Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2016, Article ID 5956713, 15 pages
http://dx.doi.org/10.1155/2016/5956713

Research Article

Unilateral Global Bifurcation from Intervals for Fourth-Order Problems and Its Applications

Wenguo Shen and Tao He

Department of Basic Courses, Lanzhou Institute of Technology, Lanzhou 730050, China

Correspondence should be addressed to Wenguo Shen; shenwg369@163.com

Received 9 November 2015; Accepted 11 January 2016

Academic Editor: Kousuke Kuto

We establish a unilateral global bifurcation result from interval for a class of fourth-order problems with nondifferentiable nonlinearity. By applying the above result, we firstly establish the spectrum for a class of half-linear fourth-order eigenvalue problems. Moreover, we also investigate the existence of nodal solutions for the following half-linear fourth-order problems: $x'''' = \alpha x^+ + \beta x^- + ra(t)f(x)$, $0 < t < 1$, $x(0) = x(1) = x''(0) = x''(1) = 0$, where $r \neq 0$ is a parameter, $a \in C([0,1],(0,\infty))$, $x^+ = \max\{x, 0\}$, $x^- = -\min\{x, 0\}$, $\alpha, \beta \in C[0,1]$, and $f \in C(\mathbb{R},\mathbb{R})$, $sf(s) > 0$, for $s \neq 0$. We give the intervals for the parameter r which ensure the existence of nodal solutions for the above fourth-order half-linear problems if $f_0 \in [0,\infty)$ or $f_\infty \in [0,\infty]$, where $f_0 = \lim_{|s|\to 0} f(s)/s$ and $f_\infty = \lim_{|s|\to+\infty} f(s)/s$. We use the unilateral global bifurcation techniques and the approximation of connected components to prove our main results.

1. Introduction

In the past twenty years, fourth-order BVP have attracted the attention of many specialists in differential equations because of their interesting applications. For example, Bai and Wang [1], Ma and Wang [2], and Chu and O'Regan [3] have investigated the fourth-order BVP by the fixed point theory in cones. Meanwhile, by applying the bifurcation techniques of Rabinowitz [4, 5], Gupta and Mawhin [6], Lazer and McKenna [7], Rynne [8], Liu and O'Regan [9], Ma et. al [10, 11], Shen [12, 13], and Ma [14] studied the existence of nodal solutions for the fourth-order BVP.

Now, consider the following operator equation:

$$u = \lambda Bu + H(\lambda, u), \tag{1}$$

where B is a compact linear operator and $H : \mathbb{R} \times E \to E$ is compact with $H = o(\|u\|)$ at $u = 0$ uniformly on bounded λ intervals, where E is a real Banach space with the norm $\|\cdot\|$. If the characteristic value μ of B has multiplicity 1 and

$$\mathcal{S} = \overline{\{(\lambda, u) : (\lambda, u) \text{ satisfies } (1),\ u \not\equiv 0\}}^{\mathbb{R}\times E}, \tag{2}$$

Dancer [15] has shown that there are two distinct unbounded continua C_μ^+ and C_μ^-, consisting of the bifurcation branch C_μ of $\mathcal{S}$ emanating from $(\mu, 0)$, where either C_μ^+ and C_μ^- are both unbounded or $C_\mu^+ \cap C_\mu^- \neq \{(\mu, 0)\}$. This result has been extended to the fourth-order problems by Dai and Han [16]. More specifically, Dai and Han [16] considered the following fourth-order problem:

$$\begin{gathered} x'''' = \lambda a(t)x + g(t, x, \lambda), \quad 0 < t < 1, \\ x(0) = x(1) = x''(0) = x''(1) = 0, \end{gathered} \tag{3}$$

where $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous satisfying $g(t, x, 0) \equiv 0$, and

$$\lim_{x\to 0} \frac{g(t, x, \lambda)}{|x|} = 0 \tag{4}$$

uniformly on $[0,1]$ and μ on bounded sets.

Let $Y = C[0,1]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$. Let $E = \{x \in C^3[0,1] \mid x(0) = x(1) = x''(0) = x''(1) = 0\}$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty + \|x''\|_\infty + \|x'''\|_\infty$. Let

$\mathbf{E} = \mathbb{R} \times E$ under the product topology. Let S_k^+ denote the set of functions in E which have exactly $k-1$ generalized simple zeros in $(0,1)$ and are positive near $t=0$, set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. They are disjoint and open in E. Finally, let $\Phi_k^{\pm} = \mathbb{R} \times S_k^{\pm}$ and $\Phi_k = \mathbb{R} \times S_k$. Let C denote the closure of the set of nontrivial solutions of (3) in $\mathbb{R} \times E$, and C_k^{ν} denote the subset of C with $u \in S_k^{\nu}$ for $\nu \in \{+,-\}$ and $C_k = C_k^+ \cup C_k^-$.

From Dancer [15], Dai and Han [16] obtained that problem (3) has two distinct unbounded subcontinua C_k^+ and C_k^-, consisting of the bifurcation branch C_k emanating from $(\lambda_k, 0)$, which satisfy the following.

Lemma 1. *Either C_k^+ and C_k^- are both unbounded or $C_k^+ \cap C_k^- \neq \{(\lambda_k, 0)\}$, and*

$$C_k^{\nu} \subset \left(\left(\mathbb{R} \times S_k^{\nu} \right) \cup \left\{ (\lambda_k, 0) \right\} \right), \tag{5}$$

where $\nu \in \{+,-\}$ and λ_k were defined as in [10] or [16].

For the abstract unilateral global bifurcation theory, we refer the reader to [4, 5, 15, 17–19] and the references therein.

However, among the above papers, the nonlinearities are linear in the zeros and infinity. The problems involving non-differentiable nonlinearities have also been investigated by applying bifurcation techniques; see Berestycki [20], Schmitt and Smith [19], Rynne [21], Ma and Dai [22], and Dai et al. [23–25] and references therein. Among them, in 1977, Berestycki [20] studied the differential equations involving nondifferentiable nonlinearity. The above Berestycki's ([20]) result has been improved partially by Schmitt and Smith [19] by applying a set-valued version of Rabinowitz global bifurcation theorem. In 1998, Rynne [21] established the interval bifurcation from $u = 0$ and $u = \infty$ and obtained sets of positive or negative solutions with the approximation technique from Berestycki [20]. Recently, Ma and Dai [22] established the global interval bifurcation for a Sturm-Liouville problem with a nonsmooth nonlinearity by [15]. Later, Dai et al. [23–25] studied the bifurcation from intervals for Sturm-Liouville problems and its applications and established the unilateral global interval bifurcation for p-Laplacian with non-$p-1$-linearization nonlinearity, respectively.

On the other hand, half-linear or half-quasilinear problems have attracted the attention of some specialists; see [20, 22, 25]. Among them, Berestycki [20] studied the bifurcation structure for the half-linear equations. Recently, Ma and Dai [22] and Dai and Ma [25] studied the existence of nodal solutions for a class of half-linear or half-quasilinear eigenvalue problems and improved Berestycki's result, respectively.

Motivated by the above papers, in this paper, we will firstly establish some Dancer-type unilateral global bifurcation results about the continuum of solutions for the fourth-order problems:

$$\begin{gathered} x'''' = \lambda a(t) x + F(t, x, \lambda), \quad 0 < t < 1, \\ x(0) = x(1) = x''(0) = x''(1) = 0, \end{gathered} \tag{6}$$

where λ is a parameter, the nonlinear term F has the form $F = f + g$, where f and g are continuous functions on $[0,1] \times \mathbb{R}^3$, and a, f, g satisfy the following conditions:

($H1$) $a \in C([0,1], (0,\infty))$.

($H2$) $|f(t,x,\lambda)/x| \leq M_1$, for all $t \in [0,1]$, $0 < |x| \leq 1$, $\forall \lambda \in \mathbb{R}$, where M_1 is a positive constant.

($H3$) $g(t,x,\lambda) = o(|x|)$ near $x = 0$, uniformly in $t \in [0,1]$ and λ on bounded sets.

Let $\mathcal{S}$ denote the closure of the set of nontrivial solutions of (6) in $\mathbb{R} \times E$, and $\mathcal{S}_k^{\nu}$ denote the subset of $\mathcal{S}$ with $u \in S_k^{\nu}$ for $\nu \in \{+,-\}$ and $\mathcal{S}_k = \mathcal{S}_k^+ \cup \mathcal{S}_k^-$.

Under assumptions ($H1$)–($H3$), we will show that $I_k = [\lambda_k - d_1, \lambda_k + d_1]$ is a bifurcation interval of problem (6) and there are two distinct unbounded subcontinua, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$, consisting of the bifurcation branch $\mathscr{C}_k$ from $I_k = [\lambda_k - d_1, \lambda_k + d_1]$, where λ_k is given in Lemma 1.

On the basis of the above unilateral global interval bifurcation result, we study the following half-linear eigenvalue problem:

$$\begin{gathered} x'''' = \lambda a(t) x + \alpha x^+ + \beta x^-, \quad 0 < t < 1, \\ x(0) = x(1) = x''(0) = x''(1) = 0, \end{gathered} \tag{7}$$

where $x^+ = \max\{x, 0\}$, $x^- = \min\{x, 0\}$ and $\alpha(t)$ and $\beta(t)$ satisfy the following:

($H4$) $\alpha(t), \beta(t) \in C[0,1]$.

We will show that there exist two sequences of simple half-eigenvalues for problem (7): $\lambda_1^+ < \lambda_2^+ \cdots < \lambda_k^+ < \cdots$ and $\lambda_1^- < \lambda_2^- \cdots < \lambda_k^- < \cdots$. The corresponding half-linear solutions are in $\{\lambda_k^+\} \times S_k^+$ and $\{\lambda_k^-\} \times S_k^-$. Furthermore, aside from these solutions and the trivial ones, there is no other solutions of problem (7).

Following the above eigenvalue theory (see Theorem 19), we will investigate the existence of nodal solutions for the following fourth-order problem:

$$\begin{gathered} x'''' = \alpha x^+ + \beta x^- + \lambda a(t) f(x), \quad 0 < t < 1, \\ x(0) = x(1) = x''(0) = x''(1) = 0. \end{gathered} \tag{8}$$

Let Σ denote the closure of the set of nontrivial solutions of (8) in $\mathbb{R} \times E$, with Σ_k^{ν} denoting the subset of Σ with $u \in S_k^{\nu}$ for $\nu \in \{+,-\}$ and $\Sigma_k = \Sigma_k^+ \cup \Sigma_k^-$. For $\alpha = \beta = 0$, Dai and Han [16] have established the existence of nodal solutions for problem (8) with crossing nonlinearity. In this paper, we assume that f satisfies the following assumptions:

($H5$) $sf(s) > 0$ for $s \neq 0$.

($H6$) $f_0 \in (0,\infty)$ and $f_\infty \in (0,\infty)$.

($H7$) $f_0 \in (0,\infty)$ and $f_\infty = \infty$.

($H8$) $f_0 \in (0,\infty)$ and $f_\infty = 0$.

($H9$) $f_0 = 0$ and $f_\infty \in (0,\infty)$.

($H10$) $f_0 = 0$ and $f_\infty = \infty$.

($H11$) $f_0 = 0$ and $f_\infty = 0$,

where

$$\lim_{|x|\to 0}\frac{f(x)}{x}=f_0,\quad \forall t\in[0,1],$$
$$\lim_{|x|\to \infty}\frac{f(x)}{x}=f_\infty,\quad \forall t\in[0,1]. \tag{9}$$

The rest of this paper is arranged as follows. In Section 2, we have given some preliminaries. In Section 3, we establish the unilateral global bifurcation result from the interval for problem (6). In Section 4, on the basis of the unilateral global interval bifurcation result, we will establish the spectrum for a class of the half-linear fourth-order eigenvalue problem (7) (see Theorem 19). In Section 5, following the above eigenvalue theory (see Theorem 19), we will investigate the existence of nodal solutions for a class of the half-linear fourth-order problem (8).

2. Hypotheses and Lemmas

We define the linear operator $L : D(L) \subset E \to Y$:

$$Lx = x'''',\quad x\in D(L), \tag{10}$$

with $D(L) = \{x \in C^4[0,1] \mid x(0) = x(1) = x''(0) = x''(1) = 0\}$.

From [26, p. 439-440], we consider the following auxiliary problem:

$$x''''=h(t),\quad t\in(0,1),$$
$$x(0)=x(1)=x''(0)=x''(1)=0, \tag{11}$$

for a given $h \in C[0,1]$. We can get that problem (11) can be equivalently written as

$$x(t)=L^{-1}(h)(t)=\int_0^1 k(t,s)h(s)\,ds, \tag{12}$$

where

$$k(t,s):=\begin{cases}\frac{1}{6}s(1-t)\left(2t-s^2-t^2\right), & s\le t,\\ \frac{1}{6}t(1-s)\left(2s-t^2-s^2\right), & s>t.\end{cases} \tag{13}$$

Then L is a closed operator and $L^{-1} : Y \to E$ is completely continuous.

Define the Nemytskii operator $H : \mathbb{R}\times E \to E$ by

$$H(\lambda,x)(t):=\lambda L^{-1}[ax]+L^{-1}[g(t,x,\lambda)]. \tag{14}$$

Then it is clear that H is continuous operator and problem (3) can be equivalently written as

$$x=H(\lambda,x). \tag{15}$$

Clearly, $H : \mathbb{R}\times E \to E$ is completely continuous and $H(\mu,0) = 0$, $\forall\mu\in\mathbb{R}$.

Let

$$\overline{g}(t,x,\lambda)=\max_{0\le|s|\le x}|g(t,s,\lambda)|$$
$$\forall t\in[0,1],\ \lambda \text{ on bounded sets}, \tag{16}$$

and then $\overline{g}$ is nondecreasing with respect to x, and

$$\lim_{x\to0^+}\frac{\overline{g}(t,x,\lambda)}{x}=0 \tag{17}$$

uniformly for $t \in [0,1]$ and λ on bounded sets. Further it follows from (17) that

$$\frac{g(t,x,\lambda)}{\|x\|}\le\frac{\overline{g}(t,|x|,\lambda)}{\|x\|}\le\frac{\overline{g}(t,\|x\|_\infty,\lambda)}{\|x\|}$$
$$\le\frac{\overline{g}(t,\|x\|,\lambda)}{\|x\|}\longrightarrow 0\quad\text{as } \|x\|\longrightarrow 0 \tag{18}$$

uniformly for $t \in [0,1]$ and λ on bounded sets.

In the following, we summarize some preliminary results from [10, 16].

Definition 2 (see [16]). Let $u \in E$ and $t_* \in I$ such that $u(t_*) = u''(t_*) = 0$. We call that t_* a generalized simple zero if $u'(t_*) \neq 0$ or $u'''(t_*) \neq 0$. Otherwise, we call that t_* a generalized double zero. If there is no generalized double zero of u, we call that u a nodal solution.

Lemma 3 (see [10] or [16]). *Let* $(H1)$ *hold. The linear eigenvalue problem*

$$x''''(t)=\lambda a(t)x,\quad 0<t<1,$$
$$x(0)=x(1)=x''(0)=x''(1)=0 \tag{19}$$

has an infinite sequence of positive eigenvalues:

$$0<\lambda_1(a)<\lambda_2(a)<\cdots<\lambda_k(a)\longrightarrow+\infty$$
$$(k\longrightarrow+\infty). \tag{20}$$

Moreover, each eigenvalue is simple. To each eigenvalue $\lambda_k(a)$ *there corresponds an essential unique eigenfunction* ψ_k *which has exactly* $k-1$ *generalized simple zeros in* $(0,1)$ *and is positive near* 0.

Lemma 4 (see [16]). *If* (λ, x) *is a nontrivial solution of (6) under assumptions* $(H2)$ *and* $(H3)$ *and* x *has a generalized double zero, then* $x \equiv 0$.

Remark 5. By Lemma 4, we can see that if (λ, x) is a nontrivial solution of (6) under assumptions $(H2)$ and $(H3)$, then $x \in \bigcup_{k=1}^{\infty} S_k^{\nu}$.

Lemma 6 (see [20, Lemma 2]). *Let* j *and* k *be two integers such that* $j \ge k \ge 2$. *Suppose that there exist two families of real numbers:*

$$\xi_0=0<\xi_1<\xi_2<\cdots<\xi_{k-1}<\xi_k=1,$$
$$\eta_0=0<\eta_1<\eta_2<\cdots<\eta_{j-1}<\eta_j=1. \tag{21}$$

Then, if $\xi_1 \le \eta_1$, there exist two integers p and q having the same parity: $1 \le p \le k-1$ and $1 \le q \le j-1$, such that

$$\xi_p \le \eta_q \le \eta_{q+1} \le \xi_{p+1}. \tag{22}$$

Definition 7 (see [27]). Let X be a Banach space and let $\{C_n \mid n = 1, 2, \ldots\}$ be a family of subsets of X. Then the superior limit $\mathbb{D}$ of $\{C_n\}$ is defined by

$$\begin{aligned}\mathbb{D} := \limsup_{n\to\infty} C_n = \Big\{x \in X \mid \exists \{n_i\} \subset \mathbb{N},\ x_{n_i} \\ \in C_{n_i},\ \text{such that } x_{n_i} \longrightarrow x\Big\}.\end{aligned} \tag{23}$$

Lemma 8 (see [27]). *Each connected subset of metric space X is contained in a component, and each connected component of X is closed.*

Lemma 9 (see [28]). *Let X be a Banach space and let $\{C_n \mid n = 1, 2, \ldots\}$ be a family of closed connected subsets of X. Assume that*

(i) *there exist $z_n \in C_n$, $n = 1, 2, \ldots$, and $z^* \in X$, such that $z_n \to z^*$;*

(ii) $r_n = \sup\{\|x\| \mid x \in C_n\} = \infty$;

(iii) *for all $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relative compact set of X, where*

$$B_R = \{x \in X \mid \|x\| \le R\}. \tag{24}$$

Then there exists an unbounded component C in $\mathbb{D}$ and $z^* \in C$.

3. Unilateral Global Bifurcation

The main result for problem (6) is the following theorem.

Theorem 10. *Let (H1), (H2), and (H3) hold. Let $d_1 = M_1/a_0$, where $a_0 = \min_{t\in[0,1]} a(t)$, and let $I_k = [\lambda_k - d_1, \lambda_k + d_1]$ for every $k \in \mathbb{N}$. The component $\mathscr{C}_k^+$ of $\mathscr{S}_k^+ \cup (I_k \times \{0\})$, containing $I_k \times \{0\}$, is unbounded and lies in $\Phi_k^+ \cup (I_k \times \{0\})$ and the component $\mathscr{C}_k^-$ of $\mathscr{S}_k^- \cup (I_k \times \{0\})$, containing $I_k \times \{0\}$, is unbounded and lies in $\Phi_k^- \cup (I_k \times \{0\})$.*

In order to prove Theorem 10, we need the following results.

Lemma 11. *If $\mathscr{C}_k^\nu$ is bounded, we can find a neighborhood $\mathscr{O}$ of $\mathscr{C}_k^\nu$ such that $\partial\mathscr{O} \cap \mathscr{S}_k^\nu = \emptyset$, where $\mathscr{C}_k^\nu$ ($\nu = +, -$) is given by Theorem 10.*

Proof. We only prove the case of $\mathscr{C}_k^+$ since the case of $\mathscr{C}_k^-$ is similar.

Let $\mathscr{U}$ be a uniform neighborhood of $\mathscr{C}_k^+$ in $\mathbb{R} \times E$.

We discuss two cases.

Case 1 (if $\partial\mathscr{U} \cap \mathscr{S}_k^+ \ne \emptyset$). Since the solutions of problem (6) are bounded in $\mathbb{R} \times E$, then $\mathscr{C}_k^+$ is compact in $\mathbb{R} \times E$. It follows that $\overline{\mathscr{U}} \cap \mathscr{S}_k^+$ is compact metric space. Obviously, $\mathscr{C}_k^+$ and $\partial\mathscr{U} \cap \mathscr{S}_k^+$ are two disjoint closed subsets of $\overline{\mathscr{U}} \cap \mathscr{S}_k^+$. Because of the maximal connectedness of $\mathscr{C}_k^+$, there does not exist a component $\mathscr{C}^*$ of $\overline{\mathscr{U}} \cap \mathscr{S}_k^+$ such that $\mathscr{C}^* \cap \mathscr{C}_k^+ \ne \emptyset$, $\mathscr{C}^* \cap (\partial\mathscr{U} \cap \mathscr{S}_k^+) \ne \emptyset$. By [27] or [4, Lemma 1.1], there exist two disjoint compact subsets K_1, K_2 of $\overline{\mathscr{U}} \cap \mathscr{S}_k^+$, such that $\overline{\mathscr{U}} \cap \mathscr{S}_k^+ = K_1 \cup K_2$, $\mathscr{C}_k^+ \subset K_1$, $\partial\mathscr{U} \cap \mathscr{S}_k^+ \subset K_2$. Evidently, $d(K_1, K_2) > 0$. Let $r = \min\{d(K_1, K_2), d(K_1, \partial\mathscr{U})\}$, and let $\mathscr{V}$ be the $r/2$-neighborhood of K_1. Set

$$\mathscr{O} = \mathscr{U} \cap \mathscr{V}, \tag{25}$$

and then

$$\begin{gathered}\mathscr{C}_k^+ \subset \mathscr{O}, \\ \partial\mathscr{O} \cap \mathscr{S}_k^+ = \emptyset.\end{gathered} \tag{26}$$

Case 2 (if $\partial\mathscr{U} \cap \mathscr{S}_k^+ = \emptyset$). In this case, we take $\mathscr{O} = \mathscr{U}$. It is obvious that the result holds.

Consider the following auxiliary approximate problem:

$$\begin{gathered}x'''' = \lambda a(t) x + f(t, x|x|^\epsilon, \lambda) + g(t, x, \lambda), \\ 0 < t < 1, \\ x(0) = x(1) = x''(0) = x''(1) = 0.\end{gathered} \tag{27}$$

For $\epsilon > 0$, it is easy to show that nonlinear term $\lambda a(t)x + f(t, x|x|^\epsilon, \lambda) + g(t, x, \lambda)$ is differ-entiable at the origin. Let

$$\mathscr{S}_\epsilon = \overline{\{(\lambda, x) : (\lambda, x) \text{ satisfies } (27),\ x \not\equiv 0\}}^{\mathbb{R}\times E}. \tag{28}$$

□

By Lemma 1, there are two unbounded continua $\mathscr{C}_{k,\epsilon}^+$ of $\mathscr{S}_\epsilon^+$ and $\mathscr{C}_{k,\epsilon}^-$ of $\mathscr{S}_\epsilon^-$ bifurcating from $(\lambda_k, 0)$, consisting of the bifurcation branch $\mathscr{C}_{k,\epsilon}$, which satisfy the following result.

Lemma 12. *$\mathscr{C}_{k,\epsilon}^+$ and $\mathscr{C}_{k,\epsilon}^-$ are both unbounded and*

$$\mathscr{C}_{k,\epsilon}^\nu \subset \Phi_k^\nu \cup \{(\lambda_k, 0)\} \quad (\nu = +, -). \tag{29}$$

Proof. By Lemma 1.24 of [4], there exists a bounded open neighborhood $\mathbb{O}_k$ of $(\lambda_k, 0)$ such that

$$\left(\mathscr{C}_{k,\epsilon}^\nu - \{(\lambda_k, 0)\}\right) \cap \mathbb{O}_k \subset \Phi_k^\nu \quad (\nu = +, -). \tag{30}$$

It follows that

$$\mathscr{C}_{k,\epsilon}^\nu \cap \mathbb{O}_k \subset \Phi_k^\nu \cup \{(\lambda_k, 0)\} \quad (\nu = +, -). \tag{31}$$

By Lemma 4, we show that $\mathscr{C}_{k,\epsilon}^\nu \setminus \{(\lambda_k, 0)\}$ cannot leave Φ_k^ν outside of a neighborhood of $(\lambda_k, 0)$. Thus, we have

$$\mathscr{C}_{k,\epsilon}^\nu \subset \Phi_k^\nu \cup \{(\lambda_k, 0)\} \quad (\nu = +, -). \tag{32}$$

Next, we will prove that both $\mathscr{C}_{k,\epsilon}^+$ and $\mathscr{C}_{k,\epsilon}^-$ are unbounded.

Without loss of generality, we may suppose that $\mathscr{C}_{k,\epsilon}^-$ is bounded. Therefore, in view of (32) and Lemma 1, there exists $(\lambda_*, u_*) \in \mathscr{C}_{k,\epsilon}^+ \cap \mathscr{C}_{k,\epsilon}^-$ such that $(\lambda_*, u_*) \ne (\lambda_k, 0)$ and $u_* \in S_k^+ \cap S_k^-$. This contradicts the definitions of S_k^+ and S_k^-. □

To prove Theorem 10, the next lemma will play a key role.

Lemma 13. *Let ϵ_n, $0 < \epsilon_n < 1$, be a sequence converging to* 0. *If there exists a sequence $(\lambda_n, x_n) \in \mathbb{R} \times S_k^{\nu}$ such that (λ_n, x_n) is a nontrivial solution of problem (27) corresponding to $\epsilon = \epsilon_n$, and (λ_n, x_n) converges to $(\lambda, 0)$ in $\mathbb{R} \times E$, then $\lambda \in I_k$.*

Proof. Without loss of generality, we may assume that $\|x_n\| \leq 1$. Let $y_n = x_n/\|x_n\|$; then y_n satisfies the problem

$$\begin{gathered} y_n'''' = \lambda_n a(t) y_n + f_n(t) + g_n(t), \quad 0 < t < 1, \\ y_n(0) = y_n(1) = y_n''(0) = y_n''(1) = 0, \end{gathered} \tag{33}$$

where

$$\begin{aligned} f_n(t) &= \frac{f\left(t, x_n |x_n|^{\epsilon}, \lambda_n\right)}{\|x_n\|}, \\ g_n(t) &= \frac{g(t, x_n, \lambda_n)}{\|x_n\|}. \end{aligned} \tag{34}$$

By (18), it follows that

$$\lim_{n\to\infty} g_n(t) = 0, \tag{35}$$

uniformly for $t \in [0,1]$ and λ on bounded sets. Furthermore, $(H2)$ implies that

$$\begin{aligned} |f_n(t)| &= \frac{\left|f\left(t, x_n |x_n|^{\epsilon}, \lambda_n\right)\right|}{\|x_n\|} \leq \frac{|x_n|^{\epsilon} M_1 |x_n|}{\|x_n\|} \\ &= |x_n|^{\epsilon} M_1 |y_n| \leq M_1 |y_n|, \end{aligned} \tag{36}$$

for all $t \in [0,1]$.

Note that $\|y_n\| = 1$ implies $\|y_n\|_{\infty} \leq 1$. Using this fact with (35) and (36), we have that $\lambda_n a(t) y_n + f_n(t) + g_n(t)$ is bounded in $C^4[0,1]$ for n large enough. The compactness of L^{-1} implies that y_n is convergence in $C^4[0,1]$. Without loss of generality, we may assume that $y_n \to y$ in E with $\|y\| = 1$. Clearly, we have $y \in \overline{S_k^{\nu}}$.

We claim that $y \in S_k^{\nu}$. On the contrary, supposing that $y \in \partial S_k^{\nu}$, by Lemma 4, then $y \equiv 0$, which is a contradiction with $\|y\| = 1$.

Now, we deduce the boundedness of λ. Let $\varphi_k^{\nu} \in S_k^{\nu}$ be an eigenfunction of problem (19) corresponding to λ_k.

Let

$$\begin{gathered} t_0 = 0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = 1, \\ s_0 = 0 < s_1 < s_2 < \cdots < s_{k-1} < s_k = 1 \end{gathered} \tag{37}$$

be, respectively, the sequences of generalized simple zeros of y_n and φ_k^{ν}.

Suppose $t_1 \leq s_1$; then we deduce from Lemma 6 existence of integers p and q having the same parity such that

$$t_p \leq s_q \leq s_{q+1} \leq t_{p+1}. \tag{38}$$

Therefore, without loss of generality, we choose $\xi_1 = t_0$, $\eta_1 = t_1$, $\xi_2 = s_q$, and $\eta_2 = s_{q+1}$ and since p and q have the same parity y_n and φ_k^{ν} do not vanish and have the same sign in both the intervals $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$. It follows that $y_n(\xi_1) = y_n(\eta_1) = y_n''(\xi_1) = y_n''(\eta_1) = 0$, and $\varphi_k^{\nu}(\xi_2) = \varphi_k^{\nu}(\eta_2) = (\varphi_k^{\nu})''(\xi_2) = (\varphi_k^{\nu})''(\eta_2) = 0$.

We can assume without loss of generality that $y_n > 0$ and $\varphi_k^{\nu} > 0$ in (ξ_1, η_1). By the Picone identity in [29, Theorem 4], we have that

$$\begin{gathered} \int_{\xi_1}^{\eta_1} \left\{ \frac{y_n}{\varphi_k^{\nu}} \left[y_n (\varphi_k^{\nu})''' - \varphi_k^{\nu} y_n''' \right] + \frac{y_n'}{\varphi_k^{\nu}} \left[\varphi_k^{\nu} y_n'' - y_n (\varphi_k^{\nu})'' \right] - y_n (\varphi_k^{\nu})'' \left(\frac{y_n}{\varphi_k^{\nu}} \right)' \right\}' dt = A_1 + B_1, \\ A_1 = \int_{\xi_1}^{\eta_1} \frac{y_n}{\varphi_k^{\nu}} \left(y_n L\varphi_k^{\nu} - \varphi_k^{\nu} L y_n \right) dt, \; B_1 = \int_{\xi_1}^{\eta_1} \left\{ \left[y_n'' - \frac{y_n}{\varphi_k^{\nu}} (\varphi_k^{\nu})'' \right]^2 - 2\varphi_k^{\nu} (\varphi_k^{\nu})'' \left[\left(\frac{y_n}{\varphi_k^{\nu}} \right)' \right]^2 \right\} dt. \end{gathered} \tag{39}$$

The left-hand side of (39) equals

$$\begin{aligned} &- \lim_{t\to\xi_1^+} \left\{ \frac{y_n}{\varphi_k^{\nu}} \left[y_n (\varphi_k^{\nu})''' - \varphi_k^{\nu} y_n''' \right] \right. \\ &\left. + \frac{y_n'}{\varphi_k^{\nu}} \left[\varphi_k^{\nu} y_n'' - y_n (\varphi_k^{\nu})'' \right] - y_n (\varphi_k^{\nu})'' \left(\frac{y_n}{\varphi_k^{\nu}} \right)' \right\} \\ &+ \lim_{t\to\eta_1^-} \left\{ \frac{y_n}{\varphi_k^{\nu}} \left[y_n (\varphi_k^{\nu})''' - \varphi_k^{\nu} y_n''' \right] \right. \\ &\left. + \frac{y_n'}{\varphi_k^{\nu}} \left[\varphi_k^{\nu} y_n'' - y_n (\varphi_k^{\nu})'' \right] - y_n (\varphi_k^{\nu})'' \left(\frac{y_n}{\varphi_k^{\nu}} \right)' \right\} \\ &=: H_{\xi_1} + H_{\eta_1}. \end{aligned} \tag{40}$$

We prove that $H_{\xi_1} \leq 0$. If $\varphi_k^{\nu}(\xi_1) \neq 0$, then $H_{\xi_1} = 0$. If $\varphi_k^{\nu}(\xi_1) = 0$, noting the conclusion of Lemma 3, then $(\varphi_k^{\nu})'(\xi_1) > 0$. By L'Hospital rule, we have that

$$
\begin{aligned}
H_{\xi_1} &= -\lim_{t\to\xi_1^+}\frac{y_n'\left[y_n(\varphi_k^\gamma)''' - \varphi_k^\gamma y_n'''\right] + y_n\left[y_n(\varphi_k^\gamma)''' - \varphi_k^\gamma y_n'''\right]'}{(\varphi_k^\gamma)'} \\
&\quad - \lim_{t\to\xi_1^+}\frac{y_n''\left[\varphi_k^\gamma y_n'' - y_n(\varphi_k^\gamma)''\right] + y_n'\left[(\varphi_k^\gamma)' y_n'' + \varphi_k^\gamma y_n''' - y_n'(\varphi_k^\gamma)'' - y_n(\varphi_k^\gamma)'''\right]}{(\varphi_k^\gamma)'} \\
&\quad + (\varphi_k^\gamma)''(\xi_1)\lim_{t\to\xi_1^+}\frac{y_n'\left[y_n'\varphi_k^\gamma - y_n(\varphi_k^\gamma)'\right] + y_n\left[y_n''\varphi_k^\gamma - y_n(\varphi_k^\gamma)''\right]}{2\varphi_k^\gamma(\varphi_k^\gamma)'} \\
&= \frac{\left[y_n'(\xi_1)\right]^2(\varphi_k^\gamma)''(\xi_1)}{(\varphi_k^\gamma)'(\xi_1)} + \frac{(\varphi_k^\gamma)''(\xi_1)}{(2\varphi_k^\gamma)'(\xi_1)}\lim_{t\to\xi_1^+}\frac{y_n''\left[y_n'\varphi_k^\gamma - y_n(\varphi_k^\gamma)'\right] + 2y_n'\left[y_n''\varphi_k^\gamma - y_n(\varphi_k^\gamma)''\right] + y_n\left[y_n''\varphi_k^\gamma - y_n(\varphi_k^\gamma)''\right]'}{(\varphi_k^\gamma)'} \\
&= \frac{\left[y_n'(\xi_1)\right]^2(\varphi_k^\gamma)''(\xi_1)}{(\varphi_k^\gamma)'(\xi_1)}.
\end{aligned} \tag{41}
$$

In the following, we will prove that

$$(\varphi_k^\gamma)''(t) \le 0, \quad \forall t \in [\xi_1, \eta_1]. \tag{42}$$

Let $u := \varphi_k^\gamma(t) + 1$. Then $u'''' = \lambda_k a \varphi_k^\gamma$ and $u \ge 1$ in (ξ_1, η_1). For some $\varepsilon > 0$ small enough, let $v \in C^4([-\varepsilon, 1+\varepsilon])$ and $b \ge 0$ be such that $v(-\varepsilon) = v(1+\varepsilon) = v''(-\varepsilon) = v''(1+\varepsilon) = 0$, $v|_I = u$, and $v'''' = \lambda_k b v$. Then we have

$$
\begin{gathered}
v'''' = \lambda_k b v, \quad t \in (-\varepsilon, 1+\varepsilon), \\
v(-\varepsilon) = v(1+\varepsilon) = v''(-\varepsilon) = v''(1+\varepsilon) = 0.
\end{gathered} \tag{43}
$$

Set $m := (2\xi_1 - \varepsilon)/2$ and $n := (2\eta_1 + \varepsilon)/2$. Let $w \in C^4([m,n])$ and $c \ge 0$ be such that $w|_I = v$, $w \ge 0$ in (m,n) and $w(m) = w(n) = w''(m) = w''(n) = 0$ and $w'''' = \lambda_k c w$. Set $z := w''$; then z should be a solution of the problem

$$
\begin{gathered}
z'' = \lambda_k c w, \quad t \in (m,n), \\
z(m) = z(n) = 0.
\end{gathered} \tag{44}
$$

The Strong Maximum Principle implies that $z < 0$ in (m,n). This follows that $(\varphi_k^\gamma)''(t) \le 0$ in $[\xi_1, \eta_1]$. It follows that $H_{\xi_1} \le 0$.

Next, we will prove that $H_{\eta_1} \le 0$. If $\varphi_k^\gamma(\eta_1) \ne 0$, then $H_{\eta_1} = 0$. If $\varphi_k^\gamma(\eta_1) = 0$, noting the conclusion of Lemma 3, then $(\varphi_k^\gamma)'(\eta_1) < 0$. By L'Hospital rule, we have that

$$H_{\eta_1} = -\frac{\left(y_n'(\eta_1)\right)^2(\varphi_k^\gamma)''(\eta_1)}{(\varphi_k^\gamma)'(\eta_1)}. \tag{45}$$

By (42), we can show that $H_{\eta_1} \le 0$. Therefore, the left-hand side of (39) ≤ 0.

By (42), we have $B_1 \ge 0$.

It follows that

$$
\begin{aligned}
0 \ge A_1 &= \int_{\xi_1}^{\eta_1}\frac{y_n}{\varphi_k^\gamma}\left[y_n \cdot L\varphi_k^\gamma - \varphi_k^\gamma \cdot Ly_n\right]dt \\
&= \int_{\xi_1}^{\eta_1}(\lambda_k - \lambda_n)a(t)y_n^2 dt \\
&\quad - \int_{\xi_1}^{\eta_1} y_n\left[f_n(t) + g_n(t)\right]dt.
\end{aligned} \tag{46}
$$

Similarly, we can also show that

$$
\begin{aligned}
0 \ge A_2 &= \int_{\xi_2}^{\eta_2}\frac{\varphi_k^\gamma}{y_n}\left[\varphi_k^\gamma \cdot Ly_n - y_n \cdot L\varphi_k^\gamma\right]dt \\
&= \int_{\xi_2}^{\eta_2}(\lambda_n - \lambda_k)a(t)(\varphi_k^\gamma)^2 dt \\
&\quad + \int_{\xi_2}^{\eta_2}\frac{(\varphi_k^\gamma)^2}{y_n}\left[f_n(t) + g_n(t)\right]dt.
\end{aligned} \tag{47}
$$

By (35), (46), and (47), taking the limit as $n \to +\infty$, we can obtain that

$$
\begin{gathered}
\int_{\xi_1}^{\eta_1}(\lambda_k - \lambda)a(t)y^2 dt - \lim_{n\to\infty}\int_{\xi_1}^{\eta_1} f_n(t)\,y\,dt \le 0, \\
\int_{\xi_2}^{\eta_2}(\lambda - \lambda_k)a(t)(\varphi_k^\gamma)^2 dt + \lim_{n\to\infty}\int_{\xi_2}^{\eta_2}\frac{(\varphi_k^\gamma)^2}{y_n}f_n(t)\,dt \\
\le 0.
\end{gathered} \tag{48}
$$

By (36), y and ϕ_k^γ have the same sign in (ξ_i, η_i) $(i = 1,2)$; one has

$$
\begin{gathered}
\left|\lim_{n\to\infty}\int_{\xi_1}^{\eta_1} f_n(t)\,y\,dt\right| \le M_1\int_{\xi_1}^{\eta_1} y^2 dt, \\
\left|\lim_{n\to\infty}\int_{\xi_2}^{\eta_2}\frac{(\varphi_k^\gamma)^2}{y_n}f_n(t)\,dt\right| \le M_1\int_{\xi_2}^{\eta_2}(\varphi_k^\gamma)^2 dt.
\end{gathered} \tag{49}
$$

So, by (48), we have that

$$\int_{\xi_1}^{\eta_1} (\lambda_k - \lambda) a(t) y^2 dt - M_1 \int_{\xi_1}^{\eta_1} y^2 dt \le 0,$$
$$\int_{\xi_2}^{\eta_2} (\lambda - \lambda_k) a(t) (\varphi_k^\nu)^2 dt - M_1 \int_{\xi_2}^{\eta_2} (\varphi_k^\nu)^2 dt \le 0. \tag{50}$$

Furthermore, it follows that

$$\text{if } \lambda \le \lambda_k,$$
$$\int_{\xi_1}^{\eta_1} [(\lambda_k - \lambda) a_0(t) - M_1] y^2 dt \le 0, \tag{51}$$
$$\text{hence } \lambda \ge \lambda_k - d_1,$$

$$\text{if } \lambda \ge \lambda_k,$$
$$\int_{\xi_2}^{\eta_2} [(\lambda - \lambda_k) a_0 dt - M_1] (\varphi_k^\nu)^2 dt \le 0, \tag{52}$$
$$\text{hence } \lambda \le \lambda_k + d_1.$$

Therefore, we have that $\lambda \in I_k$. □

Lemma 14. *For $\nu \in \{+, -\}$, the component $\mathscr{C}_k^\nu$ of $\mathcal{S}_k^\nu \cup (I_k \times \{0\})$ satisfies $\mathscr{C}_k^\nu \subset \Phi_k^\nu \cup (I_k \times \{0\})$, where $\mathscr{C}_k^\nu (\nu = +, -)$ is given by Theorem 10.*

Proof. We only prove the case of $\mathscr{C}_k^+$ since the case of $\mathscr{C}_k^-$ is similar. For any $(\lambda, x) \in \mathscr{C}_k^+$, there are two possibilities: (i) $x \in S_k^+$ or (ii) $x \in \partial S_k^+$. It is obvious that $(\lambda, x) \in \Phi_k^+$ in the case of (i). While case (ii) implies that x has at least one double zero in $[0, 1]$, Lemma 4 follows that $x \equiv 0$. Hence, there exists a sequence $(\lambda_n, x_n) \in \Phi_k^+$ such that (λ_n, x_n) is a solution of problem (27) corresponding to $\epsilon = 0$, and (λ_n, x_n) converges to $(\lambda, 0)$ in $\mathbb{R} \times E$. By Lemma 13, we have $\lambda \in I_k$: that is, $(\lambda, x) \in I_k \times \{0\}$ in the case of (ii). Hence, $\mathscr{C}_k^+ \subset \Phi_k^+ \cup (I_k \times \{0\})$. □

Proof of Theorem 10. We only prove the case of $\mathscr{C}_k^+$ since the case of $\mathscr{C}_k^-$ is similar. Let $\mathscr{C}_k^+$ be the component of $\mathcal{S}_k^+ \cup (I_k \times \{0\})$, containing $I_k \times \{0\}$. By Lemma 14, we can show that $\mathscr{C}_k^+ \subset \Phi_k^+ \cup (I_k \times \{0\})$.

Suppose on the contrary that $\mathscr{C}_k^+$ is bounded. By Lemma 11, we can find a neighborhood $\mathcal{O}$ of $\mathscr{C}_k^+$ such that $\partial\mathcal{O} \cap \mathcal{S}_k^+ = \emptyset$.

In order to complete the proof of this theorem, we consider problem (27). By Lemma 1, there are two continua $\mathscr{C}_{k,\epsilon}^+$ and $\mathscr{C}_{k,\epsilon}^-$, consisting of the bifurcation branch $\mathscr{C}_{k,\epsilon}$. By Lemma 12, we have that $\mathscr{C}_{k,\epsilon}^+$ and $\mathscr{C}_{k,\epsilon}^-$ are both unbounded and

$$\mathscr{C}_{k,\epsilon}^\nu \cap \mathbb{O}_k \subset (\Phi_k^\nu \cup \{(\lambda_k, 0)\}). \tag{53}$$

So there exists $(\lambda_\epsilon, x_\epsilon) \in \mathscr{C}_{k,\epsilon}^+ \cap \partial\mathcal{O}$ for all $\epsilon > 0$. Since $\mathcal{O}$ is bounded in $\mathbb{R} \times E$, (27) shows that $(\lambda_\epsilon, x_\epsilon)$ is bounded in $\mathbb{R} \times C^4$ independently of ϵ. By the compactness of L^{-1}, one can find a sequence $\epsilon_n \to 0$ such that $(\lambda_\epsilon, x_\epsilon)$ converges to a solution (λ, x) of (6). So $x \in \overline{S_k^+}$. If $x \in \partial S_k^+$, then from Lemma 4 or Remark 5 it follows that $x \equiv 0$. Note that $(\lambda, x) \in \mathscr{C}_{k,\epsilon}^+ \cap \partial\mathcal{O}$ since $\mathscr{C}_{k,\epsilon}^+ \cap \partial\mathcal{O}$ is closed subset of $\mathbb{R} \times E$. By Lemma 13, $\lambda \in I_k$, which contradicts the definition of $\mathcal{O}$. On the other hand, if $x \in S_k^+$, then $(\lambda, x) \in \mathcal{S}_k^+ \cap \partial\mathcal{O}$ which contradicts $\mathcal{S}_k^+ \cap \partial\mathcal{O} = \emptyset$. □

From Theorem 10 and its proof, we can easily get the following two corollaries.

Corollary 15. *There exist two unbounded subcontinua $\mathscr{D}_{k,0}^+$ and $\mathscr{D}_{k,0}^-$ of solutions of (6) in $\mathbb{R} \times E$, bifurcating from $I_k \times \{0\}$, and $\mathscr{D}_{k,0}^\nu \subset (\mathbb{R} \times S_k^\nu) \cup (I_k \times \{0\})$ for $\nu = +$ and $\nu = -$.*

We relax the assumption of a as the following:

($A1$) $a(t) \in C[0, 1]$ *is a sign-changing weight.*

Corollary 16. *Let ($A1$) hold and $f \equiv 0$. Then there exist two unbounded subcontinua $\pounds_k^+$ and $\pounds_k^-$ of solutions of (6) in $\mathbb{R} \times E$, bifurcating from $(\lambda_k, 0)$, and $\pounds_k^\nu \subset (\mathbb{R} \times S_k^\nu) \cup (I \times \{0\})$ for $\nu = +$ and $\nu = -$.*

4. Spectrum of Half-Quasi-Linear Problems

In this section, we consider the half-linear problem (7). Problem (7) is called half-linear because it is positive homogeneous in the cones $u > 0$ and $u < 0$. Similar to that of [20], we say that λ is a half-eigenvalue of problem (7) if there exists a nontrivial solution (λ, u_λ). λ is said to be simple if $\nu = cu_\lambda$, $c > 0$ for all solutions (λ, ν) of problem (7).

In order to prove our main results, we need the following Sturm type comparison result.

Lemma 17. *Let $b_2(t) \ge \max\{b_1(t), b_1(t) + \alpha(t) + \beta(t), b_1(t) - \alpha(t) - \beta(t)\}$ for $t \in (0, 1)$ and $b_i(t) \in C(0, 1)$, $i = 1, 2$. Also let u_1, u_2 be solutions of the following differential equations:*

$$u'''' = b_i(t) u + \alpha u^+ + \beta u^-, \quad t \in (0, 1), \ i = 1, 2, \tag{54}$$

respectively. If $(c, d) \subseteq (0, 1)$, $u_1(c) = u_1''(c) = 0$, and $u_1'(c) \ne 0$ or $u_1'''(c) \ne 0$ and $u_1(d) = u_1''(d) = 0$, and $u_1'(d) \ne 0$ or $u_1'''(d) \ne 0$, and $u_1(t) \ne 0$ in (c, d), then either there exists $\tau \in (c, d)$ such that $u_2(\tau) = 0$ or $b_2 = b_1$ or $b_2 = b_1 + \alpha + \beta$ or $b_2 = b_1 - \alpha - \beta$ and $u_2(t) = \mu u_1(t)$ for some constant $\mu \ne 0$.

Proof. We discuss four cases.

Case 1 ($u_1(t) > 0, u_2(t) > 0$ in (c, d)). Then an easy calculation shows that

$$\int_c^d \left(u_1'''' u_2 - u_2'''' u_1\right) dt = \int_c^d (b_1 - b_2) u_1 u_2 dt < 0. \tag{55}$$

Similar to the step of the proof of Lemma 3.1 of [16], we can obtain the left-hand side of (55) $\ge$ 0. This is a contradiction.

Case 2 ($u_1(t) > 0$, $u_2(t) < 0$ in (c,d)). Similar to (55), we can get

$$\int_c^d \left(u_1'''' u_2 - u_2'''' u_1\right) dt = \int_c^d (b_1 - b_2 + \alpha + \beta) u_1 u_2 dt > 0. \quad (56)$$

Similar to the step of the proof of Lemma 3.1 of [16], we can obtain that the left-hand side of (56) ≤ 0. This is a contradiction.

Case 3 ($u_1(t) < 0$, $u_2(t) < 0$ in (c,d)). Similar to Case 1, we can get the result.

Case 4 ($u_1(t) < 0$, $u_2(t) > 0$ in (c,d)). We can get

$$\int_c^d \left(u_1'''' u_2 - u_2'''' u_1\right) dt = \int_c^d (b_1 - b_2 - \alpha - \beta) u_1 u_2 dt > 0. \quad (57)$$

Similar to the proof of Lemma 3.1 of [16], we can obtain the left-hand side of (57) ≤ 0. This is a contradiction. □

By Lemma 17, we obtain the following result that will be used later.

Lemma 18. *Let $I = [a,b]$ be such that $I \subset [0,1]$ and*

$$\text{meas}(I) > 0. \quad (58)$$

Let $g_n \in C([0,1],\mathbb{R})$ be such that

$$\lim_{n\to+\infty} g_n(t) = +\infty \quad \text{uniformly on } I. \quad (59)$$

Let $y_n \in E$ be a solution of the equation

$$y_n''''(t) = g_n(t) y_n + \alpha y_n^+ + \beta y_n^-, \quad t \in I. \quad (60)$$

Then the number of zeros of y_n in I goes to infinity as $n \to +\infty$.

Proof. Set $\alpha^0 := \max_{t\in[0,1]}|\alpha(t)|$ and $\beta^0 := \max_{t\in[0,1]}|\beta(t)|$. By simple computation, we can show that

$$g_n(t) + \alpha\frac{y_n^+}{y_n} + \beta\frac{y_n^-}{y_n} \ge g_n(t) - \alpha^0 - \beta^0 \quad \forall t \in [0,1]. \quad (61)$$

After taking a subsequence if necessary, we may assume that

$$g_{n_j}(t) - \alpha^0 - \beta^0 \ge \lambda_j \quad \forall t \in I \quad (62)$$

as $j \to +\infty$, where λ_j is the jth eigenvalue of the following problem:

$$u'''' = \lambda u, \quad 0 < t < 1,$$
$$u(a) = u(b) = u''(a) = u''(b) = 0. \quad (63)$$

Let φ_j be the corresponding eigenfunction of λ_j. It is easy to check that the distance between any two consecutive zeros of φ_j is $(a-b)/j$ (also see [30]).

Hence, the number of zeros of $\varphi_j|_I$ goes to infinity as $j \to +\infty$. Note that the conclusion of Lemma 17 also is valid if $\alpha = \beta = 0$. Using these facts and Lemma 17, we can obtain the desired results. □

On the basis of the unilateral global interval bifurcation result, we establish the spectrum of the half-linear problem (8). More precisely, we will use Theorem 10 to prove the following result.

Theorem 19. *There exist two sequences of simple half-eigenvalues for problem (7), $\lambda_1^+ < \lambda_2^+ \cdots < \lambda_k^+ < \cdots$ and $\lambda_1^- < \lambda_2^- \cdots < \lambda_k^- < \cdots$. The corresponding half-linear solutions are in $\{\lambda_k^+\}\times S_k^+$ and $\{\lambda_k^-\}\times S_k^-$. Furthermore, aside from these solutions and the trivial ones, there is no other solutions of problem (7).*

Proof of Theorem 19. By Theorem 10, we know that there exists at least one solution of problem (7), $(\lambda_k^\nu, x_k^\nu) \in \mathbb{R}\times S_k^\nu$, for every $k = 1,2,\ldots$, $\nu = -$, and $\nu = -$. The positive of problem (7) implies that $\{(\lambda_k^\nu, c x_k^\nu), c > 0\}$ are half-linear solutions in $\{\lambda_k^\nu\} \times S_k^\nu$. Lemma 4 implies that any nontrivial solution u of problem (7) lies in some S_k^ν. We claim that, for any solution (λ, x) of problem (7) with $x \in S_k^\nu$, we have that $\lambda = \lambda_k^\nu$ and $x = h x_k^\nu$ k for some positive constant h.

Next, we only prove the case of $\nu = +$ since the case of $\nu = -$ is similar.

We divide the proof into three steps.

Step 1 (we show that $\lambda = \lambda_k^\nu$). We may assume without loss of generality that the first generalized simple zero of $x x_k^\nu$ to occur in $(0,1)$ is a generalized simple zero of x. That is, there exists $c \in (0,1]$ such that $x(c) = x''(c) = 0$ and $x'(c) \ne 0$ or $x'''(c) \ne 0$, and x and x_k^ν do not vanish and have the same sign in $(0,c)$. It follows that

$$\int_0^c \left(x'''' x_k^\nu - (x_k^\nu)'''' x\right) dt = \int_0^c (\lambda - \lambda_k^\nu) x x_k^\nu dt. \quad (64)$$

Similar to the proof of Lemma 3.1 of [16], we can obtain the left-hand side of (64) ≥ 0. So, one has that $\lambda_k^\nu \le \lambda$.

On the other hand, similar to the proof of Lemma 13, by Lemma 6, there must exist an interval $(d,e) \subset (0,1)$ such that x and x_k^ν do not vanish and have the same sign in (d,e), and $x_k^\nu(d) = x_k^\nu(e) = (x_k^\nu)''(d) = (x_k^\nu)''(e) = 0$. We have

$$\int_d^e \left((x_k^\nu)'''' x - x'''' x_k^\nu\right) dt = \int_d^e (\lambda_k^\nu - \lambda) a(t) x_k^\nu x \, dt. \quad (65)$$

Similar to the proof of Lemma 3.1 of [16], we can obtain the left-hand side of (65) ≥ 0. So, one has that $\lambda \le \lambda_k^\nu$. Hence $\lambda = \lambda_k^\nu$.

Step 2 (we will prove that $x = h x_k^\nu$ for some positive constant h). Without loss of generality, we may assume that x and x_k^ν

are positive in $(0, c)$. By the Picone identity in [29, Theorem 4], noting $\lambda = \lambda_k^\nu$, we have that

$$\int_0^c \left\{ \frac{x}{x_k^\nu} \left[x (x_k^\nu)''' - x_k^\nu x''' \right] + \frac{x'}{x_k^\nu} \left[x_k^\nu x'' - x (x_k^\nu)'' \right] - x (x_k^\nu)'' \left(\frac{x}{x_k^\nu} \right)' \right\}' dt = C_1, \quad (66)$$

where

$$C_1 = \int_0^c \left\{ \left[x'' - \frac{x}{x_k^\nu} (x_k^\nu)'' \right]^2 - 2 x_k^\nu (x_k^\nu)'' \left[\left(\frac{x}{x_k^\nu} \right)' \right]^2 \right\} dt. \quad (67)$$

Using similar methods of the proof of Lemma 13, we can show that the left-hand side of (66) ≤ 0. Hence, $C_1 \le 0$. It follows that

$$\int_0^c \left\{ \left[x'' - \frac{x}{x_k^\nu} (x_k^\nu)'' \right]^2 \right\} dt \le 2 \int_0^c \left\{ x_k^\nu (x_k^\nu)'' \left[\left(\frac{x}{x_k^\nu} \right)' \right]^2 \right\} dt \le 0. \quad (68)$$

Thus, $[(x/x_k^\nu)']^2 = 0$. Furthermore, we obtain that $x = h_1 x_k^\nu$ on $(0, c)$ for some positive constant h_1. We may assume without loss of generality that the first zero of $x x_k$ to occur in $(c, 1]$ is a generalized simple zero of x. That is, there exists $c_1 \in (c, 1]$ such that $x(c_1) = x''(c_1) = 0$ and $x'(c_1) \ne 0$ or $x'''(c_1) \ne 0$, and x and x_k^ν do not vanish and have the same sign in $(c, c_1]$. Using method similar to the above, we can show that $x = h_2 x_k^\nu$ on $(c, c_1]$ for some positive constant h_2. Clearly, $x'(c) = h_1 (x_k^\nu)'(c) = h_2 (x_k^\nu)'(c)$ and Lemma 3 imply $h_1 = h_2$. Repeating the above process k times, we can show that $x = h x_k^\nu$ for some positive constant h.

Step 3 (we prove that λ_k^ν ($\nu = +$ or $\nu = -$) are increasing). In fact, if (λ_k^ν, u) and (λ_j^ν, v) are the solutions of problem (7) with $u \in S_k^\nu$, $v \in S_j^\nu$, and $k < j$, the first generalized simple zero of uv to occur in $(0, 1)$ is a generalized simple zero of v. Indeed, if this were not, by Lemma 6, using method similar to the proof in Step 1, we could obtain $\lambda_k^\nu < \lambda_j^\nu$, which is impossible, since the half-eigenvalues were shown to be simple. Therefore, by Lemma 17, we can get $\lambda_k^\nu < \lambda_j^\nu$.

Naturally, we can consider the bifurcation structure of the perturbation of problem (7) of the form

$$x'''' = \lambda a(t) x + \alpha x^+ + \beta x^- + g(t, x, \lambda), \quad 0 < t < 1, \quad (69)$$
$$x(0) = x(1) = x''(0) = x''(1) = 0,$$

where $g(t, x, \lambda)$ satisfies (4). □

Theorem 20. *For $\nu = +, -$, $(\lambda_k^\nu, 0)$ is a bifurcation point for problem (69). Moreover, there exists an unbounded continuum $\mathscr{D}_{k,0}^\nu$ of solutions of problem (69) emanating from $(\lambda_k^\nu, 0)$, such that $\mathscr{D}_{k,0}^\nu \subset ((\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k^\nu, 0)\})$.*

Proof. Let $\alpha^0 := \max_{t\in[0,1]} |\alpha(t)|$ and $\beta^0 := \max_{t\in[0,1]} |\beta(t)|$:

$$I_k = \left[\lambda_k - \frac{\alpha^0 + \beta^0}{a_0}, \lambda_k + \frac{\alpha^0 + \beta^0}{a_0} \right]. \quad (70)$$

Corollary 15 shows that there exist two unbounded subcontinua $\mathscr{D}_{k,0}^+$ and $\mathscr{D}_{k,0}^-$ of solutions of (69) in $\mathbb{R} \times E$, bifurcating from $I_k \times \{0\}$, and $\mathscr{D}_{k,0}^\nu \subset (\mathbb{R} \times S_k^\nu) \cup (I_k \times \{0\})$ for $\nu = +$ and $\nu = -$. Let us show that $\mathscr{D}_{k,0}^\nu \cap (\mathbb{R} \times \{0\}) = (\lambda_k^\nu, 0)$; that is, $(\lambda_k^\nu, 0)$ is a bifurcation point for problem (69). Indeed, if there exists (λ_n, x_n) being a sequence of solutions of problem (69) converging to $(\lambda, 0)$, let $y_n = x_n / \|x_n\|$, and then y_n should be a solution of problem

$$y_n = L^{-1} \left(\lambda_n a y_n + \alpha y_n^+ + \beta y_n^- + \frac{g(t, x_n, \lambda)}{\|x_n\|} \right). \quad (71)$$

By (18) and the compactness of L^{-1}, we obtain that for some convenient subsequence $y_n \to y_0$ as $n \to +\infty$. Now y_0 verifies the equation

$$L y_0 = \lambda a(t) y_0 + \alpha y_0^+ + \beta y_0^- \quad (72)$$

and $\|y_0\| = 1$. This implies that $\lambda = \lambda_k^\nu$ for some $k \in N$ and $\nu \in \{+, -\}$. □

5. Nodal Solutions for Half-Linear Eigenvalue Problems

We start this section by studying the following eigenvalue problem:

$$x'''' = \alpha x^+ + \beta x^- + \lambda r a(t) f(x), \quad 0 < t < 1, \quad (73)$$
$$x(0) = x(1) = x''(0) = x''(1) = 0,$$

where $\lambda > 0$ is a parameter.

Let $\zeta(x), \xi(x) \in C(\mathbb{R}, \mathbb{R})$ be such that

$$f(x) = f_0 x + \zeta(x), \quad (74)$$
$$f(x) = f_\infty x + \xi(x)$$

with

$$\lim_{|x|\to 0}\frac{\zeta(x)}{x}=0,$$
$$\lim_{|x|\to\infty}\frac{\xi(x)}{x}=0. \tag{75}$$

Let us consider

$$x''''=\lambda ra(t)f_0x+\alpha x^++\beta x^-+\lambda ra(t)\zeta(x),\quad 0<t<1, \tag{76}$$
$$x(0)=x(1)=x''(0)=x''(1)=0$$

as a bifurcation problem from the trivial solution $x\equiv 0$, and

$$x''''=\lambda ra(t)f_\infty x+\alpha x^++\beta x^-+\lambda ra(t)\xi(x),\quad 0<t<1, \tag{77}$$
$$x(0)=x(1)=x''(0)=x''(1)=0$$

as a bifurcation problem from infinity.

Applying Theorem 20 to problem (76), we have the following result.

Lemma 21. *Let* $(H1)$, $(H4)$, $(H5)$, *and* $(H6)$ *hold. For* $\nu=+,-$, $(\lambda_k^\nu/rf_0,0)$ *is a bifurcation point for problem (76). Moreover, there exists an unbounded continuum* $\mathscr{D}_{k,0}^\nu$ *of solutions of problem (76) that joins* $(\lambda_k^\nu/rf_0,0)$ *to infinity, such that* $\mathscr{D}_{k,0}^\nu\subset((\mathbb{R}\times S_k^\nu)\cup\{(\lambda_k^\nu/rf_0,0)\})$.

We add the points $\{(\lambda,\infty)\mid\lambda\in\mathbb{R}\}$ *to space* $\mathbb{R}\times E$; *by the results of Rabinowitz [31], we have the following lemma.*

Lemma 22. *Let* $(H1)$, $(H4)$, $(H5)$, *and* $(H6)$ *hold. For* $\nu=+,-$, $(\lambda_k^\nu/rf_\infty,\infty)$ *is a bifurcation point for problem (77). Moreover, there exists an unbounded continuum* $\mathscr{D}_{k,\infty}^\nu$ *of solutions of problem (77) meeting* $(\lambda_k^\nu/rf_\infty,\infty)$, *such that* $\mathscr{D}_{k,\infty}^\nu\subset((\mathbb{R}\times S_k^\nu)\cup\{(\lambda_k^\nu/rf_\infty,\infty)\})$.

We note that problem (76) and problem (77) are the same, and each of them is equivalent to problem (73). By Lemmas 21 and 22, we obtain the following lemma.

Lemma 23. *Let* $(H1)$, $(H4)$, $(H5)$, *and* $(H6)$ *hold. There exists an unbounded continuum* $\mathscr{D}_k^\nu=\mathscr{D}_{k,0}^\nu=\mathscr{D}_{k,\infty}^\nu$ *of solutions of problem (73) emanating from* $(\lambda_k^\nu/rf_0,0)$ *or* $(\lambda_k^\nu/rf_\infty,\infty)$, *such that* $\mathscr{D}_k^\nu\subset((\mathbb{R}\times S_k^\nu)\cup\{(\lambda_k^\nu/rf_0,0)\})$ *and* $\mathscr{D}_k^\nu\subset((\mathbb{R}\times S_k^\nu)\cup\{(\lambda_k^\nu/rf_\infty,\infty)\})$, *and* $\mathscr{D}_k^\nu$ *joins* $(\lambda_k^\nu/rf_0,0)$ *to* $(\lambda_k^\nu/rf_\infty,\infty)$.

Remark 24. Any solution of (73) of the form $(1,x)$ yields a solution x of (8). In order to prove our main results, one will only show that $\mathscr{D}_k^\nu$ crosses the hyperplane $\{1\}\times E$ in $\mathbb{R}\times E$.

Remark 25. From $(H5)$ and $(H6)$, we can see that there exists a positive constant Q such that $f(s)/s\ge Q$ for all $s\ne 0$.

Theorem 26. *Let* $(H1)$, $(H4)$, $(H5)$, *and* $(H6)$ *hold. For* $\nu=+,-$, *either* $\lambda_k^\nu/f_\infty<r<\lambda_k^\nu/f_0$ *or* $\lambda_k^\nu/f_0<r<\lambda_k^\nu/f_\infty$. *Then problem (8) possesses a solution* x_k^ν *such that* x_k^ν *has exactly* $k-1$ *zeros in* $(0,1)$ *and* νx_k^ν *is positive near* 0.

Proof. By Remark 24, in order to prove Theorem 26, we only show that $\mathscr{D}_k^\nu$ joins $(\lambda_k^\nu/f_0,0)$ to $(\lambda_k^\nu/f_\infty,\infty)$. Let $(\mu_n,x_n)\in\mathscr{D}_k^\nu$ satisfy

$$\mu_n+\|x_n\|\longrightarrow\infty. \tag{78}$$

We note that $\mu_n>0$ for all $n\in\mathbb{N}$ since $(0,0)$ is the only solution of (73) for $\lambda=0$ and $\mathscr{D}_k^\nu\cap(\{0\}\times E)=\emptyset$.

Case 1 ($\lambda_k^\nu/f_\infty<r<\lambda_k^\nu/f_0$). In this case, we show that

$$\left(\frac{\lambda_k^\nu}{rf_\infty},\frac{\lambda_k^\nu}{rf_0}\right)\subseteq\{\lambda\in\mathbb{R}\mid(\lambda,x)\in\mathscr{D}_k^\nu\}. \tag{79}$$

We divide the proof into two steps.

Step 1. We show that if there exists a constant number $M>0$ such that

$$\mu_n\in(0,M], \tag{80}$$

for n large enough, then $\mathscr{D}_k^\nu$ joins $(\lambda_k^\nu/rf_0,0)$ to $(\lambda_k^\nu/rf_\infty,\infty)$.

In this case

$$\|x_n\|\longrightarrow\infty. \tag{81}$$

Let $\xi(\cdot)\in C(\mathbb{R},\mathbb{R})$ be such that

$$f(x)=f_\infty x+\xi(x) \tag{82}$$

with

$$\lim_{|x|\to\infty}\frac{\xi(x)}{x}=0. \tag{83}$$

Let $\overline{\xi}(x)=\max\{|\xi(s)|:0\le|s|\le x\}$, then $\overline{\xi}$ is nondecreasing, and

$$\lim_{x\to+\infty}\frac{\overline{\xi}(x)}{x}=0. \tag{84}$$

Moreover, from (84) and the fact that $\overline{\xi}$ is nondecreasing, we have that

$$\lim_{n\to\infty}\frac{\xi(x_n(t))}{\|x_n\|}=0,\quad\forall t\in[0,1], \tag{85}$$

since

$$\frac{\xi(x_n(t))}{\|x_n\|}\le\frac{\overline{\xi}(|x_n(t)|)}{\|x_n\|}\le\frac{\overline{\xi}(\|x_n(t)\|_\infty)}{\|x_n\|}$$
$$\le\frac{\overline{\xi}(\|x_n(t)\|)}{\|x_n\|}\longrightarrow 0,\qquad n\longrightarrow\infty\ \forall t\in[0,1]. \tag{86}$$

We divide the equation

$$\begin{aligned} x_n'''' &= \mu_n r a(t) f_\infty x_n + \alpha x_n^+ + \beta x_n^- \\ &\quad + \mu_n r a(t) \xi(x_n), \quad 0 < t < 1, \\ x_n(0) &= x_n(1) = x_n''(0) = x_n''(1) = 0 \end{aligned} \tag{87}$$

by $\|x_n\|$ and set $y_n = x_n/\|x_n\|$. Since y_n is bounded in E, choosing a subsequence and relabeling if necessary, we have that $y_n \to y$ for some $y \in E$ with $\|y\| = 1$.

By the compactness of L^{-1}, we obtain that

$$\begin{aligned} y'''' &= \mu r a(t) f_\infty y + \alpha y^+ + \beta y^-, \quad 0 < t < 1, \\ y(0) &= y(1) = y''(0) = y''(1) = 0, \end{aligned} \tag{88}$$

where $\mu := \lim_{n\to\infty}\mu_n$, again choosing a subsequence and relabeling if necessary.

It is clear that $\|y\| = 1$ and $y \in \overline{\mathscr{D}_k^\nu} \subseteq \mathscr{D}_k^\nu$ since $\mathscr{D}_k^\nu$ is closed in $\mathbb{R}\times E$. Moreover, by Theorem 19, $\mu r f_\infty = \lambda_k^\nu$, so that

$$\mu = \frac{\lambda_k^\nu}{r f_\infty}. \tag{89}$$

Thus, $\mathscr{D}_k^\nu$ joins $(\lambda_k^\nu/rf_0, 0)$ to $(\lambda_k^\nu/rf_\infty, \infty)$.

Step 2. We show that there exists a constant number $M > 0$ such that $\mu_n \in (0, M]$, for n large enough.

On the contrary, we suppose that $\lim_{n\to+\infty}|\mu_n| = +\infty$. Since $(\mu_n, x_n) \in \mathscr{D}_k^\nu$, it follows from the compactness of L^{-1} that

$$\begin{aligned} x_n'''' &= \mu_n r a(t) \widetilde{f_n}(t) x_n + \alpha x_n^+ + \beta x_n^-, \quad 0 < t < 1, \\ x_n(0) &= x_n(1) = x_n''(0) = x_n''(1) = 0, \end{aligned} \tag{90}$$

where $\widetilde{f_n} : [0,1]\times[0,\infty) \to \mathbb{R}$ by

$$\widetilde{f_n}(t) = \begin{cases} \dfrac{f(x_n)}{x_n}, & x_n \neq 0,\ t \in [0,1], \\ f_0, & x_n = 0,\ t \in [0,1]. \end{cases} \tag{91}$$

By Remark 25, we have

$$\lim_{n\to+\infty} \mu_n r \widetilde{f_n}(t) = \pm\infty. \tag{92}$$

Let φ_k^ν be an eigenfunction corresponding to λ_k^ν. But if $\lim_{n\to+\infty}\mu_n r\widetilde{f_n}(t) = -\infty$, applying Lemma 17 to x_n and φ_k^ν, we have that φ_k^ν must change sign for n large enough, which is impossible. So $\lim_{n\to+\infty}\mu_n r\widetilde{f_n}(t) = +\infty$. By Lemma 18, we get that x_n must change sign for n large enough, and this contradicts the fact that $x_n \in S_k^\nu$.

Case 2 ($\lambda_k^\nu/f_0 < r < \lambda_k^\nu/f_\infty$). In this case, if $(\mu_n, x_n) \in \mathscr{D}_k^\nu$ is such that

$$\begin{gathered} \lim_{n\to\infty}(\mu_n + \|x_n\|) = \infty, \\ \lim_{n\to\infty}\mu_n = \infty, \end{gathered} \tag{93}$$

then

$$\left(\frac{\lambda_k^\nu}{rf_0}, \frac{\lambda_k^\nu}{rf_\infty}\right) \subseteq \{\lambda \in (0,\infty) \mid (\lambda, x) \in \mathscr{D}_k^\nu\} \tag{94}$$

and, moreover, $(\{1\}\times E) \cap \mathscr{D}_k^\nu \neq \emptyset$.

Assume that there exists $M > 0$ such that, for all $n \in \mathbb{N}$,

$$\mu_n \in (0, M]. \tag{95}$$

Applying a similar argument to that used in Step 1 of Case 1, after taking a subsequence and relabeling if necessary, it follows that

$$(\mu_n, x_n) \longrightarrow \left(\frac{\lambda_k^\nu}{rf_\infty}, \infty\right), \quad n \longrightarrow \infty. \tag{96}$$

Again $\mathscr{D}_k^\nu$ joins $(\lambda_k^\nu/rf_0, 0)$ to $(\lambda_k^\nu/rf_\infty, \infty)$ and the result follows. □

Theorem 27. *Let* $(H1)$, $(H4)$, $(H5)$, *and* $(H7)$ *hold. For* $\nu \in \{+,-\}$, *assume that one of the following conditions holds.*

(i) $r \in (0, \lambda_k^\nu/f_0)$ *for* $\lambda_k^\nu > 0$.
(ii) $r \in (0, \lambda_k^+/f_0) \cup (\lambda_k^-/f_0, 0)$ *for* $\nu\lambda_k^\nu > 0$.
(iii) $r \in (0, \lambda_k^-/f_0) \cup (\lambda_k^+/f_0, 0)$ *for* $\nu\lambda_k^\nu < 0$.
(iv) $r \in (\lambda_k^\nu/f_0, 0)$ *for* $\lambda_k^\nu < 0$.

Then problem (8) possesses two solutions x_k^+ *and* x_k^- *such that* x_k^+ *has exactly* $k-1$ *zeros in* $(0,1)$ *and is positive near 0 and* x_k^- *has exactly* $k-1$ *zeros in* $(0,1)$ *and is negative near 0.*

Proof. We will only prove the case of (i) since the proofs of the cases for (ii), (iii), and (iv) are completely analogous.

Inspired by the idea of [32], we define the cut-off function of f as the following:

$$f^{[n]}(s) := \begin{cases} f(s), & s \in [-n, n], \\ \dfrac{2n^2 - f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ \dfrac{2n^2 + f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ ns, & s \in (-\infty, -2n] \cup [2n, +\infty). \end{cases} \tag{97}$$

We consider the following problem:

$$x'''' = \alpha x^+ + \beta x^- + \lambda ra(t) f^{[n]}(x), \quad 0 < t < 1,$$
$$x(0) = x(1) = x''(0) = x''(1) = 0. \tag{98}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = f_0$, and $(f^{[n]})_\infty = n$.

Similar to the proof of Theorem 26, by Lemma 23 and Remark 24, there exists an unbounded continuum $\mathcal{D}_k^{\nu[n]}$ of solutions of problem (98) emanating from $(\lambda_k^\nu/rf_0, 0)$, such that $\mathcal{D}_k^{\nu[n]} \subset (\Phi_k^\nu \cup \{(\lambda_k^\nu/rf_0, 0)\})$, and $\mathcal{D}_k^{\nu[n]}$ joins $(\lambda_k^\nu/rf_0, 0)$ to $(\lambda_k^\nu/rn, \infty)$.

Taking $z_n = (\lambda_k^\nu/rn, \infty)$ and $z^* = (0, \infty)$, we have that $z_n \to z^*$.

So condition (i) in Lemma 9 is satisfied with $z^* = (0, \infty)$.

Obviously,

$$r_n = \sup\left\{\lambda + \|x\| \mid (\lambda, x) \in \mathcal{D}_k^{\nu[n]}\right\} = \infty, \tag{99}$$

and accordingly, (ii) in Lemma 9 holds and (iii) in Lemma 9 can be deduced directly from the Arezela-Ascoli Theorem and the definition of $f^{[n]}$.

Therefore, by Lemma 9, $\limsup_{n\to\infty} \mathcal{D}_k^{\nu[n]}$ contains an unbounded connected component $\mathcal{D}_k^\nu$ with $(0, \infty) \in \mathcal{D}_k^\nu$.

From $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, (98) can be converted to the equivalent equation (73). Since $\mathcal{D}_k^{\nu[n]} \subset \Phi_k^\nu$, we conclude that $\mathcal{D}_k^\nu \subset \Phi_k^\nu$. Moreover, $\mathcal{D}_k^\nu \subset \Sigma_k^\nu$ by (8).

In the following, we show that $(\lambda_k^\nu/rf_0, 0) \in \mathcal{D}_k^\nu$.

Let $(\mu_n, x_n) \in \mathcal{D}_k^\nu \setminus \{(\lambda_k^\nu/rf_0, 0)\}$ satisfy $\lim_{n\to\infty}\|x_n\| = 0$.

Let $\zeta \in C(\mathbb{R}, \mathbb{R})$ be such that

$$f(s) = f_0 s + \zeta(s) \tag{100}$$

with $\lim_{|s|\to 0}(\zeta(s)/s) = 0$, let $\overline{\zeta}(x) = \max_{0\le|s|\le x}|\zeta(s)|$, then $\overline{\zeta}$ is nondecreasing, and

$$\lim_{x\to 0}\frac{\overline{\zeta}(x)}{x} = 0. \tag{101}$$

We divide the equation

$$x_n'''' = \mu_n ra(t) f_0 x_n + \alpha x_n^+ + \beta x_n^- + \lambda ra(t)\zeta(x_n), \quad 0 < t < 1, \tag{102}$$
$$x_n(0) = x_n(1) = x_n''(0) = x_n''(1) = 0.$$

Let $y_n = x_n/\|x_n\|$; y_n should be the solutions of problem

$$y_n'''' = \mu_n ra(t) f_0 y_n + \alpha y_n^+ + \beta y_n^- + \mu_n ra(t)\frac{\zeta(x_n)}{\|x_n\|}, \quad t \in (0,1), \tag{103}$$
$$y_n(0) = y_n(1) = y_n''(0) = y_n''(1) = 0.$$

Since y_n is bounded in E, choosing a subsequence and relabeling if necessary, we have that $y_n \to y$ for some $y \in E$.

Furthermore, from (101) and the fact that $\overline{\zeta}$ is nondecreasing, we have that

$$\lim_{n\to\infty}\frac{\zeta(x_n)}{\|x_n\|} = 0, \tag{104}$$

since

$$\frac{\zeta(x_n)}{\|x_n\|} \le \frac{\overline{\zeta}(|x_n|)}{\|x_n\|} \le \frac{\overline{\zeta}(\|x_n\|_\infty)}{\|x_n\|} \le \frac{\overline{\zeta}(\|x_n\|)}{\|x_n\|} \longrightarrow 0, \quad n \longrightarrow +\infty. \tag{105}$$

By (104) and the compactness of L^{-1}, we obtain that

$$Ly = \mu ra(t) f_0 y + \alpha y^+ + \beta y^-. \tag{106}$$

It is clear that $\|y\| = 1$ and $y \in \overline{\mathcal{D}_k^\nu} \subseteq \mathcal{D}_k^\nu$ since $\mathcal{D}_k^\nu$ is closed in $\mathbb{R}\times E$. Moreover, by Theorem 19, $\mu rf_0 = \lambda_k^\nu$, so that

$$\mu = \frac{\lambda_k^\nu}{rf_0}. \tag{107}$$

Thus, $\mathcal{D}_k^\nu$ joins $(\lambda_k^\nu/rf_0, 0)$ to $(0, \infty)$. □

Theorem 28. *Let* $(H1)$, $(H4)$, $(H5)$, *and* $(H8)$ *hold. For* $\nu \in \{+, -\}$, *assume that one of the following conditions holds.*

(i) $r \in (\lambda_k^\nu/f_0, +\infty)$ *for* $\lambda_k^\nu > 0$.

(ii) $r \in (-\infty, \lambda_k^-/f_0) \cup (\lambda_k^+/f_0, +\infty)$ *for* $\nu\lambda_k^\nu > 0$.

(iii) $r \in (-\infty, \lambda_k^+/f_0) \cup (\lambda_k^-/f_0, +\infty)$ *for* $\nu\lambda_k^\nu < 0$.

(iv) $r \in (-\infty, \lambda_k^\nu/f_0)$ *for* $\lambda_k^\nu < 0$.

Then problem (8) possesses two solutions x_k^+ *and* x_k^- *such that* x_k^+ *has exactly* $k-1$ *zeros in* $(0,1)$ *and is positive near 0 and* x_k^- *has exactly* $k-1$ *zeros in* $(0,1)$ *and is negative near 0.*

Proof. Define

$$f^{[n]}(s) := \begin{cases} \dfrac{1}{n}s, & s \in (-\infty, -2n] \cup [2n, +\infty). \\ \dfrac{2 + f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ \dfrac{2 - f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ f(s), & s \in [-n, n]. \end{cases} \tag{108}$$

We consider the following problem:

$$x'''' = \alpha x^+ + \beta x^- + \lambda ra(t) f^{[n]}(x), \quad 0 < t < 1,$$
$$x(0) = x(1) = x''(0) = x''(1) = 0. \tag{109}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = f_0$, and $(f^{[n]})_\infty = 1/n$.

Similar to the proof of Theorem 26, by Lemma 23 and Remark 24, there exists an unbounded continuum $\mathcal{D}_k^{\nu[n]}$ of solutions of problem (109) emanating from $(\lambda_k^\nu/rf_0, 0)$, such that $\mathscr{D}_k^{\nu[n]} \subset (\Phi_k^\nu \cup \{(\lambda_k^\nu/rf_0, 0)\})$, and $\mathscr{D}_k^{\nu[n]}$ joins $(\lambda_k^\nu/rf_0, 0)$ to $(n\lambda_k^\nu/r, \infty)$.

Taking $z_n = (n\lambda_k^\nu/r, \infty)$ and $z^* = (\infty, \infty)$, we have that $z_n \to z^*$.

So condition (i) in Lemma 9 is satisfied with $z^* = (\infty, \infty)$.

Obviously,

$$r_n = \sup\left\{\lambda + \|x\| \mid (\lambda, x) \in \mathscr{D}^{\nu[n]}\right\} = \infty, \tag{110}$$

and accordingly, (ii) in Lemma 9 holds and (iii) in Lemma 9 can be deduced directly from the Arezela-Ascoli Theorem and the definition of $f^{[n]}$.

Therefore, by Lemma 9, $\limsup_{n\to\infty} \mathscr{D}_k^{\nu[n]}$ contains an unbounded connected component $\mathscr{D}_k^\nu$ with $(\infty, \infty) \in \mathscr{D}_k^\nu$.

From $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, (109) can be converted to the equivalent equation (73). Since $\mathscr{D}_k^{\nu[n]} \subset \Phi_k^\nu$, we conclude $\mathscr{D}_k^\nu \subset \Phi_k^\nu$. Moreover, $\mathscr{D}_k^\nu \subset \Sigma_k^\nu$ by (8).

Similar to the proof of Theorem 27, we can obtain that $(\lambda_k^\nu/rf_0, 0) \in \mathscr{D}_k^\nu$. □

Theorem 29. *Let (H1), (H4), (H5), and (H9) hold. For $\nu \in \{+, -\}$, assume that one of the following conditions holds.*

(i) $r \in (\lambda_k^\nu/f_\infty, +\infty)$ *for* $\lambda_k^\nu > 0$.

(ii) $r \in (\lambda_k^+/f_\infty, +\infty) \cup (-\infty, \lambda_k^-/f_\infty)$ *for* $\nu\lambda_k^\nu > 0$.

(iii) $r \in (\lambda_k^-/f_\infty, +\infty) \cup (-\infty, \lambda_k^+/f_\infty)$ *for* $\nu\lambda_k^\nu < 0$.

(iv) $r \in (-\infty, \lambda_k^\nu/f_\infty)$ *for* $\lambda_k^\nu < 0$.

Then problem (8) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near 0 and x_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near 0.

Proof. We will only prove the case of (i) since the proofs of the cases for (ii), (iii), and (iv) are completely analogous.

If (λ, x) is any nontrivial solution of problem (73), dividing problem (73) by $\|x\|^2$ and setting $y = x/\|x\|^2$ yields

$$\begin{aligned} &y'''' = \alpha y^+ + \beta y^- + \lambda r a(t) \frac{f(x)}{\|x\|^2}, \quad 0 < t < 1, \\ &y(0) = y(1) = y''(0) = y''(1) = 0. \end{aligned} \tag{111}$$

Define

$$\tilde{f}(y) := \begin{cases} \|y\|^2 f\left(\dfrac{y}{\|y\|^2}\right), & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{cases} \tag{112}$$

Evidently, problem (111) is equivalent to

$$\begin{aligned} &y'''' = \alpha y^+ + \beta y^- + \lambda r a(t) \tilde{f}(y), \quad 0 < t < 1, \\ &y(0) = y(1) = y''(0) = y''(1) = 0. \end{aligned} \tag{113}$$

It is obvious that $(\lambda, 0)$ is always the solution of problem (113). By simple computation, we can show that $\widetilde{f_0} = f_\infty \in (0, \infty)$ and $\widetilde{f_\infty} = f_0 = 0$. Similar to the proof of Theorem 28, there exists an unbounded continuum $\mathscr{C}_k^\nu$ of solutions of problem (113) emanating from $(\lambda_k^\nu/r\widetilde{f_0}, 0)$, such that $\mathscr{C}_k^\nu \subset ((\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k^\nu/r\widetilde{f_0}, 0)\})$, and $\mathscr{C}_k^\nu$ joins $(\lambda_k^\nu/r\widetilde{f_0}, 0)$ to (∞, ∞).

Under the inversion $y \to y/\|y\|^2 = x$, we obtain $\mathscr{C}_k^\nu \to \mathscr{D}_k^\nu$ being an unbounded component of solutions of problem (73) emanating from $(\infty, 0)$, such that $\mathscr{D}_k^\nu \subset ((\mathbb{R} \times S_k^\nu) \cup \{(\infty, 0)\})$, and $\mathscr{D}_k^\nu$ joins $(\infty, 0)$ to $(\lambda_k^\nu/rf_\infty, \infty)$.

Moreover, by (8), we can obtain that $\mathscr{D}_k^\nu \subset \Sigma_k^\nu$.

Thus, $\mathscr{D}_k^\nu$ is an unbounded component of solutions of problem (8) such that $\mathscr{D}_k^\nu$ joins $(\infty, 0)$ to $(\lambda_k^\nu/rf_\infty, \infty)$. □

Theorem 30. *Let (H1), (H4), (H5), and (H10) hold. For $\nu \in \{+, -\}$, assume that one of the following conditions holds.*

(i) $r \in (0, +\infty)$ *for* $\lambda_k^\nu > 0$.

(ii) $r \in (0, +\infty) \cup (-\infty, 0)$ *for* $\nu\lambda_k^\nu > 0$, *or* $\nu\lambda_k^\nu < 0$.

(iii) $r \in (-\infty, 0)$ *for* $\lambda_k^\nu < 0$.

Then problem (8) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near 0 and x_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near 0.

Proof. Define

$$f^{[n]}(s) := \begin{cases} ns, & s \in (-\infty, -2n] \cup [2n, +\infty). \\ \dfrac{2n^2 + f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ \dfrac{2n^2 - f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ f(s), & s \in \left[-n, -\dfrac{2}{n}\right] \cup \left[\dfrac{2}{n}, n\right], \\ -\left[f\left(-\dfrac{2}{n}\right) + \dfrac{1}{n^2}\right](ns+2) + f\left(-\dfrac{2}{n}\right), & s \in \left(-\dfrac{2}{n}, -\dfrac{1}{n}\right), \\ \left[f\left(\dfrac{2}{n}\right) - \dfrac{1}{n^2}\right](ns-2) + f\left(\dfrac{2}{n}\right), & s \in \left(\dfrac{1}{n}, \dfrac{2}{n}\right), \\ \dfrac{1}{n}s, & s \in \left[-\dfrac{1}{n}, \dfrac{1}{n}\right]. \end{cases} \tag{114}$$

We consider the following problem:

$$x'''' = \alpha x^+ + \beta x^- + \lambda r a(t) f^{[n]}(x), \quad 0 < t < 1,$$
$$x(0) = x(1) = x''(0) = x''(1) = 0. \tag{115}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = 1/n$, and $(f^{[n]})_\infty = n$.

Applying the similar method used in the proof of Theorem 26, by Lemma 23 and Remark 24, there exists an unbounded continuum $\mathscr{D}_k^{\nu[n]}$ of solutions of problem (115) emanating from $(n\lambda^\nu/r, 0)$ or $(\lambda^\nu/nr, \infty)$, such that $\mathscr{D}_k^{\nu[n]} \subset ((\mathbb{R}\times S_k^\nu)\cup\{(n\lambda^\nu/r, 0)\})$ and $\mathscr{D}_k^{\nu[n]} \subset ((\mathbb{R}\times S_k^\nu)\cup\{(\lambda^\nu/rn, \infty)\})$, and $\mathscr{D}_k^{\nu[n]}$ joins $(n\lambda^\nu/r, 0)$ to $(\lambda^\nu/rn, \infty)$.

Taking $z_n = (n\lambda^\nu/r, 0)$ and $z^* = (\infty, 0)$ or $z_n = (\lambda^\nu/rn, \infty)$ and $z^* = (0, \infty)$, we have that $z_n \to z^*$. By Lemma 9, we obtain an unbounded component $\mathscr{D}_k^\nu$ with $(\infty, 0), (0, \infty) \in \mathscr{D}_k^\nu$.

From $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, (115) can be converted to the equivalent equation (73). Thus, $\mathscr{D}_k^\nu$ is an unbounded component of solutions of problem (73) emanating from $(\infty, 0)$ or $(0, \infty)$, such that $\mathscr{D}_k^\nu \subset ((\mathbb{R}\times S_k^\nu)\cup\{(\infty, 0)\})$ and $\mathscr{D}_k^\nu \subset (\mathbb{R}\times S_k^\nu)\cup\{(0, \infty)\})$ and $\mathscr{D}_k^\nu$ joins $(\infty, 0)$ to $(0, \infty)$.

Since $\mathscr{D}_k^{\nu[n]} \subset (\mathbb{R}\times S_k^\nu)$, we conclude $\mathscr{D}_k^\nu \subset (\mathbb{R}\times S_k^\nu)$. Moreover, by Remark 24 and (8), we can obtain that $\mathscr{D}_k^\nu \subset \Sigma_k^\nu$.

Thus, $\mathscr{D}_k^\nu$ is an unbounded component of solutions of problem (8) emanating from $(\infty, 0)$ or $(0, \infty)$, such that $\mathscr{D}_k^\nu \subset ((\mathbb{R}\times S_k^\nu)\cup\{(0, 0)\})$ and $\mathscr{D}_k^\nu \subset ((\mathbb{R}\times S_k^\nu)\cup\{(\infty, \infty)\})$, and $\mathscr{D}_k^\nu$ joins $(\infty, 0)$ to $(0, \infty)$.

Applying the similar method used in the proof of Theorem 27, we obtain an unbounded connected component $\mathscr{D}_k^\nu \subset \mathscr{C}_k^\nu$ with $(\infty, 0) \in \mathscr{D}_k^\nu$. □

Theorem 31. *Let* $(H1)$, $(H4)$, $(H5)$, *and* $(H11)$ *hold. For* $\nu \in \{+, -\}$, *assume that one of the following conditions holds.*

(i) *There exists a* $\lambda_{\nu k}^\nu > 0$ *for* $\lambda_k^\nu > 0$, *such that* $r \in (0, \lambda_{\nu k}^\nu)$.

(ii) *There exists a* $\nu\lambda_{\nu k}^\nu > 0$ *for* $\nu\lambda_k^\nu > 0$, *such that* $r \in (-\infty, \lambda_{-k}^-) \cup (\lambda_{+k}^+, +\infty)$. $r \in (\lambda_{-k}^-, 0) \cup (0, \lambda_{+k}^+)$.

(iii) *There exists a* $\nu\lambda_{\nu k}^\nu < 0$ *for* $\nu\lambda_k^\nu < 0$, *such that* $r \in (\lambda_{+k}^+, 0) \cup (0, \lambda_{-k}^-)$.

(iv) *There exists a* $\lambda_{\nu k}^\nu < 0$ *for* $\lambda_k^\nu < 0$, *such that* $r \in (\lambda_{\nu k}^\nu, 0)$.

Then problem (8) possesses two solutions x_k^+ *and* x_k^- *such that* x_k^+ *has exactly* $k-1$ *zeros in* $(0,1)$ *and is positive near 0 and* x_k^- *has exactly* $k-1$ *zeros in* $(0,1)$ *and is negative near 0.*

Proof. Define

$$f^{[n]}(s) := \begin{cases} \dfrac{1}{n}s, & s \in (-\infty, -2n] \cup [2n, +\infty). \\ \dfrac{2+f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ \dfrac{2-f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ f(s), & s \in \left[-n, -\dfrac{2}{n}\right] \cup \left[\dfrac{2}{n}, n\right], \\ -\left[f\left(-\dfrac{2}{n}\right) + \dfrac{1}{n^2}\right](ns+2) + f\left(-\dfrac{2}{n}\right), & s \in \left(-\dfrac{2}{n}, -\dfrac{1}{n}\right), \\ \left[f\left(\dfrac{2}{n}\right) - \dfrac{1}{n^2}\right](ns-2) + f\left(\dfrac{2}{n}\right), & s \in \left(\dfrac{1}{n}, \dfrac{2}{n}\right), \\ \dfrac{1}{n}s, & s \in \left[-\dfrac{1}{n}, \dfrac{1}{n}\right]. \end{cases} \tag{116}$$

We consider the following problem:

$$x'''' = \alpha x^+ + \beta x^- + \lambda r a(t) f^{[n]}(x), \quad 0 < t < 1,$$
$$x(0) = x(1) = x''(0) = x''(1) = 0. \tag{117}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = 1/n$, and $(f^{[n]})_\infty = 1/n$.

Applying the similar method used in the proof of Theorem 30, there exists an unbounded continuum $\mathscr{D}_k^{\nu[n]}$ of solutions of problem (117) emanating from $(n\lambda^\nu/r, 0)$ or $(n\lambda^\nu/r, \infty)$, such that $\mathscr{D}_k^{\nu[n]} \subset ((\mathbb{R}\times S_k^\nu)\cup\{(n\lambda^\nu/r, 0)\})$ and $\mathscr{D}_k^{\nu[n]} \subset ((\mathbb{R}\times S_k^\nu)\cup\{(n\lambda^\nu/r, \infty\})$, and $\mathscr{D}_k^{\nu[n]}$ joins $(n\lambda^\nu/r, 0)$ to $(n\lambda^\nu/r, \infty)$.

Moreover, we can obtain the desired results. □

Remark 32. If $\alpha = \beta = 0$ and $f_0, f_\infty \in (0, \infty)$, Ma et al. [10, 11] studied problem (8) under the conditions $(H5)$ and $(H6)$; in the situation, the results of Theorem 26 improve on Theorem 3.1 of [10] and Theorem 1.1 of [11], respectively.

Remark 33. When $\alpha = \beta = 0$ and $f_0, f_\infty \in (0, \infty)$, Dai and Han [16] investigated the existence of nodal solutions for problem (8). Thus, our results partially extend and improve the corresponding Theorem 4.1 of [16].

Remark 34. When $\alpha = \beta = 0$ and $f_0, f_\infty \in (0, \infty)$, Lazer and McKenna [7] investigated the existence of nodal solutions for problem (8). Thus, our results partially extend and improve the correspondingTheorem 1 of [7].

Remark 35. The nonlinear term of (8) is not necessarily homogeneous linearizable at the origin and infinity because of the influence of the term $\alpha x^{+} + \beta x^{-}$. Clearly, so the bifurcation results of [7, 8, 10–13, 16] cannot be applied directly to obtain our results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors were supported by the NSFC (no. 11561038) and Gansu Provincial National Science Foundation of China (no. 145RJZA087).

References

[1] Z. Bai and H. Wang, "On positive solutions of some nonlinear fourth-order beam equations," *Journal of Mathematical Analysis and Applications*, vol. 270, no. 2, pp. 357–368, 2002.

[2] R. Ma and H. Wang, "On the existence of positive solutions of fourth-order ordinary differential equations," *Applicable Analysis*, vol. 59, no. 1–4, pp. 225–231, 1995.

[3] J. Chu and D. O'Regan, "Positive solutions for regular and singular fourth-order boundary value problems," *Communications in Applied Analysis*, vol. 10, no. 2-3, pp. 185–199, 2006.

[4] P. H. Rabinowitz, "Some global results for nonlinear eigenvalue problems," *Journal of Functional Analysis*, vol. 7, pp. 487–513, 1971.

[5] P. H. Rabinowitz, "Some aspects of nonlinear eigenvalue problems," *The Rocky Mountain Journal of Mathematics*, vol. 3, pp. 161–202, 1973.

[6] C. P. Gupta and J. Mawhin, "Weighted eigenvalue, eigenfunctions and boundary value problems for fourth order ordinary differential equations," in *Recent Trends in Differential Equations*, vol. 1 of *World Scientific Series in Applicable Analysis*, pp. 253–267, World Scientific, 1992.

[7] A. C. Lazer and P. J. McKenna, "Global bifurcation and a theorem of Tarantello," *Journal of Mathematical Analysis and Applications*, vol. 181, no. 3, pp. 648–655, 1994.

[8] B. P. Rynne, "Infinitely many solutions of superlinear fourth order boundary value problems," *Topological Methods in Nonlinear Analysis*, vol. 19, no. 2, pp. 303–312, 2002.

[9] Y. Liu and D. O'Regan, "Bifurcation techniques for fourth order m-point boundary value problems," *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, vol. 18, no. 2, pp. 215–234, 2011.

[10] R. Y. Ma and B. Thompson, "Nodal solutions for a nonlinear fourth-order eigenvalue problem," *Acta Mathematica Sinica (English Series)*, vol. 24, no. 1, pp. 27–34, 2008.

[11] R. Ma, C. Gao, and X. Han, "On linear and nonlinear fourth-order eigenvalue problems with indefinite weight," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 18, pp. 6965–6969, 2011.

[12] W. Shen, "Existence of nodal solutions of a nonlinear fourth-order two-point boundary value problem," *Boundary Value Problems*, vol. 2012, article 31, 2012.

[13] W. Shen, "Global structure of nodal solutions for a fourth-order two-point boundary value problem," *Applied Mathematics and Computation*, vol. 219, no. 1, pp. 88–98, 2012.

[14] R. Ma, "Nodal Solutions for a fourth-order two-point boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 1, pp. 254–265, 2006.

[15] E. N. Dancer, "On the structure of solutions of non-linear eigenvalue problems," *Indiana University Mathematics Journal*, vol. 23, no. 11, pp. 1069–1076, 1974.

[16] G. Dai and X. Han, "Global bifurcation and nodal solutions for fourth-order problems with sign-changing weight," *Applied Mathematics and Computation*, vol. 219, no. 17, pp. 9399–9407, 2013.

[17] E. N. Dancer, "Bifurcation from simple eigenvalues and eigenvalues of geometric multiplicity one," *Bulletin of the London Mathematical Society*, vol. 34, no. 5, pp. 533–538, 2002.

[18] J. López-Gómez, *Spectral Theory and Nonlinear Functional Analysis*, Chapman and Hall/CRC, Boca Raton, Fla, USA, 2001.

[19] K. Schmitt and H. L. Smith, "On eigenvalue problems for nondifferentiable mappings," *Journal of Differential Equations*, vol. 33, no. 3, pp. 294–319, 1979.

[20] H. Berestycki, "On some nonlinear Sturm-Liouville problems," *Journal of Differential Equations*, vol. 26, no. 3, pp. 375–390, 1977.

[21] B. P. Rynne, "Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable," *Journal of Mathematical Analysis and Applications*, vol. 228, no. 1, pp. 141–156, 1998.

[22] R. Ma and G. Dai, "Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity," *Journal of Functional Analysis*, vol. 265, no. 8, pp. 1443–1459, 2013.

[23] G. Dai, "Global bifurcation from intervals for Sturm-Liouville problems which are not linearizable," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 65, pp. 1–7, 2013.

[24] G. Dai and R. Ma, "Bifurcation from intervals for Sturm-Liouville problems and its applications," *Electronic Journal of Differential Equations*, vol. 2014, no. 3, pp. 1–10, 2014.

[25] G. Dai and R. Ma, "Unilateral global bifurcation for p-Laplacian with non-p–1-linearization nonlinearity," *Discrete and Continuous Dynamical Systems*, vol. 35, no. 1, pp. 99–116, 2015.

[26] J. R. L. Webb, G. Infante, and D. Franco, "Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions," *Proceedings of the Royal Society of Edinburgh, Section A: Mathematics*, vol. 138, no. 2, pp. 427–446, 2008.

[27] G. T. Whyburn, *Topological Analysis*, Princeton University Press, Princeton, NJ, USA, 1958.

[28] R. Ma and Y. An, "Global structure of positive solutions for nonlocal boundary value problems involving integral conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 10, pp. 4364–4376, 2009.

[29] T. Tanigawa and N. Yoshida, "Picone identities for ordinary differential equations of fourth order," *Mathematics Journal of Toyama University*, vol. 27, pp. 91–99, 2004.

[30] P. Drábek and M. Ôani, "Global bifurcation result for the p-biharmonic operator," *Electronic Journal of Differential Equations*, vol. 2001, no. 48, pp. 1–19, 2001.

[31] P. H. Rabinowitz, "On bifurcation from infinity," *Journal of Differential Equations*, vol. 14, pp. 462–475, 1973.

[32] A. Ambrosetti, R. M. Calahorrano, and F. R. Dobarro, "Global branching for discontinuous problems," *Commentationes Mathematicae Universitatis Carolinae*, vol. 31, no. 2, pp. 213–222, 1990.