condition (5.1.3) reduces to

$$\psi(r,s;z,\zeta) \notin \Omega, \tag{5.1.4}$$

whenever $r = q(\xi),\ s = m\xi q'(\xi),\ z \in \mathcal{U},\ \xi \in \partial\mathcal{U}\backslash E(q),\ \zeta \in \overline{\mathcal{U}}$, and $m \geq n$.

We next define the class of admissible functions $\Psi'_n[\Omega, q]$ following (G. Oros, 2009).

Definition 5.1.2. Let Ω be a set in $\mathbb{C}$, $q(z) \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$, consists of those functions $\psi: \mathbb{C}^3 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\psi(r,s,t;\xi,\zeta) \in \Omega, \tag{5.1.5}$$

whenever $r = q(z),\ s = \frac{zq'(z)}{m}$, and

$$\mathrm{Re}\left\{\frac{t}{s}+1\right\} \leq \frac{1}{m}\mathrm{Re}\left\{1+\frac{zq''(z)}{q'(z)}\right\},$$

where $z \in \mathcal{U},\ \xi \in \partial\mathcal{U},\ \zeta \in \overline{\mathcal{U}}$ and $m \geq n \geq 1$. In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

In the special case when Ω is a simply connected domain, $\Omega \neq \mathbb{C}$, and h is an analytic mapping of $\mathcal{U}$ onto Ω, we denote this class by $\Psi'_n[h, q]$.

If $\psi: \mathbb{C}^2 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$, then the admissibility condition (5.1.5) reduces to

$$\psi(r,s;\xi,\zeta) \in \Omega, \tag{5.1.6}$$

whenever $r = q(z),\ s = \frac{zq'(z)}{m},\ z \in \mathcal{U},\ \xi \in \partial\mathcal{U} \backslash E(q),\ \zeta \in \overline{\mathcal{U}}$, and $m \geq n$.

For the class $\Psi_n[\Omega, q]$ of admissible functions in Definition 5.1.1, (G. Oros and Oros, 2009) proved the following result.

Lemma 5.1.3. Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $p(z) \in \mathcal{H}[a, n]$ satisfies

$$\psi\big(p(z), zp'(z), z^2p''(z); z, \zeta\big) \in \Omega\,,$$

then

$$p(z) \prec q(z).$$

On the other hand, for the class $\Psi_n'[\Omega, q]$ of admissible functions in Definition 5.1.2 (G. Oros, 2009) proved Lemma 5.1.4.

Lemma 5.1.4. Let $\psi \in \Psi_n'[\Omega, q]$ with $q(0) = a$. If $p(z) \in \mathcal{Q}(a)$ and

$$\psi\big(p(z), zp'(z), z^2p''(z); z, \zeta\big),$$

is univalent in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$, then

$$\Omega \subset \{\psi\big(p(z), zp'(z), z^2p''(z); z, \zeta\big): z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\},$$

implies

$$q(z) \prec p(z).$$

Next let us give the following result regarding the subordination for analytic functions in the unit disk following (Miller and Mocanu,2000; p.24).

Lemma 5.1.5. Let $q \in \mathcal{Q}(a)$, with $q(0) = a$ and let $p(z) = a + a_nz^n + a_{n+1}z^{n+1} + \cdots$ be analytic in $\mathcal{U}$, with $p(z) \neq a$ and $n \geq 1$. If $p(z)$ is not subordinate to $q(z)$, then there exist points $z_0 = r_0e^{i\theta_0} \in \mathcal{U}$ and $\xi_0 \in \partial\mathcal{U} \backslash E(q)$, and $m \geq n \geq 1$ such that

1. $p(z_0) = q(\xi_0)$,
2. $z_0p'(z_0) = m\xi_0q'(\xi_0)$,

3. $\mathrm{Re}\left\{\frac{z_0 p''(z_0)}{p'(z_0)}+1\right\} \geq m\mathrm{Re}\left\{\frac{\xi_0 p''(\xi_0)}{p'(\xi_0)}+1\right\}$.

Two particular cases corresponding to $q(\mathcal{U})$ being a disk and $q(\mathcal{U})$ being a half-plane, see (G. Oros, 2011)

i. The function

$$q(z) = M\frac{Mz+a}{M+\bar{a}z}$$

when $M>0$; $|a|<M$, satisfies the disk $\Delta = q(\mathcal{U}) = \{w: |w|<M\}$, $q(0)=a$, $E(q)=\emptyset$ and $q \in \mathcal{Q}$, since $q(\zeta)=Me^{i\theta}$, with $\theta \in \mathbb{R}$, when $|\zeta|=1$, the condition of admissibility (5.1.3) becomes

$$\psi(r,s,t;z,\zeta) \notin \Omega, \quad \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right), \tag{5.1.7}$$

when

$$r = Me^{i\theta}, s = m\frac{\left|M-\bar{a}\,e^{i\theta}\right|^2}{M^2-|a|^2}e^{i\theta},$$

and

$$\mathrm{Re}\left\{\frac{t}{s}+1\right\} \geq m\frac{M\left|M-\bar{a}\,e^{i\theta}\right|^2}{M^2-|a|^2}, \qquad (m \geq n).$$

If $a=0$, then the condition (5.1.7) simplifies to

$$\psi\left(Me^{i\theta}, Ke^{i\theta}, L; z, \zeta\right) \notin \Omega \ , \qquad (K \geq nM), \tag{5.1.8}$$

and

$$\mathrm{Re}\left\{Le^{-i\theta}\right\} \geq (n-1)K$$

ii. The function

$$q(z) = \frac{a+\bar{a}z}{1-z}$$

with $\operatorname{Re} a > 0$, satisfies the half plane $\Delta = q(\mathcal{U}) = \{w : \operatorname{Re} w > 0\}$, $q(0) = a$, $E(q) = \{1\}$ and $q \in \mathcal{Q}$, since $\operatorname{Re} q(\zeta) = 0$, when $\zeta \in \partial\mathcal{U} \setminus \{1\}$, the condition of admissibility (5.1.3) becomes

$$\psi(i\rho, \sigma, \mu + i\nu; z, \zeta) \notin \Omega, \quad \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right), \tag{5.1.9}$$

when $\rho, \sigma, \mu, \nu \in \mathbb{R}$ and

$$\sigma \leq -\frac{n}{2}\frac{|a - i\rho|^2}{\operatorname{Re} a}, \quad \sigma + \mu \leq 0, \qquad (n \geq 1).$$

If $a = 1$, then (5.1.9) implies

$$\psi(i\rho, \sigma, \mu + i\nu; z, \zeta) \notin \Omega, \tag{5.1.10}$$

when $\rho, \sigma, \mu, \nu \in \mathbb{R}$ and

$$\sigma \leq -\frac{n}{2}(1 + \rho^2), \quad \sigma + \mu \leq 0, \qquad (n \geq 1).$$

We also need to the following lemmas 5.1.6 due to (Miller and Mocanu, 2000; p.71) which deals with finding the best dominant from strong differential subordination for analytic functions that have coefficients are not constants but complex-valued functions.

Lemma 5.1.6. Let $h(z, \zeta)$ be convex function with $h(0, \zeta) = a$ for all $\zeta \in \overline{\mathcal{U}}$ and let $\gamma \neq 0$ be a complex number with $\operatorname{Re} \gamma \geq 0$. If $p(z, \zeta) \in \mathcal{H}^*[a, n, \zeta]$ and

$$p(z, \zeta) + \frac{1}{\gamma} z p'(z, \zeta) \prec\prec h(z, \zeta),$$

then

$$p(z, \zeta) \prec\prec q(z, \zeta) \prec\prec h(z, \zeta),$$

where

$$q(z, \zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t, \zeta) t^{\frac{\gamma}{n} - 1} dt.$$

The function q is convex and it is the best dominant.

We also need to lemma 5.1.7 following (Miller and Mocaun, 1985) which deals with finding the best dominant from strong differential subordination for analytic functions that have coefficients are not constants but complex-valued functions.

Lemma 5.1.7. Let $q(z,\zeta)$ be convex function in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$ and let $h(z,\zeta)$ be defined by

$$h(z,\zeta) = q(z,\zeta) + n\alpha z q^{'}(z,\zeta), \qquad \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right),$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z,\zeta) = q(0,\zeta) + p_n(\zeta)z^n + p_{n+1}(\zeta)z^{n+1} + \cdots \ \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right),$$

is analytic in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$, and satisfy

$$p(z,\zeta) + \alpha z p^{'}(z,\zeta) \prec\prec h(z,\zeta),$$

then

$$p(z,\zeta) \prec\prec q(z,\zeta).$$

and this result is sharp.

We also need to use the following lemmas 5.1.8 and 5.1.9 according to (Miller and Mocanu, 2003) which deal with finding the best subordinant from strong differential superordination for analytic functions that have coefficients are not constants but complex-valued functions.

Lemma 5.1.8. Let $h(z,\zeta)$ be convex with $h(0,\zeta) = a$ for all $\zeta \in \overline{\mathcal{U}}$ and let $\gamma \in \mathbb{C} \setminus \{0\}$ be a complex number with $\operatorname{Re}\gamma \geq 0$. If $p(z,\zeta) \in \mathcal{H}^*[a,1,\zeta] \cap \mathcal{Q}$. If

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} z p'(z,\zeta),$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

where

$$q(z,\zeta) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t,\zeta) t^{\gamma-1} dt.$$

The function q is convex and it is the best subordinant.

Lemma 5.1.9. Let $q(z,\zeta)$ be convex function in $\mathcal{U}$, for all $\zeta \in \overline{\mathcal{U}}$ and let $h(z,\zeta)$ be defined by

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{\gamma} z q'(z,\zeta), \quad \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}} \right),$$

where $\operatorname{Re} \gamma \geq 0$. If $p(z,\zeta) \in \mathcal{H}^*[a,1,\zeta] \cap \mathcal{Q}$, $p(z,\zeta) + \frac{1}{\gamma} z p'(z,\zeta)$ is univalent in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$, and satisfy

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} z p'(z,\zeta),$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

where

$$q(z,\zeta) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t,\zeta) t^{\gamma-1} dt.$$

The function q is the best subordinant.

5.2 Admissible functions method

In this section we obtain some new strong differential subordination results and strong differential superordination results for p-valent functions associated with the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ by investigating appropriate classes of admissible functions. Further results including strong differential sandwich-type are also considered.

5.2.1 Strong differential subordination results

Let us first define the class $\Phi_M[\Omega, q]$ of admissible functions that is required in our first result for strong differential subordination involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ according to (Amsheri and Zharkova, 2012f).

Definition 5.2.1.1. Let Ω be a set in $\mathbb{C}$, and $q(z) \in \mathcal{Q}_0 \cap \mathcal{H}[0, p]$. The class of admissible functions $\Phi_M[\Omega, q]$ consists of those functions $\phi: \mathbb{C}^3 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z, \zeta) \notin \Omega,$$

whenever $u = q(\xi)$, $v = \frac{k\xi q'(\xi) - \mu q(\xi)}{p-\mu}$, and

$$\mathrm{Re}\left\{\frac{(p-\mu)(p-\mu-1)w + \mu^2 u + (2\mu+1)(p-\mu)v}{(p-\mu)v + \mu u}\right\} \geq k \mathrm{Re}\left\{1 + \frac{\xi q''(\xi)}{q'(\xi)}\right\},$$

where $z \in \mathcal{U}$; $\xi \in \partial\mathcal{U} \backslash E(q)$; $\zeta \in \overline{\mathcal{U}}$; $\mu \neq p, p-1$ and $k \geq p$.

Let us now prove the first result for strong differential subordination by making use of Lemma 5.1.3 following (Amsheri and Zharkova, 2012f).

Theorem 5.2.1.2. Let $\phi \in \Phi_M[\Omega, q]$. If $f(z) \in \mathcal{A}(p)$ satisfies

$$\left\{\phi\left(M_{0,z}^{\lambda,\mu,\eta} f(z), M_{0,z}^{\lambda+,\mu+1,\eta+1} f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z); z, \zeta\right): z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right\} \subset \Omega, \tag{5.2.1.1}$$

then

$$M_{0,z}^{\lambda,\mu,\eta} f(z) \prec q(z).$$

Proof. Define the analytic function $q(z)$ in $\mathcal{U}$ by

$$p(z) = M_{0,z}^{\lambda,\mu,\eta} f(z). \tag{5.2.1.2}$$

Using the identity (2.2.6) in (5.2.1.2), we get

$$M_{0,z}^{\lambda+,\mu+1,\eta+1}f(z) = \frac{1}{p-\mu}\{zp'(z) - \mu p(z)\}, \qquad (5.2.1.3)$$

and

$$M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z) =$$

$$\frac{1}{(p-\mu)(p-\mu-1)}\{z^2p''(z) - 2\mu zp'(z) + \mu(\mu+1)p(z)\}. \qquad (5.2.1.4)$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \qquad v = \frac{s-\mu r}{p-\mu}, \qquad w = \frac{t-2\mu s+\mu(\mu+1)}{(p-\mu)(p-\mu-1)}. \qquad (5.2.1.5)$$

Let

$$\psi(r,s,t;z,\zeta) = \phi(u,v,w;z,\zeta)$$

$$= \phi\left(r, \frac{s-\mu r}{p-\mu}, \frac{t-2\mu s+\mu(\mu+1)r}{(p-\mu)(p-\mu-1)}; z,\zeta\right). \qquad (5.2.1.6)$$

The proof shall make use of Lemma 5.1.3, using equations (5.2.1.2) - (5.2.1.4), and from (5.2.1.6), we obtain

$$\psi\left(p(z), zp'(z), z^2p''(z); z,\zeta\right) =$$

$$\phi\left(M_{0,z}^{\lambda,\mu,\eta}f(z), M_{0,z}^{\lambda+,\mu+1,\eta+1}f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z); z,\zeta\right), \quad \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right).$$

(5.2.1.7)

Hence (5.2.1.1) becomes

$$\psi\left(p(z), zp'(z), z^2p''(z); z,\zeta\right) \in \Omega. \qquad (5.2.1.8)$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_M[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 5.1.1. Note that

$$\frac{t}{s} + 1 = \frac{(p-\mu)(p-\mu-1)w + \mu^2 u + (2\mu+1)(p-\mu)v}{(p-\mu)v + \mu u},$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Lemma 5.1.3,

$$p(z) \prec q(z),$$

or

$$M_{0,z}^{\lambda,\mu,\eta} f(z) \prec q(z).$$

This completes the proof of Theorem 5.2.1.2.

We next consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(\mathcal{U})$ where $h(z)$ is a conformal mapping of $\mathcal{U}$ onto Ω and the class $\Phi_M[h(\mathcal{U}), q]$ is written as $\Phi_M[h, q]$. The following result is an immediate consequence of Theorem 5.2.1.2 according to (Amsheri and Zharkova, 2012f).

Theorem 5.2.1.3. Let $\phi \in \Phi_M[h, q]$. If $f(z) \in \mathcal{A}(p)$ satisfies

$$\phi\left(M_{0,z}^{\lambda,\mu,\eta} f(z), M_{0,z}^{\lambda+,\mu+1,\eta+1} f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z); z, \zeta\right) \prec\prec h(z), \qquad (5.2.1.9)$$

for $z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}$, then

$$M_{0,z}^{\lambda,\mu,\eta} f(z) \prec q(z).$$

Let us now consider the particular case, the function $q(z) = Mz;\ M > 0$, corresponding to $q(\mathcal{U})$ being a disk $q(\mathcal{U}) = \Delta = \{w: |w| < M\}$. The class of admissible functions $\Phi_M[\Omega, q]$, denoted by $\Phi_M[\Omega, M]$ is described below for this particular $q(z)$ according to (Amsheri and Zharkova, 2012f).

Definition 5.2.1.4. Let Ω be a set in $\mathbb{C}$ with $\mu \neq p,\ p-1$ and $M > 0$. The class of admissible functions $\Phi_M[\Omega, M]$, consists of those functions $\phi: \mathbb{C}^3 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \left(\frac{k-\mu}{p-\mu}\right)Me^{i\theta}, \frac{L + \mu(\mu+1-2k)Me^{i\theta}}{(p-\mu)(p-\mu-1)}\ ; z, \zeta\right) \notin \Omega,$$

whenever $z \in \mathcal{U},\ \theta \in \mathbb{R}$, and $\mathrm{Re}\{Le^{-i\theta}\} \geq (k-1)kM$ for all $\zeta \in \overline{\mathcal{U}}$ and $k \geq 1$.

Now let us apply Theorem 5.2.1.2 to the special case $q(z) = Mz$; $M > 0$ following (Amsheri and Zharkova, 2012f).

Corollary 5.2.1.5. Let $\phi \in \Phi_M[\Omega, M]$. If $f(z) \in \mathcal{A}(p)$ satisfies

$$\phi\left(M_{0,z}^{\lambda,\mu,\eta} f(z), M_{0,z}^{\lambda+,\mu+1,\eta+1} f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z); z, \zeta\right) \in \Omega, \quad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

then

$$\left|M_{0,z}^{\lambda,\mu,\eta} f(z)\right| < M.$$

Let us now consider the special case $\Omega = q(\mathcal{U}) = \Delta = \{w: |w| < M\}$, the class $\Phi_M[\Omega, M]$ is simply denoted by $\Phi_M[M]$ according to (Amsheri and Zharkova, 2012f).

Corollary 5.2.1.6. Let $\phi \in \Phi_M[M]$. If $f(z) \in \mathcal{A}(p)$ satisfies

$$\left|\phi\left(M_{0,z}^{\lambda,\mu,\eta} f(z), M_{0,z}^{\lambda+,\mu+1,\eta+1} f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z); z, \zeta\right)\right| < M, \quad \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right).$$

then

$$\left|M_{0,z}^{\lambda,\mu,\eta} f(z)\right| < M.$$

To investigate further strong differential subordination results involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$, let us define further class of admissible functions, that is the class $\Phi_{M,1}[\Omega, q]$ which is required in our next result according to (Amsheri and Zharkova, 2012f).

Definition 5.2.1.7. Let Ω be a set in $\mathbb{C}$, and $q(z) \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{M,1}[\Omega, q]$ consists of those functions $\phi: \mathbb{C}^3 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z, \zeta) \notin \Omega,$$

whenever $u = q(\xi),\ v = \frac{1}{(p-\mu-1)}\left\{\frac{k\xi q'(\xi)}{q(\xi)} + (p-\mu)q(\xi) - 1\right\}$, and

$$\mathrm{Re}\left\{\frac{(p-\mu-1)[3-3(p-\mu)u+(p-\mu-2)w]v+(p-\mu)[2(p-\mu)u-3]u+1}{(p-\mu-1)v-(p-\mu)u+1}\right\}$$

$$\geq k\ \mathrm{Re}\left\{1+\frac{\xi q''(\xi)}{q'(\xi)}\right\},$$

where $z \in \mathcal{U};\ \xi \in \partial\mathcal{U}\backslash \mathrm{E}(q);\ \zeta \in \overline{\mathcal{U}}; p \in \mathbb{N};\ \mu \neq p,\ p-1$ and $k \geq 1$.

Let us prove the next result for strong differential subordination by making use of Lemma 5.1.3 following (Amsheri and Zharkova, 2012f).

Theorem 5.2.1.8. Let $\phi \in \Phi_{M,1}[\Omega, q]$ and $M_{0,z}^{\lambda,\mu,\eta} f(z) \neq 0$. If $f(z) \in \mathcal{A}(p)$ satisfies

$$\left\{\phi\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3}f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}; z, \zeta\right) : z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right\} \subset \Omega, \tag{5.2.1.10}$$

then

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} \prec q(z).$$

Proof. Define the analytic function $q(z)$ in $\mathcal{U}$ by

$$p(z) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\ . \tag{5.2.1.11}$$

Using (5.2.1.11), we get

$$\frac{zp'(z)}{p(z)} = \frac{z\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} - \frac{z\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{M_{0,z}^{\lambda,\mu,\eta}f(z)}. \tag{5.2.1.12}$$

By making use of the identity (2.2.6) in (5.2.1.12), we get

$$\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} = \frac{1}{(p-\mu-1)}\left\{\frac{zp'(z)}{p(z)} + (p-\mu)p(z) - 1\right\}. \tag{5.2.1.13}$$

Further computations show that

$$\frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3}f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)} = \frac{1}{(p-\mu-2)}\left\{\frac{zp'(z)}{p(z)} + (p-\mu)p(z) - 2 + \right.$$

$$\left.\frac{(p-\mu)zp'(z) + \frac{zp'(z)}{p(z)} + \frac{z^2p''(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2}{\frac{zp'(z)}{p(z)} + (p-\mu)p(z) - 1}\right\}. \qquad (5.2.1.14)$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \qquad v = \frac{1}{(p-\mu-1)}\left\{\frac{s}{r} + (p-\mu)r - 1\right\},$$

$$w = \frac{1}{(p-\mu-2)}\left\{\frac{s}{r} + (p-\mu)r - 2 + \frac{(p-\mu)s + \frac{s}{r} + \frac{t}{r} - \left(\frac{s}{r}\right)^2}{\frac{s}{r} + (p-\mu)r - 1}\right\}. \quad (5.2.1.15)$$

Let

$\psi(r,s,t;z,\zeta) = \phi(u,v,w;z,\zeta) =$

$$\phi\left(r, \frac{1}{(p-\mu-1)}\left\{\frac{s}{r} + (p-\mu)r - 1\right\}, \frac{1}{(p-\mu-2)}\left\{\frac{s}{r} + (p-\mu)r - 2\right.\right.$$

$$\left.\left. + \frac{(p-\mu)s + \frac{s}{r} + \frac{t}{r} - \left(\frac{s}{r}\right)^2}{\frac{s}{r} + (p-\mu)r - 1}\right\}; z, \zeta\right). \qquad (5.2.1.16)$$

The proof shall make use of Lemma 5.1.3, using equations (5.2.1.11), (5.2.1.13) and (5.2.1.14), and from (5.2.1.16), we obtain

$\psi(p(z), zp'(z), z^2p''(z); z, \zeta) =$

$$\phi\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3}f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}; z, \zeta\right). \qquad (5.2.1.17)$$

Hence (5.2.1.10) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z, \zeta) \in \Omega\,.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{M,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 5.1.1. Note that

$$\frac{t}{s} + 1$$

$$= \frac{(p-\mu-1)[3-3(p-\mu)u+(p-\mu-2)w]v+(p-\mu)[2(p-\mu)u-3]u+1}{(p-\mu-1)v-(p-\mu)u+1},$$

and hence $\psi \in \Psi[\Omega, q]$. By Lemma 5.1.3,

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec q(z).$$

The of Theorem 5.2.1.8 is complete.

We next consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(\mathcal{U})$ where $h(\mathrm{z})$ is a conformal mapping of $\mathcal{U}$ onto Ω and the class $\Phi_{M,1}[h(\mathcal{U}), q]$ is written as $\Phi_{M,1}[h, q]$. The following result is an immediate consequence of Theorem 5.2.1.8 according to (Amsheri and Zharkova, 2012f).

Theorem 5.2.1.9. Let $\phi \in \Phi_{M,1}[h, q]$ and $M_{0,z}^{\lambda,\mu,\eta} f(z) \neq 0$. If $f(z) \in \mathcal{A}(p)$ satisfies

$$\phi\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3} f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}; z, \zeta\right) \prec\prec h(z),$$

$$(5.2.1.18)$$

for $z \in \mathcal{U}$, $\zeta \in \overline{\mathcal{U}}$, then

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec q(z).$$

Let us now consider the particular case, the function $q(z) = Mz$; $M > 0$, corresponding to $q(\mathcal{U})$ being a disk $q(\mathcal{U}) = \Delta = \{w: |w| < M\}$. The class of admissible functions $\Phi_{M,1}[\Omega, q]$ denoted by $\Phi_{M,1}[\Omega, M]$ is described below for this particular $q(z)$ according to (Amsheri and Zharkova, 2012f).

Definition 5.2.1.10. Let Ω be a set in $\mathbb{C}$ with $\mu \neq p,\ p-1$ and $M > 0$. The class of admissible functions $\Phi_{M,1}[\Omega, M]$, consists of those functions $\phi: \mathbb{C}^3 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{1}{(p-\mu-1)}\left\{k+(p-\mu)Me^{i\theta}-1\right\},\right.$$
$$\frac{1}{(p-\mu-2)}\left\{k+(p-\mu)Me^{i\theta}-2\right.$$
$$\left.\left. + \frac{(p-\mu)kM^2e^{i\theta}+kM+Le^{-i\theta}-k^2M}{kM+(p-\mu)M^2e^{i\theta}-M}\right\};\ z, \zeta\right) \notin \Omega$$

whenever $z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}},\ p \in \mathbb{N},\ \mathrm{Re}\left\{Le^{-i\theta}\right\} \geq (k-1)kM$, for all real θ, and $k \geq 1$.

Now let us apply Theorem 5.2.1.9 to the special case $q(z) = Mz;\ M > 0$ following (Amsheri and Zharkova, 2012f).

Corollary 5.2.1.11. Let $\phi \in \Phi_{M,1}[\Omega, M]$. If $f(z) \in \mathcal{A}(p)$ satisfies

$$\phi\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3}f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}; z, \zeta\right) \in \Omega,$$

for $z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}$, then

$$\left|\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right| < M, \qquad (M > 0).$$

Let us now consider the special case $\Omega = q(\mathcal{U}) = \Delta = \{w: |w| < M\}$, the class $\Phi_{M,1}[\Omega, M]$ is simply denoted by $\Phi_{M,1}[M]$ following (Amsheri and Zharkova, 2012f).

Corollary 5.2.1.12. Let $\phi \in \Phi_{M,1}[M]$. If $f(z) \in \mathcal{A}(p)$ satisfies

$$\left|\phi\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3}f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}; z, \zeta\right)\right| < M,$$

for $z \in \mathcal{U}; \zeta \in \overline{\mathcal{U}}$, then

$$\left|\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right| < M, \qquad (M > 0).$$

5.2.2 Strong differential superordination results

In this subsection the dual problem of strong differential subordination, that is, strong differential superordination of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ for p-valent functions is investigated following (Amsheri and Zharkova, 2012f). For this purpose we first define the class $\Phi'_M[\Omega, q]$ of admissible functions.

Definition 5.2.2.1. Let Ω be a set in $\mathbb{C}$, and $\mu \neq p, p-1$; $q(z) \in \mathcal{H}[0, p]$. The class of admissible functions $\Phi'_M[\Omega, q]$ consists of those functions $\phi: \mathbb{C}^3 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \xi, \zeta) \in \Omega$$

whenever $u = q(z)$, $v = \frac{zq'(z) - m\mu q(z)}{m(p-\mu)}$, and

$$\mathrm{Re}\left\{\frac{(p-\mu)(p-\mu-1)w - \mu^2 u + (2\mu+1)(p-\mu)v}{(p-\mu)v + \mu u}\right\} \leq \frac{1}{m}\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\},$$

where $z \in \mathcal{U}$; $\xi \in \partial\mathcal{U}$; $\zeta \in \overline{\mathcal{U}}$; $p \in \mathbb{N}$ and $m \geq p$.

Let us now prove the first result for strong differential superordination by making use of Lemma 5.1.4 following (Amsheri and Zharkova, 2012f).

Theorem 5.2.2.2. Let $\phi \in \Phi'_M[\Omega, q]$. If $f(z) \in \mathcal{A}(p)$, $M_{0,z}^{\lambda,\mu,\eta}f(z) \in \mathcal{Q}_0$ and

$$\phi\left(M_{0,z}^{\lambda,\mu,\eta}f(z), M_{0,z}^{\lambda+,\mu+1,\eta+1}f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z); z,\zeta\right),$$

is univalent in $\mathcal{U}$, then

$$\Omega \subset \left\{\phi\left(M_{0,z}^{\lambda,\mu,\eta}f(z), M_{0,z}^{\lambda+,\mu+1,\eta+1}f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z); z,\zeta\right): z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right\},$$

$$(5.2.2.1)$$

implies

$$q(z) \prec M_{0,z}^{\lambda,\mu,\eta}f(z).$$

Proof. From (5.2.1.7) and (5.2.2.1), we have

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z); z,\zeta): z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right\}.$$

From (5.2.1.5), we see that the admissibility condition for $\phi \in \Phi'_M[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 5.1.2. Hence $\psi \in \Psi'_p[\Omega, q]$ and by Lemma 5.1.4,

$$q(z) \prec M_{0,z}^{\lambda,\mu,\eta}f(z).$$

The proof of Theorem 5.2.2.2 is complete.

We next consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(\mathcal{U})$ where $h(\mathrm{z})$ is a conformal mapping of $\mathcal{U}$ onto Ω and the class $\Phi'_M[h(\mathcal{U}), q]$ is written as $\Phi'_M[h, q]$. The following result is an immediate consequence of Theorem 5.2.2.2 according to (Amsheri and Zharkova, 2012f).

Theorem 5.2.2.3. Let $h(z)$ be analytic on $\mathcal{U}$ and $\phi \in \Phi'_M[h, q]$. If $f(z) \in \mathcal{A}(p)$, $M_{0,z}^{\lambda,\mu,\eta}f(z) \in \mathcal{Q}_0$ and

$$\phi\left(M_{0,z}^{\lambda,\mu,\eta}f(z), M_{0,z}^{\lambda+,\mu+1,\eta+1}f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z); z,\zeta\right),$$

is univalent in $\mathcal{U}$, then

$$h(z) \prec\prec \phi\left(M_{0,z}^{\lambda,\mu,\eta}f(z), M_{0,z}^{\lambda+,\mu+1,\eta+1}f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z); z,\zeta\right), \quad (5.2.2.2)$$

for $z \in \mathcal{U}; \zeta \in \overline{\mathcal{U}}$, implies

$$q(z) \prec M_{0,z}^{\lambda,\mu,\eta} f(z).$$

Next let us define further class of admissible functions, that is the class $\Phi'_{M,1}[\Omega, q]$ which is required to investigate further strong differential superoedination involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ following (Amsheri and Zharkova, 2012f).

Definition 5.2.2.4. Let Ω be a set in $\mathbb{C}$, and $q(z) \neq 0$ and $q(z) \in \mathcal{H}$. The class of admissible functions $\Phi'_{M,1}[\Omega, q]$ consists of those functions $\phi: \mathbb{C}^3 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \xi, \zeta) \in \Omega,$$

whenever $u = q(z),\ v = \frac{1}{(p-\mu-1)}\left\{\frac{zq'(z)}{mq(z)} + (p-\mu)q(z) - 1\right\}$, and

$$\operatorname{Re}\left\{\frac{(p-\mu-1)[3-3(p-\mu)u+(p-\mu-2)w]v+(p-\mu)[2(p-\mu)u-3]u+1}{(p-\mu-1)v-(p-\mu)u+1}\right\}$$

$$\leq \frac{1}{m}\operatorname{Re}\left\{1+\frac{zq''(z)}{q'(z)}\right\}.$$

where $z \in \mathcal{U};\ \xi \in \partial\mathcal{U};\ \zeta \in \overline{\mathcal{U}}; p \in \mathbb{N};\ \mu \neq p,\ p-1$ and $m \geq 1$.

Let us now prove the next result for strong differential subordination by making use of Lemma 5.1.4 following (Amsheri and Zharkova, 2012f).

Theorem 5.2.2.5. Let $\phi \in \Phi'_{M,1}[\Omega, q]$. If $f(z) \in \mathcal{A}(p)$, $\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \in \mathcal{Q}_1$ and

$$\phi\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3} f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}; z, \zeta\right),$$

is univalent in $\mathcal{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3}f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}; z, \zeta\right) : z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right\}, \tag{5.2.2.3}$$

implies

$$q(z) \prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}.$$

Proof. From (5.2.1.17) and (5.2.2.3), we have

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z, \zeta) : z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\}.$$

In view of (5.2.1.16),the admissibility condition for $\phi \in \Phi'_{M,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 5.1.2. Hence $\psi \in \Psi'[\Omega, q]$ and by Lemma 5.1.4,

$$q(z) \prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}.$$

This completes the proof of Theorem 5.2.2.5.

Next let us consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(\mathcal{U})$ for some conformal mapping $h(\mathrm{z})$ of $\mathcal{U}$ onto Ω for the class $\Phi'_{M,1}[h(\mathcal{U}), q]$ which is written as $\Phi'_{M,1}[h, q]$. The following result is an immediate consequence of Theorem 5.2.2.5 according to (Amsheri and Zharkova, 2012f).

Theorem 5.2.2.6. Let $q(z) \in \mathcal{H}$, $h(z)$ be analytic in $\mathcal{U}$ and $\phi \in \Phi'_{M,1}[h, q]$. If $f(z) \in \mathcal{A}(p)$, $\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} \in \mathcal{Q}_1$ and

$$\phi\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3}f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}; z, \zeta\right),$$

is univalent in $\mathcal{U}$, then

$h(z) \prec\prec$

$$\phi\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)},\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)},\frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3}f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)};z,\zeta\right), \tag{5.2.2.4}$$

for $z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}$, implies

$$q(z) \prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}.$$

5.2.3 Strong differential sandwich results

In this subsection we obtain the strong differential sandwich type results by combining the strong differential subordination results from the subsection 5.2.1 and the strong differential superordination results from the subsection 5.2.2. Let us begin by combining Theorem 5.2.1.3 and Theorem 5.2.2.3 to get the following sandwich theorem for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ of p-valent functions according to (Amsheri and Zharkova, 2011f).

Theorem 5.2.3.1. Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathcal{U}$, $h_2(z)$ be univalent function in $\mathcal{U}$, $q_2(z) \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_M[h_2, q_2] \cap \Phi'_M[h_1, q_1]$. If $f(z) \in \mathcal{A}(p)$, $M_{0,z}^{\lambda,\mu,\eta}f(z) \in \mathcal{H}[0,p] \cap \mathcal{Q}_0$ and

$$\phi\left(M_{0,z}^{\lambda,\mu,\eta}f(z), M_{0,z}^{\lambda+,\mu+1,\eta+1}f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z); z,\zeta\right), \qquad \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right),$$

is univalent in $\mathcal{U}$, then

$$h_1(z) \prec\prec \phi\left(M_{0,z}^{\lambda,\mu,\eta}f(z), M_{0,z}^{\lambda+,\mu+1,\eta+1}f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z); z,\zeta\right) \prec\prec h_2(z),$$

implies

$$q_1(z) \prec M_{0,z}^{\lambda,\mu,\eta}f(z) \prec q_2(z)$$

Let us establish further strong differential sandwich result by combining Theorems 5.2.1.9 and 5.2.2.6.

Theorem 5.2.3.2. Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathcal{U}$, $h_2(z)$ be univalent function in $\mathcal{U}$, $q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{M,1}[h_2, q_2] \cap \Phi'_{M,1}[h_1, q_1]$. If $f(z) \in \mathcal{A}(p)$, $\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \in \mathcal{H} \cap \mathcal{Q}_1$ and

$$\phi\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3} f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}; z, \zeta\right),$$

is univalent in $\mathcal{U}$, then

$h_1(z) \prec\prec$

$$\phi\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3} f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}; z, \zeta\right) \prec\prec h_2(z),$$

implies

$$q_1(z) \prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec q_2(z).$$

5.3 First order linear strong differential subordination

In this section, by making use of Definition 5.1.1 following (G. Oros and Oros, 2009) and the related definitions and notations described in chapter 1 section 1.8, we investigate some new first order linear strong differential subordination properties of p-valent functions associated with fractional derivative operator. We begin by defining a first order linear strong differential subordination for p-valent functions involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ according to (Amsheri and Zharkova, 2012g).

Definition 5.3.1. A strong differential subordination for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ of the form

$$A(z,\zeta)\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}+\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}-\right.$$

$$\left.(p-\mu)\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}\right)^2\right\}+B(z,\zeta)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec\prec h(z), \tag{5.3.1}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}; z \in \mathcal{U}; \zeta \in \overline{\mathcal{U}}),$$

where

$$p(z) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)},$$

and

$$\psi(r,s;z,\zeta) = A(z,\zeta)zp'(z) + B(z,\zeta)p(z).$$

is analytic in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$ and $h(z)$ is analytic in $\mathcal{U}$, is called first order linear strong differential subordination for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$.

Let us investigate the first order liner strong differential subordination result of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ by making use of lemma 5.1.5 following (Amsheri and Zharkova, 2012g).

Theorem 5.3.2. Let $\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \in \mathcal{H}[0,p]$, $A{:}\,\mathcal{U}\times\overline{\mathcal{U}} \longrightarrow \mathbb{C}$, $B{:}\,\mathcal{U}\times\overline{\mathcal{U}} \longrightarrow \mathbb{C}$ with $\psi(r,s;z,\zeta)$ analytic in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$ and

$$\operatorname{Re}\{pA(z,\zeta)+B(z,\zeta)\} \geq 1, \qquad \operatorname{Re}\{A(z,\zeta)\} \geq 0.$$

If

$$A(z,\zeta)\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}+\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-\right.$$

$$\left.(p-\mu)\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^2\right\}+B(z,\zeta)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\prec\prec(p^2+1)Mz,$$

$$(5.3.2)$$

$$(\lambda\geq 0;\ \mu<p+1;\ \eta>\max(\lambda,\mu)-p-1;\ p\in\mathbb{N};z\in\mathcal{U};\zeta\in\overline{\mathcal{U}};M>0),$$

then

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\prec Mz.$$

Proof. Let $\psi:\mathbb{C}^2\times\mathcal{U}\times\overline{\mathcal{U}}\longrightarrow\mathbb{C}$,

$\psi(r,s;z,\zeta)=$

$$A(z,\zeta)\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}+\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-\right.$$

$$\left.(p-\mu)\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^2\right\}+B(z,\zeta)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}.$$

and (5.3.2) becomes

$$\psi(r,s;z,\zeta)\prec\prec(p^2+1)Mz. \tag{5.3.3}$$

Since $h(z)=(p^2+1)Mz$, it gives $h(z)=\mathcal{U}(0,(p^2+1)M)$. In this case (5.3.3) is equivalent to

$$\psi(r,s;z,\zeta)\in\mathcal{U}(0,(p^2+1)M). \tag{5.3.4}$$

Suppose that

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)},$$

is not subordinate to $q(z)=Mz$. Then by using Lemma 5.1.5, we have that there exist $z_0\in\mathcal{U}$ and $\zeta_0\in\partial\mathcal{U}$ such that

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z_0)}{M_{0,z}^{\lambda,\mu,\eta} f(z_0)} = q(\zeta_0) = Me^{i\theta_0},$$

where $\theta_0 \in \mathbb{R}$ when $|\zeta_0| = 1$, and

$$(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z_0)}{M_{0,z}^{\lambda,\mu,\eta} f(z_0)} + \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z_0)}{M_{0,z}^{\lambda,\mu,\eta} f(z_0)} -$$

$$(p-\mu)\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z_0)}{M_{0,z}^{\lambda,\mu,\eta} f(z_0)}\right)^2 = m\zeta_0 h'(\zeta_0) = ke^{i\theta_0}, \qquad (k \geq pM).$$

Hence we obtain

$$\begin{aligned}
|\psi(r,s;z_0,\zeta)| &= \left|A(z_0,\zeta)ke^{i\theta_0} + B(z_0,\zeta)Me^{i\theta_0}\right| \\
&= |A(z_0,\zeta)k + B(z_0,\zeta)M| \\
&\geq \mathrm{Re}\,\{A(z_0,\zeta)k + B(z_0,\zeta)M\} \\
&\geq k\mathrm{Re}\,A(z_0,\zeta) + M\mathrm{Re}\,B(z_0,\zeta) \\
&\geq M\{p\mathrm{Re}\,A(z_0,\zeta) + \mathrm{Re}\,B(z_0,\zeta)\} \\
&\geq M.
\end{aligned}$$

Since this result contradicts (5.3.4), we conclude that that assumption made concerning the subordination relation between $p(z)$ and $q(z)$ is false, hence

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec Mz.$$

This completes the proof of Theorem 5.3.2.

Let us now establish the first order liner strong differential subordination of p-valent functions by letting $\lambda = \mu = 0$ in Theorem 5.3.2 according to (Amsheri and Zharkova, 2012g).

Corollary 5.3.3. Let $\frac{zf\prime(z)}{pf(z)} \in \mathcal{H}[0,p]$, $A: \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$, $B: \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$ with

$$A(z,\zeta)\left\{\frac{z^2 f''(z)}{pf(z)} + \frac{zf'(z)}{pf(z)} - \frac{1}{\mathrm{p}}\left(\frac{zf'(z)}{f(z)}\right)^2\right\} + B(z,\zeta)\frac{zf'(z)}{pf(z)},$$

analytic function in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$ and

$$\operatorname{Re}\{pA(z,\zeta)+B(z,\zeta)\} \geq 1, \qquad \operatorname{Re}\{A(z,\zeta)\} \geq 0.$$

If

$$A(z,\zeta)\left\{\frac{z^2 f''(z)}{pf(z)}+\frac{zf'(z)}{pf(z)}-\frac{1}{\mathrm{p}}\left(\frac{zf'(z)}{f(z)}\right)^2\right\}+B(z,\zeta)\frac{zf'(z)}{pf(z)} \prec\prec (p^2+1)Mz,$$

$$(\,p \in \mathbb{N};\ z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}};M>0),$$

then

$$\frac{zf'(z)}{pf(z)} \prec Mz.$$

Next let us investigate further first order linear strong differential subordination of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ by making use of lemma 5.1.5 following (Amsheri and Zharkova, 2012g).

Theorem 5.3.4. Let $\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} \in \mathcal{H}[1,p]$, $A:\mathcal{U}\times\overline{\mathcal{U}}\longrightarrow\mathbb{C}$, $B:\mathcal{U}\times\overline{\mathcal{U}}\longrightarrow\mathbb{C}$ with $\psi(r,s;z,\zeta)$ analytic in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$ and

$$\operatorname{Re} A(z,\zeta) \geq 0, \qquad \operatorname{Im} B(z,\zeta) \leq p\operatorname{Re} A(z,\zeta).$$

If

$$\operatorname{Re}\left(A(z,\zeta)\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}+\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-\right.\right.$$

$$\left.\left.(p-\mu)\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^2\right\}+B(z,\zeta)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)>0, \quad (5.3.5)$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu)-p-1;\ p \in \mathbb{N}; z \in \mathcal{U}; \zeta \in \overline{\mathcal{U}}; M>0),$$

then

$$\operatorname{Re}\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)>0.$$

Proof. Let $\psi:\mathbb{C}^2\times\mathcal{U}\times\overline{\mathcal{U}}\longrightarrow\mathbb{C}$,

$\psi(r,s;z,\zeta) =$

$$A(z,\zeta)\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}+\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-\right.$$

$$\left.(p-\mu)\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^2\right\}+B(z,\zeta)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}.$$

In this case (5.3.5) becomes

$$\mathrm{Re}\,\{\psi(r,s;z,\zeta)\} > 0. \tag{5.3.6}$$

Since $h(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, and $h(\mathcal{U}) = \{w \in \mathbb{C}\colon \mathrm{Re}\, w(z) > 0\}$, hence (5.3.6) becomes

$$\psi(r,s;z,\zeta) \prec\prec \frac{1+Az}{1+Bz}, \qquad (-1 \le B < A \le 1).$$

Suppose

$$\mathrm{Re}\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right) < 0,$$

Meaning $p(z) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}$ is not subordinate to $h(z) = \frac{1+Az}{1+Bz}, -1 \le B < A \le 1$. Using Lemma 5.1.5, we have that there exist $z_0 \in \mathcal{U}$ and $\zeta_0 \in \partial\mathcal{U}$ with $|\zeta_0| = 1$ such that

$$p(z_0) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z_0)}{M_{0,z}^{\lambda,\mu,\eta}f(z_0)} = h(\zeta_0) = \rho i,$$

and

$$(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z_0)}{M_{0,z}^{\lambda,\mu,\eta}f(z_0)}+\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z_0)}{M_{0,z}^{\lambda,\mu,\eta}f(z_0)}$$

$$-(p-\mu)\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z_0)}{M_{0,z}^{\lambda,\mu,\eta}f(z_0)}\right)^2 = m\zeta_0 h'(\zeta_0) = \sigma,$$

where $\rho,\sigma \in \mathbb{R}$ and $\sigma \le -\frac{p}{2}(1+\rho^2), p \ge 1$. Then we obtain

$$\operatorname{Re}\psi(r,s;z_0,\zeta)=\operatorname{Re}\psi(\rho i,\sigma;z_0,\zeta)$$
$$=\operatorname{Re}\{A(z_0,\zeta)\sigma+B(z_0,\zeta)\rho i\}$$
$$=\operatorname{Re}\{A(z_0,\zeta)\sigma+[B_1(z_0,\zeta)+iB_2(z_0,\zeta)]\rho i\}$$
$$=\sigma\operatorname{Re}A(z_0,\zeta)-\rho\operatorname{Im}B(z_0,\zeta)$$
$$\leq -\frac{p}{2}(1+\rho^2)\operatorname{Re}A(z_0,\zeta)-\rho\operatorname{Im}B(z_0,\zeta)$$
$$\leq -\frac{p}{2}\rho^2\operatorname{Re}A(z_0,\zeta)-\rho\operatorname{Im}B(z_0,\zeta)-\frac{p}{2}\operatorname{Re}A(z_0,\zeta)\leq 0.$$

Hence $\operatorname{Re}\psi(r,s;z_0,\zeta)\leq 0$ which contradicts (5.3.6), and we conclude that

$$\operatorname{Re}\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)>0.$$

This completes the proof.

Let us now establish further result for first order linear strong differential subordination of p-valent functions by letting $\lambda=\mu=0$ in Theorem 5.3.4 according to (Amsheri and Zharkova, 2012g).

Corollary 5.3.5. Let $\frac{zf\prime(z)}{pf(z)}\in\mathcal{H}[1,p]$, $A:\mathcal{U}\times\overline{\mathcal{U}}\longrightarrow\mathbb{C}$, $B:\mathcal{U}\times\overline{\mathcal{U}}\longrightarrow\mathbb{C}$ with $\psi(r,s;z,\zeta)$ analytic function in $\mathcal{U}$ for all $\zeta\in\overline{\mathcal{U}}$ and

$$\operatorname{Re}A(z,\zeta)\geq 0,\qquad \operatorname{Im}B(z,\zeta)\leq p\operatorname{Re}A(z,\zeta).$$

If

$$\operatorname{Re}\left(A(z,\zeta)\left\{\frac{z^2f''(z)}{pf(z)}+\frac{zf'(z)}{pf(z)}-\frac{1}{\mathrm{p}}\left(\frac{zf'(z)}{f(z)}\right)^2\right\}+B(z,\zeta)\frac{zf'(z)}{pf(z)}\right)>0,$$

$$(p\in\mathbb{N};\ z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}),$$

then

$$\operatorname{Re}\left(\frac{zf'(z)}{pf(z)}\right)>0.$$

Next let us investigate further result for first order linear strong differential subordination of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ by making use of lemma 5.1.5 following (Amsheri and Zharkova, 2012g).

Theorem 5.3.6. Let $\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} \in \mathcal{H}[1,p]$, $A: \mathcal{U}\times\overline{\mathcal{U}} \longrightarrow \mathbb{C}$, $B: \mathcal{U}\times\overline{\mathcal{U}} \longrightarrow \mathbb{C}$ with $\psi(r,s;z,\zeta)$ analytic in $\mathcal{U}$ for all $\zeta \in \overline{\mathcal{U}}$ and

$$\mathrm{Re}\, A(z,\zeta) \geq 0, \qquad \mathrm{Im}\, B(z,\zeta) \leq p\mathrm{Re}\, A(z,\zeta)[p\mathrm{Re}A(z,\zeta) - 2].$$

If

$$A(z,\zeta)\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} + \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} - (p-\mu)\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^2\right\} + B(z,\zeta)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} \prec\prec Mz, \quad (5.3.7)$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}; z \in \mathcal{U}; \zeta \in \overline{\mathcal{U}}; M > 0),$$

then

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} \prec \frac{1+Az}{1+Bz}, \qquad (-1 \leq B < A \leq 1).$$

Proof. Let $\psi: \mathbb{C}^2 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$,

$\psi(r,s;z,\zeta) =$

$$A(z,\zeta)\left\{(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} + \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} - (p-\mu)\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^2\right\} + B(z,\zeta)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}.$$

and (5.3.7) becomes

$$\psi(r,s;z,\zeta) \prec\prec Mz. \qquad (5.3.8)$$

Since $h(z) = Mz$, it gives $h(z) = \mathcal{U}(0,M)$. Thus

$$\psi(r,s;z,\zeta) \in \mathcal{U}, \qquad (z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}).$$

Suppose that

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)},$$

is not subordinated to $q(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$. Then by using Lemma 5.1.5, we have that there exist $z_0 \in \mathcal{U}$ and $\zeta_0 \in \partial\mathcal{U}$ such that

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z_0)}{M_{0,z}^{\lambda,\mu,\eta} f(z_0)} = q(\zeta_0) = \rho i,$$

and

$$(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z_0)}{M_{0,z}^{\lambda,\mu,\eta} f(z_0)} + \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z_0)}{M_{0,z}^{\lambda,\mu,\eta} f(z_0)}$$

$$-(p-\mu)\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z_0)}{M_{0,z}^{\lambda,\mu,\eta} f(z_0)}\right)^2 = m\zeta_0 q'(\zeta_0) = \sigma,$$

where $\rho, \sigma \in \mathbb{R}$ and $\sigma \leq -\frac{p}{2}(1+\rho^2)$, $p \geq 1$. Then we obtain

$$\begin{aligned}
\operatorname{Re}\psi(r,s;z_0,\zeta) &= \operatorname{Re}\psi(\rho i,\sigma;z_0,\zeta) \\
&= \operatorname{Re}\{A(z_0,\zeta)\sigma + B(z_0,\zeta)\rho i\} \\
&= \sigma \operatorname{Re} A(z_0,\zeta) - \rho \operatorname{Im} B(z_0,\zeta) \\
&\leq -\frac{p}{2}(1+\rho^2)\operatorname{Re} A(z_0,\zeta) - \rho \operatorname{Im} B(z_0,\zeta) \\
&\leq -\frac{p}{2}\rho^2 \operatorname{Re} A(z_0,\zeta) - \rho \operatorname{Im} B(z_0,\zeta) - \frac{p}{2}\operatorname{Re} A(z_0,\zeta) \leq -1.
\end{aligned}$$

Hence we have

$$\operatorname{Re}\psi(r,s;z_0,\zeta) \leq -1,$$

which contradicts (5.3.8), we conclude that

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec \frac{1+Az}{1+Bz}, \qquad (-1 \leq B < A \leq 1).$$

This completes the proof.

Let us now establish further result for first order linear strong differential subordination of p-valent functions by letting $\lambda=\mu=0$ and $q(z)=\frac{1+z}{1-z}$ in Theorem 5.3.6 according to (Amsheri and Zharkova, 2012g).

Corollary 5.3.7. Let $\frac{zf\prime(z)}{pf(z)}\in\mathcal{H}[1,p]$, $A:\mathcal{U}\times\overline{\mathcal{U}}\longrightarrow\mathbb{C}$, $B:\mathcal{U}\times\overline{\mathcal{U}}\longrightarrow\mathbb{C}$ with $\psi(r,s;z,\zeta)$ analytic function in $\mathcal{U}$ for any $\zeta\in\overline{\mathcal{U}}$ and

$$\mathrm{Re}\,A(z,\zeta)\geq 0,\qquad \mathrm{Im}\,B(z,\zeta)\ \leq p\mathrm{Re}\,A(z,\zeta)[p\mathrm{Re}A(z,\zeta)-2].$$

If

$$A(z,\zeta)\left\{\frac{z^2f''(z)}{pf(z)}+\frac{zf'(z)}{pf(z)}-\frac{1}{\mathrm{p}}\left(\frac{zf'(z)}{f(z)}\right)^2\right\}+B(z,\zeta)\frac{zf'(z)}{pf(z)}\prec\prec z,$$

$$(\,p\in\mathbb{N};\ z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}),$$

then

$$\frac{zf'(z)}{pf(z)}\prec\frac{1+z}{1-z}.$$

5.4 On new strong differential subordination and superordination

This section is based on the fact that the coefficients of the functions in those classes $\mathcal{H}^*[a,n,\zeta]$ and $\mathcal{H}_u(\mathcal{U})$ given in chapter 1 section 1.8, are not constants but complex-valued functions. Using these classes, a new approach in the studying strong subordination and superordination can be seen for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)$ defined for $f(z,\zeta)\in\mathcal{A}_\zeta^*(p)$ according to (Amsheri and Zharkova, 2012h) and (Amsheri and Zharkova, 2012i). Let $\mathcal{A}_\zeta^*(p)$ be the class of functions $f\in\mathcal{H}(\mathcal{U}\times\overline{\mathcal{U}})$ of the form

$$f(z,\zeta) = z^p + \sum_{n=1}^{\infty} a_{p+n}(\zeta) z^{p+n}, \qquad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}})$$

and set $\mathcal{A}_\zeta^*(1) \equiv \mathcal{A}_\zeta^*$. We define the modification of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}$ for $f(z,\zeta) \in \mathcal{A}_\zeta^*(p)$ by

$$M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta) = \frac{\Gamma(1-\mu+p)\Gamma(1+\eta-\lambda+p)}{\Gamma(1+p)\Gamma(1+\eta-\mu+p)}\, z^\mu\, J_{0,Z}^{\lambda,\mu,\eta} f(z,\zeta), \tag{5.4.1}$$

or

$$M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta) = z^p + \sum_{n=1}^{\infty} \delta_n(\lambda,\mu,\eta,p)\, a_{p+n}(\zeta) z^{p+n}, \tag{5.4.2}$$

where $z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}; \lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}$ and

$$\delta_n(\lambda,\mu,\eta,p) = \frac{(1+p)_n(1+\eta-\mu+p)_n}{(1-\mu+p)_n(1+\eta-\lambda+p)_n}\,. \tag{5.4.3}$$

It is easily verified from (5.4.2) that

$$z\left(M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)\right)' = (p-\mu)\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta) + \mu\, M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta). \tag{5.4.4}$$

This identity plays a critical role in obtaining information about functions defined by use of the fractional derivative operator. Notice that

$$M_{0,z}^{0,0,\eta} f(z,\zeta) = f(z,\zeta),$$

and

$$M_{0,z}^{1,1,\eta} f(z,\zeta) = \frac{z f_z'(z,\zeta)}{p}.$$

5.4.1 Strong differential subordination results

In this subsection we investigate some new strong differential subordination for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)$ by making use of Lemmas 5.1.6 and 5.1.7. The next result deals with finding the best dominant from strong differential subordination by making use of Lemmas 5.1.6 following (Amsheri and Zharkova, 2012h).

Theorem 5.4.1.1. Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $f \in \mathcal{A}_\zeta^*(p)$ and the strong differential subordination

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)_z' \prec\prec h(z,\zeta), \quad \left(z \in \mathcal{U}; \zeta \in \overline{\mathcal{U}}\right), \tag{5.4.1.1}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p} \prec\prec q(z,\zeta) \prec\prec h(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best dominant.

Proof. Consider

$$p(z,\zeta) = \frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p} = 1 + p_1(\zeta)\, z + p_2(\zeta)\, z^2 + \cdots,$$

and $p(0,\zeta)=1,\ z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}$, we have

$$p_z'(z,\zeta) = \frac{\left(M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)\right)_z'}{z^p} - p\,\frac{p(z,\zeta)}{z}.$$

we obtain

$$p(z,\zeta) + z p_z'(z,\zeta) = \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)_z', \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

Then (5.4.1.1) becomes

$$p(z,\zeta) + z p_z'(z,\zeta) \prec\prec h(z,\zeta), \qquad \left(z \in \mathcal{U}; \zeta \in \overline{\mathcal{U}}\right).$$

Since $p(z,\zeta) \in \mathcal{H}^*[1,1,\zeta]$, using Lemma 5.1.6 for $n=1$ and $\gamma=1$, we have

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta),$$

or

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p} \prec\prec q(z,\zeta) \prec\prec h(z,\zeta), \qquad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}),$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best dominant.

Let us next find the best dominant from strong differential subordination by making use of Lemmas 5.1.7 following (Amsheri and Zharkova, 2012h).

Theorem 5.4.1.2. Let $q(z,\zeta)$ be a convex function such that $q(0,\zeta) = 1$ and h be the function defined by

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta),$$

If $f \in \mathcal{A}_\zeta^*(p)$ and the strong differential subordination

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)'_z \prec\prec h(z,\zeta), \qquad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\ ), \tag{5.4.1.2}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p} \prec\prec q(z,\zeta), \qquad (z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}).$$

and this result is sharp.

Proof. Following the same steps as in the proof of Theorem 5.4.1.1 and considering

$$p(z,\zeta) = \frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p},$$

we have

$$p_z'(z,\zeta) = \frac{\left(M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)\right)_z'}{z^p} - p\,\frac{p(z,\zeta)}{z},$$

then

$$p(z,\zeta) + z p_z'(z,\zeta) = \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)_z'.$$

The strong differential subordination (5.4.1.2) becomes

$$p(z,\zeta) + z p_z'(z,\zeta) \prec\prec q(z,\zeta) + z q_z'(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

By using Lemma 5.1.7, we have

$$p(z,\zeta) \prec\prec q(z,\zeta),$$

or

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p} \prec\prec q(z,\zeta), \qquad \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right).$$

Let us now find the best dominant from strong differential subordination by making use of Lemmas 5.1.6, when $h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z}$ following (Amsheri and Zharkova, 2012h).

Theorem 5.4.1.3. Let $h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z}$ be a convex function in $\mathcal{U} \times \overline{\mathcal{U}}$, $0 \leq \beta < 1$. If $f \in \mathcal{A}_\zeta^*(p)$ and the strong differential subordination

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)_z' \prec\prec h(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right), \tag{5.4.1.3}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p} \prec\prec q(z,\zeta) \prec\prec h(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where q is given by

$$q(z,\zeta) = 2\beta - \zeta + \frac{2(\zeta - \beta)}{z}\ln(1+z), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 5.4.1.1 and considering

$$p(z,\zeta) = \frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p}.$$

The strong differential subordination (5.4.1.3) becomes

$$p(z,\zeta) + zp_z'(z,\zeta) \prec\prec h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z}, \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

By using Lemma 5.1.6, for $n = 1$ and $\gamma = 1$, we have

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta),$$

or

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p} \prec\prec q(z,\zeta)$$

$$= \frac{1}{z}\int_0^z h(t,\zeta)dt$$

$$= \frac{1}{z}\int_0^z \frac{\zeta + (2\beta - \zeta)t}{1+t} dt$$

$$= 2\beta - \zeta + \frac{2(\zeta - \beta)}{z}\ln(1+z), \quad \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right).$$

The function q is convex and it is the best dominant.

Let us now investigate further strong differential subordination result of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)$ to find best dominant by making use of Lemma 5.1.6 according to (Amsheri and Zharkova, 2012h).

Theorem 5.4.1.4. Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta) = 1$. If $f \in \mathcal{A}_\zeta^*(p)$ and the strong differential subordination

$$\left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}\right)_z' \prec\prec h(z,\zeta), \quad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right), \qquad (5.4.1.4)$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)} \prec\prec q(z,\zeta) \prec\prec h(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best dominant.

Proof. Consider

$$p(z,\zeta) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}, \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

we have

$$p_z'(z,\zeta) = \frac{\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)\right)_z'}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)} - p(z,\zeta)\frac{\left(M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)\right)_z'}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)},$$

and we obtain

$$p(z,\zeta) + zp_z'(z,\zeta) = \left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}\right)_z'.$$

Then (5.4.1.4) becomes

$$p(z,\zeta) + zp_z'(z,\zeta) \prec\prec h(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

By using Lemma 5.1.6 for $n = 1$ and $\gamma = 1$, we have

$$p(z,\zeta) \prec\prec q(z,\zeta),$$

or

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)} \prec\prec q(z,\zeta) \prec\prec h(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best dominant.

Let us now find the best dominant from strong differential subordination by making use of Lemma 5.1.7 according to (Amsheri and Zharkova, 2012h).

Theorem 5.4.1.5. Let $q(z,\zeta)$ be a convex function such that $q(0,\zeta) = 1$ and h be the function defined by

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta).$$

If $f \in \mathcal{A}_\zeta^*(p)$ and the strong differential subordination

$$\left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}\right)'_z \prec\prec h(z,\zeta)\ , \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right), \qquad (5.4.1.5)$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)} \prec\prec q(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

and this result is sharp.

Proof. Following the same steps as in the proof of Theorem 5.4.1.4 and considering

$$p(z,\zeta) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)},$$

The strong differential subordination (5.4.1.5) becomes

$$p(z,\zeta) + zp'_z(z,\zeta) \prec\prec q(z,\zeta) + zq'_z(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

By using Lemma 5.1.7, we have

$$p(z,\zeta) \prec\prec q(z,\zeta),$$

or

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)} \prec\prec q(z,\zeta), \qquad \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right).$$

Let us next investigate further strong differential subordination result of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)$ to find best dominant by making use of Lemma 5.1.6 when $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$ according to (Amsheri and Zharkova, 2012h).

Theorem 5.4.1.6. Let $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $\mathcal{U}\times\overline{\mathcal{U}}, 0 \leq \beta < 1$. If $f \in \mathcal{A}_{\zeta}^{*}(p)$ and the strong differential subordination

$$\left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}\right)_z' \prec\prec h(z,\zeta), \quad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right), \qquad (5.4.1.6)$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)} \prec\prec q(z,\zeta) \prec\prec h(z,\zeta), \quad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where q is given by

$$q(z,\zeta) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z), \qquad \left(z \in \mathcal{U};\ \ \zeta \in \overline{\mathcal{U}}\right).$$

The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 5.4.1.4 and considering

$$p(z,\zeta) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}.$$

The strong differential subordination (5.4.1.6) becomes

$$p(z,\zeta) + zp_z'(z,\zeta) \prec\prec h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}, \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

By using Lemma 5.1.6, for $n = 1$ and $\gamma = 1$, we have

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta),$$

or

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)} \prec\prec q(z,\zeta)$$

$$= \frac{1}{z}\int_0^z h(t,\zeta)dt$$

$$= \frac{1}{z}\int_0^z \frac{\zeta + (2\beta - \zeta)t}{1+t} dt$$

$$= 2\beta - \zeta + \frac{2(\zeta - \beta)}{z}\ln(1+z), \quad \left(z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\right).$$

The function q is convex and it is the best dominant.

Let us investigate further strong differential subordination of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)$ by making use of lemma 5.1.6 following (Amsheri and Zharkova, 2012h).

Theorem 5.4.1.7. Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta) = 1$. If $f \in \mathcal{A}_\zeta^*(p)$ and the strong differential subordination

$$(p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z,\zeta)}{z^p} +$$

$$(p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{z^p} +$$

$$(\mu-p+1)^2\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p} \prec\prec h(z,\zeta), \quad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right), \qquad (5.4.1.7)$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)_z' \prec\prec q(z,\zeta) \prec\prec h(z,\zeta), \quad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best dominant.

Proof. Consider the function

$$p(z,\zeta)=\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_z'=1+p_1(\zeta)\,z+p_2(\zeta)\,z^2+\cdots,\quad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right).$$

we have

$$p(z,\zeta)+zp_z'(z,\zeta)=(p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z,\zeta)}{z^p}+$$

$$(p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{z^p}+(\mu-p+1)^2\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p}.$$

Then (5.4.1.7) becomes

$$p(z,\zeta)+zp_z'(z,\zeta)\prec\prec h(z,\zeta),\qquad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right).$$

By using Lemma 5.1.6, for $n=1$ and $\gamma=1$, we have

$$p(z,\zeta)\prec\prec q(z,\zeta)\prec\prec h(z,\zeta),$$

or

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_z'\prec\prec q(z,\zeta),\qquad (z\in\mathcal{U},\zeta\in\overline{\mathcal{U}})$$

where

$$q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best dominant.

Next result deals with finding the best dominant from strong differential subordination by making use of Lemma 5.1.7 following (Amsheri and Zharkova, 2012h).

Theorem 5.4.1.8. Let $q(z,\zeta)$ be a convex function such that $q(0,\zeta)=1$ and h be the function defined by

$$h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta), \qquad \left(z\in\mathcal{U},\zeta\in\overline{\mathcal{U}}\right).$$

If $f\in\mathcal{A}^*_\zeta(p)$ and the strong differential subordination

$$(p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z,\zeta)}{z^p}+$$

$$(p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{z^p}+$$

$$(\mu-p+1)^2\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p}\prec\prec h(z,\zeta), \quad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right), \qquad (5.4.1.8)$$

$$(\lambda\ge 0;\ \ \mu<p+1;\ \eta>\max(\lambda,\mu)-p-1;\ p\in\mathbb{N}),$$

holds, then

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)'_z\prec\prec q(z,\zeta), \qquad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right).$$

and this result is sharp.

Proof. Following the same steps as in the proof of Theorem 5.4.1.7 and considering

$$p(z,\zeta)=\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)'_z.$$

The strong differential subordination (5.1.4.8) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec q(z,\zeta)+zq'_z(z,\zeta), \quad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right).$$

By using Lemma 5.1.7, we have

$$p(z,\zeta)\prec\prec q(z,\zeta),$$

or

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)'_z\prec\prec q(z,\zeta), \qquad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right).$$

Let us establish further result that deals with finding the best dominant from strong differential subordination when $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$, by making use of Lemma 5.1.6 following (Amsheri and Zharkova, 2012h).

Theorem 5.4.1.9. Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $\mathcal{U}\times\overline{\mathcal{U}}$, $0\leq\beta<1$. If $f\in\mathcal{A}_{\zeta}^{*}(p)$ and the strong differential subordination

$$(p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z,\zeta)}{z^{p}}+$$

$$(p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{z^{p}}+$$

$$(\mu-p+1)^{2}\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p}}\prec\prec h(z,\zeta),\quad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right),\quad (5.4.1.9)$$

$$(\lambda\geq 0;\ \mu<p+1;\ \eta>\max(\lambda,\mu)-p-1;\ p\in\mathbb{N}),$$

holds, then

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_{z}^{\prime}\prec\prec q(z,\zeta)\prec\prec h(z,\zeta),\ \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right),$$

where q is given by

$$q(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(1+z),\quad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right).$$

The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 6.4.1.7 and considering

$$p(z,\zeta)=\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_{z}^{\prime}.$$

The strong differential subordination (5.4.1.9) becomes

$$p(z,\zeta)+zp_{z}^{\prime}(z,\zeta)\prec\prec h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z},\qquad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right).$$

By using Lemma 5.1.6, for $n = 1$ and $\gamma = 1$, we have

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta),$$

or

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)'_z \prec\prec q(z,\zeta)$$

$$= \frac{1}{z}\int_0^z h(t,\zeta)dt$$

$$= \frac{1}{z}\int_0^z \frac{\zeta + (2\beta - \zeta)t}{1+t} dt$$

$$= 2\beta - \zeta + \frac{2(\zeta - \beta)}{z} \ln(1+z), \quad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}})$$

The function q is convex and it is the best dominant.

5.4.2 Strong differential superordination results

In this subsection we investigate some new strong differential superordination results for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)$ by making use of Lemmas 5.1.8 and 5.1.9. The next result deals with finding the best subordinant from strong differential superordination by making use of Lemmas 5.1.8 following (Amsheri and Zharkova, 2012i).

Theorem 5.4.2.1. Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = 1$. If $f \in \mathcal{A}_\zeta^*(p)$ and suppose that $\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)'_z$ is univalent and

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p} \in \mathcal{H}^*[1,1,\zeta] \cap \mathcal{Q}.$$

If the strong differential superordination

$$h(z,\zeta) \prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)'_z, \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right), \tag{5.4.2.1}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$q(z,\zeta) \prec\prec \frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p}\,, \qquad \left(z \in \mathcal{U}\,; \zeta \in \overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best subordinant.

Proof. Consider the function

$$p(z,\zeta) = \frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p} = 1 + p_1(\zeta)\, z + p_2(\zeta)\, z^2 + \cdots, \ \left(z \in \mathcal{U}\,;\ \zeta \in \overline{\mathcal{U}}\right).$$

we obtain

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)_z' = p(z,\zeta) + zp_z'(z,\zeta), \quad \left(z \in \mathcal{U}\,;\ \zeta \in \overline{\mathcal{U}}\right).$$

Then (5.4.2.1) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \qquad \left(z \in \mathcal{U}\,;\ \zeta \in \overline{\mathcal{U}}\right).$$

By using Lemma 5.1.8, for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

or

$$q(z,\zeta) \prec\prec \frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p},$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best subordinant.

Let us find the best subordinat from strong differential superordination by making use of lemma 5.1.9 according to (Amsheri and Zharkova, 2012i).

Theorem 5.4.2.2. Le $q(z,\zeta)$ be a convex function and h be the function defined by

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta).$$

If $f \in \mathcal{A}_\zeta^*(p)$ and suppose that $\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)'_z$ is univalent and

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p} \in \mathcal{H}^*[1,1,\zeta] \cap \mathcal{Q},$$

and the strong differential superordination

$$h(z,\zeta) \prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)'_z, \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right), \tag{5.4.2.2}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$q(z,\zeta) \prec\prec \frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p}, \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 5.4.2.1 and considering

$$p(z,\zeta) = \frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p}.$$

The strong differential superordination (5.4.2.2) becomes

$$q(z,\zeta) + zq'_z(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

By using Lemma 5.1.9, for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

or

$$q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt \prec\prec \frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p}\ ,\quad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right).$$

The function q is the best subordinant.

Let us now find the best subordinat from strong differential superordination when $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ by making use of lemma 5.1.8 following (Amsheri and Zharkova, 2012i).

Theorem 5.4.2.3. Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $\mathcal{U}\times\overline{\mathcal{U}}$, $0\le\beta<1$. If $f\in\mathcal{A}_\zeta^*(p)$ and suppose that $\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_z'$ is univalent and

$$\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p}\in\mathcal{H}^*[1,1,\zeta]\cap\mathcal{Q}.$$

If the strong differential superordination

$$h(z,\zeta)\prec\prec\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_z'\ ,\qquad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right),\tag{5.4.2.3}$$

$$(\lambda\ge 0;\ \ \mu<p+1;\ \eta>\max(\lambda,\mu)-p-1;\ p\in\mathbb{N}),$$

holds, then

$$q(z,\zeta)\prec\prec\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p},\qquad \left(z\in\mathcal{U};\zeta\in\overline{\mathcal{U}}\right),$$

where q is given by

$$q(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(1+z),\qquad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right).$$

The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 5.4.2.1 and considering

$$p(z,\zeta)=\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p}.$$

The strong differential superordination (5.4.2.3) becomes

$$h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp_z'(z,\zeta),\qquad \left(z\in\mathcal{U};\zeta\in\overline{\mathcal{U}}\right).$$

By using Lemma 5.1.8, for $\gamma=1$, we have

$$q(z,\zeta)\prec\prec p(z,\zeta),$$

or

$$\begin{aligned} q(z,\zeta) &= \frac{1}{z}\int_0^z h(t,\zeta)dt \\ &= \frac{1}{z}\int_0^z \frac{\zeta+(2\beta-\zeta)t}{1+t}dt \\ &= 2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(1+z) \\ &\prec\prec \frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p},\qquad \left(z\in\mathcal{U},\zeta\in\overline{\mathcal{U}}\right). \end{aligned}$$

The function q is convex and it is the best subordinant.

Let us next investigate further strong differential superordination result for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)$ by making use of Lemma 5.1.8 following (Amsheri and Zharkova, 2012i).

Theorem 5.4.2.4. Let $h(z,\zeta)$ be a convex function with $h(0,\zeta)=1$. If $f\in\mathcal{A}_\zeta^*(p)$ and suppose that $\left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}\right)_z'$ is univalent and

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}\in\mathcal{H}^*[1,1,\zeta]\cap\mathcal{Q},$$

If the strong differential superordination

$$h(z,\zeta)\prec\prec\left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}\right)_z',\qquad \left(z\in\mathcal{U};\zeta\in\overline{\mathcal{U}}\right),\tag{5.4.2.4}$$

$$(\lambda\geq 0;\ \mu<p+1;\ \eta>\max(\lambda,\mu)-p-1;\ p\in\mathbb{N}),$$

holds, then

$$q(z,\zeta) \prec\prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}, \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best subordinant.

Proof. Consider the function

$$p(z,\zeta) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}, \qquad \left(z \in \mathcal{U}; \zeta \in \overline{\mathcal{U}}\right).$$

we have

$$p_z'(z,\zeta) = \frac{\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)\right)_z'}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)} - p(z,\zeta)\frac{\left(M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)\right)_z'}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)},$$

and we obtain

$$p(z,\zeta) + zp_z'(z,\zeta) = \left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}\right)_z'.$$

Then (5.4.2.4) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

By using Lemma 5.1.8, for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

or

$$q(z,\zeta) \prec\prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}, \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best subordinant.

Let us now find the best subordinant from strong differential superordination by making use of Lemma 5.1.9 following (Amsheri and Zharkova, 2012i).

Theorem 5.4.2.5. Let $q(z,\zeta)$ be a convex function and h be the function defined by

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta).$$

If $f \in \mathcal{A}_\zeta^*(p)$ and suppose that $\left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}\right)'_z$ is univalent and

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)} \in \mathcal{H}^*[1,1,\zeta] \cap \mathcal{Q}.$$

If the strong differential superordination

$$h(z,\zeta) \prec\prec \left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}\right)'_z, \quad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right), \tag{5.4.2.5}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$q(z,\zeta) \prec\prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}, \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 5.4.2.4 and considering

$$p(z,\zeta) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}.$$

The strong differential superordination (5.4.2.5) becomes

$$q(z,\zeta) + zq'_z(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

By using Lemma 5.1.9, for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

or

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt \ \prec\prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}, \qquad \left(z \in \mathcal{U}; \zeta \in \overline{\mathcal{U}}\right).$$

The function q is the best subordinant.

Let us next investigate further strong differential superordination result when $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$ by making use of Lemma 5.1.8 following (Amsheri and Zharkova, 2012i).

Theorem 5.4.2.6. Let $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $\mathcal{U}\times\overline{\mathcal{U}}$, $0 \leq \beta < 1$. If $f \in \mathcal{A}_\zeta^*(p)$ and suppose that $\left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}\right)'_z$ is univalent and

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)} \in \mathcal{H}^*[1,1,\zeta] \cap \mathcal{Q},$$

If the strong differential superordination

$$h(z,\zeta) \prec\prec \left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}\right)'_z, \qquad \left(z \in \mathcal{U}; \zeta \in \overline{\mathcal{U}}\right), \tag{5.4.2.6}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$q(z,\zeta) \prec\prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}, \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where q is given by

$$q(z,\zeta) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 5.4.2.4 and considering

$$p(z,\zeta) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}.$$

The strong differential superordination (5.4.2.6) becomes

$$h(z,\zeta) = \frac{\zeta + (2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \quad \left(z \in \mathcal{U}; \zeta \in \overline{\mathcal{U}}\right),$$

By using Lemma 5.1.8, for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

or

$$\begin{aligned} q(z,\zeta) &= \frac{1}{z}\int_0^z h(t,\zeta)dt \\ &= \frac{1}{z}\int_0^z \frac{\zeta + (2\beta-\zeta)t}{1+t} dt \\ &= 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z) \\ &\prec\prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}. \end{aligned}$$

The function q is convex and it is the best subordinant.

The next result deals with finding the best subordinat from strong differential superordination by making use of Lemma 5.1.8 according to (Amsheri and Zharkova, 2012i).

Theorem 5.4.2.7. Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $f\in\mathcal{A}_{\zeta}^{*}(p)$ and

$$(p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z,\zeta)}{z^p}+$$

$$(p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{z^p}+(\mu-p+1)^2\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p},$$

is univalent and

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_z' \in \mathcal{H}^*[1,1,\zeta]\cap\mathcal{Q}.$$

If the strong differential superordination

$$h(z,\zeta)\prec\prec(p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z,\zeta)}{z^p}+$$

$$(p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{z^p}+(\mu-p+1)^2\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p},$$

$$(5.4.2.7)$$

$$(\lambda\ge 0;\ \ \mu<p+1;\ \eta>\max(\lambda,\mu)-p-1;\ p\in\mathbb{N}),$$

holds, then

$$q(z,\zeta)\prec\prec\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_z',\qquad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best subordinant.

Proof. Consider the function

$$p(z,\zeta)=\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_z'=1+p_1(\zeta)\,z+p_2(\zeta)\,z^2+\cdots,\ \ (z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}})$$

we have

$$p(z,\zeta)+zp'_z(z,\zeta)=(p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z,\zeta)}{z^p}+$$

$$(p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{z^p}+(\mu-p+1)^2\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p}.$$

Then (5.4.2.7) becomes

$$h(z,\zeta)\prec\prec p(z,\zeta)+zp'_z(z,\zeta), \qquad \left(z\in\mathcal{U};\zeta\in\overline{\mathcal{U}}\right).$$

By using Lemma 5.1.8, for $\gamma=1$, we have

$$q(z,\zeta)\prec\prec p(z,\zeta),$$

or

$$q(z,\zeta)\prec\prec\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)'_z, \qquad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best subordinant.

Let us next establish further strong differential superordination by making use of Lemma 5.1.9 following (Amsheri and Zharkova, 2012i).

Theorem 5.4.2.8. Let $q(z,\zeta)$ be a convex function and h be the function defined by

$$h(z,\zeta)=q(z,\zeta)+\frac{1}{\gamma}zq'(z,\zeta), \qquad \left(z\in\mathcal{U};\ \zeta\in\overline{\mathcal{U}}\right).$$

If $f\in\mathcal{A}_\zeta^*(p)$ and suppose that

$$(p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z,\zeta)}{z^p}+$$

$$(p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{z^p}+(\mu-p+1)^2\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p},$$

is univalent and

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_z' \in \mathcal{H}^*[1,1,\zeta] \cap \mathcal{Q}\,.$$

If the strong differential superordination

$$h(z,\zeta) \prec\prec (p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z,\zeta)}{z^p} +$$

$$(p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta)}{z^p} + (\mu-p+1)^2\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^p}, \tag{5.4.2.8}$$

$$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$q(z,\zeta) \prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_z', \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 5.4.2.7 and considering

$$p(z,\zeta) = \left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)}{z^{p-1}}\right)_z'.$$

The strong differential superordination (5.4.2.8) becomes

$$q(z,\zeta) + zq_z'(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right),$$

By using Lemma 5.1.9, for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

or

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt \prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)_z^{\prime}, \quad (z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}).$$

The function q is the best subordinant.

In the next result let us find the best subordinant from strong differential superordination when $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$ by making use of Lemma 5.1.8 according to (Amsheri and Zharkova, 2012i).

Theorem 5.4.2.9. Let $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $\mathcal{U}\times\overline{\mathcal{U}},\ 0 \leq \beta < 1$. If $f \in \mathcal{A}_\zeta^*(p)$ and suppose that

$$(p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z,\zeta)}{z^p} +$$

$$(p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{z^p} + (\mu-p+1)^2\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p},$$

is univalent and

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)_z^{\prime} \in \mathcal{H}^*[1,1,\zeta] \cap \mathcal{Q}.$$

If the strong differential superordination

$$h(z,\zeta) \prec\prec (p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z,\zeta)}{z^p} +$$

$$(p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{z^p} + (\mu-p+1)^2\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p}, \tag{5.4.2.9}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in \mathbb{N}),$$

holds, then

$$q(z,\zeta) \prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)_z^{\prime}, \quad (z \in \mathcal{U};\zeta \in \overline{\mathcal{U}}),$$

where q is given by

$$q(z,\zeta) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z), \qquad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 5.4.2.7 and considering

$$p(z,\zeta) = \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)'_z.$$

The strong differential superordination (5.4.2.9) becomes

$$h(z,\zeta) = \frac{\zeta + (2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad \left(z \in \mathcal{U};\ \zeta \in \overline{\mathcal{U}}\right).$$

By using Lemma 5.1.8, for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

or

$$\begin{aligned} q(z,\zeta) &= \frac{1}{z}\int_0^z h(t,\zeta)dt \\ &= \frac{1}{z}\int_0^z \frac{\zeta + (2\beta-\zeta)t}{1+t} dt \\ &= 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z) \\ &\prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}}\right)'_z . \end{aligned}$$

The function q is convex and it is the best subordinant.

Conclusions

This research is mainly concerned with the analytic functions defined in the open unit disk. In this thesis, by making use of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$, certain new classes of analytic and p-valent (or multivalent) functions with negative coefficients such as $T_{\lambda,\mu,\eta}^{*}(p,\alpha,\beta,\gamma)$, $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$, $S_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$, $C_{\lambda,\mu,\eta}^{p}(a,c,\alpha,\beta)$ and $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ were introduced and their properties were investigated. These classes generalized the concepts of starlike and convex, prestarlike, and uniformly starlike and convex functions. Several new sufficient conditions for starlikeness and convexity of the operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ by using certain results of (Owa, 1985a), convolution, Jack's Lemma and Nunokakawa' Lemma were obtained. The technique of subordination was employed to introduce new classes involving the operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ such as $S_{b,p,\lambda,\mu,\eta}^{*}(\phi)$, $R_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$, $M_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ and $N_{p,b,\alpha,\beta}^{\lambda,\mu,\eta}(\phi)$ in order to obtain the bounds of the coefficient functional $|a_{p+2}-\theta a_{p+1}^{2}|$. These classes generalized the concepts of starlike, Bazilevič and non-Bazilevič functions of complex order. Several differential subordination, superordination and sandwich type results were investigated for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$. By making use of the notations of strong differential subordination and superordination, new classes of admissible functions were introduced such that subordination and superordination implications of functions involving the operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ hold. First order linear strong differential subordination properties were investigated. Several strong differential subordination and superordination results based on the fact that the coefficients of the functions $M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)$ are not constants but complex-valued functions. This thesis is composed of five chapters in which the research have been carried out.

(i) First chapter is an introduction where we presented review of literature to provide background for certain classes of analytic functions. Some elementary concepts of univalent and p-valent functions, analytic functions with positive real part, special classes of analytic functions, fractional derivative operators, differential subordination and superordination, strong differential subordination and superordination are defined. The motivations and outlines of this research are also considered.

(ii) Chapter 2 is dedicated for the application of fractional derivative operator to analytic and p-valent functions with negative coefficients in the open unit disk. More precisely, we introduced new classes $T_{\lambda,\mu,\eta}^{*}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ of p-valent starlike functions with negative coefficients by using fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$. We obtained the sufficient conditions for functions to be the these classes by using the results of (Owa, 1985a) and investigated a number of distortion properties which determine how large the modulus of p-valent function together with its derivatives can be in these classes. Further distortion properties involving generalized fractional derivative operator $J_{0,Z}^{\lambda,\mu,\eta}f(z)$ of p-valent functions are also studied. The radii of convexity problem for the classes $T_{\lambda,\mu,\eta}^{*}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ which determine the largest disk $|z|<r$ such that each function

belonging to these classes is convex in $|z| < r$ are also considered. The well-known results according to (Aouf and Hossen, 2006), (Srivastava and Owa, 1991a) (Srivastava and Owa, 1991b) and (Gupta, 1984) follow as particular cases from the generalized results of the classes $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ which are presented in this chapter by specialising the parameters.

Moreover, by using the Hadamard product (or convolution) involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ we introduced new classes $S^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ and $C^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ of p –valent starlike and convex functions with negative coefficients. The necessary and sufficient conditions for a function to be in such these classes are obtained. Further results including distortion properties, extreme points, modified Hadamard product and inclusion properties are also studied. We determined the radii of close-to-convexity, starlikeness and convexity. Relevant connections of the newly derived generalized results of the classes $S^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ and $C^p_{\lambda,\mu,\eta}(a,c,\alpha,\beta)$ which are presented in this chapter with various earlier results, for example, (Aouf, 1988), (Gupta and Jain, 1976), (Owa, 1985a), (Silverman, 1975), (Aouf, 2007) and (Aouf and Silverman, 2007) are also studied by specialising the parameters.

In addition, we introduced the new class $k-TUCV^{\lambda,\mu,\eta}_{\beta,\gamma,\xi}(p,\alpha)$ of k-uniformly p-valent starlike and convex functions in the open unit disk associated with fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$. We obtained coefficient estimates, distortion theorems and extreme points for functions belonging to such these classes. We established a number of closure properties. The radii of starlikeness, convexity and close-to-convexity are also determined. We remark that several results given the coefficient estimates, distortion properties, extreme points, closure and , inclusion properties, and radii of convexity and starlikeness of functions which belong to various subclasses of $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ can be obtained by suitable choices of parameters, including some of the results obtained by (AL-Kharsani and AL-Hajiry, 2006), (Owa, 1998), (Rønning, 1991), (Goodman, 1991b) and (Partil and Thakare, 1983).

(iii) In chapter 3, we studied two types of problems. The first type deals sufficient conditions for starlikeness and convexity of p-valent functions associated with fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$. We found the sufficient conditions by using the results of (Owa, 1985a) and the Hadamard product. Further sufficient conditions for starlikeness and convexity by using Jack's Lemma and Nunokakawa's Lemma are also obtained. We remark that several characterization properties given the starlikeness and convexity properties of fractional derivative operator can be obtained by suitable choices of the parameters. Our results obtained here extend the previous results obtained by (Owa and Shen, 1998), (Raina and Nahar, 2000) and (Imark and Piejko, 2005).

The second type is concerned with the coefficient bounds for some subclasses of p-valent functions of complex order defined by fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$. We obtained the bounds of the coefficient functional $\left|a_{p+2}-\theta a_{p+1}^2\right|$ and bounds for the coefficient a_{p+3}

of function belonging to the new classes $S^*_{p,\lambda,\mu,\eta}(\phi)$ and $S^*_{b,p,\lambda,\mu,\eta}(\phi)$ of p-valent functions. We studied the similar problem for more general new classes $R^{\lambda,\mu,\eta}_{p,1,\alpha,\beta}(\phi)$, $R^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$, $M^{\lambda,\mu,\eta}_{p,1,\alpha,\beta}(\phi)$ and $M^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$ of Bazilevič functions and for the new classes $N^{\lambda,\mu,\eta}_{p,1,\alpha,\beta}(\phi)$, $N^{\lambda,\mu,\eta}_{p,b,\alpha,\beta}(\phi)$ of non-Bazilevič functions. Relevant connections of the newly results obtained here with those in earlier papers, for example, (Ali, et al., 2007), (Ma and Minda, 1994), (Ravichandram et al. 2004), (Ravichandran et al. 2005), (Ramachandran et al. 2007), (Srivastava and Mishra, 2000), (Dixit and Pal,1995), (Rosy et al., 2009), (Obradović, 1998), (Shanmugam et al., 2006) and (Tuneski and Darus, 2002) are also provided.

(iv) In chapter 4, the classical notations of differential subordinations and its dual, differential superordinations were introduced by (Miller and Mocaun,1981) and (Miller and Mocaun,1985) and developed in (Miller and Mocaun,2000) are the starting point for new differential subordinations and superordinations obtained by using certain fractional derivative operator $M^{\lambda,\mu,\eta}_{0,z}f(z)$ of p-valent functions in the open unit disk. We investigated some new differential subordination and superordination results for the operator $M^{\lambda,\mu,\eta}_{0,z}f(z)$. Several differential sandwich results are also obtained.

(v) In chapter 5, we investigated new classes of admissible functions of strong differential subordination and strong differential superordination involving the fractional derivative operator $M^{\lambda,\mu,\eta}_{0,z}f(z)$, so that the subordination as well as superordination implications of functions associated with the fractional derivative operator hold. New strong differential sandwich type results are also obtained. Moreover, we derived several first order linear strong differential subordination properties for the operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$. Further new strong differential subordination and superordination properties were obtained for the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)$ on the fact that the coefficients of the functions are not constants but complex-valued functions.

Overall, the careful research carried out earlier and in this thesis shows that the fractional calculus operator (that is; fractional derivative operator) has many extensive and interesting applications in the theory of analytic and multivalent functions. We observed that some well known results are reduced as special cases from our main results signifying the work presented in this thesis

Future work

The scope of this thesis has caused several limitations, which however provide basis for future research along the path to fractional calculus in several areas. These areas include:

- application of fractional calculus to analytic functions theory,
- application of fractional calculus to special functions,
- application of fractional calculus to physics, and
- application of fractional calculus to engineering.

The following sections discuss each of these areas in more detail.

1. Application of fractional calculus to analytic functions theory

The future improvement of this thesis to analytic functions theory can be developed in several ways. One possible extension is to investigate a more general linear operator that involving fractional calculus operator (that is, derivative or integral) or other linear operator such as Ruscheweh derivative operator, Multiplier differential operator and Salagean differential operator. The current framework requires that the linear operator be specified explicitly. It would be preferred that an initial linear operator be suggested and framework allowed to adapt or extend. Another possibility would be to use the fractional derivative operator which was studied in the present thesis to other fields of analytic functions such as high-order derivatives of multivalent functions, harmonic functions and meromorphic functions.

2. Application of fractional calculus to special functions

The field of special functions is ripe for further work, as there are many special functions appear as solutions of differential equations or integrals of elementary functions. For example, the Riemann zeta function is a function of complex variable defined by infinite series. The fractional calculus operators will be applied to the summation of the series and evaluation to definite integrals in corresponding zeta function. Some of properties will be derived such as the fractional derivative operator of zeta function is again zeta function. Moreover, by extending The Riemann zeta function and obtaining some properties such as analytic continuation and integral representation of the extended function. The connections between the extended function with other functions in the literature will be considered. It will expect that some of the results may find applications in the solution of certain fractional order differential and integral equations.

3. Application of fractional calculus to physics

Development of solving problems in physics is an important area for future research. One possible direction of research is to fluid mechanics which studies fluids (liquids, gases, and plasmas) and the forces on them, i.e. work based on Mathematical Physics. The scope of future work in this area will deal with obtaining the solution of time-dependent viscous-diffusion problem of a semi-infinite fluid bounded by a flat plate by using fractional derivative operator. It can be obtained that, by making use of the equation describing the time-dependent of viscous-diffusion which is a partial differential equation of first order in time and second order in space. The initial and boundary conditions corresponding to the problem will be used. Together with the Laplace transform method of the equation, the application of fractional derivative operator to the equation in a semi-infinite space will be useful to reduce the order of the differential equation to yield explicit analytical (fractional) solutions.

Another possible application of fractional calculus in fluid mechanics will deal with obtaining the solution of the instability phenomenon in fluid flow through porous media with mean capillary pressure. When water is injected into oil saturated porous medium, as a result perturbation (instability) occurs and develops the finger flow. It can be obtained that, by making use of the equation describing the instability phenomenon which is a partial differential equation of fractional order. The solution of the problem will yield by making use of the initial and boundary conditions and fractional calculus together with Fourier and Laplace transforms method.

4. Application of fractional calculus to engineering

The fractional calculus can be applied to other scientific areas such as engineering, and more particularly in electric. For example, Ultracapacitors (aka supercapacitors) are electrical devices which are used to store energy and offer high power density that is not possible to achieve with traditional capacitors. Nowadays, ultracapacitors have many industrial applications and are used wherever a high current in a short time is needed. They are able to store or yield a lot of energy in a short period of time. One of the most prominent is the ultracapacitor application in hybrid cars when a hybrid car is decelerating the electric motor acts as a generator producing a short, but high value energy impulse. This is used to charge the ultracapacitor. Charging the conventional batteries with such a short impulse would be extremely ineffective. Similarly, during start-up of the electric motor a short-time but substantial in value increase of the source power is needed. This is achieved by using the ultracapacitor. It is essential to have a fairly detailed model of ultracapacitor. This model makes the design of control systems possible. The more accurate model we have, the more advanced control schema can be achieved. Control systems are needed to stabilise the ultracapacitor voltage which tends to fluctuate significantly.

Ultracapacitors are large capacity and power density electrical energy storage devices. This large capacity is the effect of a very complicated internal structure. This structure also has a significant impact on the dynamic behaviour of the ultracapacitor. The scope of future works in this area will deal with describing the performance of the ultracapacitors by using fractional order model to give high accurate results of modelling over a wider range of frequencies. This will be made by using the fractional-order integrator which is based on the fractional calculus dealing with derivatives of arbitrary order. To define fractional order ultracapacitor models as functions and find frequency and time domain modelling of ultracapacitors which then will allow comparing fractional order models with the integer model for a better description of the behaviour. This model of the ultracapacitor will be used in either time or frequency domains. Also, by building a model of the ultracapacitor which will be composed of the part responsible for the integer order capacitor and the fractional order part responsible for a better description of the behaviour.

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