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Research Article

Robust Fusion Filtering for Multisensor Time-Varying Uncertain Systems: The Finite Horizon Case

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The robust $H\infty$ fusion filtering problem is considered for linear time-varying uncertain systems observed by multiple sensors. A performance index function for this problem is defined as an indefinite quadratic inequality which is solved by the projection method in Krein space. On this basis, a robust centralized finite horizon $H\infty$ fusion filtering algorithm is proposed. However, this centralized fusion method is with poor real time property, as the number of sensors increases. To resolve this difficulty, within the sequential fusion framework, the performance index function is described as a set of quadratic inequalities including an indefinite quadratic inequality. And a sequential robust finite horizon $H\infty$ fusion filtering algorithm is given by solving this quadratic inequality group. Finally, two simulation examples for time-varying/time-invariant multisensor systems are exploited to demonstrate the effectiveness of the proposed methods in the respect of the real time property and filtering accuracy.

1. Introduction

In many advanced systems, multiple homogeneous or heterogeneous sensors are spatially distributed to provide large coverage, diverse viewing angles of the things of interest [1, 2]. How to deal with large amounts of overlapping and complementary data sampled by these sensors is a crucial issue. The fusion filtering technology could effectively integrate these data to estimate the signal of interest.

In the existing literature, the research of fusion filter has already become a focus in recent years. Most of the existing results are usually developed from Kalman filter [3–7], on the basis of two necessary assumptions: the system parameters are given, and the system noises satisfy the Gaussian distributions with given statistic characteristics. These assumptions, however, are usually too idealistic to obtain in practice. Recently, several fusion filtering methods are also developed with different assumptions. While the system parameters include some uncertainties and the system noises are Gaussian, by describing the system parameter uncertainties as multiplicative noises or the norm-bounded uncertainties, several robust Kalman fusion filters are deduced in [8–10]. While the system parameters are given and the statistic characteristic of system noises is unknown, some centralized $H\infty$ fusion filters are presented based on the linear matrix inequality (LMI) technology or the Riccati equation technology [11–13]. For the linear time-invariant multisensor system with energy-bounded noises and norm-bounded uncertain parameters, the centralized robust $H\infty$ fusion filters are proposed in [14, 15]. In [16], a centralized distributed $H\infty$-consensus filtering method is proposed for discrete time-varying systems by solving a set of different linear matrix inequalities in each filtering period and further extended for two kinds of uncertain systems.

However, there are still some performance and application deficiencies in the (robust) $H\infty$ fusion filters mentioned above. The deficiencies on performance are mainly embodied in the real time property. This is due to the centralized fusion structure of these filters [11–15], in which the measurement functions of different sensors are augmented into a high-dimensional measurement function, whose dimension increases with the increase of sensors. Therefore, the running time of these fusion filters can be affected by the implicit high-dimensional operation. What is more, these fusion filters are designed on the basis of the measurements sampled by all sensors. It is implied that the estimate of the signal to be estimated cannot be obtained until all measurements sampled by different sensors in a fusion period arrive at the information processer. Particularly when measurements are transmitted with random delayed phenomenon, the real time performance of the centralized fusion methods is usually very poor. The deficiency on application refers to the fact that most available literature concerning the $H\infty$ fusion filtering problems has been limited to time-invariant systems, and the state estimation problem for the corresponding time-varying systems has not been paid adequate research attention to despite its clear engineering significance.

In this paper, we aim to investigate the robust $H\infty$ fusion filtering method for time-varying multisensor uncertain systems. The research work in this paper mainly includes the following parts:

(i) In this paper, the impacts of the parameter uncertainty and the system noises on the fusion estimate errors are expressed by an indefinite quadratic inequality, whose stationary can be given by a projection method in Krein space. On this basis, a robust centralized finite horizon $H\infty$ fusion filtering algorithm is designed.

(ii) In order to improve the real time property of the above robust fusion filtering algorithm, the performance index function is reformulated into a set of quadratic inequalities. By sequentially solving these quadratic inequalities, a real time robust finite horizon $H\infty$ fusion filtering algorithm is developed.

The remainder of this paper is organized as follows. In Section 2, the time-varying multisensor system is formulated. Two robust finite horizon $H\infty$ fusion filtering algorithms are proposed in Section 3, respectively, according to the centralized fusion strategy and the sequential one. Simulation results and comparisons are presented in Section 4, and some conclusions are given in Section 5.

Notation. The notation used here is fairly standard except where otherwise stated. The superscript "T" stands for matrix transposition, $\mathfrak{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbf{I}$ denotes the identity matrix with appropriate dimension. The notation $\mathbf{P} > 0$, where $\mathbf{P}$ is positive definite. The vectors in Hilbert space are denoted by bold face letters, such as "$\mathbf{x}(i)$," and the ones in Krein space are denoted by the bold face letters with bar, such as "$\overline{\mathbf{x}}(i)$." $\langle \mathbf{A}, \mathbf{B} \rangle$ stands for the inner product in Krein space.

2. Problem Formulation

Consider the following time-varying N-sensor system with uncertain parameters:

$$\begin{aligned}\mathbf{x}(k+1) &= (\mathbf{F}(k+1,k) + \Delta\mathbf{F}(k+1,k))\,\mathbf{x}(k) \\ &\quad + \mathbf{w}(k+1,k),\\ \mathbf{y}_i(k) &= \mathbf{H}_i(k)\,\mathbf{x}(k) + \mathbf{v}_i(k), \quad i = 1,2,\ldots,N,\\ \mathbf{z}(k) &= \mathbf{L}(k)\,\mathbf{x}(k),\end{aligned} \tag{1}$$

where $\mathbf{x}(k) \in \mathfrak{R}^n$ is the state vector. $\mathbf{y}_i(k) \in \mathfrak{R}^{m_i}$ is the measurement output of sensor i. $\mathbf{w}(k,k-1) \in l_2[1,N)$ is the process noise and $\mathbf{v}_i(k) \in l_2[1,N)$ is the corresponding measurement noise of sensor i. $\mathbf{z}(k) \in \mathfrak{R}^p$ is the signal to be estimated. $\mathbf{F}(k+1,k), \mathbf{H}_i(k), \mathbf{L}(k)$ are the given matrices with compatible dimensions. $\Delta\mathbf{F}(k+1,k)$ is a real-valued uncertain matrix satisfying $\Delta\mathbf{F}(k+1,k) = \mathbf{D}(k+1,k)\Lambda(k)\mathbf{M}(k)$, in which $\mathbf{D}(k+1,k), \mathbf{M}(k)$ are known time-varying matrices and $\Lambda(k)$ is time-varying uncertainty satisfying $\|\Lambda(k)\| \le 1$.

3. Robust Finite Horizon $H\infty$ Fusion Filtering Algorithms

Define the following auxiliary variables [17]:

$$\begin{aligned}\mathbf{s}(k) &:= \mathbf{M}(k)\,\mathbf{x}(k),\\ \boldsymbol{\xi}(k) &:= \Lambda(k)\,\mathbf{s}(k) = \Lambda(k)\,\mathbf{M}(k)\,\mathbf{x}(k).\end{aligned} \tag{2}$$

Then the system shown in (1) can be rewritten as

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{F}(k+1,k)\,\mathbf{x}(k) + \mathbf{D}(k+1,k)\,\boldsymbol{\xi}(k)\\ &\quad + \mathbf{w}(k+1,k).\end{aligned} \tag{3}$$

Due to $\|\Lambda(k)\| \le 1$, $\mathbf{s}^T(k)\mathbf{s}(k) \ge \boldsymbol{\xi}^T(k)\boldsymbol{\xi}(k)$, we can also get

$$\sum_{i=1}^{k}\mathbf{s}^T(i)\,\mathbf{s}(i) \ge \sum_{i=1}^{k}\boldsymbol{\xi}^T(i)\,\boldsymbol{\xi}(i). \tag{4}$$

Denote $\widehat{\mathbf{z}}(k \mid k)$ as the fusion filtering result of $\mathbf{z}(k)$, based on the measurements $\{\mathbf{y}_i(j) \mid i = 1,\ldots,N;\ j = 1,\ldots,k\}$, and $\mathbf{e}_z(k) = \widehat{\mathbf{z}}(k \mid k) - \mathbf{z}(k)$. The transfer function of the system noises and $\mathbf{e}_z(k)$ can be expressed as

$$\begin{aligned}&\|T_k(s)\|_\infty^2\\ &= \sup_{\mathbf{w},\mathbf{v}\in l_2}\frac{\sum_{i=1}^{k}\mathbf{e}_z^T(i)\,\mathbf{e}_z(i)}{\sum_{i=1}^{k}\mathbf{v}_i^T(j)\,\mathbf{v}_i(j) + \sum_{i=1}^{k}\mathbf{w}^T(i,i-1)\,\mathbf{w}(i,i-1) + \widetilde{\mathbf{x}}_0^T\mathbf{P}_0^{-1}\widetilde{\mathbf{x}}_0},\end{aligned} \tag{5}$$

where $\widetilde{\mathbf{x}}_0 = \mathbf{x}(0) - \widehat{\mathbf{x}}_0$ and $\widehat{\mathbf{x}}_0$ is an initial estimate of $\mathbf{x}(0)$. $\mathbf{P}_0$ is a given positive definite matrix.

For a given scalar $\gamma > 0$, the following constraint is given to map the system noises to the filtering error:

$$\begin{aligned}&\sup_{\mathbf{w},\mathbf{v}\in l_2}\frac{\sum_{i=1}^{k}\mathbf{e}_z^T(i)\,\mathbf{e}_z(i)}{\sum_{i=1}^{k}\mathbf{v}^T(i)\,\mathbf{v}(i) + \sum_{i=1}^{k}\mathbf{w}^T(i,i-1)\,\mathbf{w}(i,i-1) + \widetilde{\mathbf{x}}_0^T\mathbf{P}_0^{-1}\widetilde{\mathbf{x}}_0}\\ &\quad < \gamma^2.\end{aligned} \tag{6}$$

It is obvious from (6) that

$$\begin{aligned}&\sum_{i=1}^{k}\mathbf{w}^T(i,i-1)\,\mathbf{w}(i,i-1) + \widetilde{\mathbf{x}}_0^T\mathbf{P}_0^{-1}\widetilde{\mathbf{x}}_0\\ &\quad + \sum_{i=1}^{k}\left(\sum_{j=1}^{N}\mathbf{v}_j^T(i)\,\mathbf{v}_j(i) - \gamma^{-2}\mathbf{e}_z^T(i)\,\mathbf{e}_z(i)\right) > 0.\end{aligned} \tag{7}$$

Combining the above constraints (4) and (7), we can obtain the following performance index function for the robust fusion filtering process:

$$J(k) = \tilde{\mathbf{x}}_0^T \mathbf{P}_0^{-1} \tilde{\mathbf{x}}_0 + \sum_{i=1}^{k} \mathbf{w}^T(i,i-1)\mathbf{w}(i,i-1) + \sum_{i=1}^{k} \boldsymbol{\xi}^T(i)\boldsymbol{\xi}(i) - \sum_{i=1}^{k} \mathbf{s}^T(i)\mathbf{s}(i) + \sum_{i=1}^{k}\left(\sum_{j=1}^{N} \mathbf{v}_j^T(i)\mathbf{v}_j(i) - \gamma^{-2}\mathbf{e}_z^T(i)\mathbf{e}_z(i)\right) > 0. \quad (8)$$

Rewrite (8) as

$$J(k) = \tilde{\mathbf{x}}_0^T \mathbf{P}_0^{-1} \tilde{\mathbf{x}}_0 + \sum_{i=1}^{k} \left[\mathbf{w}^T(i,i-1) \;\; \boldsymbol{\xi}^T(i)\right] \begin{bmatrix} \mathbf{w}(i,i-1) \\ \boldsymbol{\xi}(i) \end{bmatrix} + \sum_{i=1}^{k}\left(\sum_{j=1}^{N} \mathbf{v}_j^T(i)\mathbf{v}_j(i) - \gamma^{-2}\mathbf{e}_z^T(i)\mathbf{e}_z(i) - \mathbf{s}^T(i)\mathbf{s}(i)\right) > 0. \quad (9)$$

The above performance index means

(1) there is a minimum $\mathbf{J}^*(k)$ of $J(k)$ at a stationary point $\hat{\mathbf{z}}(k \mid k)$;

(2) the minimum $\mathbf{J}^*(k) > 0$.

The stationary point of indefinite quadratic forms in Hilbert space corresponds to a projection in Krein space [17–20]. In the remainder of this section, the projections in Krein space will be solved to obtain the stationary point of $J(k)$ and further to yield the estimates of the signal to be estimated.

3.1. Centralized Robust Finite Horizon H∞ Fusion Filtering Algorithm. Define the following augmented matrices:

$$\widehat{\mathbf{Y}}(k) = \begin{bmatrix} \mathbf{y}_1(k) \\ \vdots \\ \mathbf{y}_N(k) \end{bmatrix}, \quad \widehat{\mathbf{H}}(k) = \begin{bmatrix} \mathbf{H}_1(k) \\ \vdots \\ \mathbf{H}_N(k) \end{bmatrix}, \quad \widehat{\mathbf{V}}(k) = \begin{bmatrix} \mathbf{v}_1(k) \\ \vdots \\ \mathbf{v}_N(k) \end{bmatrix},$$

$$\mathbf{Y}(k) = \begin{bmatrix} \widehat{\mathbf{Y}}(k) \\ 0 \\ \hat{\mathbf{z}}(k \mid k) \end{bmatrix}, \quad \overleftrightarrow{\mathbf{H}}(k) = \begin{bmatrix} \widehat{\mathbf{H}}(k) \\ \mathbf{M}(k) \\ \mathbf{L}(k) \end{bmatrix}, \quad \mathbf{V}(k) = \begin{bmatrix} \widehat{\mathbf{V}}(k) \\ \mathbf{s}(k) \\ \mathbf{e}_z(k) \end{bmatrix}, \quad \overleftrightarrow{\mathbf{w}}(k,k-1) = \begin{bmatrix} \mathbf{w}(k,k-1) \\ \boldsymbol{\xi}(k) \end{bmatrix}. \quad (10)$$

Then we can rewrite (1) in the following augmented matrix form:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{F}(k+1,k)\mathbf{x}(k) + \mathbf{D}(k+1,k)\boldsymbol{\xi}(k) + \mathbf{w}(k+1,k) \\ &= \mathbf{F}(k+1,k)\mathbf{x}(k) + \mathbf{G}(k)\overleftrightarrow{\mathbf{w}}(k+1,k), \\ \mathbf{Y}(k) &= \overleftrightarrow{\mathbf{H}}(k)\mathbf{x}(k) + \mathbf{V}(k); \end{aligned} \quad (11)$$

here $\mathbf{G}(k) = [\mathbf{I}, \mathbf{D}(k+1,k)]$. And the performance index function shown in (9) can be further expressed as

$$J(k) = \tilde{\mathbf{x}}_0^T \mathbf{P}_0^{-1} \tilde{\mathbf{x}}_0 + \sum_{i=1}^{k} \overleftrightarrow{\mathbf{w}}^T(i,i-1)\overleftrightarrow{\mathbf{w}}(i,i-1) + \sum_{i=1}^{k} \mathbf{V}^T(i)\mathbf{R}^{-1}(i)\mathbf{V}(i) > 0 \quad (12)$$

in which $\mathbf{R}(i) = \mathrm{diag}\{\mathbf{I}, -\mathbf{I}, -\gamma^{-2}\mathbf{I}\}$.

Theorem 1. *Given a positive scalar γ, for the augmented matrix system shown in (11), the following robust finite horizon H∞ fusion filter can be given to satisfy the performance index function (9), based on the centralized fusion strategy:*

$$\begin{aligned} \hat{\mathbf{z}}(k \mid k) &= \mathbf{L}(k)\hat{\mathbf{x}}(k \mid k), \\ \hat{\mathbf{x}}(k \mid k) &= \hat{\mathbf{x}}(k \mid k-1) + \mathbf{P}(k)\begin{bmatrix} \widehat{\mathbf{H}}(k) \\ \mathbf{M}(k) \end{bmatrix}^T \mathbf{A}^{-1}(k) \cdot \begin{pmatrix} \widehat{\mathbf{Y}}(k) - \widehat{\mathbf{H}}(k)\hat{\mathbf{x}}(k \mid k-1) \\ -\mathbf{M}(k)\hat{\mathbf{x}}(k \mid k-1) \end{pmatrix}, \\ \hat{\mathbf{x}}(k \mid k-1) &= \mathbf{F}(k,k-1)\hat{\mathbf{x}}(k-1 \mid k-1), \end{aligned} \quad (13)$$

where

$$\mathbf{A}(k)=\begin{bmatrix}\widehat{\mathbf{H}}(k)\\ \mathbf{M}(k)\end{bmatrix}\mathbf{P}(k)\begin{bmatrix}\widehat{\mathbf{H}}(k)\\ \mathbf{M}(k)\end{bmatrix}^T+\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I}\end{bmatrix},$$
$$\mathbf{P}(k+1)=\mathbf{F}(k+1,k)\mathbf{P}(k)\mathbf{F}^T(k+1,k)+\mathbf{G}(k+1)\cdot\mathbf{G}^T(k+1)-\mathbf{F}(k+1,k)\mathbf{P}(k)\overleftrightarrow{\mathbf{H}}^T(k)\mathbf{P}_{eY}^{-1}(k)\cdot\overleftrightarrow{\mathbf{H}}(k)\mathbf{P}(k)\mathbf{F}^T(k+1,k),\tag{14}$$
$$\mathbf{P}_{eY}(k)=\overleftrightarrow{\mathbf{H}}(k)\mathbf{P}(k)\overleftrightarrow{\mathbf{H}}^T(k)+\mathbf{R}(k).$$

The existing condition of this robust H∞ fusion filter is that $\mathbf{P}_{eY}(k)$ *and* $\mathbf{R}(k)$ *have the same inertia index.*

Proof. The performance index function (9) can be expressed by the indefinite quadratic augmented matrix inequality (12), in which the stationary point of $J(k)$ corresponds to a projection in the following Krein subspace:

$$\overline{\mathbf{x}}(i)=\mathbf{F}(i,i-1)\overline{\mathbf{x}}(i-1)+\mathbf{G}(i)\overline{\mathbf{w}}(i,i-1),$$
$$\overline{\mathbf{Y}}(i)=\overleftrightarrow{\mathbf{H}}(i)\overline{\mathbf{x}}(i)+\overline{\mathbf{V}}(i),\quad i=1,\ldots,k\tag{15}$$

with

$$\left\langle\begin{bmatrix}\widetilde{\mathbf{x}}_0\\ \overline{\mathbf{w}}(j_1,j_1-1)\\ \overline{\mathbf{V}}(j_1)\end{bmatrix},\begin{bmatrix}\widetilde{\mathbf{x}}_0\\ \overline{\mathbf{w}}(j_2,j_2-1)\\ \overline{\mathbf{V}}(j_2)\end{bmatrix}\right\rangle=\begin{bmatrix}\mathbf{P}_0 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\delta_{j_1,j_2} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{R}(j_1)\delta_{j_1,j_2}\end{bmatrix},\quad 1\le j_1,j_2\le k.\tag{16}$$

Denote $\overline{\mathbf{W}}(k):=[\overline{\mathbf{w}}^T(1,0),\ldots,\overline{\mathbf{w}}^T(k,k-1)]^T$, $\overline{\boldsymbol{\xi}}(k):=[\overline{\mathbf{x}}^T(0),\overline{\mathbf{W}}^T(k)]^T$. The stationary point of the indefinite quadratic form (12) corresponds to the projection of $\overline{\boldsymbol{\xi}}(k)$ into the Krein subspace L_Y^k spanned by $\{\overline{\mathbf{Y}}(i)\mid i=1,\ldots,k\}$. Let $\widehat{\overline{\mathbf{Y}}}(i\mid i-1)$ be the projection of $\overline{\mathbf{Y}}(i)$ into L_Y^k, and let $\overline{\mathbf{e}}_Y(i)$ be $\overline{\mathbf{Y}}(i)-\widehat{\overline{\mathbf{Y}}}(i\mid i-1)$. Then $\{\overline{\mathbf{e}}_Y(1),\ldots,\overline{\mathbf{e}}_Y(k-1),\overline{\mathbf{e}}_Y(k)\}$ is an orthogonal basis of L_Y^k, and the projection of $\overline{\boldsymbol{\xi}}(k)$ into the Krein subspace L_Y^k is given by

$$\widehat{\overline{\boldsymbol{\xi}}}(k\mid k)=\sum_{i=1}^{k}\left\langle\overline{\boldsymbol{\xi}}(k),\overline{\mathbf{e}}_Y(i)\right\rangle\left\langle\overline{\mathbf{e}}_Y(i),\overline{\mathbf{e}}_Y(i)\right\rangle^{-1}\overline{\mathbf{e}}_Y(i).\tag{17}$$

The projection of $\overline{\mathbf{x}}(k)$ into L_Y^k is

$$\widehat{\overline{\mathbf{x}}}(k\mid k)=\sum_{i=1}^{k}\left\langle\overline{\mathbf{x}}(k),\overline{\mathbf{e}}_Y(i)\right\rangle\left\langle\overline{\mathbf{e}}_Y(i),\overline{\mathbf{e}}_Y(i)\right\rangle^{-1}\overline{\mathbf{e}}_Y(i):=\widehat{\overline{\mathbf{x}}}(k\mid k-1)+\overleftrightarrow{\mathbf{K}}(k)\overline{\mathbf{e}}_Y(k),\tag{18}$$

where

$$\widehat{\overline{\mathbf{x}}}(k\mid k-1)=\sum_{i=1}^{k-1}\left\langle\overline{\mathbf{x}}(k),\overline{\mathbf{e}}_Y(i)\right\rangle\left\langle\overline{\mathbf{e}}_Y(i),\overline{\mathbf{e}}_Y(i)\right\rangle^{-1}\overline{\mathbf{e}}_Y(i)=\mathbf{F}(k,k-1)\widehat{\overline{\mathbf{x}}}(k-1\mid k-1),\tag{19}$$
$$\overleftrightarrow{\mathbf{K}}(k)=\left\langle\overline{\mathbf{x}}(k),\overline{\mathbf{e}}_Y(k)\right\rangle\left\langle\overline{\mathbf{e}}_Y(k),\overline{\mathbf{e}}_Y(k)\right\rangle^{-1}.$$

Denote $\overline{\mathbf{e}}_x(k)=\overline{\mathbf{x}}(k)-\widehat{\overline{\mathbf{x}}}(k\mid k-1)$, $\mathbf{P}(k):=\langle\overline{\mathbf{e}}_x(k),\overline{\mathbf{e}}_x(k)\rangle$; then

$$\overline{\mathbf{e}}_Y(k)=\overline{\mathbf{Y}}(k)-\widehat{\overline{\mathbf{Y}}}(k\mid k-1)=\overleftrightarrow{\mathbf{H}}(k)\left[\overline{\mathbf{x}}(k)-\widehat{\overline{\mathbf{x}}}(k\mid k-1)\right]+\overline{\mathbf{V}}(k),$$
$$\mathbf{P}_{eY}(k)=\left\langle\overline{\mathbf{e}}_Y(k),\overline{\mathbf{e}}_Y(k)\right\rangle=\overleftrightarrow{\mathbf{H}}(k)\mathbf{P}(k)\overleftrightarrow{\mathbf{H}}^T(k)+\mathbf{R}(k),\tag{20}$$
$$\overleftrightarrow{\mathbf{K}}(k)=\left\langle\overline{\mathbf{x}}(k),\overline{\mathbf{e}}_Y(k)\right\rangle\left\langle\overline{\mathbf{e}}_Y(k),\overline{\mathbf{e}}_Y(k)\right\rangle^{-1}=\mathbf{P}(k)\overleftrightarrow{\mathbf{H}}^T(k)\left(\overleftrightarrow{\mathbf{H}}(k)\mathbf{P}(k)\overleftrightarrow{\mathbf{H}}^T(k)+\mathbf{R}(k)\right)^{-1}.$$

The projection of $\overline{\boldsymbol{\xi}}(k)$ in (17) corresponds to a stationary point of the indefinite quadratic form $J(k)$ in (12), and the value of $J(k)$ at this stationary point is

$$\mathbf{J}^*(k)=\sum_{i=1}^{k}\mathbf{e}_Y^T(i)\mathbf{P}_{eY}^{-1}(i)\mathbf{e}_Y(i)=\sum_{i=1}^{k-1}\mathbf{e}_Y^T(i)\mathbf{P}_{eY}^{-1}(i)\mathbf{e}_Y(i)+\mathbf{e}_Y^T(k)\mathbf{P}_{eY}^{-1}(k)\mathbf{e}_Y(k)=\mathbf{J}^*(k-1)+\mathbf{e}_Y^T(k)\mathbf{P}_{eY}^{-1}(k)\mathbf{e}_Y(k)\tag{21}$$

in which

$$\mathbf{e}_Y(k)=\mathbf{Y}(k)-\widehat{\mathbf{Y}}(k\mid k-1)=\begin{bmatrix}\widehat{\mathbf{Y}}(k)\\ 0\\ \widehat{\mathbf{z}}(k\mid k)\end{bmatrix}-\begin{bmatrix}\widehat{\mathbf{H}}(k)\\ \mathbf{M}(k)\\ \mathbf{L}(k)\end{bmatrix}\widehat{\mathbf{x}}(k\mid k-1),\tag{22}$$
$$\widehat{\mathbf{x}}(k\mid k-1)=\mathbf{F}(k,k-1)\widehat{\mathbf{x}}(k-1\mid k-1).$$

Due to the fact that $\mathbf{P}_{eY}^{-1}(k)$ can be expressed as the form in (23), $\mathbf{J}^*(k)$ can also be given by (24):

$$\mathbf{P}_{eY}^{-1}(k) = \begin{bmatrix} \mathbf{A}(k) & \mathbf{B}(k) \\ \mathbf{B}^T(k) & \mathbf{C}(k) \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1}(k) + \mathbf{A}^{-1}(k)\mathbf{B}(k)\boldsymbol{\varepsilon}^{-1}(k)\mathbf{B}^T(k)\mathbf{A}^{-1}(k) & -\mathbf{A}^{-1}(k)\mathbf{B}(k)\boldsymbol{\varepsilon}^{-1}(k) \\ -\boldsymbol{\varepsilon}^{-1}(k)\mathbf{B}^T(k)\mathbf{A}^{-1}(k) & \boldsymbol{\varepsilon}^{-1}(k) \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{B}^T(k)\mathbf{A}^{-1}(k) & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{A}^{-1}(k) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}^{-1}(k) \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{B}^T(k)\mathbf{A}^{-1}(k) & \mathbf{I} \end{bmatrix} \quad (23)$$
$$= \begin{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{12}^T(k)\mathbf{A}_{11}^{-1}(k) & \mathbf{I} \end{bmatrix} & \mathbf{0} \\ -\mathbf{B}^T(k)\mathbf{A}^{-1}(k) & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \begin{bmatrix} \mathbf{A}_{11}^{-1}(k) & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_3^{-1}(k) \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}^{-1}(k) \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{12}^T(k)\mathbf{A}_{11}^{-1}(k) & \mathbf{I} \end{bmatrix} & \mathbf{0} \\ -\mathbf{B}^T(k)\mathbf{A}^{-1}(k) & \mathbf{I} \end{bmatrix},$$

$$\begin{aligned} \mathbf{J}^*(k) &= \mathbf{J}^*(k-1) + \mathbf{e}_Y^T(k)\mathbf{P}_{eY}^{-1}(k)\mathbf{e}_Y(k) \\ &= \mathbf{J}^*(k-1) + \widetilde{\widetilde{\mathbf{Y}}}^T(k \mid k-1)\mathbf{A}_{11}^{-1}(k)\widetilde{\widetilde{\mathbf{Y}}}(k \mid k-1) + \widehat{\widetilde{\mathbf{x}}}^T(k \mid k-1)\mathbf{M}^T(k)\mathbf{A}_3^{-1}(k)\mathbf{M}(k)\widehat{\widetilde{\mathbf{x}}}(k \mid k-1) \\ &\quad + [\widehat{\mathbf{z}}(k \mid k) - \widehat{\mathbf{z}}^*(k \mid k)]^T \boldsymbol{\varepsilon}^{-1}(k)[\widehat{\mathbf{z}}(k \mid k) - \widehat{\mathbf{z}}^*(k \mid k)]. \end{aligned} \quad (24)$$

Here,

$$\begin{aligned} \mathbf{A}(k) &= \begin{bmatrix} \mathbf{A}_{11}(k) & \mathbf{A}_{12}(k) \\ \mathbf{A}_{12}^T(k) & \mathbf{A}_{22}(k) \end{bmatrix} \\ &= \begin{bmatrix} \widehat{\mathbf{H}}(k) \\ \mathbf{M}(k) \end{bmatrix} \mathbf{P}(k) \begin{bmatrix} \widehat{\mathbf{H}}(k) \\ \mathbf{M}(k) \end{bmatrix}^T + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}, \\ \mathbf{B}(k) &= \begin{bmatrix} \widehat{\mathbf{H}}(k) \\ \mathbf{M}(k) \end{bmatrix} \mathbf{P}(k)\mathbf{L}^T(k), \\ \mathbf{C}(k) &= \mathbf{L}(k)\mathbf{P}(k)\mathbf{L}^T(k) - \gamma^{-2}\mathbf{I}, \\ \mathbf{A}_3(k) &= \mathbf{A}_{22}(k) - \mathbf{A}_{12}^T(k)\mathbf{A}_{11}^{-1}(k)\mathbf{A}_{12}(k), \\ \boldsymbol{\varepsilon}(k) &= \mathbf{C}(k) - \mathbf{B}^T(k)\mathbf{A}^{-1}(k)\mathbf{B}(k), \\ \widetilde{\widetilde{\mathbf{Y}}}(k \mid k-1) &= \widehat{\mathbf{Y}}(k) - \widehat{\mathbf{H}}(k)\widehat{\mathbf{x}}(k \mid k-1), \\ \widehat{\mathbf{z}}^*(k \mid k) &= \mathbf{L}(k)\widehat{\mathbf{x}}(k \mid k-1) \\ &\quad + \mathbf{B}^T(k)\mathbf{A}^{-1}(k) \begin{pmatrix} \widehat{\mathbf{Y}}(k) - \widehat{\mathbf{H}}(k)\widehat{\mathbf{x}}(k \mid k-1) \\ -\mathbf{M}(k)\widehat{\mathbf{x}}(k \mid k-1) \end{pmatrix}, \\ \widehat{\widetilde{\mathbf{x}}}(k \mid k-1) &= -\mathbf{P}(k)\widehat{\mathbf{H}}^T(k)\mathbf{A}_{11}^{-1}(k)\widetilde{\widetilde{\mathbf{Y}}}(k \mid k-1). \end{aligned} \quad (25)$$

According to [19, 20], $\mathbf{J}^*(k)$ is the minimum of $\mathbf{J}(k)$ if and only if $\mathbf{P}_{eY}(k)$ and $\mathbf{R}(k)$ have the same inertia. Considering the block triangular factorization of $\mathbf{P}_{eY}^{-1}(k)$ as shown in (23), the sufficient condition of the minimum is $\mathbf{A}_3(k) < 0$, $\boldsymbol{\varepsilon}(k) < 0$. Therefore, a choice of $\widehat{\mathbf{z}}(k \mid k)$ to ensure $\mathbf{J}^*(k) > 0$ is $\widehat{\mathbf{z}}(k \mid k) = \widehat{\mathbf{z}}^*(k \mid k)$; then the estimate of the signal to be estimated is

$$\widehat{\mathbf{z}}(k \mid k) = \widehat{\mathbf{z}}^*(k \mid k) = \mathbf{L}(k)\widehat{\mathbf{x}}(k \mid k) \quad (26)$$

in which

$$\begin{aligned} \widehat{\mathbf{x}}(k \mid k) &= \widehat{\mathbf{x}}(k \mid k-1) + \mathbf{P}(k) \begin{bmatrix} \widehat{\mathbf{H}}(k) \\ \mathbf{M}(k) \end{bmatrix}^T \mathbf{A}^{-1}(k) \\ &\quad \cdot \begin{pmatrix} \widehat{\mathbf{Y}}(k) - \widehat{\mathbf{H}}(k)\widehat{\mathbf{x}}(k \mid k-1) \\ -\mathbf{M}(k)\widehat{\mathbf{x}}(k \mid k-1) \end{pmatrix}, \end{aligned} \quad (27)$$

$$\mathbf{A}(k) = \begin{bmatrix} \widehat{\mathbf{H}}(k) \\ \mathbf{M}(k) \end{bmatrix} \mathbf{P}(k) \begin{bmatrix} \widehat{\mathbf{H}}(k) \\ \mathbf{M}(k) \end{bmatrix}^T + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}.$$

The existing condition of this filter is that $\mathbf{P}_{eY}(k)$ and $\mathbf{R}(k)$ have the same inertia index.

The Riccati equation is given by

$$\begin{aligned} \mathbf{P}(k+1) &= \mathbf{F}(k+1,k)\mathbf{P}(k)\mathbf{F}^T(k+1,k) + \mathbf{G}(k+1) \\ &\quad \cdot \mathbf{G}^T(k+1) - \mathbf{F}(k+1,k)\mathbf{P}(k)\overleftrightarrow{\mathbf{H}}^T(k)\mathbf{P}_{eY}^{-1}(k) \\ &\quad \cdot \overleftrightarrow{\mathbf{H}}(k)\mathbf{P}(k)\mathbf{F}^T(k+1,k), \\ \mathbf{P}_{eY}(k) &= \overleftrightarrow{\mathbf{H}}(k)\mathbf{P}(k)\overleftrightarrow{\mathbf{H}}^T(k) + \mathbf{R}(k). \end{aligned} \quad (28)$$

□

Remark 2. Given a positive scalar γ, the value of $\mathbf{R}(k)$ and $\mathbf{P}_{eY}(k)$ can be obtained in each fusion period. Then compare the inertia index (the number of positive eigenvalues) of $\mathbf{R}(k)$ and $\mathbf{P}_{eY}(k)$. If they have the same inertia index, the fusion filters in Theorem 1 are applied to the corresponding fusion period. Otherwise, the value of γ should be changed, and the fusion filter needs to be resolved for the changed performance index function.

In Theorem 1, a robust finite horizon $H\infty$ fusion filtering algorithm is proposed based on the centralized fusion strategy. The estimate of the signal to be estimated cannot be obtained until all measurements sampled by different sensors arrive at the fusion center, by a high-dimensional operation. Obviously, the real time property of the centralized fusion methods is usually lost to some extent in this fusion filtering process.

Motivated by this situation, in the next subsection, an equivalent real time robust finite horizon $H\infty$ fusion filtering algorithm is also proposed for the time-varying N-sensor system with uncertain parameter, on the basis of the sequential fusion strategy.

3.2. Sequential Robust Finite Horizon $H\infty$ Fusion Filtering Algorithm. Without loss of generality, assume that the arrival sequence of the N measurements is just the sequence of the sensors; namely, the measurements arrive at the fusion center in the sequence $\mathbf{y}_1(k), \mathbf{y}_2(k), \ldots, \mathbf{y}_N(k)$. These measurements could be dealt with sequentially in the fusion center, according to the sequential fusion strategy. Denote the corresponding system state of $\mathbf{y}_i(k)$ by $\mathbf{x}_i(k)$; then one gets

$$\begin{aligned}\mathbf{x}_N(k) &= \cdots = \mathbf{x}_1(k) = \mathbf{x}(k)\\ &= \mathbf{F}(k,k-1)\mathbf{x}(k-1) + \mathbf{G}(k)\overleftrightarrow{\mathbf{w}}(k,k-1).\end{aligned} \tag{29}$$

Lemma 3. *The performance index function shown in (9) can also be expressed as the following set of quadratic inequalities:*

$$\begin{aligned}\mathbf{J}_1(k) &= \mathbf{J}_N(k-1) + \overleftrightarrow{\mathbf{w}}^T(k,k-1)\overleftrightarrow{\mathbf{w}}(k,k-1)\\ &\quad + \mathbf{v}_1^T(k)\mathbf{v}_1(k) > 0,\end{aligned} \tag{30a}$$

$$\mathbf{J}_i(k) = \mathbf{J}_{i-1}(k) + \mathbf{v}_i^T(k)\mathbf{v}_i(k) > 0, \quad i = 2,\ldots,N-1, \tag{30b}$$

$$\begin{aligned}\mathbf{J}_N(k) &= \mathbf{J}_{N-1}(k) + \mathbf{v}_N^T(k)\mathbf{v}_N(k) - \gamma^{-2}\mathbf{e}_z^T(k)\mathbf{e}_z(k)\\ &\quad - \mathbf{s}^T(k)\mathbf{s}(k) > 0.\end{aligned} \tag{30c}$$

Theorem 4. *For the augmented matrix system shown in (11), the following sequential robust finite horizon $H\infty$ fusion filter can be given to satisfy the performance index functions (30a), (30b), and (30c):*

$$\hat{\mathbf{z}}_i(k \mid k) = \mathbf{L}(k)\hat{\mathbf{x}}_i(k \mid k), \quad i = 1,\ldots,N \tag{31}$$

in which

$$\begin{aligned}
\hat{\mathbf{x}}_i(k \mid k) &= \begin{cases}\hat{\mathbf{x}}_{i-1}(k \mid k) + \mathbf{P}_i(k)\mathbf{H}_i^T(k)\left(\mathbf{H}_i(k)\mathbf{P}_i(k)\mathbf{H}_i^T(k) + \mathbf{I}\right)^{-1}\left(\mathbf{y}_i(k) - \mathbf{H}_i(k)\hat{\mathbf{x}}_{i-1}(k \mid k)\right), & i < N\\ \hat{\mathbf{x}}_{N-1}(k \mid k) + \mathbf{P}_N(k)\begin{bmatrix}\mathbf{H}_N(k)\\ \mathbf{M}(k)\end{bmatrix}^T \widehat{\mathbf{A}}^{-1}(k)\begin{pmatrix}\mathbf{y}_N(k) - \mathbf{H}_N(k)\hat{\mathbf{x}}_{N-1}(k \mid k)\\ -\mathbf{M}(k)\hat{\mathbf{x}}_{N-1}(k \mid k)\end{pmatrix}, & i = N,\end{cases}\\
&\mathbf{P}_i(k)\\
&= \begin{cases}\mathbf{F}(k,k-1)\mathbf{P}_N(k-1)\mathbf{F}^T(k,k-1) + \mathbf{G}(k)\mathbf{G}^T(k) - \mathbf{F}(k,k-1)\mathbf{P}_N(k-1)\mathbf{H}_N^T(k-1)\mathbf{P}_{ey,N}^{-1}(k-1)\mathbf{H}_N(k-1)\mathbf{P}_N(k-1)\mathbf{F}^T(k,k-1), & i = 1\\ \mathbf{P}_{i-1}(k) - \mathbf{P}_{i-1}(k)\mathbf{H}_{i-1}^T(k)\left(\mathbf{H}_{i-1}(k)\mathbf{P}_{i-1}(k)\mathbf{H}_{i-1}^T(k) + \mathbf{I}\right)^{-1}\mathbf{H}_{i-1}(k)\mathbf{P}_{i-1}(k), & i = 1,\end{cases}\\
\widehat{\mathbf{A}}(k) &= \begin{bmatrix}\mathbf{H}_N(k)\\ \mathbf{M}(k)\end{bmatrix}\mathbf{P}_N(k)\begin{bmatrix}\mathbf{H}_N(k)\\ \mathbf{M}(k)\end{bmatrix}^T + \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I}\end{bmatrix},\\
\hat{\mathbf{x}}_0(k \mid k) &= \hat{\mathbf{x}}(k \mid k-1) = \mathbf{F}(k,k-1)\hat{\hat{\mathbf{x}}}(k-1 \mid k-1),\\
\mathbf{P}_{ey,N}(k) &= \mathbf{R}_N(k) + \begin{bmatrix}\mathbf{H}_N(k)\\ \mathbf{M}(k)\\ \mathbf{L}(k)\end{bmatrix}\mathbf{P}_N(k)\left[\mathbf{H}_N^T(k)\ \ \mathbf{M}^T(k)\ \ \mathbf{L}^T(k)\right].
\end{aligned} \tag{32}$$

The existence condition of this robust $H\infty$ fusion filter is that $\mathbf{P}_{ey,N}(k)$ and $\mathbf{R}_N(k)$ have the same inertia index.

An analytical proof of this theorem is given in the remainder of this section.

(I) When $\mathbf{y}_1(k)$ arrives at the fusion center, the corresponding subsystem is

$$\begin{aligned}\mathbf{x}_1(k) &= \mathbf{x}(k)\\ &= \mathbf{F}(k,k-1)\mathbf{x}(k-1) + \mathbf{G}(k)\overleftrightarrow{\mathbf{w}}(k,k-1),\\ \mathbf{y}_1(k) &= \mathbf{H}_1(k)\mathbf{x}_1(k) + \mathbf{v}_1(k),\\ \mathbf{z}_1(k) &= \mathbf{L}(k)\mathbf{x}_1(k).\end{aligned} \tag{33}$$

The corresponding performance index function is (30a), the stationary point of which corresponds to a projection in the following Krein subspace:

$$\overline{\mathbf{x}}(i) = \mathbf{F}(i,i-1)\overline{\mathbf{x}}(i-1) + \mathbf{G}(i)\overline{\mathbf{w}}(i,i-1),$$

$$\begin{aligned}\overline{\mathbf{Y}}(i) &= \overleftrightarrow{\mathbf{H}}(i)\,\overline{\mathbf{x}}(i) + \overline{\mathbf{V}}(i), \quad i = 1,\ldots,k-1,\\ \overline{\mathbf{x}}_1(k) &= \overline{\mathbf{x}}(k)\\ &= \mathbf{F}(k,k-1)\,\overline{\mathbf{x}}(k-1) + \mathbf{G}(k)\,\overline{\mathbf{w}}(k,k-1),\\ \overline{\mathbf{y}}_1(k) &= \mathbf{H}_1(k)\,\overline{\mathbf{x}}_1(k) + \overline{\mathbf{v}}_1(k)\end{aligned} \tag{34}$$

with

$$\begin{aligned}&\left\langle \begin{bmatrix}\tilde{\mathbf{x}}_0\\ \overline{\mathbf{w}}(j_1,j_1-1)\\ \overline{\mathbf{v}}_1(k)\\ \overline{\mathbf{V}}(j_1)\end{bmatrix}, \begin{bmatrix}\tilde{\mathbf{x}}_0\\ \overline{\mathbf{w}}(j_2,j_2-1)\\ \overline{\mathbf{v}}_1(k)\\ \overline{\mathbf{V}}(j_2)\end{bmatrix}\right\rangle\\ &= \begin{bmatrix}\mathbf{P}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\delta_{j_1,j_2} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}(j_1)\,\delta_{j_1,j_2}\end{bmatrix}, \quad 1 \le j_1, j_2 \le k.\end{aligned} \tag{35}$$

The stationary point of (30a) corresponds to the projection of $\overline{\boldsymbol{\xi}}(k)$ in the Krein subspace $L_{y,1}^k$ spanned by $\{\overline{\mathbf{Y}}(1),\ldots,\overline{\mathbf{Y}}(k-1),\overline{\mathbf{y}}_1(k)\}$. Denote the projection of $\overline{\mathbf{y}}_1(k)$ in L_Y^{k-1} by $\hat{\overline{\mathbf{y}}}_1(k \mid k-1)$, and the error $\overline{\mathbf{e}}_{y,1}(k) := \overline{\mathbf{y}}_1(k) - \hat{\overline{\mathbf{y}}}_1(k \mid k-1)$. Obviously, $\{\overline{\mathbf{e}}_Y(1),\ldots,\overline{\mathbf{e}}_Y(k-1),\overline{\mathbf{e}}_{y,1}(k)\}$ is an orthogonal basis of $L_{y,1}^k$. Therefore, the projection of $\overline{\boldsymbol{\xi}}(k)$ in $L_{y,1}^k$ is given by

$$\begin{aligned}&\hat{\overline{\boldsymbol{\xi}}}_1(k \mid k)\\ &= \sum_{i=1}^{k-1}\left\langle \overline{\boldsymbol{\xi}}(k), \overline{\mathbf{e}}_Y(i)\right\rangle \left\langle \overline{\mathbf{e}}_Y(i), \overline{\mathbf{e}}_Y(i)\right\rangle^{-1} \overline{\mathbf{e}}_Y(i)\\ &\quad + \left\langle \overline{\boldsymbol{\xi}}(k), \overline{\mathbf{e}}_{y,1}(k)\right\rangle \left\langle \overline{\mathbf{e}}_{y,1}(k), \overline{\mathbf{e}}_{y,1}(k)\right\rangle^{-1} \overline{\mathbf{e}}_{y,1}(k).\end{aligned} \tag{36}$$

The projection of $\overline{\mathbf{x}}(k)$ in $L_{y,1}^k$ is given by

$$\begin{aligned}&\hat{\overline{\mathbf{x}}}_1(k \mid k)\\ &= \sum_{i=1}^{k-1}\left\langle \overline{\mathbf{x}}(k), \overline{\mathbf{e}}_Y(i)\right\rangle \left\langle \overline{\mathbf{e}}_Y(i), \overline{\mathbf{e}}_Y(i)\right\rangle^{-1} \overline{\mathbf{e}}_Y(i)\\ &\quad + \left\langle \overline{\mathbf{x}}(k), \overline{\mathbf{e}}_{y,1}(k)\right\rangle \left\langle \overline{\mathbf{e}}_{y,1}(k), \overline{\mathbf{e}}_{y,1}(k)\right\rangle^{-1} \overline{\mathbf{e}}_{y,1}(k)\\ &= \hat{\overline{\mathbf{x}}}(k \mid k-1)\\ &\quad + \left\langle \overline{\mathbf{x}}(k), \overline{\mathbf{e}}_{y,1}(k)\right\rangle \left\langle \overline{\mathbf{e}}_{y,1}(k), \overline{\mathbf{e}}_{y,1}(k)\right\rangle^{-1} \overline{\mathbf{e}}_{y,1}(k),\end{aligned} \tag{37}$$

where

$$\begin{aligned}\overline{\mathbf{e}}_{y,1}(k) &= \overline{\mathbf{y}}_1(k) - \hat{\overline{\mathbf{y}}}_1(k \mid k-1)\\ &= \mathbf{H}_1(k)\left[\overline{\mathbf{x}}(k) - \hat{\overline{\mathbf{x}}}(k \mid k-1)\right] + \overline{\mathbf{v}}_1(k).\end{aligned} \tag{38}$$

One gets

$$\begin{aligned}\mathbf{P}_{ey,1}(k) &= \left\langle \overline{\mathbf{e}}_{y,1}(k), \overline{\mathbf{e}}_{y,1}(k)\right\rangle\\ &= \mathbf{H}_1(k)\,\mathbf{P}_1(k)\,\mathbf{H}_1^T(k) + \mathbf{I},\\ &\left\langle \overline{\mathbf{x}}(k), \overline{\mathbf{e}}_{y,1}(k)\right\rangle \left\langle \overline{\mathbf{e}}_{y,1}(k), \overline{\mathbf{e}}_{y,1}(k)\right\rangle^{-1}\\ &= \mathbf{P}_1(k)\,\mathbf{H}_1^T(k)\left(\mathbf{H}_1(k)\,\mathbf{P}_1(k)\,\mathbf{H}_1^T(k) + \mathbf{I}\right)^{-1}.\end{aligned} \tag{39}$$

$\hat{\overline{\boldsymbol{\xi}}}_1(k \mid k)$ corresponds to the stationary point of (30a), the value of $\mathbf{J}_1(k)$ at which point is

$$\begin{aligned}\mathbf{J}_1^*(k) &= \sum_{i=1}^{k-1}\mathbf{e}_Y^T(i)\,\mathbf{P}_{eY}^{-1}(i)\,\mathbf{e}_Y(i)\\ &\quad + \mathbf{e}_{y,1}^T(k)\,\mathbf{P}_{ey,1}^{-1}(k)\,\mathbf{e}_{y,1}(k)\\ &= \mathbf{J}_1^*(k-1) + \mathbf{e}_{y,1}^T(k)\,\mathbf{P}_{ey,1}^{-1}(k)\,\mathbf{e}_{y,1}(k),\end{aligned} \tag{40}$$

where $\mathbf{e}_{y,1}(k) = \mathbf{y}_1(k) - \mathbf{H}_1(k)\hat{\mathbf{x}}(k \mid k-1)$, $\hat{\mathbf{x}}(k \mid k-1) = \mathbf{F}(k,k-1)\hat{\hat{\mathbf{x}}}(k-1 \mid k-1)$.

Because $\mathbf{P}_{ey,1}(k) = \mathbf{H}_1(k)\mathbf{P}_1(k)\mathbf{H}_1^T(k) + \mathbf{I} > 0$, $\mathbf{J}_1^*(k) > 0$. The estimate of the signal to be estimated is given by

$$\hat{\mathbf{z}}_1(k \mid k) = \mathbf{L}(k)\,\hat{\mathbf{x}}_1(k \mid k) \tag{41a}$$

in which

$$\begin{aligned}\hat{\mathbf{x}}_1(k \mid k) &= \hat{\mathbf{x}}(k \mid k-1) + \mathbf{P}_1(k)\,\mathbf{H}_1^T(k)\\ &\quad\cdot \left(\mathbf{H}_1(k)\,\mathbf{P}_1(k)\,\mathbf{H}_1^T(k) + \mathbf{I}\right)^{-1}\\ &\quad\cdot \left(\mathbf{y}_1(k) - \mathbf{H}_1(k)\,\hat{\mathbf{x}}(k \mid k-1)\right),\end{aligned} \tag{41b}$$

$$\hat{\mathbf{x}}(k \mid k-1) = \mathbf{F}(k,k-1)\,\hat{\hat{\mathbf{x}}}(k-1 \mid k-1). \tag{41c}$$

Because $\mathbf{x}_1(k) = \mathbf{x}_2(k)$, the corresponding Riccati equation is given by

$$\begin{aligned}\mathbf{P}_2(k) &= \mathbf{P}_1(k) - \mathbf{P}_1(k)\,\mathbf{H}_1^T(k)\\ &\quad\cdot \left(\mathbf{H}_1(k)\,\mathbf{P}_1(k)\,\mathbf{H}_1^T(k) + \mathbf{I}\right)^{-1}\mathbf{H}_1(k)\\ &\quad\cdot \mathbf{P}_1(k).\end{aligned} \tag{41d}$$

(II) When $\mathbf{y}_i(k)$ $(i = 2,\ldots,N-1)$ arrives at the fusion center, the corresponding subsystem is

$$\begin{aligned}\mathbf{x}_i(k) &= \mathbf{x}_{i-1}(k),\\ \mathbf{y}_i(k) &= \mathbf{H}_i(k)\,\mathbf{x}_i(k) + \mathbf{v}_i(k),\\ \mathbf{z}_i(k) &= \mathbf{L}(k)\,\mathbf{x}_i(k).\end{aligned} \tag{42}$$

The corresponding performance index function is (30b), the stationary point of which corresponds to a projection in the following Krein subspace:

$$\begin{aligned}\bar{\mathbf{x}}(l) &= \mathbf{F}(l,l-1)\bar{\mathbf{x}}(l-1)+\mathbf{G}(l)\bar{\mathbf{w}}(l,l-1),\\ \bar{\mathbf{Y}}(l) &= \overleftrightarrow{\mathbf{H}}(l)\bar{\mathbf{x}}(l)+\bar{\mathbf{V}}(l),\quad l=1,\ldots,k-1,\\ \bar{\mathbf{x}}_i(k) &= \cdots = \bar{\mathbf{x}}_1(k)=\bar{\mathbf{x}}(k)\\ &= \mathbf{F}(k,k-1)\bar{\mathbf{x}}(k-1)+\mathbf{G}(k)\bar{\mathbf{w}}(k,k-1),\\ \bar{\mathbf{y}}_j(k) &= \mathbf{H}_j(k)\bar{\mathbf{x}}_j(k)+\bar{\mathbf{v}}_j(k),\quad j=1,\ldots,i\end{aligned} \tag{43}$$

with

$$\left\langle \begin{bmatrix}\tilde{\mathbf{x}}_0\\ \bar{\mathbf{w}}(j_1,j_1-1)\\ \bar{\mathbf{v}}_{j_3}(k)\\ \bar{\mathbf{V}}(j_5)\end{bmatrix}, \begin{bmatrix}\tilde{\mathbf{x}}_0\\ \bar{\mathbf{w}}(j_2,j_2-1)\\ \bar{\mathbf{v}}_{j_4}(k)\\ \bar{\mathbf{V}}(j_6)\end{bmatrix}\right\rangle = \begin{bmatrix}\mathbf{P}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\delta_{j_1,j_2} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I}\delta_{j_3,j_4} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}(j_5)\delta_{j_5,j_6}\end{bmatrix}, \tag{44}$$

$$1\le j_1,j_2\le k;\ 1\le j_3,j_4\le i;\ 1\le j_5,j_6<k.$$

The stationary point of (30b) corresponds to the projection of $\bar{\boldsymbol{\xi}}(k)$ in the Krein subspace $L^k_{y,i}$ spanned by $\{\bar{\mathbf{Y}}(1),\ldots,\bar{\mathbf{Y}}(k-1),\bar{\mathbf{y}}_1(k),\ldots,\bar{\mathbf{y}}_i(k)\}$. Denote the projection of $\bar{\mathbf{y}}_j(k)$ in $L^k_{y,j-1}$ as $\hat{\bar{\mathbf{y}}}_j(k\mid k-1)$, and $\bar{\mathbf{e}}_{y,j}(k) := \bar{\mathbf{y}}_j(k)-\hat{\bar{\mathbf{y}}}_j(k\mid k-1)$. Obviously, $\{\bar{\mathbf{e}}_Y(1),\ldots,\bar{\mathbf{e}}_Y(k-1),\bar{\mathbf{e}}_{y,1}(k),\ldots,\bar{\mathbf{e}}_{y,i}(k)\}$ is an orthogonal basis of $L^k_{y,i}$. Therefore, the projection of $\bar{\boldsymbol{\xi}}(k)$ in $L^k_{y,i}$ is given by

$$\begin{aligned}\hat{\bar{\boldsymbol{\xi}}}_i(k\mid k) &= \sum_{l=1}^{k-1}\left\langle \bar{\boldsymbol{\xi}}(k),\bar{\mathbf{e}}_Y(l)\right\rangle \left\langle \bar{\mathbf{e}}_Y(l),\bar{\mathbf{e}}_Y(l)\right\rangle^{-1}\bar{\mathbf{e}}_Y(l)\\ &+\sum_{j=1}^{i}\left\langle \bar{\boldsymbol{\xi}}(k),\bar{\mathbf{e}}_{y,j}(k)\right\rangle \left\langle \bar{\mathbf{e}}_{y,j}(k),\bar{\mathbf{e}}_{y,j}(k)\right\rangle^{-1}\bar{\mathbf{e}}_{y,j}(k).\end{aligned} \tag{45}$$

The corresponding projection of $\bar{\mathbf{x}}(k)$ is given by

$$\begin{aligned}&\hat{\bar{\mathbf{x}}}_i(k\mid k)\\ &\quad=\hat{\bar{\mathbf{x}}}_{i-1}(k\mid k)\\ &\qquad+\left\langle \bar{\mathbf{x}}(k),\bar{\mathbf{e}}_{y,i}(k)\right\rangle \left\langle \bar{\mathbf{e}}_{y,i}(k),\bar{\mathbf{e}}_{y,i}(k)\right\rangle^{-1}\bar{\mathbf{e}}_{y,i}(k)\end{aligned} \tag{46}$$

in which

$$\begin{aligned}\bar{\mathbf{e}}_{y,i}(k) &= \bar{\mathbf{y}}_i(k)-\hat{\bar{\mathbf{y}}}_i(k\mid k-1)\\ &= \mathbf{H}_i(k)\left[\bar{\mathbf{x}}(k)-\hat{\bar{\mathbf{x}}}_{i-1}(k\mid k)\right]+\bar{\mathbf{v}}_i(k).\end{aligned} \tag{47}$$

Then,

$$\begin{aligned}\mathbf{P}_{ey,i}(k) &= \left\langle \bar{\mathbf{e}}_{y,i}(k),\bar{\mathbf{e}}_{y,i}(k)\right\rangle\\ &= \mathbf{H}_i(k)\mathbf{P}_{i-1}(k)\mathbf{H}_i^T(k)+\mathbf{I},\\ \left\langle \bar{\mathbf{x}}(k),\bar{\mathbf{e}}_{y,i}(k)\right\rangle &\left\langle \bar{\mathbf{e}}_{y,i}(k),\bar{\mathbf{e}}_{y,i}(k)\right\rangle^{-1}\\ &= \mathbf{P}_i(k)\mathbf{H}_i^T(k)\left(\mathbf{H}_i(k)\mathbf{P}_i(k)\mathbf{H}_i^T(k)+\mathbf{I}\right)^{-1}.\end{aligned} \tag{48}$$

$\hat{\bar{\boldsymbol{\xi}}}_i(k\mid k)$ corresponds to the stationary point of (30b), the value of $\mathbf{J}_i(k)$ at which point is

$$\begin{aligned}\mathbf{J}_i^*(k) &= \sum_{l=1}^{k-1}\mathbf{e}_Y^T(l)\mathbf{P}_{eY}^{-1}(l)\mathbf{e}_Y(l)\\ &+\sum_{j=1}^{i}\mathbf{e}_{y,j}^T(k)\mathbf{P}_{ey,j}^{-1}(k)\mathbf{e}_{y,j}(k)\\ &= \mathbf{J}_{i-1}^*(k)+\mathbf{e}_{y,i}^T(k)\mathbf{P}_{ey,i}^{-1}(k)\mathbf{e}_{y,i}(k),\end{aligned} \tag{49}$$

where $\mathbf{e}_{y,i}(k) = \mathbf{y}_i(k)-\mathbf{H}_i(k)\hat{\mathbf{x}}_i(k\mid k-1)$, $\mathbf{P}_{ey,i}(k) = \mathbf{H}_i(k)\mathbf{P}_i(k)\mathbf{H}_i^T(k)+\mathbf{I}>0$. Therefore, $\mathbf{J}_i^*(k)>0$.

Then the estimate of the signal to be estimated is given by

$$\hat{\mathbf{z}}_i(k\mid k) = \mathbf{L}(k)\hat{\mathbf{x}}_i(k\mid k), \tag{50a}$$

$$\begin{aligned}\hat{\mathbf{x}}_i(k\mid k) &= \hat{\mathbf{x}}_{i-1}(k\mid k)+\mathbf{P}_i(k)\mathbf{H}_i^T(k)\\ &\cdot\left(\mathbf{H}_i(k)\mathbf{P}_i(k)\mathbf{H}_i^T(k)+\mathbf{I}\right)^{-1}\\ &\cdot\left(\mathbf{y}_i(k)-\mathbf{H}_i(k)\hat{\mathbf{x}}_{i-1}(k\mid k)\right).\end{aligned} \tag{50b}$$

Because $\mathbf{x}_{i+1}(k) = \mathbf{x}_i(k)$, the corresponding Riccati equation is given by

$$\begin{aligned}\mathbf{P}_{i+1}(k) &= \mathbf{P}_i(k)-\mathbf{P}_i(k)\mathbf{H}_i^T(k)\\ &\cdot\left(\mathbf{H}_i(k)\mathbf{P}_i(k)\mathbf{H}_i^T(k)+\mathbf{I}\right)^{-1}\mathbf{H}_i(k)\\ &\cdot\mathbf{P}_i(k).\end{aligned} \tag{50c}$$

(III) When $\mathbf{y}_N(k)$ arrives at the fusion center, the corresponding subsystem is

$$\begin{aligned}\mathbf{x}_N(k) &= \mathbf{x}_{N-1}(k),\\ \mathbf{y}_N(k) &= \mathbf{H}_N(k)\mathbf{x}_N(k)+\mathbf{v}_N(k),\\ \mathbf{z}_N(k) &= \mathbf{L}(k)\mathbf{x}_N(k).\end{aligned} \tag{51}$$

The corresponding performance index function is (30c), the stationary point of which corresponds to a projection in the following Krein subspace:

$$\begin{aligned}\bar{\mathbf{x}}(l) &= \mathbf{F}(l,l-1)\bar{\mathbf{x}}(l-1)+\mathbf{G}(l)\bar{\mathbf{w}}(l,l-1)\\ \bar{\mathbf{Y}}(l) &= \overleftrightarrow{\mathbf{H}}(l)\bar{\mathbf{x}}(l)+\bar{\mathbf{V}}(l),\quad l=1,\ldots,k\end{aligned} \tag{52}$$

with

$$\left\langle \begin{bmatrix} \tilde{\mathbf{x}}_0 \\ \overline{\mathbf{w}}(j_1, j_1-1) \\ \overline{\mathbf{V}}(j_3) \end{bmatrix}, \begin{bmatrix} \tilde{\mathbf{x}}_0 \\ \overline{\mathbf{w}}(j_2, j_2-1) \\ \overline{\mathbf{V}}(j_4) \end{bmatrix} \right\rangle = \begin{bmatrix} \mathbf{P}_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\delta_{j_1,j_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}(j_3)\delta_{j_3,j_4} \end{bmatrix}, \quad 1 \le j_1, j_2 \le k;\ 1 \le j_3, j_4 \le k. \tag{53}$$

The stationary point of (30c) corresponds to the projection of $\overline{\boldsymbol{\xi}}(k)$ in the Krein subspace L_Y^k. Denote

$$\overline{\mathbf{e}}_{y,N}(k) = \begin{bmatrix} \overline{\mathbf{y}}_N(k) \\ 0 \\ \hat{\mathbf{z}}_N(k \mid k) \end{bmatrix} - \begin{bmatrix} \mathbf{H}_N(k) \\ \mathbf{M}(k) \\ \mathbf{L}(k) \end{bmatrix} \hat{\mathbf{x}}_N(k \mid k-1). \tag{54}$$

Then $\{\overline{\mathbf{e}}_Y(1), \ldots, \overline{\mathbf{e}}_Y(k-1), \overline{\mathbf{e}}_{y,1}(k), \ldots, \overline{\mathbf{e}}_{y,N}(k)\}$ is an orthogonal basis of L_Y^k. And we can obtain the projection of $\overline{\boldsymbol{\xi}}(k)$ in L_Y^k as

$$\begin{aligned} \hat{\overline{\boldsymbol{\xi}}}_N(k \mid k) &= \sum_{l=1}^{k-1} \left\langle \overline{\boldsymbol{\xi}}(k), \overline{\mathbf{e}}_Y(l) \right\rangle \left\langle \overline{\mathbf{e}}_Y(l), \overline{\mathbf{e}}_Y(l) \right\rangle^{-1} \overline{\mathbf{e}}_Y(l) \\ &\quad + \sum_{j=1}^{N} \left\langle \overline{\boldsymbol{\xi}}(k), \overline{\mathbf{e}}_{y,j}(k) \right\rangle \left\langle \overline{\mathbf{e}}_{y,j}(k), \overline{\mathbf{e}}_{y,j}(k) \right\rangle^{-1} \overline{\mathbf{e}}_{y,j}(k) \\ &= \sum_{l=1}^{k} \left\langle \overline{\boldsymbol{\xi}}(k), \overline{\mathbf{e}}_Y(l) \right\rangle \left\langle \overline{\mathbf{e}}_Y(l), \overline{\mathbf{e}}_Y(l) \right\rangle^{-1} \overline{\mathbf{e}}_Y(l). \end{aligned} \tag{55}$$

The corresponding projection of $\overline{\mathbf{x}}(k)$ is given by

$$\begin{aligned} \hat{\overline{\mathbf{x}}}_N(k \mid k) &= \hat{\overline{\mathbf{x}}}_{N-1}(k \mid k) + \left\langle \overline{\mathbf{x}}(k), \overline{\mathbf{e}}_{y,N}(k) \right\rangle \\ &\quad \cdot \left\langle \overline{\mathbf{e}}_{y,N}(k), \overline{\mathbf{e}}_{y,N}(k) \right\rangle^{-1} \overline{\mathbf{e}}_{y,N}(k). \end{aligned} \tag{56}$$

Denote $\mathbf{R}_N(k) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\gamma^2\mathbf{I} \end{bmatrix}$; then one gets

$$\begin{aligned} &\left\langle \overline{\mathbf{x}}(k), \overline{\mathbf{e}}_{y,N}(k) \right\rangle \left\langle \overline{\mathbf{e}}_{y,N}(k), \overline{\mathbf{e}}_{y,N}(k) \right\rangle^{-1} \\ &\quad = \mathbf{P}_N(k) \left[\mathbf{H}_N^T(k) \ \ \mathbf{M}^T(k) \ \ \mathbf{L}^T(k)\right] \mathbf{P}_{ey,N}^{-1}(k), \\ &\mathbf{P}_{ey,N}(k) = \left\langle \overline{\mathbf{e}}_{y,N}(k), \overline{\mathbf{e}}_{y,N}(k) \right\rangle \\ &\quad = \mathbf{R}_N(k) \\ &\qquad + \begin{bmatrix} \mathbf{H}_N(k) \\ \mathbf{M}(k) \\ \mathbf{L}(k) \end{bmatrix} \mathbf{P}_N(k) \left[\mathbf{H}_N^T(k) \ \ \mathbf{M}^T(k) \ \ \mathbf{L}^T(k)\right]. \end{aligned} \tag{57}$$

$\hat{\overline{\boldsymbol{\xi}}}_N(k \mid k)$ corresponds to the stationary point of (30c), the value of $\mathbf{J}_N(k)$ at which point is

$$\begin{aligned} \mathbf{J}_N^*(k) &= \sum_{l=1}^{k-1} \mathbf{e}_Y^T(l) \mathbf{P}_{eY}^{-1}(l) \mathbf{e}_Y(l) \\ &\quad + \sum_{j=1}^{N} \mathbf{e}_{y,j}^T(k) \mathbf{P}_{ey,j}^{-1}(k) \mathbf{e}_{y,j}(k) \\ &= \mathbf{J}_{N-1}^*(k) + \mathbf{e}_{y,N}^T(k) \mathbf{P}_{ey,N}^{-1}(k) \mathbf{e}_{y,N}(k) \end{aligned} \tag{58}$$

in which

$$\begin{aligned} \mathbf{P}_{ey,N}^{-1}(k) &= \begin{bmatrix} \hat{\mathbf{A}}(k) & \hat{\mathbf{B}}(k) \\ \hat{\mathbf{B}}^T(k) & \hat{\mathbf{C}}(k) \end{bmatrix}^{-1} = \begin{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\hat{\mathbf{A}}_{12}^T(k)\hat{\mathbf{A}}_{11}^{-1}(k) & \mathbf{I} \end{bmatrix} & \mathbf{0} \\ -\hat{\mathbf{B}}^T(k)\hat{\mathbf{A}}^{-1}(k) & \mathbf{I} \end{bmatrix}^T \\ &\quad \cdot \begin{bmatrix} \begin{bmatrix} \hat{\mathbf{A}}_{11}^{-1}(k) & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{A}}_3^{-1}(k) \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \hat{\boldsymbol{\varepsilon}}^{-1}(k) \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\hat{\mathbf{A}}_{12}^T(k)\hat{\mathbf{A}}_{11}^{-1}(k) & \mathbf{I} \end{bmatrix} & \mathbf{0} \\ -\hat{\mathbf{B}}^T(k)\hat{\mathbf{A}}^{-1}(k) & \mathbf{I} \end{bmatrix}, \end{aligned} \tag{59}$$

$$\begin{aligned} \hat{\mathbf{A}}(k) &= \begin{bmatrix} \hat{\mathbf{A}}_{11}(k) & \hat{\mathbf{A}}_{12}(k) \\ \hat{\mathbf{A}}_{12}^T(k) & \hat{\mathbf{A}}_{22}(k) \end{bmatrix} = \begin{bmatrix} \mathbf{H}_N(k) \\ \mathbf{M}(k) \end{bmatrix} \mathbf{P}_N(k) \begin{bmatrix} \mathbf{H}_N(k) \\ \mathbf{M}(k) \end{bmatrix}^T \\ &\quad + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}, \\ \hat{\mathbf{C}}(k) &= \mathbf{L}(k)\mathbf{P}_N(k)\mathbf{L}^T(k) - \gamma^{-2}\mathbf{I}, \\ \hat{\mathbf{A}}_3(k) &= \hat{\mathbf{A}}_{22}(k) - \hat{\mathbf{A}}_{12}^T(k)\hat{\mathbf{A}}_{11}^{-1}(k)\hat{\mathbf{A}}_{12}(k), \\ \hat{\boldsymbol{\varepsilon}}(k) &= \hat{\mathbf{C}}(k) - \hat{\mathbf{B}}^T(k)\hat{\mathbf{A}}^{-1}(k)\hat{\mathbf{B}}(k), \\ \hat{\mathbf{B}}(k) &= \begin{bmatrix} \mathbf{H}_N(k) \\ \mathbf{M}(k) \end{bmatrix} \mathbf{P}_N(k)\mathbf{L}^T(k). \end{aligned} \tag{60}$$

Therefore,

$$\begin{aligned} \mathbf{J}_N^*(k) &= \mathbf{J}_{N-1}^*(k) + \mathbf{e}_{y,N}^T(k)\mathbf{P}_{ey,N}^{-1}(k)\mathbf{e}_{y,N}(k) \\ &= \mathbf{J}_{N-1}^*(k) + \tilde{\overline{\mathbf{y}}}_N^T(k)\mathbf{A}_{11}^{-1}(k)\tilde{\overline{\mathbf{y}}}_N(k) + \hat{\overline{\mathbf{x}}}^T(k \mid k-1) \\ &\quad \cdot \mathbf{M}^T(k)\mathbf{A}_3^{-1}(k)\mathbf{M}(k)\hat{\overline{\mathbf{x}}}(k \mid k-1) \\ &\quad + \left[\hat{\mathbf{z}}_N(k \mid k) - \hat{\mathbf{z}}_N^*(k \mid k)\right]^T \boldsymbol{\varepsilon}^{-1}(k) \\ &\quad \cdot \left[\hat{\mathbf{z}}_N(k \mid k) - \hat{\mathbf{z}}_N^*(k \mid k)\right], \end{aligned} \tag{61}$$

where

$$\begin{aligned} \hat{\overline{\mathbf{x}}}(k \mid k-1) &= -\mathbf{P}_N(k)\mathbf{H}_N^T(k)\hat{\mathbf{A}}_{11}^{-1}(k)\tilde{\overline{\mathbf{y}}}_N(k) \\ \hat{\mathbf{z}}_N^*(k \mid k) & \\ &= \mathbf{L}(k)\hat{\mathbf{x}}_{N-1}(k \mid k) \\ &\quad + \hat{\mathbf{B}}^T(k)\hat{\mathbf{A}}^{-1}(k)\begin{pmatrix} \mathbf{y}_N(k) - \mathbf{H}_N(k)\hat{\mathbf{x}}_{N-1}(k \mid k) \\ -\mathbf{M}(k)\hat{\mathbf{x}}_{N-1}(k \mid k) \end{pmatrix}, \\ \tilde{\overline{\mathbf{y}}}_N(k) &= \mathbf{y}_N(k) - \mathbf{H}_N(k)\hat{\mathbf{x}}_{N-1}(k \mid k). \end{aligned} \tag{62}$$

$\mathbf{J}_N^*(k)$ is the minimum of $\mathbf{J}_N(k)$, if and only if $\mathbf{P}_{ey,N}(k)$ and $\mathbf{R}_N(k)$ have the same inertia. Considering (59), one gets the sufficient condition of the minimum that is equivalent to $\widehat{\mathbf{A}}_3(k) < 0$, $\widehat{\boldsymbol{\varepsilon}}(k) < 0$. A choice of $\widehat{\mathbf{z}}(k \mid k)$ to ensure $\mathbf{J}^*(k) > 0$ is $\widehat{\mathbf{z}}(k \mid k) = \widehat{\mathbf{z}}^*(k \mid k)$; then the estimate of the signal to be estimated is

$$\widehat{\mathbf{z}}_N(k \mid k) = \widehat{\mathbf{z}}_N^*(k \mid k) = \mathbf{L}(k)\widehat{\mathbf{x}}_N(k \mid k) \tag{63a}$$

$$\widehat{\mathbf{x}}_N(k \mid k) = \widehat{\mathbf{x}}_{N-1}(k \mid k) + \mathbf{P}_N(k)\begin{bmatrix}\mathbf{H}_N(k)\\ \mathbf{M}(k)\end{bmatrix}^T \cdot \widehat{\mathbf{A}}^{-1}(k)\begin{pmatrix}\mathbf{y}_N(k) - \mathbf{H}_N(k)\widehat{\mathbf{x}}_{N-1}(k \mid k)\\ -\mathbf{M}(k)\widehat{\mathbf{x}}_{N-1}(k \mid k)\end{pmatrix}, \tag{63b}$$

$$\widehat{\mathbf{A}}(k) = \begin{bmatrix}\mathbf{H}_N(k)\\ \mathbf{M}(k)\end{bmatrix}\mathbf{P}_N(k)\begin{bmatrix}\mathbf{H}_N(k)\\ \mathbf{M}(k)\end{bmatrix}^T + \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I}\end{bmatrix}. \tag{63c}$$

Because $\mathbf{x}_1(k+1) = \mathbf{F}(k+1,k)\mathbf{x}_N(k) + \mathbf{G}(k+1)\overleftrightarrow{\mathbf{w}}(k+1,k)$, the corresponding Riccati equation is given by

$$\mathbf{P}_1(k+1) = \mathbf{F}(k+1,k)\mathbf{P}_N(k)\mathbf{F}^T(k+1,k) + \mathbf{G}(k+1)\mathbf{G}^T(k+1) - \mathbf{F}(k+1,k)\mathbf{P}_N(k) \cdot \mathbf{H}_N^T(k)\mathbf{P}_{ey,N}^{-1}(k)\mathbf{H}_N(k)\mathbf{P}_N(k)\mathbf{F}^T(k+1,k), \tag{63d}$$

$$\mathbf{P}_{ey,N}(k) = \mathbf{R}_N(k) + \begin{bmatrix}\mathbf{H}_N(k)\\ \mathbf{M}(k)\\ \mathbf{L}(k)\end{bmatrix}\mathbf{P}_N(k) \cdot \begin{bmatrix}\mathbf{H}_N^T(k) & \mathbf{M}^T(k) & \mathbf{L}^T(k)\end{bmatrix}. \tag{63e}$$

Based on the analysis of the above three cases, a novel robust finite horizon $H\infty$ fusion filter is given by (41a), (41b), (41c), (41d), (50a), (50b), and (50c) and (63a), (63b), (63c), (63d), and (63e), which could deal with the measurements sequentially.

Remark 5. The existing robust finite horizon $H\infty$ fusion filters are mostly designed for the time-invariant systems, while the two proposed robust finite horizon $H\infty$ fusion filters not only apply to the time-varying systems, but also could deal with the fusion filtering problem for time-invariant ones.

Remark 6. As shown in Section 3.1, the centralized robust finite horizon $H\infty$ fusion filter deals with all N measurements in a filtering process. It means that the estimate of the signal to be estimated cannot be obtained until these measurements are all received by the fusion center. It is implied that the real time performance of the centralized fusion methods is usually poor, especially for the scene that the fusion center asynchronously receives measurements or with delay phenomenon. By contrast, the sequential robust finite horizon $H\infty$ fusion filter proposed in Section 3.2 sequentially handles measurements in time, once it is received by the fusion center, and avoids waiting for other measurements. In this sense, this sequential fusion filter is real time.

Remark 7. The two proposed methods could obtain the same fusion filtering accuracy, which is illustrated in the next section.

Remark 8. In a fusion period, the centralized robust finite horizon $H\infty$ fusion filter needs strong matrix computing ability to deal with a filtering process of a $\sum_{i=1}^{N} m_i$-dimensional measurement, while the sequential robust finite horizon $H\infty$ fusion filter needs to deal with N filtering processes of m_i-dimensional measurement. The filtering time of the proposed robust fusion filters is comparatively analyzed in the simulation in the next section.

4. Simulation

In this section, two numerical examples are exploited to illustrate the effectiveness and the equivalence of the two proposed robust fusion filters for linear time-invariant systems and linear time-varying systems. For convenience, the centralized robust finite horizon $H\infty$ fusion filter proposed in Section 3.1 is marked as "Filter 1," while the sequential one proposed in Section 3.2 is " Filter 2."

4.1. The Linear Time-Invariant System Case. In order to verify the effectiveness of the two proposed robust fusion filters for linear time-invariant systems, the following linear discrete system is considered:

$$\mathbf{x}(k) = \left(\begin{bmatrix}0.71 & 1\\ 0 & 0.81\end{bmatrix} + \begin{bmatrix}0\\ 1.2\end{bmatrix}\Lambda(k)\begin{bmatrix}0 & 0.25\end{bmatrix}\right)\mathbf{x}(k-1) + \mathbf{w}(k,k-1), \tag{64}$$

where $\mathbf{x}(k)$ is the state and $\mathbf{w}(k,k-1) \in l_2[1,N)$ is the process noise. $\Lambda(k)$ represents the unknown real-valued time-varying matrix, and $\|\Lambda(k)\| \leq 1$. In this section, $\Lambda(k) = \sin(k)$.

Consider the system running process observed by three sensors, the observation equations of which can be given by $\mathbf{y}_i(k) = \mathbf{H}\mathbf{x}(k) + \mathbf{v}_i(k)$, $i = 1,2,3$, in which $\mathbf{H}$ could be $\mathbf{H}_1 = [1,0]$, $\mathbf{H}_2 = [1,0;0\,1;0.5\,0.5;0.3\,0.7;1,0;0\,1;0.5\,0.5;0.3\,0.7]$, or $\mathbf{H}_3 = [1,0;0\,1;0.5\,0.5;0.3\,0.7;1,0;0\,1;0.5,0.5;0.3\,0.7;1,0;0\,1;0.5\,0.5;0.3\,0.7;1,0;0\,1;0.5\,0.5;0.3\,0.7]$. The signal to be estimated is $\mathbf{z}(k) = [1,0]\mathbf{x}(k)$. The initial conditions are $\mathbf{x}_0 = \begin{bmatrix}0.1 & -0.5\end{bmatrix}^T$, $\mathbf{P}_0 = \begin{bmatrix}1/2 & 1/2\\ 1/2 & 1\end{bmatrix}$, and $\gamma = 0.25$.

In each fusion period, the global fusion filtering result is obtained when the last received measurement is filtered. Considering the asynchronously and delay phenomenon mentioned in Remark 6, the filtering time of the last measurement in each fusion period of different methods is statistically compared. What is more, the whole filtering time of different methods is also compared. Using Monte-Carlo method of 100 runs, we statistically analyze the mean of the filtering time of the last measurement in each fusion period of different methods and the mean of the filtering time of different methods in Table 1. By this way, we compare the real value and the estimates of the signal to be estimated by different methods in each fusion period in Figure 1 and the absolute

Table 1: The mean absolute estimation error of the signal to be estimated, when $\mathbf{H}$ is $\mathbf{H}_1$, $\mathbf{H}_2$, and $\mathbf{H}_3$.

$\mathbf{H}$	$\mathbf{H}_1$		$\mathbf{H}_2$		$\mathbf{H}_3$	
Filter	Filter 1	Filter 2	Filter 1	Filter 2	Filter 1	Filter 2
The mean absolute estimation error	0.0035	0.0035	0.0087	0.0087	0.0087	0.0087
Mean filtering time	0.7864	1.1232	1.4820	1.6068	3.6348	2.4336
Mean filtering time of the last measurement in each fusion period	0.7864	0.4992	1.4820	0.6552	3.6348	1.1700

Table 2: The mean absolute estimation error of the signal to be estimated.

	Filter 1	Filter 2	Filter 3	Filter 4
The mean absolute estimation error	0.1176	0.1176	0.1209	0.1890
Mean filtering time of the last measurement in each fusion period	0.0156	0.0084	0.0140	3.1871

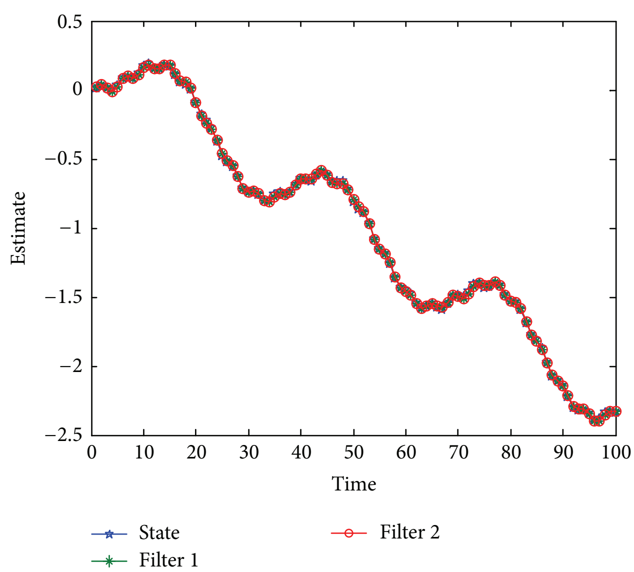

Figure 1: The real value and the estimates of the signal to be estimated, when $\mathbf{H}$ is $\mathbf{H}_1$.

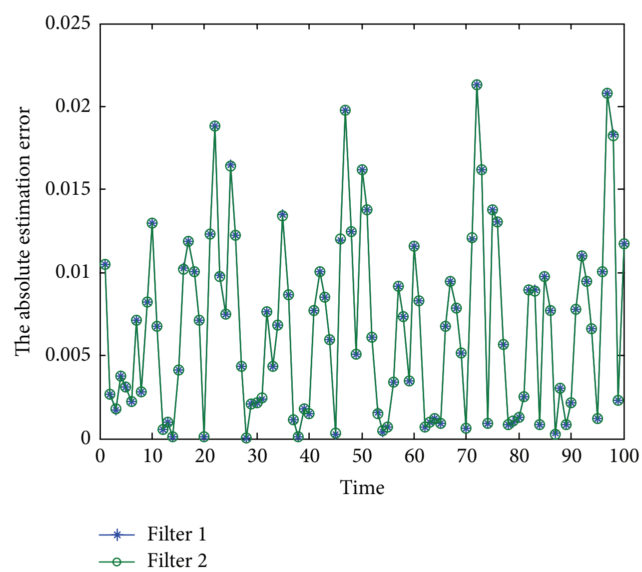

Figure 2: The absolute estimation error curves of the signal to be estimated, when $\mathbf{H}$ is $\mathbf{H}_1$.

value of estimation error values by different methods in each fusion period in Figure 2, the mean values of which are further given in Table 1.

As shown in the simulation results above, the two proposed robust fusion filters are able to effectively deal with the robust fusion filtering problems for the linear time-invariant system with unknown system parameter as given by (1). The functional equivalence of the two proposed robust fusion filters is verified from the fact that they have the same estimated accuracy as shown in Figure 2 and Table 1.

Furthermore, due to the sequential fusion strategy, Filter 2 could estimate the interested signal with each measurement once it arrives at the fusion center, without waiting for all the measurements received by the fusion center. It means that Filter 2 is a real time robust fusion filter. In each fusion period, although the whole filtering time of Filter 2 is longer than the one of Filter 1 when $\mathbf{H}$ is $\mathbf{H}_1$ and $\mathbf{H}_2$, the filtering time to deal with the last measurement by Filter 2 is shorter than the one by Filter 1, as shown in Table 1.

It is indicated that the mean filtering time of Filter 1 in a fusion period increases with the increase of the dimension of the measurement matrix $\mathbf{H}$, and it is over the mean filtering time of Filter 1 when $\mathbf{H}$ is $\mathbf{H}_3$. This is because Filter 1 is designed based on the augmented measurement function, while the augmented operation is avoided in Filter 2. In the fusion filtering process, it is inevitable to perform high-dimensional matrix inversion operation. Therefore, the sequential fusion filter is more effective when the dimension

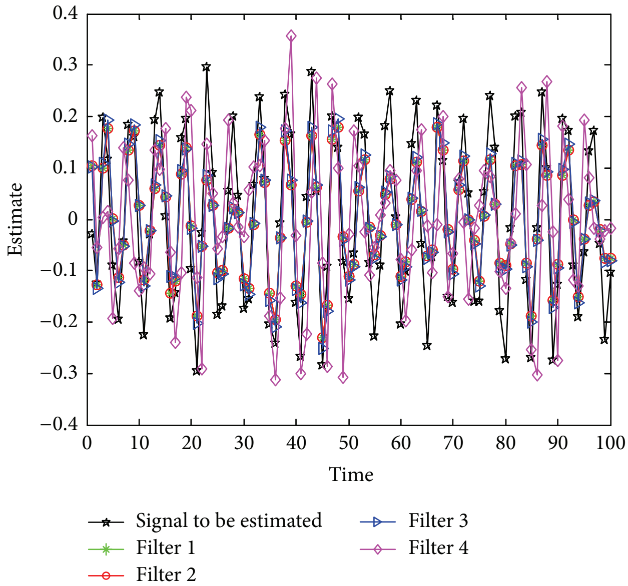

Figure 3: The real value and the estimates of the signal to be estimated.

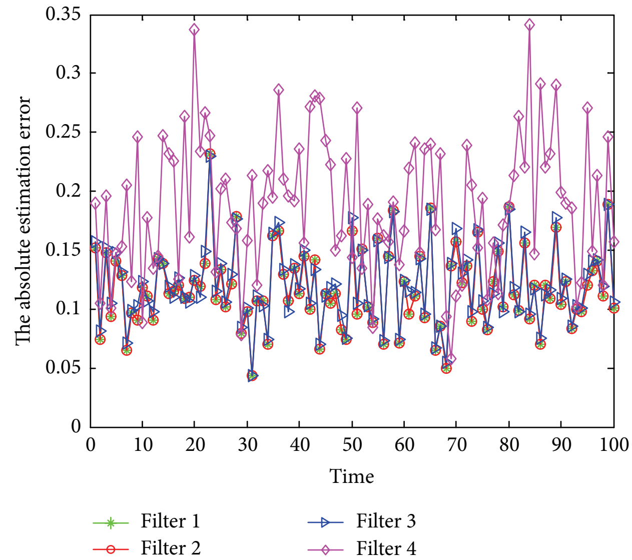

Figure 4: The absolute estimation error curves of the signal to be estimated.

of the measurement is higher. For the scenario that all measurements are sampled by different sensors at the same time and received by fusion filter synchronously, the centralized fusion filter is a good choice.

4.2. The Linear Time-Varying System Case. In this subsection, the following linear time-varying system is considered. And the two proposed fusion filters are compared with the fusion filter which ignores the uncertainty of system parameters and the centralized robust fusion filtering method evolved from [16], which are separately marked as "Filter 3" and "Filter 4":

$$\mathbf{x}(k) = \left(\begin{bmatrix} 0 & -0.4 \\ 0.6 & 0.7 * \sin(6 * k) \end{bmatrix} + \begin{bmatrix} 0 \\ -0.2 \end{bmatrix} \Lambda(k) \begin{bmatrix} 0.2 & 0 \end{bmatrix}\right) \mathbf{x}(k-1) + \mathbf{w}(k, k-1), \quad (65)$$

$$\mathbf{y}_i(k) = [0.3 + 0.2 * \sin(6 * k), 1] \mathbf{x}(k) + \mathbf{v}_i(k), \quad i = 1, 2, 3,$$

where $\Lambda(k) = \cos(0.1 * k)$. The signal to be estimated is $\mathbf{z}(k) = [1, 1]\mathbf{x}(k)$. The initial conditions are $\mathbf{x}_0 = [0.2 \ \ -0.1]^T$, $\mathbf{P}_0 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and $\gamma = 1$. Using Monte-Carlo method of 100 runs, the statistical analysis results are shown in Figures 3 and 4 and Table 2.

As shown in the simulation results, the two proposed fusion filters have the same filtering results. Compared with the centralized $H\infty$ fusion filter ignoring the system parameter uncertainty (Filter 3), both of the proposed fusion filters have higher filtering accuracy. This is because the information of the uncertain parameter is effectively utilized by the proposed fusion filters, while it is ignored by Filter 3. Filter 4 is a priori robust fusion filter, while the proposed robust fusion filters are a posteriori ones. Therefore, the absolute values of the estimate error by Filter 1 and Filter 2 are lower than the ones of Filter 4, as shown in Figure 4 and Table 2. What is more, in each fusion period, a recursive optimization process is required to solve the robust fusion filter parameters by the LMI toolbox. That is why the filtering time of Filter 4 is longer than others.

In each fusion period, the mean filtering time of the last measurement by Filter 2 is shorter than the one by others. This is because of the sequential fusion strategy, according to which the measurements are sequentially dealt with by fusion filter once they arrive, but not dealt with by the others together.

5. Conclusions

In this paper, two robust finite horizon $H\infty$ fusion filtering algorithms are proposed for linear time-varying uncertain systems. The linear estimation method in Krein spaces is utilized to solve the performance index function which is defined as an indefinite quadratic inequality. Firstly, the stationary of the indefinite quadratic form is given by a projection method in Krein space. Based on the projection method, a robust centralized finite horizon $H\infty$ fusion filtering algorithm is designed. Then, the performance index function is substituted by a set of quadratic inequalities. A sequential robust fusion filtering method is developed by solving these quadratic inequalities. The simulations illustrate the effectiveness of the two proposed algorithms.

Appendix

Proof of Lemma 3

Proof. Assume that $\mathbf{J}(k-1) = \mathbf{J}_N(k-1) > 0$.

Because of the fact that $\overleftrightarrow{\mathbf{w}}^T(k,k-1)\overleftrightarrow{\mathbf{w}}(k,k-1) \geq 0$, $\mathbf{v}_i^T(k)\mathbf{v}_i(k) \geq 0$, $i = 1,\dots,N-1$, then

$$\begin{aligned}\mathbf{J}_1(k) &= \mathbf{J}_N(k-1) + \overleftrightarrow{\mathbf{w}}^T(k,k-1)\overleftrightarrow{\mathbf{w}}(k,k-1) \\ &\quad + \mathbf{v}_1^T(k)\mathbf{v}_1(k) > 0, \\ \mathbf{J}_i(k) &= \mathbf{J}_{i-1}(k) + \mathbf{v}_i^T(k)\mathbf{v}_i(k) > 0, \\ &\qquad\qquad i = 2,\dots,N-1.\end{aligned} \tag{A.1}$$

In (9),

$$\begin{aligned}\mathbf{J}(k) &= \mathbf{J}(k-1) \\ &\quad + \begin{bmatrix}\mathbf{w}^T(k,k-1) & \boldsymbol{\xi}^T(k)\end{bmatrix}\begin{bmatrix}\mathbf{w}(k,k-1) \\ \boldsymbol{\xi}(k)\end{bmatrix} \\ &\quad + \sum_{j=1}^{N}\mathbf{v}_j^T(k)\mathbf{v}_j(k) - \gamma^{-2}\mathbf{e}_z^T(k)\mathbf{e}_z(k) \\ &\quad - \mathbf{s}^T(k)\mathbf{s}(k) \\ &= \mathbf{J}_{N-1}(k) + \mathbf{v}_N^T(k)\mathbf{v}_N(k) - \gamma^{-2}\mathbf{e}_z^T(k)\mathbf{e}_z(k) \\ &\quad - \mathbf{s}^T(k)\mathbf{s}(k) = \mathbf{J}_N(k).\end{aligned} \tag{A.2}$$

Therefore, $\mathbf{J}(k) > 0 \Leftrightarrow \mathbf{J}_N(k) > 0$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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