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EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, using a simple and classical application of the Leray-Schauder degree theory, we study the existence of solutions of the following boundary value problem for functional differential equations

$$x''(t)+f(t,x_t,x'(t)) = 0, \quad t \in [0,T]$$

$$x_0+\alpha x'(0) = h$$

$$x(T)+\beta x'(T) = \eta$$

where $f \in C([0,T]\times C_r\times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C_r$, $\eta \in \mathbb{R}^n$ and α,β are real constants.

KEY WORDS AND PHRASES. Boundary value problem, functional differential equations.
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1. INTRODUCTION

Let $\mathbb{R}^n$ be the real euclidean space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. Let also, C_r be the space of all continuous functions $x : [-r,0] \to \mathbb{R}^n$, $r > 0$, endowed with the sup-norm

$$\|x\| = \sup\{|x(t)| : t \in [-r,0]\}.$$

For every continuous function $x : [-r,T] \to \mathbb{R}^n$, $T > 0$ and every $t \in [0,T]$, we denote by x_t the element of C_r defined by

$$x_t(\vartheta) = x(t+\vartheta), \quad \vartheta \in [-r,0].$$

The main purpose of this paper is to discuss when the functional differential equation

$$x''(t)+f(t,x_t,x'(t))=0, \quad t\in[0,T], \tag{1.1}$$

admits a solution x on $[0,T]$ such that the boundary value conditions

$$x_0+\alpha x'(0)=h \tag{1.2a}$$

$$x(T)+\beta x'(T)=\eta \tag{1.2b}$$

to be satisfied. Here, $f:[0,T]\times C_r\times\mathbb{R}^n\to\mathbb{R}^n$ is a continuous function, $h\in C_r$, $\eta\in\mathbb{R}^n$ and α,β are real constants such that

$$\alpha\leq 0\leq\beta \tag{1.2c}$$

By $x'(0)$ and $x'(T)$ we mean $x'(0^+)$ and $x(T^-)$, respectively. In the next, the boundary value problem (B.V.P.) which constitutes from the equation (1.1) and the boundary conditions (1.2a),(1.2b),(1.2c), will be mentioned briefly as B.V.P. (1.1)-(1.2).

Analogous boundary value problems for ordinary differential equations has been studied by many authors, who used the Leray-Schauder continuation theorem (see Lasota and Yorke [1], Szmanda [2], Traple [3] and others). Usually, in these problems the authors derive a priori estimates of solutions by using inequalities of Wirtinger and Opial type.

Our work is motivated by the recent papers of Fabry and Habets [4], Fabry [5] and Ntouyas [6]. In [6] the author generalizes the results of Fabry and Habets [4] to the functional equation (1.1) with boundary conditions

$$x_0=h, \quad h(0)=0,$$
$$x(T)=0.$$

Here, following Fabry [5] we extend the results of Ntouyas [6].

2. MAIN RESULTS

Before stating our main results we refer some lemmas which simplify the proof of the theorem bellow.

LEMMA 2.1. [4, pp 187]. Let X be a Banach space, $A:X\to X$ be a completely continuous mapping such that I-A is one to one, and let Ω be a bounded set such that $0\in(I-A)(\Omega)$. Then the completely continuous mapping $S:\Omega\to X$ has a fixed point in Ω if for any $\lambda\in(0,1)$, the equation

$$x=\lambda Sx+(1-\lambda)Ax \tag{2.1}$$

has no solution on the boundary $\partial\Omega$ of Ω.

LEMMA 2.2. [5, pp 133]. Let $X:[0,T]\to\mathbb{R}^n$ be a twise differentiable function and let $R>0$ be such that

$$\|x\|\leq R. \tag{2.2}$$

Assume that positive constants c,d exist, with $c<1$, such that

$$-\langle x(t),x''(t)\rangle\leq c|x'(t)|^2+d, \quad t\in[0,T]. \tag{2.3}$$

Moreover, assume that positive constants c',d' exist with $c'<(1-c)^2/8R$ such that

$$|\langle x'(t),x''(t)\rangle|\leq(c'|x'(t)|^2+d')|x'(t)|, \quad t\in[0,T]. \tag{2.4}$$

Then there exists a number K nondepending on x, such that

$$\|x'(t)\|\leq K.$$

LEMMA 3.2. If $\alpha\leq 0\leq\beta$ the B.V.P

$$x''(t) = kx(t), \quad k > 0$$
$$x(0)+\alpha x'(0) = 0, \quad x(T)+\beta x'(T) = 0$$

has the unique solution $x = 0$.

PROOF. The general solution of the above equation has the form

$$x(t) = c_1 e^{\sqrt{k}t} + c_2 e^{-\sqrt{k}t}.$$

On account of the above boundary conditions we obtain

$$\frac{(1+\alpha\sqrt{k})(1-\beta\sqrt{k})}{(1-\alpha\sqrt{k})(1+\beta\sqrt{k})} \neq e^{2\sqrt{k}\,T}.$$

Since $e^{2\sqrt{k}T} > 1$, $k > 0$, the last expression is true for every $k > 0$, provided the left hand side is less than or equal to one. But this is clear since $\alpha \leq 0 \leq \beta$.

The next Theorem guarantees existence of solutions for the B.V.P. (1.1)-(1.2) which are bounded by an a priori given function φ. Moreover, the first derivative of a such solution is also bounded by a constant ρ not depending on this solution.

THEOREM. Let $f : [0,T]\times C_r \times \mathbb{R}^n$ be a continuous function which maps bounded sets of $[0,T]\times C_r\times\mathbb{R}^n$ into bounded sets of $\mathbb{R}^n$. Assume that $\varphi : [0,T] \to (0,\infty)$ is a twice continuously differentiable function such that

$$\begin{aligned} -\varphi(0)-|\alpha|\ \varphi'(0) &> |h(0)|, \text{ if } \alpha \neq 0 \\ \varphi(0) &> |h(0)|, \text{ if } \alpha = 0 \end{aligned} \tag{2.5α}$$

and

$$\begin{aligned} -\varphi(T)+|\beta|\ \varphi'(T) &> |n|, \text{ if } \beta \neq 0 \\ \varphi(T) &> |n|, \text{ if } \beta = 0. \end{aligned} \tag{2.5β}$$

Also, we suppose that

$$\varphi(t)\varphi''(t)+\langle u(0),f(t,u,v)\rangle \leq 0 \tag{2.6}$$

for any $(t,u,v) \in [0,T]\times C_r\times\mathbb{R}^n$ with $\varphi(t) = |u(0)|$ and $\langle u(0),v\rangle = |u(0)|\varphi'(t)$.

Moreover, assume that there exist positive numbers k_1,k_2 with $k_1 < 1$ and positive numbers k_1',k_2' with

$$k_1' < \frac{1}{8m}(1-k_1)^2, \quad m = \max_{t\in[0,T]} |\varphi(t)|$$

such that

$$\langle u(0),f(t,u,v)\rangle \leq k_1|v|^2+k_2, \tag{2.7}$$

$$|\langle v,f(t,u,v)\rangle| \leq (k_1'|v|^2+k_2')|v| \tag{2.8}$$

for any $(t,u,v) \in [0,T]\times C_r\times\mathbb{R}^n$ with $|u(0)| \leq \varphi(t)$.

Then the problem (1.1)-(1.2) has at least one solution x such that $|x(t)| \leq \varphi(t)$, $t \in [0,T]$ and $|x'(t)| \leq \rho$, $t \in [0,T]$.

PROOF. Let $k > 0$ be a constant, such that $k > \max\left\{\frac{\varphi''(t)}{\varphi(t)},\ t \in [0,T]\right\}$ and x a solution of the equation

$$x''(t)+\lambda f(t,x_t,x'(t)) = (1-\lambda)kx(t), \quad \lambda \in (0,1) \tag{2.9}$$

with $t \in [0,T]$ and $|x(t)| \leq \varphi(t)$.

Multiplying both sides of (2.9) by $x(t)$ and using (2.7) we deduce that

$$\begin{aligned} -\langle x(t),x''(t)\rangle &= \lambda\langle x_t(0),f(t,x_t,x'(t))\rangle-(1-\lambda)k|x(t)|^2 \\ &\leq \lambda(k_1|x'(t)|^2+k_2) \end{aligned}$$

$$\leq k_1|x'(t)|^2+k_2$$

Similarly, condition (2.8) yields

$$|\langle x'(t),x''(t)\rangle| \leq (k_1'|x'(t)|^2+k_2')|x'(t)|+k|x'(t)|m$$
$$\leq (k_1'|x'(t)|^2+\hat{c})|x'(t)|$$

where $\hat{c} = k_2'+km$.

Thus the conditions of Lemma 2.2 are fulfilled and hence there exists a number K not depending on x, such that $|x'(t)| \leq K$.

Let us now consider the Banach space B of all continuous functions $x : [0,T] \to \mathbb{R}^n$, which are continuously differentiable on $[0,T]$, endowed with the norm

$$\|x\|_1 = \max \left\{ \sup_{t\in[0,T]} |x(t)|, \sup_{t\in[0,T]} |x'(t)| \right\}.$$

Also, for any $x \in B$ we set

$$Sx(t) = \int_0^T G(t,s)f(s,x_s,x'(s))ds + \frac{1}{\ell}\left[(T-t)h(0)+\beta h(0)-\alpha\eta+t\eta\right], \ t \in [0,T] \qquad (2.10\alpha)$$

where

$$x_s(\vartheta) = \begin{cases} x(s+\vartheta), & \text{if } \vartheta \geq -s \\ h(s+\vartheta)-\alpha x'(0), & \text{if } \vartheta < -s. \end{cases} \qquad (2.10\beta)$$

Here, G is the Green function for the B.V.P.

$$y'' = 0$$
$$y(0)+\alpha y'(0) = 0, \ y(T)+\beta y'(T) = 0$$

and is given by the formula

$$G(t,s) = \frac{1}{\ell}\begin{cases} (t-T-\beta)(s-\alpha), & s \leq T \\ (t-\alpha)(s-T-\beta), & t \leq s, \end{cases}$$

where $\ell = T+\beta-\alpha \neq 0$ because of (1.2c).

Obviously, the operator S is a compact operator defined on B and taking values in B.

Since the B.V.P. (1.1)-(1.2) is equivalent to (2.10α) and (2.10β), the purpose of the following proof is to show that the mapping S has a fixed point.

To this end we define an operator $A : B \to B$, and a subset Ω of B as follows:

$$(Ax)(t) = -\int_0^T G(t,s)k\,x(t)dt, \ k \neq 0 \qquad (2.11)$$

and

$$\Omega = \{x \in B : \forall t \in [0,T], \ |x(t)| < \varphi(t), \ |x'(t)| < K+1\}, \qquad (2.12)$$

where k and K are defined as above.

It is clear that Ω is open and bounded in B and A is a completely continuous operator

First we prove that the operator I-A is one to one. Let $(I-A)x = (I-A)y$. If $z(t) = x(t)-y(t)$ then $(I-A)z = 0$ and $z(0)+\alpha z'(0) = 0$, $z(T)+\beta z'(T) = 0$. Hence, z is a solution of the B.V.P.

$$z''(t) = k\,z(t)$$
$$z(0)+\alpha z'(0) = 0$$
$$z(T)+\beta z'(T) = 0.$$

By Lemma 2.3 the last problem has the unique solution $z=0$, and consequently I-A is one to one.

Next, we show that for any $\lambda \in [0,1]$ and $x \in \partial\Omega$ it is the case that

$$x \neq \lambda Sx+(1-\lambda)Ax$$

Indeed, if there exists $\lambda \in [0,1]$ and $x \in \partial\Omega$ satisfying

$$x = \lambda Sx+(1-\lambda)Ax,$$

then the equation

$$x''(t)+\lambda f(t,x_t,x'(t)) = (1-\lambda)kx(t),$$

has a solution $x : [0,T] \to \mathbb{R}^n$ satisfying

$$x_0+\alpha x'(0) = h$$

$$x(T)+\beta x'(T) = \eta \tag{2.13α}$$

$$x \in \bar{\Omega}. \tag{2.13β}$$

Hence there exist $\xi, r \in [0,T]$ such that either

$$|x(\xi)| = \varphi(\xi) \text{ or } |x'(r)| = K+1. \tag{2.14}$$

Now, we shall prove that, in view of (2.13α), (2.13β), the relations in (2.14) cannot hold. Since x is a solution of (2.9) for some $\lambda \in [0,1]$, the computation following (2.9) show that $|x'(t)| \leq K$ and hence $|x'(t)| < K+1$, $0 \leq t \leq T$. Hence, the second case in (2.14) cannot hold. Thus it remains to eliminate the first possibility of (2.14). We shall prove that if $x \in \partial\Omega$ is a solution of (2.9), then there exists no $\xi \in [0,T]$ such that $|x(t)|^2-\varphi^2(t)$ reaches maximum value zero at $t = \xi \in [0,T]$.

Assume the contrary. Then, if $\xi \in (0,T)$, we have the following relations

$$|x(\xi)| = \varphi(\xi) \tag{2.15}$$

$$<x(\xi),x'(\xi)> = \varphi(\xi)\varphi'(\xi) \tag{2.16α}$$

$$<x_\xi(0),x'(\xi)> = \varphi(\xi)\varphi'(\xi)$$

or

$$<x_\xi(0),x'(\xi)> = \varphi(\xi)\varphi'(\xi) \tag{2.16β}$$

$$J \equiv <x_\xi(0),x''(\xi)>+|x'(\xi)|^2-\varphi(\xi)\varphi''(\xi)-\varphi'^2(\xi) \leq 0. \tag{2.17}$$

Now assume that x is a solution of (2.9). Then by (2.6), (2.15), (2.16β) we obtain

$$\begin{aligned} J &= -\lambda<x_\xi(0),f(t,x_\xi,x'(\xi))>+(1-\lambda)k|x(\xi)|^2+|x'(\xi)|^2-\varphi(\xi)\varphi''(\xi)-\varphi'^2(\xi) \\ &\geq (1-\lambda)\{|x'(\lambda)|^2-\varphi'^2(\xi)-\varphi(\xi)\varphi''(\xi)+k|x(\xi)|^2\} \\ &\geq (1-\lambda)\varphi(\xi)\{k\varphi(\xi)-\varphi''(\xi)\}. \end{aligned}$$

Since $k > \frac{\varphi''(t)}{\varphi(t)}$, $t \in (0,T)$, we get $J > 0$, $\lambda \in [0,1]$, contradicting (2.17).

Next we show that $\xi \neq T$. If $\xi = T$ and $g(t) = |x(t)|^2-\varphi^2(t)$ then the following must hold:

$$g'(T) = 2<x(T),x'(T)>-2\varphi(T)\varphi'(T) \geq 0$$

and

$$g(T) = 0.$$

Then $|x(T)| = \varphi(T)$ and $\varphi'(T) \leq |x'(T)|$. But, by the boundary condition (1.2b), we have

$$|\beta||x'(T)| \leq |\eta|+\varphi(T).$$

Hence

$$|\beta| \varphi'(T) \leq |\eta|+\varphi(T), \text{ if } \beta \neq 0$$

or

$$\varphi(T) \leq |\eta|, \text{ if } \beta = 0$$

which contradicts (2.5β). Therefore $\xi \neq T$ as required.

Finally, we show that $\xi \neq 0$. Assume on the contrary that $\xi = 0$. It is straightforward to see that

$$g(0) = 0 \text{ and } g'(0) \leq 0,$$

imply

$$|x(0)| = \varphi(0) \text{ and } -|x'(0)| \leq \varphi'(0)$$

From the boundary condition (1.2a) we obtain

$$-\varphi(0) \leq |h(0)|+|\alpha| \varphi'(0) \text{ , if } \alpha \neq 0$$

or

$$\varphi(0) \leq |h(0)|, \text{ if } \alpha = 0,$$

contradicting (2.5α).

Consequently, no solutions of (2.9) can belong to $\partial\Omega$ for $\lambda \in [0,1)$, completing the proof of the theorem.

3. APPLICATIONS

As an application of the Theorem we consider the equation

$$x''(t)+\ell(t,x_t)x'(t)+p(t,x_t)x(t)+q(t,x_t) = 0, \quad t \in [0,T] \tag{3.1}$$

where ℓ and p are bounded real valued functions defined on $[0,T]\times C_r$ and q is also bounded $\mathbb{R}^n$-valued function defined on $[0,T]\times C_r$.

We set

$$\tilde{\ell} = \sup_{(t,u)\in[0,T]\times C_r} |\ell(t,u)|, \quad \tilde{p} = \sup_{(t,u)\in[0,T]\times C_r} |p(t,u)|, \quad \tilde{q} = \sup_{(t,u)\in[0,T]\times C_r} |q(t,u)|.$$

Then we have the following

PROPOSITION. If there exists a constant M,

$$M \geq \max\{\tilde{\ell},\tilde{p},\tilde{q}\}$$

such that the inequality

$$\varphi''(t)+M[|\varphi'(t)|+\varphi(t)+1] \leq 0, \quad t \in [0,T] \tag{3.2}$$

has a strictly positive solution φ, subject to the conditions (2.5α), (2,5β), then the B.V.P. (3.1)-(1.2) has at least one solution satisfying

$$|x(t)| \leq \varphi(t), \quad t \in [0,T].$$

Moreover, there exists ρ not depending on x with

$$|x'(t)| \leq \rho, \quad t \in [0,T].$$

PROOF. It is enough to check the conditions of the theorem for the function

$$f(t,u,v) = \ell(t,u)v+p(t,u)u(0)+q(t,u), (t,u,v) \in [0,T]\times C_r\times\mathbb{R}^n .$$

Indeed, for any $(t,u,v) \in [0,T]\times C_r\times\mathbb{R}^n$, with $|u(0)| = \varphi(t)$ and $\langle u(0),v\rangle = |u(0)|\varphi'(t)$, we obtain

$$\langle u(0),f(t,u,v)\rangle = \ell(t,u)\langle u(0),v\rangle+p(t,u)|u(0)|^2+\langle u(0),q(t,u)\rangle$$

$$\leq |\ell(t,u)||u(0)||\varphi'(t)+p(t,u)|u(0)|^2+|u(0)||q(t,u)|$$
$$=|\ell(t,u)|\varphi(t)\varphi'(t)+p(t,u)\varphi^2(t)+\varphi(t)|q(t,u)|$$
$$\leq \tilde{\ell}\varphi(t)|\varphi'(t)|+\tilde{p}\varphi^2(t)+\tilde{q}\varphi(t)$$
$$\leq M\varphi(t)(|\varphi'(t)|+\varphi(t)+1).$$

In view of (3.2), the above relation shows that (2.6) holds.

Also, for any $(t,u,v)\in[0,T]\times C_r\times \mathbb{R}^n$ with $|u(0)|\leq\varphi(t)$ we get, obviously,

$$<u(0),f(t,u,v)>\leq \tilde{\ell}\varphi(t)|v|+\tilde{p}\varphi^2(t)+\tilde{q}\varphi(t)$$
$$\leq c_1+c_2|v|,$$

where $c_1=\sup_{t\in[0,T]}(\tilde{p}\varphi^2(t)+\tilde{q}\varphi(t))$ and $c_2=\sup_{t\in[0,T]}(\tilde{\ell}\varphi(t))$.

Moreover,

$$<v,f(t,u,v)>\leq \tilde{\ell}|v|^2+\tilde{p}|v|\varphi(t)+\tilde{q}|v|$$
$$\leq c_1'|v|+\tilde{\ell}|v|^2,$$

where $c_1'=\sup_{t\in[0,T]}(\tilde{p}\varphi(t)+\tilde{q})$. Now, if $|v|\geq 1$ then we have $c_1'|v|+\tilde{\ell}|v|^2\leq(c_1'+\tilde{\ell}|v|^2)|v|$. If $|v|<1$ then (2.8) follows from the inequality

$$\tilde{\ell}\geq\tilde{\ell}|v|-\ell_1|v|^2, \text{ for each } \ell_1\geq 0.$$

Indeed, we have

$$c_1'+\tilde{\ell}|v|=c_1'+\ell_1|v|^2+\tilde{\ell}|v|-\ell_1|v|^2\leq c_1'+\ell_1|v|^2+\tilde{\ell}.$$

Hence (2.8) is satisfied for $k_1'=\ell_1$ and $k_2'=c_1'+\tilde{\ell}$.

EXAMPLE. The B.V.P.

$$x''(t)+\frac{x(t)}{1+\|x_t\|}x'(t)=0,\quad t\in[0,1]$$
$$x_0=h$$
$$x(1)+\beta x'(1)=\eta$$

has at least one solution x such that

$$|x(t)|\leq 2-e^{-t}$$

provided that function h and constants β and η are such that,

$$|h(0)|<1 \text{ and } |\beta|+1>e(2+|\eta|).$$

We remark that in this case $\tilde{\ell}=1$ (and hence $M=1$) and (3.2) becomes $\varphi''(t)+|\varphi'(t)|\leq 0$, $t\in[0,1]$.

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