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ON AN ISOLATION AND A GENERALIZATION OF HÖLDER'S INEQUALITY

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ABSTRACT. We generalize the well-known Hölder inequality and give a condition at which the equality holds.

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Let us consider the famous Hölder inequality [1]

$$\sum_{i=1}^m \prod_{j=1}^n X_{ij} \leq \prod_{j=1}^n \left(\sum_{i=1}^m X_{ij}^{p_j} \right)^{1/p_j}, \quad (1)$$

where $X_{ij} > 0$ ($1 \leq i \leq m$, $1 \leq j \leq n$), $p_j > 1$, $\sum_{j=1}^n (1/p_j) = 1$.

We give a generalization of (1), first, we need the following lemmas.

LEMMA 1. Let $y_i = \prod_{j=1}^n X_{ij}$, $i = 1, 2, \dots, m$, then $\sum_{k=1}^n 1/p_k (\sum_{i=1}^m y_i \ln z_{ik}) = 0$, where $z_{ik} = X_{ik}^{p_k} / y_i$, $i = 1, 2, \dots, m$.

PROOF. Since $y_i = \prod_{j=1}^n X_{ij}$, we have $\ln y_i = \sum_{j=1}^n \ln X_{ij}$ and

$$\begin{aligned} \sum_{k=1}^n \frac{1}{p_k} \ln z_{ik} &= \sum_{k=1}^n \frac{1}{p_k} \ln \left(\frac{X_{ik}^{p_k}}{y_i} \right) = \sum_{k=1}^n \frac{1}{p_k} (p_k \ln X_{ik} - \ln y_i) \\ &= \sum_{k=1}^n \ln X_{ik} - \sum_{k=1}^n \frac{1}{p_k} \ln y_i = \ln \prod_{k=1}^n X_{ik} - \ln y_i = 0, \quad i = 1, 2, \dots, m \end{aligned} \quad (2)$$

here we have used the fact $\sum_{k=1}^n (1/p_k) = 1$, from the above calculations we obtain

$$\left(\sum_{i=1}^m y_i \right) \left(\sum_{k=1}^n \frac{1}{p_k} \ln z_{ik} \right) = \sum_{k=1}^n \frac{1}{p_k} \left(\sum_{i=1}^m y_i \ln z_{ik} \right) = 0. \quad (3)$$

□

LEMMA 2. If a, b are positive numbers, then

$$\begin{aligned} (\ln a - \ln b)(a^t - b^t) &\geq 0, \quad \text{if } t \geq 0, \\ (\ln a - \ln b)(a^t - b^t) &\leq 0, \quad \text{if } t \leq 0, \end{aligned} \quad (4)$$

and the equalities hold if and only if $(a - b)t = 0$.

PROOF. If $a = b$, (4) holds, let $a \neq b$, by the mean value theorem $f(a) - f(b) = (a - b)f'(z)$ with $z \in (a, b)$, we have

$$\begin{aligned} \ln a - \ln b &= (a - b) \frac{1}{z_1}, \quad z_1 \in (a, b), \\ (a^t - b^t) &= (a - b)t z_2^{t-1}, \quad z_2 \in (a, b) \end{aligned} \quad (5)$$

hence

$$(\ln a - \ln b)(a^t - b^t) = \frac{(a - b)^2 t z_2^{t-1}}{z_1} \begin{cases} \geq 0, & \text{if } t \geq 0 \\ \leq 0, & \text{if } t \leq 0 \end{cases} \quad (6)$$

and the equality holds if and only if $t = 0$. \square

The main result of our paper is the following theorem.

THEOREM 3. If $X_{ij} > 0$ ($1 \leq i \leq m$, $1 \leq j \leq n$), $p_i > 1$, $\sum_{i=1}^n (1/p_i) = 1$, let

$$h(t) = \prod_{k=1}^n \left[\sum_{i=1}^m y_i \left(\frac{X_{ik}^{p_k}}{y_i} \right)^t \right]^{1/p_k} = \prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n X_{ij} \right)^{(1-t)} (X_{ik}^{p_k})^t \right]^{1/p_k}. \quad (7)$$

Then

$$\begin{aligned} h'(t) &\geq 0, \quad \text{if } t \geq 0, \\ h'(t) &\leq 0, \quad \text{if } t \leq 0, \end{aligned} \quad (8)$$

the equality holds if and only if $t = 0$, or $X_{ik}^{p_k} / \prod_{\tau=1}^n X_{i\tau} = X_{jk}^{p_k} / \prod_{\tau=1}^n X_{j\tau}$ for $1 \leq i, j \leq m$, $k = 1, 2, \dots, n$, that is, $X_{1k}^{p_k} / \prod_{j=1}^n X_{1j} = X_{2k}^{p_k} / \prod_{j=1}^n X_{2j} = \dots = X_{mk}^{p_k} / \prod_{j=1}^n X_{mj}$.

PROOF. Let $H(t) = \ln h(t)$, $t \in \mathbb{R}$, then by Lemmas 1 and 2 we obtain

$$\begin{aligned} H'(t) &= \frac{h'(t)}{h(t)} = \sum_{k=1}^n \frac{1}{p_k} \frac{(\sum_{i=1}^m y_i z_{ik}^t \ln z_{ik})}{(\sum_{i=1}^m y_i z_{ik}^t)} \\ &= \sum_{k=1}^n \frac{1}{p_k} \frac{(\sum_{i=1}^m y_i z_{ik}^t \ln z_{ik})}{(\sum_{i=1}^m y_i z_{ik}^t)} - \sum_{k=1}^n \frac{1}{p_k} \frac{(\sum_{i=1}^m y_i \ln z_{ik})}{(\sum_{i=1}^m y_i)} \\ &= \sum_{k=1}^n \frac{1}{p_k} \frac{\sum_{1 \leq i < j \leq m} y_i y_j (\ln z_{ik} - \ln z_{jk})(z_{ik}^t - z_{jk}^t)}{(\sum_{i=1}^m y_i)(\sum_{i=1}^m y_i z_{ik}^t)} \begin{cases} \geq 0, & \text{if } t \geq 0, \\ \leq 0, & \text{if } t \leq 0. \end{cases} \end{aligned} \quad (9)$$

The equality holds if and only if $t = 0$ or

$$z_{1k} = z_{2k} = \dots = z_{mk}, \quad k = 1, 2, \dots, n, \quad (10)$$

where $y_i = \prod_{j=1}^n X_{ij}$, $z_{ik} = X_{ik}^{p_k} / \prod_{j=1}^n X_{ij}$, $i = 1, 2, \dots, m$, $k = 1, 2, \dots, n$. This completes the proof. \square

COROLLARY 4. Let $t_2 < t_1 \leq 0$ or $0 \leq t_1 < t_2$, we have

$$\prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n X_{ij} \right)^{1-t_1} X_{ik}^{p_k t_1} \right]^{1/p_k} \leq \prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n X_{ij} \right)^{1-t_2} X_{ik}^{p_k t_2} \right]^{1/p_k}, \quad (1')$$

equality holds if and only if for any $k = 1, 2, \dots, n$,

$$\frac{X_{ik}^{p_k}}{\prod_{\tau=1}^n X_{i\tau}} = \frac{X_{jk}^{p_k}}{\prod_{\tau=1}^n X_{j\tau}}, \quad \text{for } 1 \leq i \leq j \leq m, \quad (2')$$

and for all $0 < t < 1$,

$$\sum_{i=1}^m \prod_{j=1}^n X_{ij} \leq \prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n X_{ij} \right)^{1-t} X_{ik}^{p_k t} \right]^{1/p_k} \leq \prod_{i=1}^n \left(\sum_{j=1}^m X_{ij}^{p_j} \right)^{1/p_j} \quad (3')$$

equality holds only (2') holds.

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