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Research Article

Convergence Analysis of Generalized Jacobi-Galerkin Methods for Second Kind Volterra Integral Equations with Weakly Singular Kernels

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We develop a generalized Jacobi-Galerkin method for second kind Volterra integral equations with weakly singular kernels. In this method, we first introduce some known singular nonpolynomial functions in the approximation space of the conventional Jacobi-Galerkin method. Secondly, we use the Gauss-Jacobi quadrature rules to approximate the integral term in the resulting equation so as to obtain high-order accuracy for the approximation. Then, we establish that the approximate equation has a unique solution and the approximate solution arrives at an optimal convergence order. One numerical example is presented to demonstrate the effectiveness of the proposed method.

1. Introduction

In this paper we present a generalized Jacobi-Galerkin method for solving Volterra integral equations of second kind with weakly singular kernels. Specifically, for a given function $K \in C(I^2)$ with $I := [-1, 1]$ and a parameter $\mu \in (0, 1)$, we define a Volterra integral operator $\mathscr{K} : C(I) \to C(I)$ by

$$(\mathscr{K}\phi)(x) := \int_{-1}^{x} (x-t)^{-\mu} K(x,t)\,\phi(t)\,dt, \quad x \in I, \quad (1)$$

and then consider the Volterra integral equation of the form

$$u + \mathscr{K}u = f, \quad (2)$$

where $f \in C(I)$ is a given function and $u \in C(I)$ is the unknown to be determined.

In view of the singularity of the kernel function in the operator $\mathscr{K}$, the solution u of (2) exhibits a singularity at the point -1 in its derivative even if the forcing term f is a smooth function. There are many numerical attempts based on the spline approximation to overcome the difficulty caused by the singularity of the solution of (2) (see [1–8]). Recently, spectral methods using Jacobi polynomial basis have received considerable attention to approximating the solution of integral equations due to their high accuracy and easy implementation (see [9–17]). In particular, Chen and Tang in [11] proposed a Jacobi-collocation spectral method for second kind Volterra integral equations with weakly singular kernels. Some function transformations and variable transformations are employed to change the equation into new Volterra integral equations possessing better regularity so that the orthogonal polynomial theory can be applied accordingly. In [12], they proposed a spectral Jacobi-Galerkin approach for solving (2). A rigorous error estimate was given in both the infinite norm and the weighted square norm. As far as I am concerned, all existing spectral methods always either suppose that the original equation has a sufficiently smooth solution or convert the equation into a new one with a solution of better regularity than that of the original equation (2) so that the spectral method can be applied. It goes without saying that the function transformation makes the resulting equations and approximations more complicated, which leads us to consider the generalized spectral method involved.

We organize this paper as follows. In Section 2, we develop a generalized Jacobi-Galerkin method for solving (2) and then show the stability and convergence of this algorithm.

For the semidiscrete system proposed in previous section, we construct the efficient numerical integration scheme so as to obtain the fully discrete linear system. In Sections 3 and 4, we give a few technical results and analyze the stability and convergence analysis, respectively. In Section 5, one numerical example is presented to illustrate the efficiency and accuracy of this method. In addition, a conclusion is drawn.

2. A Generalized Jacobi-Galerkin Method

In this section, we first introduce some index sets: $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ with $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_r := \mathbb{Z}_r^+ \cup \{0\}$ with $\mathbb{Z}_r^+ := \{1, 2, \ldots, r\}$ for $r \in \mathbb{N}$. We let $w^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, be a Jacobi weight function and let $L^2_{w^{\alpha,\beta}}(I)$ denote the space of measurable functions whose square is Lebesgue integrable in I relative to the Jacobi weight function $w^{\alpha,\beta}$. The inner product and norm of this space are given by

$$(\phi, \psi)_{w^{\alpha,\beta}} := \int_I w^{\alpha,\beta}(x)\,\phi(x)\,\psi(x)\,dx, \qquad \|\phi\|_{w^{\alpha,\beta}} := (\phi, \phi)^{1/2}_{w^{\alpha,\beta}}. \tag{3}$$

For $n \in \mathbb{N}_0$, we let $J_n^{\alpha,\beta}$ be the Jacobi orthonormal polynomial of degree n relative to the weight function $w^{\alpha,\beta}$.

The following result regarding the regularity of the solution of (2) comes from [6].

Theorem 1. *Suppose that $K \in C^m(I^2)$ with $m \in \mathbb{N}$. Then the original equation (14) has a unique solution $u \in C(I)$. Moreover, if the function f is expressed as*

$$f(x) = \sum_{(i,j)\in(i,j)_m^\mu} e_{i,j}(\mu)(1+x)^{i+j(1-\mu)} + h(x), \quad x \in I, \tag{4}$$

where $h \in C^m(I)$, then the solution u can be written in the form

$$u(x) = \sum_{(i,j)\in(i,j)_m^\mu} c_{i,j}(\mu)(1+x)^{i+j(1-\mu)} + v(x), \quad x \in I. \tag{5}$$

Here $(i,j)_m^\mu := \{(i,j) : i, j \in \mathbb{N}_0,\ i + j(1-\mu) < m\}$ and $h, v \in C^m(I)$, and the coefficients $c_{i,j}(\mu)$ and $f_{i,j}$ are some constants.

Now we define an index set $\mathbb{W}$ by

$$\mathbb{W} := \left\{\lambda_k := i + j(1-\mu) < m : \lambda_k \in (i,j)_m^\mu,\ \lambda_k \notin \mathbb{N}\right\} \tag{6}$$

and suppose that L is the cardinality of the set $\mathbb{W}$, and then we define a nonpolynomial function set W by

$$W := \operatorname{span}\left\{w_k(x) : w_k(x) := (1+x)^{\lambda_k},\ \lambda_k \in \mathbb{W},\ x \in I\right\}. \tag{7}$$

It follows from the notations above that Theorem 1 is rewritten as follows.

Corollary 2. *Suppose that the kernel function $K \in C^m(I^2)$. If there exist some constants g_i, $i \in \mathbb{Z}_L^+$, such that*

$$f(x) := g(x) + h(x), \qquad g(x) := \sum_{i\in\mathbb{Z}_L^+} g_i w_i(x), \qquad x \in I, \tag{8}$$

where $h \in C^m(I)$, then there exist some constants c_i, $i \in \mathbb{Z}_L^+$, such that the solution u has the similar decomposition

$$u(x) = w(x) + v(x), \qquad w(x) := \sum_{i\in\mathbb{Z}_L^+} c_i w_i(x), \qquad x \in I, \tag{9}$$

where $v \in C^m(I)$.

Now we introduce another finite dimensional space Y_n given by

$$Y_n := \operatorname{span}\left\{J_i^{\alpha,\beta} : i \in \mathbb{Z}_n\right\}, \tag{10}$$

and then let

$$X_n := W \oplus Y_n. \tag{11}$$

The generalized spectral Galerkin method for solving (2) is to seek a vector $\mathbf{u}_n := [a_{1,1}, \ldots, a_{L,1}, a_{0,2}, \ldots, a_{n,2}]^T$ such that

$$u_n(x) := \sum_{i\in\mathbb{Z}_L^+} a_{i,1} w_i(x) + \sum_{i\in\mathbb{Z}_n} a_{i,2} J_i^{\alpha,\beta}(x), \quad x \in I, \tag{12}$$

satisfying the equation

$$\left((\mathcal{I} + \mathcal{K})\,u_n, \phi\right)_{w^{\alpha,\beta}} = (f, \phi)_{w^{\alpha,\beta}}, \quad \text{for } \phi \in X_n. \tag{13}$$

If we use $\mathcal{P}_n$ which is the orthogonal projection operator from $L^2_{w^{\alpha,\beta}}(I)$ to X_n, then the equation mentioned above has the operator form

$$(\mathcal{I} + \mathcal{P}_n\mathcal{K})\,u_n = \mathcal{P}_n f. \tag{14}$$

By expression (8), $\mathcal{P}_n f$ can be written as

$$\mathcal{P}_n f = g + \mathcal{P}_n h. \tag{15}$$

The conventional Jacobi-Galerkin method is to choose Y_n as the approximation space and the test space, but when the original solution has a singularity, the approximation solution suffers from possessing lower-order accuracy. In order to overcome this difficulty, the same as in [7], we include the set W of the known nonpolynomial functions reflecting the singularity of the original solution in the usual Jacobi-Galerkin approximation space Y_n. Hence, we call this method the generalized spectral Galerkin method.

Next we are going to analyze this generalized Chebyshev-Galerkin method. We first show the stability of the original operator $\mathcal{I} + \mathcal{K} : L^2_{w^{\alpha,\beta}}(I) \to L^2_{w^{\alpha,\beta}}(I)$.

Theorem 3. *Suppose that $K \in C^m(I^2)$; then there exists a positive constant ρ such that for $\phi \in L^2_{w^{\alpha,\beta}}(I)$*

$$\left\|(\mathcal{I}+\mathcal{K})\phi\right\|_{w^{\alpha,\beta}} \geq \rho\left\|\phi\right\|_{w^{\alpha,\beta}}. \quad (16)$$

Proof. First, it follows from $K \in C(I^2)$ and Theorems 2.21 and 4.12 in [18] that the integral operator $\mathcal{K}: L^2_{w^{\alpha,\beta}}(I) \to L^2_{w^{\alpha,\beta}}(I)$ is compact. On the other hand, by the fact that -1 is not the eigenvalue of the integral operator $\mathcal{K}$ we conclude that $\mathcal{I}+\mathcal{K}$ is injective from $L^2_{w^{\alpha,\beta}}(I)$ into itself. Thus, using Theorem 3.4 in [18], the inverse operator $(\mathcal{I}+\mathcal{K})^{-1}: L^2_{w^{\alpha,\beta}}(I) \to L^2_{w^{\alpha,\beta}}(I)$ exists and is bounded. This completes result (16). □

On the other hand, for a single function, let $\mathcal{D}^i$ denote the ith generalized usual differential operator, and for any function of several variables, let ∂^i_x denote the ith partial generalized differential operator on the variable x. We introduce the nonuniformly weighted Sobolev space $H^r_{w^{\alpha,\beta}}(I)$, $r \in \mathbb{N}$, by

$$H^r_{w^{\alpha,\beta}}(I) := \left\{\phi : \mathcal{D}^i\phi \in L^2_{w^{\alpha+i,\beta+i}}(I),\ i \in \mathbb{Z}_r\right\}, \quad (17)$$

with the norm

$$\left\|\phi\right\|_{w^{\alpha,\beta},r} := \sum_{i\in\mathbb{Z}_r}\left\|\mathcal{D}^i\phi\right\|_{w^{\alpha+i,\beta+i}}. \quad (18)$$

If we use $\mathcal{Q}_n$ which is the orthogonal projection operator from $L^2_{w^{\alpha,\beta}}(I)$ to Y_n, it is clear that there holds

$$\mathcal{P}_n\mathcal{Q}_n = \mathcal{Q}_n. \quad (19)$$

Throughout the remainder of this paper, we use the symbol c to denote a positive constant which may take different values on different occurrences. Moreover, it follows from [19] that, for $\psi \in H^r_{w^{\alpha,\beta}}(I)$, there exists a positive constant c such that, for $i \in \mathbb{Z}_r$,

$$\left\|\mathcal{D}^i\left(\psi-\mathcal{Q}_n\psi\right)\right\|_{w^{\alpha,\beta}} \leq c\left\|\mathcal{D}^r\psi\right\|_{w^{\alpha+r,\beta+r}} n^{i-r}, \quad (20)$$

which implies that

$$\left\|\mathcal{D}^r\mathcal{Q}_n\psi\right\|_{w^{\alpha,\beta}} \leq c\left\|\mathcal{D}^r\psi\right\|_{w^{\alpha+r,\beta+r}}. \quad (21)$$

In particular, if the function ψ has the decomposition $\psi := \psi_1+\psi_2$ with $\psi_1 \in W$ and $\psi_2 \in H^r_{w^{\alpha,\beta}}(I)$, then using (20) yields that

$$\left\|\psi-\mathcal{P}_n\psi\right\|_{w^{\alpha,\beta}} \leq c\left\|\mathcal{D}^r\psi_2\right\|_{w^{\alpha+r,\beta+r}} n^{-r}. \quad (22)$$

In the following we consider the stability and convergence result regarding approximation equation (14).

Theorem 4. *If $K \in C^m(I^2)$, there exists a positive integer n_0 such that $n \geq n_0$ and for $\phi \in X_n$,*

$$\left\|(\mathcal{I}+\mathcal{P}_n\mathcal{K})\phi\right\|_{w^{\alpha,\beta}} \geq \frac{\rho}{2}\left\|\phi\right\|_{w^{\alpha,\beta}}, \quad (23)$$

where ρ appears in (16). Moreover, there exists a positive constant c such that

$$\left\|u-u_n\right\|_{w^{\alpha,\beta}} \leq cn^{-m}\left\|v\right\|_{w^{\alpha+m,\beta+m}}. \quad (24)$$

Proof. Since $\mathcal{K}$ is a compact operator from $L^2_{\alpha,\beta}(I)$ into itself and $\mathcal{P}_n\phi \to \phi$ for all $\phi \in L^2_{w^{\alpha,\beta}}(I)$ as n tends to ∞, we conclude that there exists a positive integer n_0 such that, for $n \geq n_0$ and for $v \in X_n$,

$$\left\|(\mathcal{K}-\mathcal{P}_n\mathcal{K})\phi\right\|_{w^{\alpha,\beta}} \leq \frac{\rho}{2}\left\|\phi\right\|_{w^{\alpha,\beta}}, \quad (25)$$

where it and (16) and the triangle inequality

$$\begin{aligned}\left\|(\mathcal{I}+\mathcal{P}_n\mathcal{K})\phi\right\|_{w^{\alpha,\beta}} \leq{}& \left\|(\mathcal{I}+\mathcal{K})\phi\right\|_{w^{\alpha,\beta}} \\ &- \left\|(\mathcal{K}-\mathcal{P}_n\mathcal{K})\phi\right\|_{w^{\alpha,\beta}}\end{aligned} \quad (26)$$

yield conclusion (23).

On the other hand, subtracting (2) from (14) obtains

$$u_n-u+\mathcal{P}_n\mathcal{K}u_n-\mathcal{K}u = \mathcal{P}_nf-f. \quad (27)$$

By applying the operator $\mathcal{P}_n$ to both sides of (2), we have

$$\mathcal{P}_nu+\mathcal{P}_n\mathcal{K}u = \mathcal{P}_nf. \quad (28)$$

Thus,

$$\mathcal{P}_nf-f = \mathcal{P}_n\mathcal{K}u-\mathcal{K}u+\mathcal{P}_nu-u. \quad (29)$$

A combination of (27) and (29) gives

$$u_n-u = \left(\mathcal{I}+\mathcal{P}_n\mathcal{K}\right)^{-1}\left(\mathcal{P}_nu-u\right), \quad (30)$$

where it and (16) imply that

$$\left\|u-u_n\right\|_{w^{\alpha,\beta}} \leq \frac{2}{\rho}\left\|u-\mathcal{P}_nu\right\|_{w^{\alpha,\beta}}. \quad (31)$$

Hence, by the solution expansion of (9) and (22) with $\psi := u$, $\psi_1 := w$, and $\psi_2 := v$, we conclude that

$$\left\|u-\mathcal{P}_nu\right\| \leq \gamma n^{-m}\left\|v\right\|_{w^{\alpha+m,\beta+m}}. \quad (32)$$

A combination of (31) and (32) presents the desired conclusion. □

In the remainder of the section, we write the matrix form of (14). To this end, for $i, j \in \mathbb{N}_0$, by introducing

$$\begin{aligned}
a_{i,j,1} &:= \left(w_i, w_j\right)_{w^{\alpha,\beta}}, \\
b_{i,j,1} &:= \left(w_i, \mathcal{K}w_j\right)_{w^{\alpha,\beta}}, \\
a_{i,j,2} &:= \left(w_i, J^{\alpha,\beta}_j\right)_{w^{\alpha,\beta}}, \\
b_{i,j,2} &:= \left(w_i, \mathcal{K}J^{\alpha,\beta}_j\right)_{w^{\alpha,\beta}}, \\
a_{i,j,3} &:= \left(J^{\alpha,\beta}_i, w_j\right)_{w^{\alpha,\beta}}, \\
b_{i,j,3} &:= \left(J^{\alpha,\beta}_i, \mathcal{K}w_j\right)_{w^{\alpha,\beta}}, \\
a_{i,j,4} &:= \left(J^{\alpha,\beta}_i, J^{\alpha,\beta}_j\right)_{w^{\alpha,\beta}}, \\
b_{i,j,4} &:= \left(J^{\alpha,\beta}_i, \mathcal{K}J^{\alpha,\beta}_j\right)_{w^{\alpha,\beta}},
\end{aligned} \quad (33)$$

we then define four block matrices by

$$\mathbf{A}' := \left[a_{i,j,1} : i, j \in \mathbb{Z}_L^+\right],$$
$$\mathbf{A}'' := \left[a_{i,j,2} : i \in \mathbb{Z}_L^+,\ j \in \mathbb{Z}_n\right],$$
$$\mathbf{A}''' := \left[a_{i,j,3} : i \in \mathbb{Z}_n, j \in \mathbb{Z}_{m-1}\right],$$
$$\mathbf{A}^* := \left[a_{i,j,4} : i, j \in \mathbb{Z}_n\right]. \quad (34)$$

It is clear that

$$\mathbf{A}^* = \mathbf{I}_{n+1},$$
$$\mathbf{A}''' = \mathbf{A}''^T. \quad (35)$$

Likewise, we define the matrices $\mathbf{B}'$, $\mathbf{B}''$, $\mathbf{B}'''$, and $\mathbf{B}^*$. Using these notations, we define $\mathbf{A}_n$ and $\mathbf{B}_n$ by

$$\mathbf{A}_n := \begin{bmatrix} \mathbf{A}' & \mathbf{A}'' \\ \mathbf{A}''^T & \mathbf{I}_{n+1} \end{bmatrix},$$
$$\mathbf{B}_n := \begin{bmatrix} \mathbf{B}' & \mathbf{B}'' \\ \mathbf{B}''' & \mathbf{B}^* \end{bmatrix}. \quad (36)$$

Associated with $\mathscr{P}_n f$, by letting

$$f_{i,1} := (w_i, g)_{w^{\alpha,\beta}} + (w_i, h)_{w^{\alpha,\beta}}, \quad i \in \mathbb{Z}_L^+,$$
$$f_{i,2} := \left(J_i^{\alpha,\beta}, g\right)_{w^{\alpha,\beta}} + \left(J_i^{\alpha,\beta}, h\right)_{w^{\alpha,\beta}}, \quad i \in \mathbb{Z}_n, \quad (37)$$

we define the vector $\mathbf{f}_n$ as

$$\mathbf{f}_n := \left[f_{1,1}, \ldots, f_{L,1}, f_{0,2}, \ldots, f_{0,n}\right]^T. \quad (38)$$

Thus using the matrices and vectors above, the matrix form of (14) is written as

$$\left(\mathbf{A}_n + \mathbf{B}_n\right) \mathbf{u}_n = \mathbf{f}_n. \quad (39)$$

In order to solve system (39) in previous section, the matrix entries of integral form in (39) must be computed. Hence, the main purpose of this section is going to approximate the integral operator and the inner product based on the Gauss-Jacobi quadrature rule. To this end, for $n \in \mathbb{N}$ and $i \in \mathbb{Z}_n^+$, we denote by $x_{i,n}^{\alpha,\beta}$ and $w_{i,n}^{\alpha,\beta}$ the set of n Jacobi-Gauss points and the corresponding weights relative to the weight function $w^{\alpha,\beta}$. We use the notation P_n to denote the set of all polynomials of degree not more than n. Moreover, the classical Gauss-Jacobi quadrature rule is given by

$$(\phi, 1)_{w^{\alpha,\beta}} = \sum_{i \in \mathbb{Z}_n^+} w_{i,n}^{\alpha,\beta} \phi\left(x_{i,n}^{\alpha,\beta}\right), \quad \phi \in P_{2n-1}. \quad (40)$$

Thus, upon relation (35) we only need to give the fully discrete form of $\mathbf{A}'$ and $\mathbf{A}''$. A direct computation using the Gauss-Jacobi quadrature rule (40) yields that

$$a_{i,j,1} = w_{1,1}^{\alpha,\beta+\lambda_i+\lambda_j}, \quad i, j \in \mathbb{Z}_L^+;$$
$$a_{i,j,2} := \sum_{k \in \mathbb{Z}_n^+} w_{k,n}^{\alpha,\beta+\lambda_i} J_j^{\alpha,\beta}\left(x_{k,n}^{\alpha,\beta+\lambda_i}\right), \quad i \in \mathbb{Z}_L^+,\ j \in \mathbb{Z}_n. \quad (41)$$

In order to give the fully discrete form of matrix $\mathbf{B}_n$, we first approximate the integral operator $\mathscr{K}$. For this purpose, for $x \in I$, we introduce a variable transformation

$$t := g(x, \tau) = \frac{x+1}{2}\tau + \frac{x-1}{2}, \quad x, \tau \in I, \quad (42)$$

which converts the interval $[-1, x]$ into I. Thus, the operator $\mathscr{K}$ has the following form:

$$(\mathscr{K}\phi)(x) = 2^{\mu-1} w^{0,1-\mu}(x) \cdot \int_I w^{-\mu,0}(\tau) K(x, g(x,\tau)) \phi(g(x,\tau)) \, d\tau, \quad x \in I. \quad (43)$$

In particular, when $\phi := w_i \in W$, then

$$(\mathscr{K} w_i)(x) = 2^{\mu-1-\lambda_i} w^{0,1-\mu+\lambda_i}(x) \int_I w^{-\mu,\lambda_i}(\tau) K(x, g(x,\tau)) \, d\tau, \quad x \in I. \quad (44)$$

Next we define an integral operator $\mathscr{G}^{\xi,\eta}$ by

$$\left(\mathscr{G}^{\xi,\eta}\phi\right)(x) := \int_I w^{\xi,\eta}(\tau) b(x, \tau; \phi) \, d\tau, \quad x \in I, \quad (45)$$

where

$$b(x, \tau; \phi) := \begin{cases} K(x, g(x,\tau)), & \phi = w_i \in W \\ K(x, g(x,\tau)) \phi(g(x,\tau)), & \text{otherwise} \end{cases} \quad x, \tau \in I. \quad (46)$$

Consequently, the integral operator $\mathscr{K}$ is rewritten as

$$(\mathscr{K}\phi)(x) = \begin{cases} 2^{\mu-1-\lambda_i} w^{0,1-\mu+\lambda_i}(x) \left(\mathscr{G}^{-\mu,\lambda_i} 1\right)(x), & w = w_i \in W, \\ 2^{\mu-1} w^{0,1-\mu}(x) \left(\mathscr{G}^{-\mu,0}\phi\right)(x), & \text{otherwise} \end{cases} \quad x \in I. \quad (47)$$

In order to discretize the operator $\mathscr{K}$, we first discretize the operator $\mathscr{G}^{\xi,\eta}$. To this end, for $\phi \in C(I)$, we define the Lagrange interpolation polynomial $\mathscr{L}_n^{\alpha,\beta}\phi \in P_n$ by

$$\left(\mathscr{L}_n^{\alpha,\beta}\phi\right)(x) := \sum_{i \in \mathbb{Z}_n^+} \phi\left(x_{i,n}^{\alpha,\beta}\right) L_{i,n}^{\alpha,\beta}(x), \quad x \in I, \quad (48)$$

where

$$L_{i,n}^{\alpha,\beta}(x) := \frac{\pi(x)}{\left(x - x_{i,n}^{\alpha,\beta}\right)\pi'\left(x_{i,n}^{\alpha,\beta}\right)},$$
$$\pi(x) := \left(x - x_{1,n}^{\alpha,\beta}\right)\left(x - x_{2,n}^{\alpha,\beta}\right) \cdots \left(x - x_{n,n}^{\alpha,\beta}\right), \quad x \in I. \quad (49)$$

Thus, for $N \in \mathbb{N}$, let $b_N(x,\tau)$ denote the N-degree interpolation polynomial of b about the variable τ relative to the weight $w^{\alpha,\beta}$; that is,

$$b_N(x,\tau;\phi) := \sum_{i\in\mathbb{Z}_N^+} b\left(x, x_{i,N}^{\alpha,\beta};\phi\right) L_{i,N}^{\alpha,\beta}(\tau), \quad x,\tau \in I, \tag{50}$$

and then replacing b by b_N in (45) yields the discrete form of $\mathscr{G}_N^{\xi,\eta}$ as follows:

$$\left(\mathscr{G}_N^{\xi,\eta}\phi\right)(x) := \int_I w^{\xi,\eta}(\tau)\, b_N(x,\tau;\phi)\, d\tau, \quad x \in I. \tag{51}$$

It follows from the Gauss-Jacobi quadrature rule (40) that

$$\begin{aligned}&\left(\mathscr{G}_N^{\xi,\eta}\phi\right)(x)\\ &\quad= \sum_{i\in\mathbb{Z}_N^+}\sum_{k\in\mathbb{Z}_N^+} w_{i,N}^{\xi,\eta} b\left(x, x_{k,N}^{\alpha,\beta};\phi\right) L_{k,N}^{\alpha,\beta}\left(x_{i,N}^{\xi,\eta}\right), \quad x \in I.\end{aligned} \tag{52}$$

Subsequently, we can obtain the fully discrete form $\mathscr{K}_N$ of the operator $\mathscr{K}$,

$$\begin{aligned}&(\mathscr{K}_N\phi)(x)\\ &\quad= \begin{cases} 2^{\mu-1-\lambda_i} w^{0,1-\mu+\lambda_i}(x)\left(\mathscr{G}_N^{-\mu,\lambda_i}1\right)(x), & w = w_i \in W,\\ 2^{\mu-1} w^{0,1-\mu}(x)\left(\mathscr{G}_N^{-\mu,0}\phi\right)(x), & \text{otherwise}\end{cases}\\ &\qquad x \in I.\end{aligned} \tag{53}$$

In order to approximate the inner product and easily analyze the stability and convergence of fully discrete equation, we define the operator $\widetilde{\mathscr{K}}_N$ by

$$\begin{aligned}&\left(\widetilde{\mathscr{K}}_N\phi\right)(x)\\ &\quad= \begin{cases} 2^{\mu-1-\lambda_i} w^{0,1-\mu+\lambda_i}(x)\left(\mathscr{L}_N^{\alpha,\beta+1-\mu}\mathscr{G}_N^{-\mu,\lambda_i}1\right)(x), & \phi = w_i \in W,\\ 2^{\mu-1} w^{0,1-\mu}(x)\left(\mathscr{L}_N^{\alpha,\beta+1-\mu}\mathscr{G}_N^{-\mu,0}\phi\right)(x), & \text{otherwise}\end{cases}\\ &\qquad x \in I.\end{aligned} \tag{54}$$

On the other hand, let

$$\widetilde{\mathscr{L}}_n^{\alpha,\beta} f := g + \mathscr{L}_n^{\alpha,\beta} h. \tag{55}$$

Using these notations above, we replace $\mathscr{K}$ and $\mathscr{P}_n f$ by $\widetilde{\mathscr{K}}_N$ and $\widetilde{\mathscr{L}}_n^{\alpha,\beta} f$, obtaining

$$\left(\mathscr{I} + \mathscr{P}_n\widetilde{\mathscr{K}}_N\right)\tilde{u}_n = \widetilde{\mathscr{L}}_n^{\alpha,\beta} f, \tag{56}$$

where $\tilde{u}_n$ is given by

$$\tilde{u}_n := \sum_{i\in\mathbb{Z}_L^+} \tilde{a}_{i,1} w_i + \sum_{i\in\mathbb{Z}_n} \tilde{a}_{i,2} J_i^{\alpha,\beta}. \tag{57}$$

In order to observe that (56) is a fully discrete form, we have to write its matrix. To this end, suppose $N \ge n+1$; replacing the operator $\mathscr{K}$ in (33) by the operator $\widetilde{\mathscr{K}}_N$ given in (54) and then using Gauss-Jacobi quadrature (40) produce that

$$\begin{aligned}\tilde{b}_{i,j,1} &:= 2^{\mu-1-\lambda_j} \sum_{k\in\mathbb{Z}_N^+}\sum_{l\in\mathbb{Z}_N^+} w_{k,N}^{\alpha,\beta+1-\mu+\lambda_i+\lambda_j}\\ &\quad\cdot\left(\left(\mathscr{G}_N^{-\mu,\lambda_i}1\right)\left(x_{l,N}^{\alpha,\beta+1-\mu}\right)\right) L_{l,N}^{\alpha,\beta+1-\mu}\left(x_{k,N}^{\alpha,\beta+1-\mu+\lambda_i+\lambda_j}\right),\\ \tilde{b}_{i,j,2} &:= 2^{\mu-1} \sum_{k\in\mathbb{Z}_N^+}\sum_{l\in\mathbb{Z}_N^+} w_{k,N}^{\alpha,\beta+1-\mu+\lambda_i}\\ &\quad\cdot\left(\left(\mathscr{G}_N^{-\mu,\lambda_i} J_j^{\alpha,\beta}\right)\left(x_{l,n}^{\alpha,\beta+1-\mu}\right)\right) L_{l,N}^{\alpha,\beta+1-\mu}\left(x_{k,N}^{\alpha,\beta+1-\mu+\lambda_i}\right),\\ \tilde{b}_{i,j,3} &:= 2^{\mu-1-\lambda_j} \sum_{k\in\mathbb{Z}_N^+}\sum_{l\in\mathbb{Z}_N^+} w_{k,N}^{\alpha,\beta+1-\mu+\lambda_j} J_i^{\alpha,\beta}\left(x_{k,N}^{\alpha,\beta+1-\mu+\lambda_j}\right)\\ &\quad\cdot\left(\left(\mathscr{G}_N^{-\mu,\lambda_j}1\right)\left(x_{l,n}^{\alpha,\beta+1-\mu}\right)\right) L_{l,N}^{\alpha,\beta+1-\mu}\left(x_{k,N}^{\alpha,\beta+1-\mu+\lambda_j}\right),\\ \tilde{b}_{i,j,4} &:= 2^{\mu-1} \sum_{k\in\mathbb{Z}_N^+} w_{k,N}^{\alpha,\beta+1-\mu} J_i^{\alpha,\beta}\left(x_{k,N}^{\alpha,\beta+1-\mu}\right)\\ &\quad\cdot\left(\left(\mathscr{G}_N^{-\mu,0} J_j^{\alpha,\beta}\right)\left(x_{k,N}^{\alpha,\beta+1-\mu}\right)\right).\end{aligned} \tag{58}$$

The same as before, we define four matrices $\widetilde{\mathbf{B}}'$, $\widetilde{\mathbf{B}}''$, $\widetilde{\mathbf{B}}'''$, and $\widetilde{\mathbf{B}}^*$ and then set

$$\widetilde{\mathbf{B}}_n := \begin{bmatrix} \widetilde{\mathbf{B}}' & \widetilde{\mathbf{B}}'' \\ \widetilde{\mathbf{B}}''' & \widetilde{\mathbf{B}}^* \end{bmatrix}. \tag{59}$$

On the other hand, replacing the function h in (37) by the interpolation polynomial $\mathscr{L}_N^{\alpha,\beta} h$ and then using Gauss-Jacobi quadrature (40) produce that

$$\begin{aligned}\tilde{f}_{i,1} &:= \sum_{j\in\mathbb{Z}_L^+} g_j w_{1,1}^{\alpha,\beta+\lambda_i+\lambda_j}\\ &\quad+ \sum_{j\in\mathbb{Z}_M^+}\sum_{k\in\mathbb{Z}_M^+} w_{j,M}^{\alpha,\beta+\lambda_i} h\left(x_{k,M}^{\alpha,\beta}\right) L_{k,M}^{\alpha,\beta}\left(w_{j,M}^{\alpha,\beta+\lambda_i}\right),\\ \tilde{f}_{i,2} &:= \sum_{j\in\mathbb{Z}_L^+}\sum_{k\in\mathbb{Z}_n^+} g_j w_{k,n}^{\alpha,\beta+\lambda_j} J_i^{\alpha,\beta}\left(x_{k,n}^{\alpha,\beta+\lambda_j}\right)\\ &\quad+ \sum_{k\in\mathbb{Z}_M^+} w_{k,M}^{\alpha,\beta} J_i^{\alpha,\beta}\left(x_{k,M}^{\alpha,\beta}\right) h\left(x_{k,M}^{\alpha,\beta}\right).\end{aligned} \tag{60}$$

Using the notations above, in a similar manner, we let

$$\tilde{\mathbf{f}}_n := \left[\tilde{f}_{1,1},\ldots,\tilde{f}_{L,1},\tilde{f}_{0,2},\ldots,\tilde{f}_{n,2}\right]^T. \tag{61}$$

Hence, we have the following matrix form of (56):

$$\left(\widetilde{\mathbf{A}}_n + \widetilde{\mathbf{B}}_n\right)\tilde{\mathbf{u}}_n = \tilde{\mathbf{f}}_n, \tag{62}$$

where

$$\tilde{\mathbf{u}}_n := \left[\tilde{a}_{1,1},\ldots,\tilde{a}_{L,i},\tilde{a}_{1,2},\ldots,\tilde{a}_{n,2}\right]^T. \tag{63}$$

3. Some Useful Results

In this section we are going to give some technical results so as to analyze the fully discrete equation (56).

Lemma 5. *Suppose the kernel function $K \in C^m(I^2)$. If $\phi \in C^m(I)$, then there exists a positive constant c such that, for $x, \tau \in I$,*

$$\left(\partial_\tau^m b(x,\tau;\phi)\right)^2 \le c w^{0,2m}(x) \sum_{i\in\mathbb{Z}_m} \left(\mathscr{D}_g^i \phi(g(x,\tau))\right)^2,$$
$$\left(\partial_x^m b(x,\tau;\phi)\right)^2 \le c \sum_{i\in\mathbb{Z}_m} w^{0,2i}(\tau) \left(\mathscr{D}_g^i \phi(g(x,\tau))\right)^2. \tag{64}$$

Proof. We only need to show the first inequality in (64) since the other is the same. In fact, a direct application of the high-order derivative formula to b yields that

$$\partial_\tau^m b(x,\tau;\phi) = \sum_{i\in\mathbb{Z}_m} C_m^i \partial_\tau^i v(g(x,\tau)) \partial_\tau^{m-i} K(x, g(x,\tau)), \quad x,\tau \in I, \tag{65}$$

where C_m^i is the binomial coefficient given by $C_m^i := m(m-1)\cdots(m-i+1)$. Clearly,

$$\partial_\tau^i v(g(x,\tau)) = 2^{-i} w^{0,i}(x) \mathscr{D}_g^i v(g(x,\tau)),$$
$$\partial_\tau^{m-i} K(x, g(x,\tau)) = 2^{i-m} w^{0,m-i}(x) \mathscr{D}_g^{m-i} K(x, g(x,\tau)). \tag{66}$$

Substituting the above result into the right hand side of (65) yields that

$$\partial_\tau^m b(x,\tau;\phi) = 2^{-m} w^{0,m}(x) \sum_{i\in\mathbb{Z}_m} C_m^i \mathscr{D}_g^i v(g(x,\tau)) \cdot \mathscr{D}_g^{m-i} K(x, g(x,\tau)), \quad x,\tau \in I. \tag{67}$$

Using the Cauchy-Schwartz inequality to the right hand side of (67) yields the desired conclusion with c being given by $c := (\sum_{i\in\mathbb{Z}_m} C_m^i)^2 \max_{i\in\mathbb{Z}_m} \|\partial_t^i K\|_\infty^2$. □

Now we give the difference between $\mathscr{G}^{-\mu,0}\phi$ and $\mathscr{G}^{-\mu,0}\phi$ for $\phi \in C^m(I)$. To this end, we introduce the result in [8]: for $\psi \in H_{w^{\alpha,\beta}}^r(I)$, there exists a positive constant c such that, for $i \in \mathbb{Z}_r$,

$$\left\|\mathscr{D}^i\left(\psi - \mathscr{L}_n^{\alpha,\beta}\psi\right)\right\|_{w^{\alpha,\beta}} \le c \left\|\mathscr{D}^r \psi\right\|_{w^{\alpha+r,\beta+r}} n^{i-r}. \tag{68}$$

Lemma 6. *Suppose the kernel function $K \in C^m(I^2)$. If three parameters α, β, and μ satisfy $-1 < \alpha < 1-2\mu$, $-1 < \beta < 1$, and $\alpha + \beta < 1 - 2\mu$, then there exists a positive constant c such that, for $\phi \in C^m(I)$,*

$$\left\|\mathscr{G}^{-\mu,0}\phi - \mathscr{G}_N^{-\mu,0}\phi\right\|_\infty \le cN^{-m} \|\phi\|_{w^{\alpha,\beta},m}. \tag{69}$$

Similarly, there also exists a positive constant c such that, for $i \in \mathbb{Z}_L^+$,

$$\left\|\mathscr{G}^{-\mu,\lambda_i} 1 - \mathscr{G}_N^{-\mu,\lambda_i} 1\right\|_\infty \le cN^{-m}. \tag{70}$$

Proof. We only prove the first result (69), and the other is the same. We first observe that

$$\left(\mathscr{G}^{-\mu,0}\phi\right)(x) - \left(\mathscr{G}_N^{-\mu,0}\phi\right)(x) = \int_I w^{-\mu,0}(\tau)\left((b(x,\tau;\phi) - b_N(x,\tau;\phi))\right) d\tau, \quad x \in I. \tag{71}$$

Associated with the above equation, we define $S_1(x)$ by the left hand side of (71) and then define S_2 and S_3 by

$$S_2(\tau) := w^{-\mu-\alpha/2,-\beta/2}(\tau),$$
$$S_3(\tau;x) := w^{\alpha/2,\beta/2}(\tau)\left(b(x,\tau;\phi) - b_N(x,\tau;\phi)\right). \tag{72}$$

It is clear that

$$S_1(x) = \int_I S_2(\tau) S_3(\tau) d\tau. \tag{73}$$

Applying Cauchy inequality to the right hand side of (73) produces that

$$S_1^2(x) \le \int_I S_2^2(\tau) d\tau \int_I S_3^2(\tau;x) d\tau. \tag{74}$$

It follows from the hypothesis that $\alpha + 2\mu < 1$ and $-1 < \beta < 1$ yield that

$$\int_I S_2^2(\tau) d\tau \le \|1\|_{w^{-\alpha-2\mu,-\beta}} < +\infty. \tag{75}$$

On the other hand, an application of (68) produces that

$$\int_I S_3^2(\tau;x) d\tau \le cN^{-2m} \left\|\partial_\tau^m b\right\|_{w^{\alpha+m,\beta+m}}^2. \tag{76}$$

It follows from the first estimation in (64) that

$$\left\|\partial_\tau^m b\right\|_{w^{\alpha+m,\beta+m}}^2 \le c w^{0,2m}(x) \sum_{i\in\mathbb{Z}_m} \left\|\mathscr{D}_g^i \phi\right\|_{w^{\alpha+m,\beta+m}}^2. \tag{77}$$

For $i \in \mathbb{Z}_m$, a direct observation for $\|\mathscr{D}_g^i\phi\|_{w^{\alpha+m,\beta+m}}^2$ using the condition that $\alpha + \beta < 1 - 2\mu$ can obtain that

$$w^{0,2m}(x) \left\|\mathscr{D}_g^i \phi\right\|_{w^{\alpha+m,\beta+m}}^2 \le c w^{0,2m-2i+1-\alpha-\beta-2\mu}(x) \sum_{i\in\mathbb{Z}_m} \left\|\mathscr{D}^i \phi\right\|_{w^{\alpha+i,\beta+i}}^2. \tag{78}$$

A combination of (73)–(78) yields the desired conclusion (69). □

Now we introduce the operator $\overline{\mathscr{K}}_N$ as

$$\left(\overline{\mathscr{K}}_N\phi\right)(x) = \begin{cases} 2^{\mu-1-\lambda_i} w^{0,1-\mu+\lambda_i}(x)\left(\mathscr{L}_N^{\alpha,\beta+2-2\mu+2\lambda_i}\mathscr{G}^{-\mu,\lambda_i}1\right)(x), & \phi = w_i,\ i \in \mathbb{Z}_L, \\ 2^{\mu-1} w^{0,1-\mu}(x)\left(\mathscr{L}_N^{\alpha,\beta+2-2\mu}\mathscr{G}^{-\mu,0}\phi\right)(x), & \text{otherwise} \end{cases} \quad x \in I, \tag{79}$$

and then we estimate the difference between $\mathscr{K}\phi$ and $\overline{\mathscr{K}}_N\phi$.

Lemma 7. *Suppose the kernel function $K \in C^m(I^2)$. If three parameters α, β, and μ satisfy $-1 < \alpha, \beta < 1 - 2\mu$, $\alpha + \beta < 1 - 2\mu$, then there exists a positive constant c such that, for $\phi \in C^m(I)$,*

$$\left\|\mathscr{K}\phi - \overline{\mathscr{K}}_N\phi\right\|_{w^{\alpha,\beta}} \le cN^{-m}\|\phi\|_{w^{\alpha,\beta},m}. \tag{80}$$

Similarly, there also exists a positive constant c such that, for $i \in \mathbb{Z}_L^+$,

$$\left\|\mathscr{K}w_i - \overline{\mathscr{K}}_N w_i\right\|_{w^{\alpha,\beta}} \le cN^{-m}\|w_i\|_{w^{\alpha,\beta},m}. \tag{81}$$

Proof. The same as Lemma 6, we only need to show that (80) holds. For $\phi \in C^m(I)$, by the definition of the operators $\mathscr{K}$ and $\overline{\mathscr{K}}_n$,

$$\begin{aligned}(\mathscr{K}\phi)(x) - \left(\overline{\mathscr{K}}_N\phi\right)(x) &= 2^{\mu-1} w^{0,1-\mu}(x)\left(\mathscr{G}^{-\mu,0}\phi\right)(x) \\ &\quad - \left(\left(\mathscr{L}_N^{\alpha,\beta+2-2\mu}\right)\mathscr{G}^{-\mu,0}\phi\right)(x), \quad x \in I.\end{aligned} \tag{82}$$

A direct estimation yields that

$$\begin{aligned}&\left\|\mathscr{K}\phi - \overline{\mathscr{K}}_N\phi\right\|_{w^{\alpha,\beta}}^2 \\ &\quad \le \left\|\mathscr{G}^{-\mu,0}\phi - \mathscr{L}_N^{\alpha,\beta+2-2\mu}\mathscr{G}^{-\mu,0}\phi\right\|_{w^{\alpha,\beta+2-2\mu}}^2,\end{aligned} \tag{83}$$

in which combining result (68) with $\psi \coloneqq \mathscr{G}^{-\mu,0}\phi$ implies that

$$\begin{aligned}&\left\|\mathscr{K}\phi - \overline{\mathscr{K}}_N\phi\right\|_{w^{\alpha,\beta}}^2 \\ &\quad \le cn^{-2m}\left\|\mathscr{D}^m\left(\mathscr{G}^{-\mu,0}\phi\right)\right\|_{w^{\alpha+m,\beta+2-2\mu+m}}^2.\end{aligned} \tag{84}$$

In the following we estimate $\|\mathscr{D}^m(\mathscr{G}^{-\mu,0}\phi)\|_{w^{\alpha+m,\beta+2-2\mu+m}}^2$. If we let

$$S(x) \coloneqq \int_I w^{-\mu,0}(\tau)\left(\partial_x^m b\right)(x,\tau;\phi)\,d\tau, \quad x \in I, \tag{85}$$

then

$$\begin{aligned}&\left\|\mathscr{D}^m\left(\mathscr{G}^{-\mu,0}\phi\right)\right\|_{w^{\alpha+m,\beta+2-2\mu+m}}^2 \\ &\quad = \int_I w^{\alpha+m,\beta+2-2\mu+m}(x)\,S^2(x)\,dx.\end{aligned} \tag{86}$$

By the assumption that $\alpha+\beta, \beta < 1-2\mu$, there exists a positive constant δ such that $\max\{-1, \alpha+\beta, \beta\} < \delta < 1 - 2\mu$. Thus we define two functions S_1 and S_2 by

$$\begin{aligned}S_1(\tau) &\coloneqq w^{-\mu-\alpha/2,-\delta/2}(\tau), \\ S_2(\tau, x; \phi) &\coloneqq w^{\alpha/2,\delta/2}(\tau)\left(\partial_x^m b\right)(x,\tau;\phi), \\ &\qquad \tau \in I.\end{aligned} \tag{87}$$

It is obvious that

$$S(x) = \int_I S_1(\tau)\,S_2(\tau, x; \phi)\,d\tau. \tag{88}$$

Applying Cauchy inequality to the right hand side of (88) yields that

$$S(x)^2 \le \int_I S_1^2(\tau)\,d\tau \int_I S_2(\tau, x; \phi)\,d\tau, \quad x \in I. \tag{89}$$

Clearly,

$$\int_I S_1^2(\tau)\,d\tau \le \|1\|_{w^{-\alpha-2\mu,-\delta}} < +\infty. \tag{90}$$

By using the second estimation in (64), we have

$$\int_I S_2^2(\tau)\,d\tau \le c\sum_{i\in\mathbb{Z}_m}\left\|\mathscr{D}_g^i\phi\right\|_{\alpha,\delta+2i}^2. \tag{91}$$

Substituting results (89)–(91) into (86) can obtain that

$$\left\|\mathscr{D}^m\left(\mathscr{G}^{-\mu,0}\phi\right)\right\|_{w^{\alpha+m,\beta+2-2\mu+m}}^2 \le c\sum_{i\in\mathbb{Z}_m} I_i, \tag{92}$$

where I_i is defined by

$$\begin{aligned}I_i \coloneqq &\int_I w^{\alpha,\delta+2i}(\tau)\,d\tau \\ &\cdot \int_I w^{\alpha+m,\beta+2-2\mu+m}(x)\left(\mathscr{D}_x^i\phi\left(g(x,\tau)\right)\right)^2 dx, \\ &\qquad i \in \mathbb{Z}_m.\end{aligned} \tag{93}$$

The left thing is to estimate I_i for $i \in \mathbb{Z}_m$. To this end, we let

$$\epsilon_i \coloneqq \begin{cases}\dfrac{\delta-\beta}{2}, & i = 0 \\ 0, & \text{otherwise.}\end{cases} \tag{94}$$

Then, for $i \in \mathbb{Z}_m$, a direct estimation for I_i yields that

$$I_i \le c \int_I w^{\alpha,\delta-\alpha-\beta-\epsilon_i-1}(\tau)\, d\tau \cdot \int_I w^{\alpha+i,\beta+i}(t) \left(\left(\mathscr{D}^i \phi\right)(t)\right)^2 dt, \tag{95}$$

which implies that

$$I_i \le c \left\| \mathscr{D}^i \phi \right\|^2_{w^{\alpha+i,\beta+i}}. \tag{96}$$

A consequence of (83), (92), and (96) produces the desired conclusion. □

The next result is concerned with the difference between $\mathscr{K}\phi$ and $\widetilde{\mathscr{K}}_N\phi$. To this end, we introduce the result proposed in [9–11]: for any $\psi \in C(I)$, there exists a positive constant c independent of n,

$$\left\| \mathscr{L}_n^{\alpha,\beta} \psi \right\|_{w^{\alpha,\beta}} \le c \|\psi\|_\infty. \tag{97}$$

Lemma 8. *Suppose the kernel function $K \in C^m(I^2)$. If three parameters α, β, and μ satisfy $-1 < \alpha, \beta < 1-2\mu, \alpha+\beta < 1-2\mu$, then there exists a positive constant c such that, for $\phi \in C^m(I)$,*

$$\left\| \mathscr{K}\phi - \widetilde{\mathscr{K}}_N \phi \right\|_{w^{\alpha,\beta}} \le c N^{-m} \|\phi\|_{w^{\alpha,\beta},m}. \tag{98}$$

Moreover, if $\phi := w_i$ for $i \in \mathbb{Z}_L^+$, then there exists a positive constant c such that

$$\left\| \mathscr{K} w_i - \widetilde{\mathscr{K}}_N w_i \right\|_{w^{\alpha,\beta}} \le c N^{-m}. \tag{99}$$

Proof. We only show that (98) holds, since the proof of the result (99) is the same. For $\phi \in C^m(I)$, using triangle inequality produces that

$$\left\| \mathscr{K}\phi - \widetilde{\mathscr{K}}_N \phi \right\|_{w^{\alpha,\beta}} \le \left\| \mathscr{K}\phi - \overline{\mathscr{K}}_N \phi \right\|_{w^{\alpha,\beta}} + \left\| \overline{\mathscr{K}}_N \phi - \widetilde{\mathscr{K}}_N \phi \right\|_{w^{\alpha,\beta}}. \tag{100}$$

It follows from result (80) in Lemma 7 that we only need to estimate the second term in the right hand side of (100). By definition of $\overline{\mathscr{K}}_N\phi$ and $\widetilde{\mathscr{K}}_N\phi$

$$\left\| \widehat{\mathscr{K}}_N \phi - \widetilde{\mathscr{K}}_N \phi \right\|_{w^{\alpha,\beta}} = 2^{\mu-1} \left\| \mathscr{L}_n^{\alpha,\beta+2-2\mu} \mathscr{G}^{-\mu,0} \phi - \mathscr{L}_n^{\alpha,\beta+2-2\mu} \mathscr{G}_N^{-\mu,0} \phi \right\|_{w^{\alpha,\beta+2-2\mu}}, \tag{101}$$

in which combining (97) produces that there exists a positive constant c such that

$$\left\| \widehat{\mathscr{K}}_N \phi - \widetilde{\mathscr{K}}_N \phi \right\|_{w^{\alpha,\beta}} \le c \left\| \mathscr{G}^{-\mu,0} \phi - \mathscr{G}_N^{-\mu,0} \phi \right\|_\infty. \tag{102}$$

This and (69) conclude the desired conclusion (98). □

As a consequence of Lemma 8, for $v \in P_n$, by using the inverse inequality relative to two norms weighted with different Jacobi weight functions in Theorem 3.31 in [19], we can easily obtain the following.

Corollary 9. *Suppose the conditions in Lemma 8 hold. Then there exists a positive constant c such that, for $\phi \in P_n$,*

$$\left\| \mathscr{K}\phi - \widetilde{\mathscr{K}}_N \phi \right\|_{w^{\alpha,\beta}} \le c \|\phi\|_{w^{\alpha,\beta}} \left(\frac{n}{N}\right)^m. \tag{103}$$

4. Convergence Analysis

In the section, we are going to analyze the convergence of the approximate solution of the fully discrete generalized Jacobi-Galerkin method. First we give the stability analysis of the operator $(\mathscr{I} + \mathscr{P}_n \widetilde{\mathscr{K}}_N)\phi : X_n \to X_n$.

Theorem 10. *Suppose the kernel function $K \in C^m(I^2)$. If three parameters α, β, and μ satisfy $-1 < \alpha, \beta < 1-2\mu, \alpha+\beta < 1-2\mu$ and if we choose N as*

$$N \ge N_{\min} := \left[n \log^{1/m} n \right] + 1, \tag{104}$$

then there exists a positive integer n_0 such that $n \ge n_0$ and for $\phi \in X_n$,

$$\left\| \left(\mathscr{I} + \mathscr{P}_n \widetilde{\mathscr{K}}_N\right) \phi \right\|_{w^{\alpha,\beta}} \ge \frac{\rho}{6} \|\phi\|_{w^{\alpha,\beta}}, \tag{105}$$

where ρ appears in (16).

Proof. For $\phi \in X_n$, there exists functions $\eta_i \in W$ for $i \in \mathbb{Z}_L^+$ and a polynomial function $\psi \in Y_n$ such that

$$\phi(x) := \sum_{i \in \mathbb{Z}_L^+} \eta_i(x) + \psi(x), \quad \eta_i(x) := c_i w_i(x) \;\; x \in I. \tag{106}$$

By using (99) and (104), there exists a positive constant c such that, for $i \in \mathbb{Z}_L^+$,

$$\left\| \mathscr{P}_n \mathscr{K} \eta_i - \mathscr{P}_n \widetilde{\mathscr{K}}_N \eta_i \right\|_{w^{\alpha,\beta}} \le c n^{-m} \|\eta_i\|_{w^{\alpha,\beta}}. \tag{107}$$

Because of the result

$$\lim_{n \to \infty} n^{-m} = 0, \tag{108}$$

we conclude that there exists a positive integer n_0 such that, for $n \ge n_0$,

$$\left\| \mathscr{P}_n \mathscr{K} \eta_i - \mathscr{P}_n \widetilde{\mathscr{K}}_N \eta_i \right\|_{w^{\alpha,\beta}} \le \frac{\rho}{6L} \|\eta_i\|_{w^{\alpha,\beta}}, \tag{109}$$

where L denotes the cardinality of the set $\mathbb{W}$ as in Section 2.

On the other hand, by estimation (103) in Corollary 9 and the choice of N in (104) there exists a positive constant c such that

$$\left\| \mathscr{P}_n \mathscr{K} \psi - \mathscr{P}_n \widetilde{\mathscr{K}}_N \psi \right\|_{w^{\alpha,\beta}} \le c \log^{-1} n \|\psi\|_{w^{\alpha,\beta}}. \tag{110}$$

It follows from the fact that $\log^{-1} n$ tends to zero as $n \to \infty$ that there exists a positive integer constant n_1 such that, for $n \ge n_1$,

$$\left\| \mathscr{P}_n \mathscr{K} \psi - \mathscr{P}_n \widetilde{\mathscr{K}}_N \psi \right\|_{w^{\alpha,\beta}} \le \frac{\rho}{6} \|\psi\|_{w^{\alpha,\beta}}. \tag{111}$$

Hence, when $n \ge \max\{n_0, n_1\}$, a combination of (109) and (111) produces that

$$\left\| \mathscr{P}_n \mathscr{K} \phi - \mathscr{P}_n \widetilde{\mathscr{K}}_N \phi \right\|_{w^{\alpha,\beta}} \le \frac{\rho}{6} \left(\sum_{i \in \mathbb{Z}_L^+} \|\eta_i\|_{w^{\alpha,\beta}} + \|\psi\|_{w^{\alpha,\beta}} \right). \tag{112}$$

This combining (23) and the next inequality

$$\begin{aligned}&\left\|\left(\mathcal{I}+\mathcal{P}_n\widetilde{\mathcal{K}}_n\right)\phi\right\|_{w^{\alpha,\beta}}\\&\geq\left\|\left(\mathcal{I}+\mathcal{P}_n\mathcal{K}\right)\phi\right\|_{w^{\alpha,\beta}}-\left\|\mathcal{P}_n\mathcal{K}\phi-\mathcal{P}_n\widetilde{\mathcal{K}}_N\phi\right\|_{w^{\alpha,\beta}}\end{aligned}\quad(113)$$

produces

$$\begin{aligned}&\left\|\left(\mathcal{I}+\mathcal{P}_n\widetilde{\mathcal{K}}_n\right)\phi\right\|_{w^{\alpha,\beta}}\\&\geq\frac{\rho}{2}\|\phi\|_{w^{\alpha,\beta}}-\frac{\rho}{6}\left(\sum_{i\in\mathbb{Z}_L^+}\|\eta_i\|_{w^{\alpha,\beta}}+\|\psi\|_{w^{\alpha,\beta}}\right)\\&\geq\frac{\rho}{6}\|\phi\|_{w^{\alpha,\beta}}.\end{aligned}\quad(114)$$

Hence, we draw the desired conclusion. □

Theorem 10 ensures that, for sufficiently large n, if we select N as in (104), then the fully discrete system possesses (62) which possesses a unique solution $\tilde{\mathbf{u}}_n$. The next result is concerned with the convergence of the approximation solution $\tilde{u}_n$.

Theorem 11. *Suppose the kernel function $K\in C^m(I^2)$ and f is given by (8). Three parameters α, β, and μ satisfy $-1<\alpha,\beta<1-2\mu,\alpha+\beta<1-2\mu$ and N is given in (104). Then there exist a positive constant c and a positive integer n_0 such that, for $n\geq n_0$,*

$$\|u-\tilde{u}_n\|_{w^{\alpha,\beta}}\leq cn^{-m}.\quad(115)$$

Proof. By using the triangle inequality, we have

$$\|u-\tilde{u}_n\|_{w^{\alpha,\beta}}\leq\|u-\mathcal{P}_nu\|_{w^{\alpha,\beta}}+\|\mathcal{P}_nu-\tilde{u}_n\|_{w^{\alpha,\beta}}.\quad(116)$$

Based on expression (9) of the solution u, an application of relation (19) yields that

$$\|u-\mathcal{P}_nu\|_{w^{\alpha,\beta}}\leq\|v-\mathcal{Q}_nv\|_{w^{\alpha,\beta}},\quad(117)$$

in which combining estimation (20) with $\phi:=v$ produces

$$\|u-\mathcal{P}_nu\|_{w^{\alpha,\beta}}\leq c\|\mathcal{D}^mv\|_{w^{\alpha+m,\beta+m}}n^{-m}.\quad(118)$$

Hence, we only need to estimate the second term in the right hand side of (116). In fact, employing $\mathcal{P}_n$ to both sides of (2) yields that

$$\mathcal{P}_nu+\mathcal{P}_n\mathcal{K}u=\mathcal{P}_nf.\quad(119)$$

A combination of the above equation and (55) and (56) confirms that

$$\begin{aligned}&\left(\mathcal{I}+\mathcal{P}_n\widetilde{\mathcal{K}}_N\right)(\tilde{u}_n-\mathcal{P}_nu)\\&=\mathcal{P}_n\mathcal{K}u-\mathcal{P}_n\widetilde{\mathcal{K}}_N\mathcal{P}_nu+\mathcal{P}_nh-\mathcal{L}_n^{\alpha,\beta}h.\end{aligned}\quad(120)$$

By Theorem 10, there exist a positive constant c and a positive integer n_0 such that $n\geq n_0$,

$$\begin{aligned}\|\tilde{u}_n-\mathcal{P}_nu\|_{w^{\alpha,\beta}}\leq c\Big(&\left\|\mathcal{P}_n\mathcal{K}u-\mathcal{P}_n\widetilde{\mathcal{K}}_N\mathcal{P}_nu\right\|_{w^{\alpha,\beta}}\\&+\left\|\mathcal{P}_nh-\mathcal{L}_M^{\alpha,\beta}h\right\|_{w^{\alpha,\beta}}\Big).\end{aligned}\quad(121)$$

Clearly, using (68) with $\psi:=h$ and $i:=0$ produces

$$\left\|\mathcal{P}_n\phi-\mathcal{L}_M^{\alpha,\beta}\phi\right\|_{w^{\alpha,\beta}}\leq c\|\mathcal{D}^mh\|_{w^{\alpha+m,\beta+m}}n^{-m}.\quad(122)$$

On the other hand, we let

$$\begin{aligned}I_1&:=\|\mathcal{K}u-\mathcal{K}\mathcal{P}_nu\|_{w^{\alpha,\beta}},\\I_2&:=\left\|\mathcal{K}\mathcal{P}_nu-\widetilde{\mathcal{K}}_N\mathcal{P}_nu\right\|_{w^{\alpha,\beta}}.\end{aligned}\quad(123)$$

Obviously,

$$\left\|\mathcal{P}_n\mathcal{K}u-\mathcal{P}_n\widetilde{\mathcal{K}}_N\mathcal{P}_nu\right\|_{w^{\alpha,\beta}}\leq I_1+I_2.\quad(124)$$

It follows from the compactness of the operator $\mathcal{K}$ and estimate (118) that there exists a positive constant c such that

$$I_1\leq c\|\mathcal{D}^mv\|_{w^{\alpha+m,\beta+m}}n^{-m}.\quad(125)$$

Again using the solution decomposition of Theorem 1, we observe that there exist positive constants c_i such that

$$u(x)=\sum_{i\in\mathbb{Z}_L^+}\eta_i(x)+v(x),\quad\eta_i(x):=c_iw_i(x),\ x\in I.\quad(126)$$

Thus, applying estimate (99) leads to the fact that there exists a positive constant c such that, for $i\in\mathbb{Z}_L^+$,

$$\left\|\mathcal{K}\eta_i-\widetilde{\mathcal{K}}_N\eta_i\right\|_{w^{\alpha,\beta}}\leq c\|\eta_i\|_{w^{\alpha,\beta}}N^{-m}.\quad(127)$$

In a similar manner, using result (98) produces

$$\left\|\mathcal{K}v-\widetilde{\mathcal{K}}_Nv\right\|_{w^{\alpha,\beta}}\leq cN^{-m}\|v\|_{w^{\alpha,\beta},m}.\quad(128)$$

A combination of (127) and (128) with the help of choice (104) yields that

$$I_2\leq cn^{-m}\log^{-1}n.\quad(129)$$

As a consequence, a combination of (121), (122), (123), (124), (126), (127), (128), and (129) completes the proof. □

Theorem 11 illustrates that the approximation solution arrives at the optimal convergence order.

5. One Numerical Example

In this section, we present a numerical example to demonstrate the approximation accuracy, the order convergence, and the stability of the proposed method. We also compare it with the conventional Jacobi-Galerkin method. Here, we compute the Gauss-Jacobi quadrature rule nodes and weights by Theorems 3.4 and 3.6 discussed in [19].

In this example, for simplicity we choose $K(x,t):=1$. For the numerical comparison purpose, we choose the right hand side function f so that

$$u(x):=(1+x)^{1/2}\cos(1+x),\quad x\in I,\quad(130)$$

is the exact solution of the equation. Note that the first derivative of this solution has a singularity at $x=-1$. The purpose

Table 1: The comparison of numerical results of the two methods.

	$n = 4$	$n = 6$	$n = 8$	$n = 10$	$n = 12$	$n = 14$
Conventional method	$3.99e-2$	$2.53e-2$	$1.60e-2$	$1.11e-2$	$8.37e-3$	$6.58e-3$
Our method	$1.59e-3$	$6.42e-4$	$2.53e-4$	$1.24e-4$	$7.01e-5$	$4.33e-5$

of these numerical experiments is to compare the numerical performance between the generalized spectral method with the conventional spectral method. For both of the methods we let $L^2_{w^{-1/2,-1/2}}(I)$ be the solution space of the original solution u. It is obvious that the solution u belongs to the space $H^1_{w^{-1/2,-1/2}}(I)$ and $u \notin H^m_{w^{-1/2,-1/2}}(I)$ for $m \geq 2$. Now we choose the corresponding conventional finite dimensional subspace Y_n as

$$Y_n := \operatorname{span}\{\cos(m \arccos x),\ m \in \mathbb{Z}_n,\ x \in I\}. \quad (131)$$

In this numerical example, our generalized spectral method chooses the nonpolynomial function set W given by

$$W := \operatorname{span}\left\{(1+x)^{1/2} : x \in I\right\}. \quad (132)$$

The comparison of numerical results between the conventional method and our proposed method is given in Table 1.

From these numerical results we observe that our method is superior to the conventional spectral method, which is consistent with the theoretical results. In summary, the conventional Jacobi-Galerkin method for solving Volterra integral equations with nonsmooth solutions may be of low-order accuracy. In order to obtain the high-order accuracy, in this paper we introduce the set W of nonpolynomial functions in the conventional Jacobi-Galerkin approximation space Y_n of Jacobi polynomial basis and then develop a generalized Jacobi-Galerkin method. The price we pay to obtain this optimal convergence rate is the dimension of the increase of the approximation subspace. But we observe that the additional cost to achieve the optimal convergence rate is insignificant in comparison with the acceleration convergence that we obtain.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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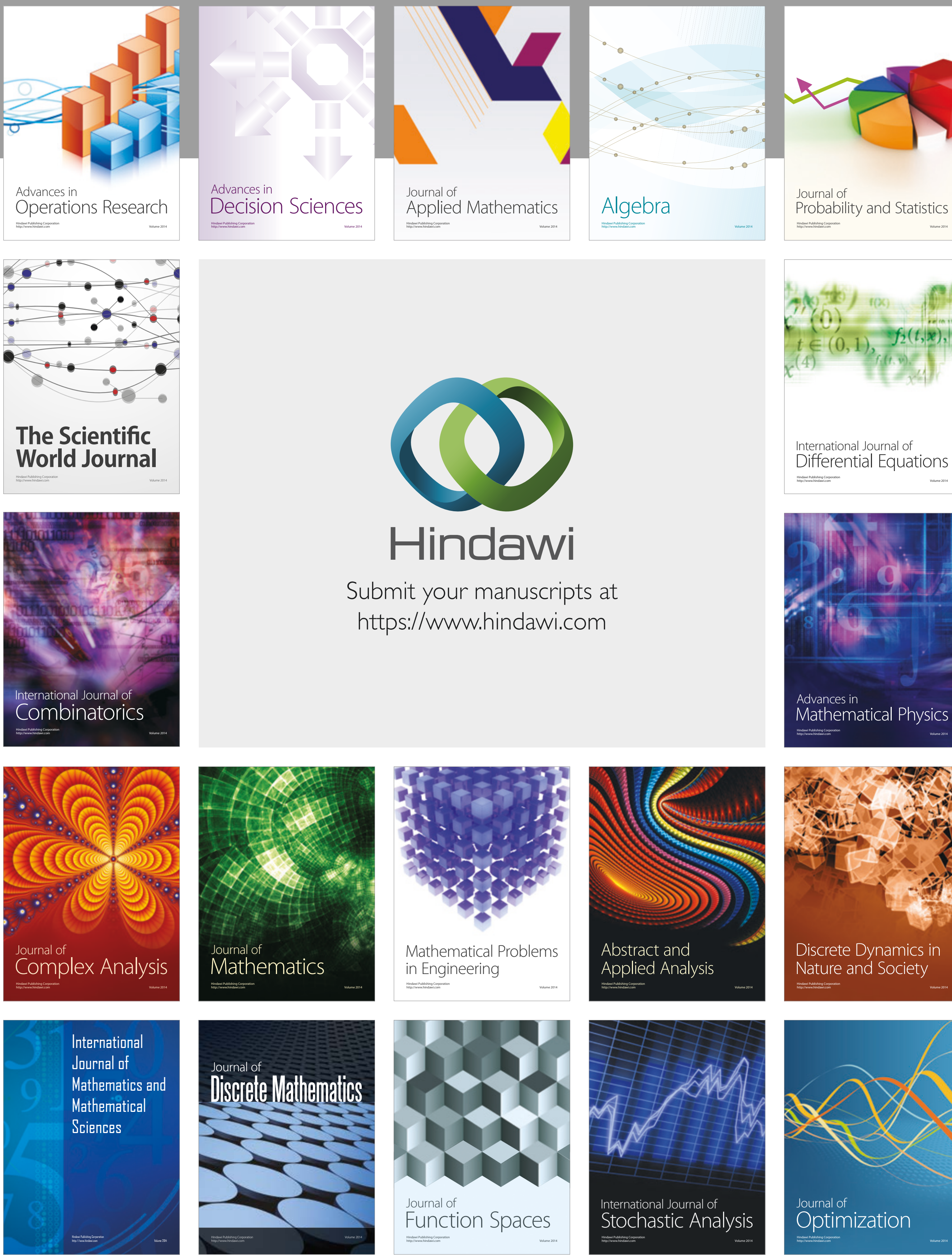
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