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Research Article

Dynamics of a Stochastic Functional System for Wastewater Treatment

Xuehui Ji and Sanling Yuan

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China

Correspondence should be addressed to Sanling Yuan; sanling@usst.edu.cn

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The dynamics of a delayed stochastic model simulating wastewater treatment process are studied. We assume that there are stochastic fluctuations in the concentrations of the nutrient and microbes around a steady state, and introduce two distributed delays to the model describing, respectively, the times involved in nutrient recycling and the bacterial reproduction response to nutrient uptake. By constructing Lyapunov functionals, sufficient conditions for the stochastic stability of its positive equilibrium are obtained. The combined effects of the stochastic fluctuations and delays are displayed.

1. Introduction

In the last few years, the use of mathematical models describing wastewater treatment is gaining attention as a promising method [1–6]. A basic chemostat model describing substrate-microbe interaction in an activated sludge process is as follows:

$$\begin{aligned}\frac{dS}{dt} &= \frac{Q\left(S^0 - S\right)}{V} - \frac{kxS}{K_S + S}\frac{D_O}{K_O + D_O},\\ \frac{dx}{dt} &= x\left(\frac{kYS}{K_S + S} - K_d\right)\frac{D_O}{K_O + D_O} - \frac{Q_w x}{V},\end{aligned} \tag{1}$$

where $S(t)$ and $x(t)$ represent the concentrations of the substrate (biochemical oxygen demand) and microbes in an aeration tank at time t, respectively. Q is the washout rate, S^0 is the input concentration of the substrate, and V is the effective volume of the aeration tank; k is the maximum uptake rate of the substrate; K_S and K_O are the half-saturation constants of the substrate and oxygen; respectively, K_d is the decay rate of microbes and Q_w is the emission rate of the sludge; D_O is the concentration of the dissolved oxygen and $D_O/(K_O+D_O)$ is a switching function describing the effect of D_O on the uptake rate k and the decay rate K_d; $Y \in (0,1)$ is the ratio of the concentration of mixed liquor suspended solids to the substrate. Some extensions and generalizations of the model have been proposed by many researchers (see [7–27], etc.).

Even though deterministic model (1) has a stable positive equilibrium (S^*, x^*) under certain conditions, oscillations have been observed frequently in the growth of microbes during the experiments [28, 29], which have also been confirmed by many mathematical works for some extended chemostat models incorporating factors such as time delay [15–18, 30–32], periodic nutrient input [19–21, 33–35], feedback control [22–24], and stochastic environmental perturbations [25–27]. For a better understanding of microbial population dynamics in the activated sludge process, we take two steps towards developing model (1).

On the one hand, we take into account time delays that may exist in the process of wastewater treatment. By the death regeneration theory of Dold and Marais [36], the active biomass dies at a certain rate; of the biomass lost, the biodegradable portion adds to the slowly biodegradable organic matter which passes through the various stages to be utilised for active biomass synthesis, which requires some time for the completion of the regeneration. Also there is a time delay that accounts for the time lapse between the uptakes of substrates and the incorporation of these substrates, which has ever been observed from chemostat experiments with microalgae *Chlamidomonas Reinhardii* even when the limiting nutrient is at undetectable small concentration (see [37, 38], etc.). In the recent years, chemostat models with such time delays have been given much attention (see, e.g., [9, 14, 16–18, 39], etc.). In this paper, we will use distributed delays to describe the nutrient recycling and the time lapse between the uptakes of nutrient and the incorporation of this nutrient with delay kernels $f(s)$ and $g(s)$, respectively.

On the other hand, in a real process of wastewater treatment there will be fluctuations in concentration of the substrate and microbe population due to stochastic perturbations from external sources such as temperature, light, and the like, or inherent sources in the chemical-physical and biological processes [40]. So we assume that model (1) is exposed by stochastic perturbations which are of white noise type and are proportional to the distances $S(t), x(t)$ from values of the positive equilibrium S^*, x^*, influence on the $\dot{S}(t)$ and $\dot{x}(t)$, respectively. By this way, model (1) becomes in the following form:

$$\begin{aligned} dS = & \left[\frac{Q\left(S^0 - S\right)}{V} - kU(S)\frac{D_O}{K_O + D_O} \right. \\ & \left. + \mu K_d \frac{D_O}{K_O + D_O}\int_0^\infty f(s)\,x(t-s)\,ds \right] dt \\ & + \sigma_1 \left(S - S^*\right) dB_1(t), \\ dx = & \left[x\left(Yk\int_0^\infty g(s)\,U(S(t-s))\,ds - K_d\right) \right. \\ & \left. \times \frac{D_O}{K_O + D_O} - \frac{Q_w x}{V} \right] dt + \sigma_2\left(x - x^*\right) dB_2(t), \end{aligned} \tag{2}$$

where $B_i(t)$ $(i = 1, 2)$ are standard independent Wiener processes and $\sigma_i \geq 0$ $(i = 1, 2)$ represent the intensities of the noises. $\mu \in (0, 1)$ is the fraction of the substrate regenerated from the dead biomass; $U(S)$ is a general specific growth function.

Recently, stochastic biological systems and stochastic epidemic models have been studied by many authors; see, for example, Mao et al. [41, 42], Jiang et al. [43, 44], Liu and Wang [45, 46], and the references cited therein. But, as far as we know, there are few works on model (2). In this paper, our main purpose is to study the combined effect of the noises and delays on the dynamics of model (2), that is, whether and how the noises and delays affect the stability of E^*. By the construction of appropriate Lyapunov functionals, we will show that the positive equilibrium keeps stochastically stable if the noises and delays are small. Furthermore, the sensitivities of the stability of E^* with respect to the delays and noises are also discussed.

The paper is organized as follows. We first establish some preliminary results in Section 2. By constructing Lyapunov function(al)s, sufficient conditions for the stochastic stability of the positive equilibrium of the model without and with delays are obtained, respectively, in Sections 3 and 4. Numerical simulations and discussions are finally presented in Section 5.

2. Some Preliminaries

Define $Q/V = D$, $Q_w/V = D_w$, $k(D_O/(K_O + D_O)) = m$, $K_d(D_O/(K_O + D_O)) = D_1$, and $Y = \gamma$. Then model (2) can be simplified as follows:

$$\begin{aligned} dS = & \left[D\left(S^0 - S\right) - mU(S)\,x \right. \\ & \left. + \mu D_1 \int_0^\infty f(s)\,x(t-s)\,ds \right] dt \\ & + \sigma_1\left(S - S^*\right) dB_1, \\ dx = & \, x\left[-\left(D_w + D_1\right) + \gamma m \int_0^\infty g(s)\,U(S(t-s))\,ds \right] dt \\ & + \sigma_2\left(x - x^*\right) dB_2 \end{aligned} \tag{3}$$

with initial value conditions

$$S(\theta,\omega) = \varphi_1(\theta) \geq 0, \qquad x(\theta,\omega) = \varphi_2(\theta) \geq 0, \qquad \theta \in (-\infty, 0], \tag{4}$$

where $\varphi_1(\theta)$, $\varphi_2(\theta) \in \mathscr{BC}((-\infty, 0], \mathbb{R}_+)$, the families of bounded continuous functions from $(-\infty, 0]$ to $\mathbb{R}_+$.

The corresponding deterministic model of (3) is

$$\begin{aligned} \dot{S} &= D\left(S^0 - S\right) - mU(S)\,x + \mu D_1 \int_0^\infty f(s)\,x(t-s)\,ds, \\ \dot{x} &= -\left(D_w + D_1\right)x + \gamma m x \int_0^\infty g(s)\,U(S(t-s))\,ds, \end{aligned} \tag{5}$$

the special case of which when $D = D_w$ has ever been investigated by He et al. [18]. It is easy to see that model (5) has a positive equilibrium $E^*(S^*, x^*)$ provided that

$$D_w + D_1 < \gamma m, \qquad S^0 > S^*, \tag{6}$$

where

$$S^* = U^{-1}\left(\frac{D_w + D_1}{\gamma m}\right), \qquad x^* = \frac{D\left(S^0 - S^*\right)}{mU(S^*) - \mu D_1}. \tag{7}$$

$E^*(S^*, x^*)$ is globally asymptotically stable provided that the average delays are sufficiently small. Obviously, E^* is still an equilibrium of stochastic model (3) if condition (6) holds.

We assume that function $U(S)$ is nonnegative satisfying

$$\begin{aligned} & U(0) = 0, \quad U'(S) > 0, \\ & U''(S) < 0 \quad \text{for } S > 0, \quad \lim_{S\to\infty} U(S) = 1. \end{aligned} \tag{8}$$

And we extend the function $U(S)$ by defining

$$U(S) = U'(0)\,S + \frac{1}{2}U''(0)\,S^2 \quad \text{for } S \leq 0, \tag{9}$$

so that U is well defined in $\mathbb{R}$ and is still of class $\mathscr{C}^2$ in $\mathbb{R}$. Thus one can write

$$U(S) = a + b\left(S - S^*\right) + F\left(S - S^*\right), \tag{10}$$

where F represents terms of order ≥ 2 in $S-S^*$. Noting also that $a = U(S^*)$ and $b = U'(S^*)$, by condition (6), it follows that $ma > \mu D_1$.

Introduce new variables $u_1 = S - S^*$, $u_2 = x - x^*$; then model (3) can be rewritten as follows:

$$\begin{aligned} du_1 &= \left[-\left(D + mbx^*\right) u_1 + \mu D_1 \int_0^\infty f(s)\, u_2(t-s)\, ds \right.\\ &\qquad \left. - mau_2 + F_1(u_1, u_2)\right] dt + \sigma_1 u_1 dB_1, \\ du_2 &= \left[\gamma mbx^* \int_0^\infty g(s)\, u_1(t-s)\, ds \right.\\ &\qquad \left. + \widetilde{F}_2(u_1, u_2)\right] dt + \sigma_2 u_2 dB_2, \end{aligned} \tag{11}$$

where

$$\begin{aligned} F_1 &= -mbu_1u_2 - mF(u_1)\, u_2 - mx^* F(u_1), \\ \widetilde{F}_2 &= \gamma mbu_2 \int_0^\infty g(s)\, u_1(t-s)\, ds \\ &\quad + \left(u_2 + x^*\right) \gamma m \int_0^\infty g(s)\, F\left(u_1(t-s)\right) ds. \end{aligned} \tag{12}$$

Note that if $f(s) = g(s) = \delta(0)$, then model (11) has the form

$$\begin{aligned} du_1 &= \left[-\left(D + mbx^*\right) u_1 + \left(\mu D_1 - ma\right) u_2 \right.\\ &\qquad \left. + F_1(u_1, u_2)\right] dt + \sigma_1 u_1 dB_1, \\ du_2 &= \left[\gamma mbx^* u_1 + F_2(u_1, u_2)\right] dt + \sigma_2 u_2 dB_2, \end{aligned} \tag{13}$$

where

$$F_2 = \gamma mbu_1u_2 + \left(u_2 + x^*\right) \gamma mF(u_1). \tag{14}$$

Obviously, model (13) has the same equilibrium $(0,0)$ as model (11), and the stochastic stability of the positive equilibrium E^* of model (3) is equivalent to the zero solution of model (11). We wonder how the stochastic perturbations and delays affect the dynamics of model (3) or (11).

Before starting our analysis, we first give some basic theories in stochastic differential equations and stochastic functional differential equations [47–49]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let B_i $(i = 1, 2, \ldots, n)$ be the Brownian motions defined on this probability space. Consider the following n-dimensional stochastic differential equation:

$$dx(t) = f(x(t), t)\, dt + g(x(t), t)\, dB(t), \quad t \geq t_0. \tag{15}$$

Definition 1. The trivial solution of system (15) is said to be as follows:

(i) stochastically stable or stable in probability if for every pair of $\varepsilon \in (0,1)$ and $r > 1$, there exists a $\delta = \delta(\varepsilon, r, t_0) > 0$ such that

$$P\left\{\left|x(t; t_0, x_0)\right| < r \ \forall t \geq t_0\right\} \geq 1 - \varepsilon, \tag{16}$$

whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically unstable,

(ii) stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\varepsilon \in (0,1)$, there exists a $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ such that

$$P\left\{\lim_{t\to\infty} x(t; t_0, x_0) = 0\right\} \geq 1 - \varepsilon, \tag{17}$$

whenever $|x_0| < \delta_0$,

(iii) globally asymptotically stable in probability if it is stochastically asymptotically stable and, moreover, for all $x_0 \in \mathbb{R}^n$

$$P\left\{\lim_{t\to\infty} x(t; t_0, x_0) = 0\right\} = 1. \tag{18}$$

Lemma 2. *If there exists a nonnegative function $V(x,t) \in C^{2,1}(\mathbb{R}^n \times [t_0, \infty]; \mathbb{R}_+)$, two continuous functions ψ_1, ψ_2 : $\mathbb{R}_+^0 \to \mathbb{R}_+^0$, and a positive constant K such that, for $|x| < K$,*

$$\psi_1(|x|) \leq V(x,t) \leq \psi_2(|x|) \tag{19}$$

hold.

(i) *If*

$$LV \leq 0, \quad \text{for } |x| \in [0, K], \tag{20}$$

then the trivial solution of system (A.1) is stochastically stable.

(ii) *If there exists a continuous function $\psi_3 : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ such that*

$$LV \leq -\psi_3(|x|) \tag{21}$$

holds, then the trivial solution of system (15) *is stochastically asymptotically stable.*

(iii) *If (ii) holds and moreover*

$$\lim_{r\to\infty} \psi_1(r) = +\infty, \tag{22}$$

then the trivial solution of system (15) *is globally asymptotically stable in probability.*

For the stability of the equilibrium of a nonlinear stochastic system, it can be reduced to problems concerning stability of solutions of the linear associated system. The linear form of (15) is defined as follows:

$$dx(t) = F(t) \cdot x(t)\, dt + G(t) \cdot x(t)\, dB(t), \quad t \geq t_0. \tag{23}$$

Lemma 3. *If the trivial solution is stochastically stable for the linear system* (23) *with constant coefficients ($F(t) = F$, $G(t) = G$) and the coefficients of systems* (15) *and* (23) *satisfy the following inequality:*

$$\left|f(x,t) - F \cdot x\right| + \left|g(x,t) - G \cdot x\right| < \rho |x| \tag{24}$$

in a sufficiently small neighborhood of $x = 0$, with a sufficiently small constant ρ, then the trivial solution of system (15) *is asymptotically stable in probability.*

Consider the following n-dimensional stochastic functional differential equation

$$dx = f(t, x_t)\, dt + g(t, x_t)\, dB(t) \tag{25}$$

with initial condition $x_0 = \varphi \in \mathscr{H}$, where $\mathscr{H}$ is the space of $\mathscr{F}_0$-adapted random variables φ, with $\varphi(s) \in \mathbb{R}^n$ for $s \le 0$, and

$$\|\varphi\| = \sup_{s\le 0} |\varphi(s)|, \qquad \|\varphi\|_1^2 = \sup E\left(|\varphi(s)|^2\right). \tag{26}$$

Definition 4. The trivial solution of system (25) is said to be

(i) mean square stable if, for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for any initial process $\varphi(\theta)$,

$$E\left(|x(t, \varphi(\theta))|^2\right) < \varepsilon, \tag{27}$$

for any $t \ge 0$ provided that $\sup_{\theta\le 0} E(|\varphi(\theta)|^2) < \delta(\varepsilon)$,

(ii) asymptotically mean square stable if it is mean square stable and

$$\lim_{t\to\infty} E\left(|x(t, \varphi)|^2\right) = 0, \tag{28}$$

(iii) stochastically stable if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists a $\delta > 0$ such that

$$P\left\{\sup_{t\ge 0} |x(t, \varphi)| \le \varepsilon_1\right\} \ge 1 - \varepsilon_2 \tag{29}$$

provided that $P\{\|\varphi\| \le \delta\} = 1$.

3. Dynamical Behavior of the System without Delays

We first study the stochastic stability of the equilibria (0, 0) of model (13). Throughout the paper, we assume that the basic hypotheses given in the Section 2 are satisfied. The linearized system of model (13) is

$$\begin{aligned} du_1 &= \left[-(D + mbx^*) u_1 + (\mu D_1 - ma) u_2\right] dt \\ &\quad + \sigma_1 u_1 dB_1, \\ du_2 &= \gamma mbx^* u_1 dt + \sigma_2 u_2 dB_2. \end{aligned} \tag{30}$$

For convenience, let

$$p = \frac{\gamma mbx^*}{2(ma - \mu D_1)}, \qquad q = \frac{\gamma mbx^* - p(ma - \mu D_1)}{\gamma^2 (ma - \mu D_1) + \gamma D}. \tag{31}$$

For linearized system (30), we have the following theorem.

Theorem 5. *Let condition* (6) *hold. If*

$$\sigma_1^2 < 2D + 2mbx^*, \qquad \sigma_2^2 < \frac{2q}{1+q}\gamma (ma - \mu D_1), \tag{32}$$

then the trivial solution of system (30) *is globally asymptotically stable in probability.*

Proof. Define a smooth function $V : \mathbb{R}^2 \to \mathbb{R}_+$ by

$$V(u_1, u_2) = pu_1^2 + u_2^2 + q(\gamma u_1 + u_2)^2. \tag{33}$$

Then using Itô's formula, for all $(u_1, u_2) \neq (0, 0)$, we have

$$\begin{aligned} dV(u_1, u_2) &= 2pu_1 du_1 + p(du_1)^2 + 2u_2 du_2 + (du_2)^2 \\ &\quad + 2q(\gamma u_1 + u_2)\, d(\gamma u_1 + u_2) \\ &\quad + q(d(\gamma u_1 + u_2))^2 \\ &= LV(u_1, u_2)\, dt + 2p\sigma_1 u_1^2 dB_1 + 2\sigma_2 u_2^2 dB_2 \\ &\quad + 2q(\gamma u_1 + u_2)(\gamma \sigma_1 u_1 dB_1 + \sigma_2 u_2 dB_2), \end{aligned} \tag{34}$$

where

$$\begin{aligned} LV(u_1, u_2) &= 2pu_1\left[-(D + mbx^*) u_1 + (\mu D_1 - ma) u_2\right] \\ &\quad + p\sigma_1^2 u_1^2 + 2\gamma mbx^* u_1 u_2 + \sigma_2^2 u_2^2 \\ &\quad + 2q(\gamma u_1 + u_2) \\ &\quad \times\left[-\gamma D u_1 + \gamma(\mu D_1 - ma) u_2\right] \\ &\quad + q\left(\gamma^2 \sigma_1^2 u_1^2 + \sigma_2^2 u_2^2\right) \\ &= -\Big[2p(D + mbx^*) - p\sigma_1^2 \\ &\qquad + 2q\gamma^2 D - q\gamma^2 \sigma_1^2\Big] u_1^2 \\ &\quad - \left[2q\gamma(ma - \mu D_1) - (1+q)\sigma_2^2\right] u_2^2 \\ &\quad - 2\Big[p(ma - \mu D_1) - \gamma mbx^* \\ &\qquad + q\gamma^2 (ma - \mu D_1) + q\gamma D\Big] u_1 u_2. \end{aligned} \tag{35}$$

By (31), we obtain

$$\begin{aligned} &LV(u_1, u_2) \\ &\quad = -\left[2p(D + mbx^*) - p\sigma_1^2 + 2q\gamma^2 D - q\gamma^2\sigma_1^2\right] u_1^2 \\ &\qquad - \left[2q\gamma(ma - \mu D_1) - (1+q)\sigma_2^2\right] u_2^2. \end{aligned} \tag{36}$$

We take $\psi_i : R_+^0 \to R_+^0$ $(i = 1, 2, 3)$ by

$$\begin{aligned} \psi_1(|u|) &= \min\{p, 1, q\} |u|^2, \\ \psi_2(|u|) &= \max\{p, 1, q\} |u|^2, \\ \psi_3(|u|) &= \min\Big\{2p(D + mbx^*) - p\sigma_1^2 + 2q\gamma^2 D \\ &\qquad - q\gamma^2\sigma_1^2, 2q\gamma(ma - \mu D_1) - (1+q)\sigma_2^2\Big\} |u|^2; \end{aligned} \tag{37}$$

thus the thesis follows by Lemma 2. This completes the proof of Theorem 5. □

Now, we are in a position to prove the stability of the trivial solution $(0,0)$ of model (13).

Theorem 6. *Let condition* (6) *hold. If the conditions in* (32) *are satisfied, then the trivial solution of model* (13) *is stochastically asymptotically stable.*

Proof. For a sufficiently small constant $\epsilon > 0$, $(u_1,u_2) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$, we have

$$\begin{aligned} &\left|f(t,X) - F\cdot X\right| + \left|g(t,X) - G\cdot X\right| \\ &\quad = \sqrt{F_1^2(u_1,u_2) + F_2^2(u_1,u_2)}. \end{aligned} \tag{38}$$

Note that F_1, F_2 are the terms of order ≥ 2 in u_1 and u_2; then we have

$$\lim_{u_1^2+u_2^2\to 0} \frac{F_1^2(u_1,u_2) + F_2^2(u_1,u_2)}{u_1^2+u_2^2} = 0. \tag{39}$$

Thus for a sufficiently small constant $\rho > 0$, we have

$$F_1^2(u_1,u_2) + F_2^2(u_1,u_2) < \rho^2\left(u_1^2+u_2^2\right) \tag{40}$$

provided $u_1^2+u_2^2 < \epsilon^2$. Therefore,

$$\left|f(t,X) - F\cdot X\right| + \left|g(t,X) - G\cdot X\right| < \rho\,|u|. \tag{41}$$

Applying Lemma 3 and Theorem 5, we obtain the conclusion. □

4. Dynamical Behavior of the System with Delays

We now study the stability in probability of the equilibria $(0,0)$ of system (11). Its corresponding linearized system is

$$\begin{aligned} du_1 &= \left[-\left(D + mbx^*\right)u_1 \right. \\ &\quad \left. + \mu D_1 \int_0^\infty f(s)\,u_2(t-s)\,ds - mau_2\right]dt \\ &\quad + \sigma_1 u_1 dB_1, \\ du_2 &= \gamma mbx^* \int_0^\infty g(s)\,u_1(t-s)\,ds dt + \sigma_2 u_2 dB_2. \end{aligned} \tag{42}$$

Define the average time lags as

$$T_f = \int_0^\infty sf(s)\,ds, \qquad T_g = \int_0^\infty sg(s)\,ds, \tag{43}$$

and let q, p be defined in (31). For linearized system (42) we have the following theorem.

Theorem 7. *Let condition* (6) *hold. If*

$$\begin{aligned} &\sigma_1^2 + 2\mu D_1\gamma mbx^* T_f + \frac{1+q}{p+q\gamma^2}\left(D + mbx^*\right)\gamma mbx^* T_g \\ &\quad < 2D + \frac{2pmbx^*}{p+q\gamma^2}, \\ &\sigma_2^2 + \left(D + mbx^* + 2ma + 2\mu D_1\right)\gamma mbx^* T_g \\ &\quad < \frac{2q}{1+q}\gamma\left(ma - \mu D_1\right), \end{aligned} \tag{44}$$

then the trivial solution of system (42) *is asymptotically mean square stable.*

Proof. Consider the function $V_1(u_1,u_2)$ defined in (33). It follows from (42) and Itô's formula that

$$\begin{aligned} dV_1(u_1,u_2) &= 2pu_1du_1 + p(du_1)^2 + 2u_2du_2 \\ &\quad + (du_2)^2 + 2q(\gamma u_1 + u_2)\,d(\gamma u_1 + u_2) \\ &\quad + q(d(\gamma u_1 + u_2))^2 \\ &= \left\{2pu_1\left[-\left(D + mbx^*\right)u_1 \right.\right. \\ &\quad + \mu D_1 \int_0^\infty f(s)\,u_2(t-s)\,ds \\ &\quad \left. - mau_2\right] \\ &\quad + p\sigma_1^2u_1^2 + 2\gamma mbx^* u_2 \int_0^\infty g(s)\,u_1(t-s)\,ds \\ &\quad + \sigma_2^2u_2^2 + 2q(\gamma u_1 + u_2) \\ &\quad \times\left[-\gamma\left(D + mbx^*\right)u_1 \right. \\ &\quad + \gamma\mu D_1 \int_0^\infty f(s)\,u_2(t-s)\,ds - \gamma mau_2 \\ &\quad \left. + \gamma mbx^* \int_0^\infty g(s)\,u_1(t-s)\,ds\right] \\ &\quad \left. + q\left(\gamma^2\sigma_1^2u_1^2 + \sigma_2^2u_2^2\right)\right\}dt \\ &\quad + 2\sigma_2u_2^2dB_2 + 2q(\gamma u_1 + u_2) \\ &\quad \times\left(\gamma\sigma_1u_1dB_1 + \sigma_2u_2dB_2\right) + 2p\sigma_1u_1^2dB_1. \end{aligned} \tag{45}$$

Straightforward computations lead to

$$\begin{aligned} LV_1(u_1,u_2) = & -\left[2p\left(D+mbx^*\right)-p\sigma_1^2 \right.\\ & \left. + 2q\gamma^2\left(D+mbx^*\right)-q\gamma^2\sigma_1^2\right]u_1^2 \\ & -\left[2q\gamma ma-(1+q)\sigma_2^2\right]u_2^2 \\ & -2\left[pma+q\gamma^2 ma+q\gamma\left(D+mbx^*\right)\right]u_1u_2 \\ & +2(1+q)\gamma mbx^* u_2\int_0^\infty g(s)u_1(t-s)\,ds \\ & +2q\gamma^2 mbx^* u_1\int_0^\infty g(s)u_1(t-s)\,ds \\ & +2\left(p+q\gamma^2\right)\mu D_1 u_1\int_0^\infty f(s)u_2(t-s)\,ds \\ & +2q\gamma\mu D_1 u_2\int_0^\infty f(s)u_2(t-s)\,ds. \end{aligned} \tag{46}$$

From the terms of the right-hand side of (46), we have

$$\begin{aligned} u_1\int_0^\infty g(s)u_1(t-s)\,ds &\le \frac{1}{2}\left(u_1^2+\int_0^\infty g(s)u_1^2(t-s)\,ds\right), \\ u_2\int_0^\infty f(s)u_2(t-s)\,ds &\le \frac{1}{2}\left(u_2^2+\int_0^\infty f(s)u_2^2(t-s)\,ds\right). \end{aligned} \tag{47}$$

For the term $u_1\int_0^\infty f(s)u_2(t-s)ds$, it is clear that

$$\begin{aligned} & u_1\int_0^\infty f(s)u_2(t-s)\,ds \\ & \quad = u_1u_2-u_1\int_0^t f(s)\int_{t-s}^t du_2(\tau)\,ds+h_1(t) \\ & \quad = u_1u_2-\gamma mbx^* H_1(u_1,u_2)+h_1(t) \\ & \qquad -u_1\int_0^t f(s)\int_{t-s}^t \sigma_2u_2(\tau)\,dB_2(\tau)\,ds, \end{aligned} \tag{48}$$

where

$$h_1(t) = -u_1\int_t^\infty f(s)\left(u_2(t)-u_2(t-s)\right)ds, \tag{49}$$

$$\begin{aligned} H_1(u_1,u_2) &= u_1\int_0^t f(s)\int_{t-s}^t\int_0^\infty g(v)u_1(\tau-v)\,dv\,d\tau\,ds \\ &\le \frac{1}{2}\int_0^\infty f(s) \\ &\quad \times\int_{t-s}^t\int_0^\infty g(v) \\ &\quad \times\left(u_1^2(t)+u_1^2(\tau-v)\right)dv\,d\tau\,ds \\ &= \frac{1}{2}T_f u_1^2+\frac{1}{2}\int_0^\infty f(s) \\ &\quad \times\int_{t-s}^t\int_0^\infty g(v) \\ &\quad \times u_1^2(\tau-v)\,dv\,d\tau\,ds. \end{aligned} \tag{50}$$

For the term $u_2\int_0^\infty g(s)u_1(t-s)ds$, we have that

$$\begin{aligned} & u_2\int_0^\infty g(s)u_1(t-s)\,ds \\ & \quad = u_1u_2-u_2\int_0^t g(s)\int_{t-s}^t du_1(\tau)\,ds+h_2(t) \\ & \quad = u_1u_2+\left(D+mbx^*\right)H_2(u_1,u_2) \\ & \qquad +maH_3(u_1,u_2)-\mu D_1H_4(u_1,u_2) \\ & \qquad +u_2\int_0^\infty g(s)\int_{t-s}^t\sigma_2u_2(\tau)\,dB_2(\tau)\,ds+h_2(t), \end{aligned} \tag{51}$$

where

$$h_2(t) = -u_2\int_t^\infty g(s)\left(u_1(t)-u_1(t-s)\right)ds, \tag{52}$$

$$\begin{aligned} H_2(u_1,u_2) &= u_2\int_0^t g(s)\int_{t-s}^t u_1(\tau)\,d\tau\,ds \\ &\le \frac{1}{2}T_g u_2^2+\frac{1}{2}\int_0^\infty g(s)\int_{t-s}^t u_1^2(\tau)\,d\tau\,ds, \\ H_3(u_1,u_2) &= u_2\int_0^t g(s)\int_{t-s}^t u_2(\tau)\,d\tau\,ds \\ &\le \frac{1}{2}T_g u_2^2+\frac{1}{2}\int_0^\infty g(s)\int_{t-s}^t u_2^2(\tau)\,d\tau\,ds, \\ H_4(u_1,u_2) &= u_2\int_0^t g(s)\int_{t-s}^t\int_0^\infty f(v)u_2(\tau-v)\,dv\,d\tau\,ds \\ &\le \frac{1}{2}T_g u_2^2+\frac{1}{2}\int_0^\infty g(s) \\ &\quad \times\int_{t-s}^t\int_0^\infty f(v) \\ &\quad \times u_2^2(\tau-v)\,dv\,d\tau\,ds. \end{aligned} \tag{53}$$

Substituting (47)–(48) together with (51) into (46), we obtain

$$\begin{aligned} LV_1(u_1,u_2) \le & -\left[2p\left(D+mbx^*\right)-p\sigma_1^2 \right.\\ & +2q\gamma^2\left(D+mbx^*\right)-q\gamma^2\sigma_1^2-q\gamma^2 mbx^* \\ & \left. -\left(p+q\gamma^2\right)\mu D_1\gamma mbx^* T_f\right]u_1^2 \\ & -\left[2q\gamma ma-(1+q)\sigma_2^2-(1+q)\right. \end{aligned}$$

$$\begin{aligned}
&\times \left(D + mbx^* + ma + \mu D_1\right) \gamma mbx^* T_g \\
&- q\gamma\mu D_1 \Big] u_2^2 \\
&- 2 \left[pma + q\gamma^2 ma + q\gamma \left(D + mbx^*\right) \right. \\
&\quad \left. - (1+q)\, \gamma mbx^* - \left(p + q\gamma^2\right) \mu D_1 \right] u_1 u_2 \\
&+ (1+q)\, \gamma mbx^* \\
&\times \left[\mu D_1 \int_0^\infty g(s) \right. \\
&\qquad \times \int_{t-s}^t \int_0^\infty f(v)\, u_2^2 (\tau - v)\, dv\, d\tau\, ds \\
&\quad + \left(D + mbx^*\right) \int_0^\infty g(s) \int_{t-s}^t u_1^2(\tau)\, d\tau\, ds \\
&\quad \left. + ma \int_0^\infty g(s) \int_{t-s}^t u_2^2(\tau)\, d\tau\, ds \right] \\
&+ q\gamma^2 mbx^* \int_0^\infty g(s)\, u_1^2(t-s)\, ds \\
&+ q\gamma\mu D_1 \int_0^\infty f(s)\, u_2^2(t-s)\, ds \\
&+ \left(p + q\gamma^2\right) \mu D_1 \gamma mbx^* \\
&\times \int_0^\infty f(s) \int_{t-s}^t \int_0^\infty g(v)\, u_1^2(\tau - v)\, dv\, d\tau\, ds \\
&+ 2(1+q)\, \gamma mbx^* h_2(t) \\
&+ 2\left(p + q\gamma^2\right) \mu D_1 h_1(t).
\end{aligned} \tag{54}$$

For technical reasons, we assume that $\int_0^\infty s^2 f(s) ds < \infty$ and $\int_0^\infty s^2 g(s) ds < \infty$. Then the function

$$\begin{aligned}
&V_2(u_1, u_2) \\
&\quad = (1+q)\, \gamma mbx^* \\
&\quad \times \left[\mu D_1 \int_0^\infty g(s) \right. \\
&\qquad \times \int_{t-s}^t \int_r^t \int_0^\infty f(v)\, u_2^2(\tau - v)\, dv\, d\tau\, dr\, ds \\
&\quad + \left(D + mbx^*\right) \\
&\quad \times \int_0^\infty g(s) \int_{t-s}^t \int_r^t u_1^2(\tau)\, d\tau\, dr\, ds \\
&\quad \left. + ma \int_0^\infty g(s) \int_{t-s}^t \int_r^t u_2^2(\tau)\, d\tau\, dr\, ds \right] \\
&\quad + q\gamma^2 mbx^* \int_0^\infty g(s) \int_{t-s}^t u_1^2(\tau)\, d\tau\, ds \\
&\quad + q\gamma\mu D_1 \int_0^\infty f(s) \int_{t-s}^t u_2^2(\tau)\, d\tau\, ds \\
&\quad + \left(p + q\gamma^2\right) \mu D_1 \gamma mbx^* \\
&\quad \times \int_0^\infty f(s) \\
&\qquad \times \int_{t-s}^t \int_r^t \int_0^\infty g(v)\, u_1^2(\tau - v)\, dv\, d\tau\, dr\, ds
\end{aligned} \tag{55}$$

is well defined. Using Itô's formula, we have

$$\begin{aligned}
L\left(V_1 + V_2\right) \le\; & - \left[2p\left(D + mbx^*\right) - p\sigma_1^2 \right. \\
&\quad + 2q\gamma^2 \left(D + mbx^*\right) - q\gamma^2 \sigma_1^2 - 2q\gamma^2 mbx^* \\
&\quad - \left(p + q\gamma^2\right) \mu D_1 \gamma mbx^* T_f \\
&\quad \left. - (1+q) \left(D + mbx^*\right) \gamma mbx^* T_g \right] u_1^2 \\
& - \left[2q\gamma ma - (1+q)\, \sigma_2^2 - (1+q) \right. \\
&\quad \times \left(D + mbx^* + 2ma + \mu D_1\right) \gamma mbx^* T_g \\
&\quad \left. - 2q\gamma\mu D_1 \right] u_2^2 \\
& - 2 \left[pma + q\gamma^2 ma + q\gamma \left(D + mbx^*\right) \right. \\
&\quad \left. - (1+q)\, \gamma mbx^* - \left(p + q\gamma^2\right) \mu D_1 \right] u_1 u_2 \\
& + (1+q)\, \gamma mbx^* \mu D_1 T_g \\
& \times \int_0^\infty f(s)\, u_2^2(t-s)\, ds \\
& + \left(p + q\gamma^2\right) \mu D_1 \gamma mbx^* T_f \\
& \times \int_0^\infty g(s)\, u_1^2(t-s)\, ds \\
& + 2(1+q)\, \gamma mbx^* h_2(t) \\
& + 2\left(p + q\gamma^2\right) \mu D_1 h_1(t).
\end{aligned} \tag{56}$$

We now consider the function

$$\begin{aligned}
V_3(u_1, u_2) =\; & (1+q)\, \gamma mbx^* \mu D_1 T_g \\
& \times \int_0^\infty f(s) \int_{t-s}^t u_2^2(\tau)\, d\tau\, ds \\
& + \left(p + q\gamma^2\right) \mu D_1 \gamma mbx^* T_f \\
& \times \int_0^\infty g(s) \int_{t-s}^t u_1^2(\tau)\, d\tau\, ds.
\end{aligned} \tag{57}$$

It follows from (56) and (57) that

$$\begin{aligned}
&L\left(V_1+V_2+V_3\right)\\
&\quad\leq -\Big[2p\left(D+mbx^*\right)-p\sigma_1^2\\
&\qquad +2q\gamma^2\left(D+mbx^*\right)-q\gamma^2\sigma_1^2-2q\gamma^2 mbx^*\\
&\qquad -2\left(p+q\gamma^2\right)\mu D_1\gamma mbx^* T_f\\
&\qquad -(1+q)\left(D+mbx^*\right)\gamma mbx^* T_g\\
&\qquad -2\left(p+q\gamma^2\right)\mu D_1\int_t^\infty f(s)\,ds\\
&\qquad -(1+q)\gamma mbx^*\int_t^\infty g(s)\,ds\Big]u_1^2\\
&\quad -\Big[2q\gamma ma-(1+q)\sigma_2^2-(1+q)\\
&\qquad \times\left(D+mbx^*+2ma+2\mu D_1\right)\gamma mbx^* T_g\\
&\qquad -2q\gamma\mu D_1-\left(p+q\gamma^2\right)\mu D_1\int_t^\infty f(s)\,ds\\
&\qquad -2(1+q)\gamma mbx^*\int_t^\infty g(s)\,ds\Big]u_2^2\\
&\quad -2\Big[pma+q\gamma^2 ma+q\gamma\left(D+mbx^*\right)\\
&\qquad -(1+q)\gamma mbx^*-\left(p+q\gamma^2\right)\mu D_1\Big]u_1u_2\\
&\quad +\left(p+q\gamma^2\right)\mu D_1\int_t^\infty f(s)\,u_2^2(t-s)\,ds\\
&\quad +(1+q)\gamma mbx^*\int_t^\infty g(s)\,u_1^2(t-s)\,ds.
\end{aligned}\tag{58}$$

Therefore, for the function

$$V\left(u_1,u_2\right)=V_1\left(u_1,u_2\right)+V_2\left(u_1,u_2\right)+V_3\left(u_1,u_2\right),\tag{59}$$

we have

$$\begin{aligned}
LV\leq\ &-\Big[2p\left(D+mbx^*\right)-p\sigma_1^2\\
&\quad +2q\gamma^2\left(D+mbx^*\right)-q\gamma^2\sigma_1^2-2q\gamma^2 mbx^*\\
&\quad -2\left(p+q\gamma^2\right)\mu D_1\gamma mbx^* T_f\\
&\quad -(1+q)\left(D+mbx^*\right)\gamma mbx^* T_g\\
&\quad -2\left(p+q\gamma^2\right)\mu D_1\int_t^\infty f(s)\,ds\\
&\quad -(1+q)\gamma mbx^*\int_t^\infty g(s)\,ds\Big]u_1^2\\
&-\Big[2q\gamma ma-(1+q)\sigma_2^2-(1+q)\\
&\quad \times\left(D+mbx^*+2ma+2\mu D_1\right)\gamma mbx^* T_g\\
&\quad -2q\gamma\mu D_1-\left(p+q\gamma^2\right)\mu D_1\int_t^\infty f(s)\,ds\\
&\quad -2(1+q)\gamma mbx^*\int_t^\infty g(s)\,ds\Big]u_2^2\\
&+\left(p+q\gamma^2\right)\mu D_1\int_t^\infty f(s)\,u_2^2(t-s)\,ds\\
&+(1+q)\gamma mbx^*\int_t^\infty g(s)\,u_1^2(t-s)\,ds.
\end{aligned}\tag{60}$$

By (44), we choose $\varepsilon>0$ such that

$$\begin{aligned}
&2p\left(D+mbx^*\right)+2q\gamma^2 D\\
&\quad>\left(p+q\gamma^2\right)\sigma_1^2+2\left(p+q\gamma^2\right)\mu D_1\gamma mbx^* T_f\\
&\qquad +(1+q)\left(D+mbx^*\right)\gamma mbx^* T_g\\
&\qquad +2\left(p+q\gamma^2\right)\mu D_1\varepsilon+(1+q)\gamma mbx^*\varepsilon,\\
&\frac{2q}{1+q}\gamma\left(ma-\mu D_1\right)\\
&\quad>\sigma_2^2+\left(D+mbx^*+2ma+2\mu D_1\right)\gamma mbx^* T_g\\
&\qquad +\left(p+q\gamma^2\right)\mu D_1\varepsilon+2(1+q)\gamma mbx^*\epsilon.
\end{aligned}\tag{61}$$

Let $T=T(\varepsilon)$ such that $\int_t^\infty f(s)ds<\varepsilon$ and $\int_t^\infty g(s)ds<\varepsilon$ for all $t\geq T$. Then for all $t\geq T$, one has

$$\begin{aligned}
LV\leq\ &-\Big[2p\left(D+mbx^*\right)-p\sigma_1^2+2q\gamma^2\left(D+mbx^*\right)\\
&\quad -q\gamma^2\sigma_1^2-2q\gamma^2 mbx^*\\
&\quad -2\left(p+q\gamma^2\right)\mu D_1\gamma mbx^* T_f-(1+q)\\
&\quad \times\left(D+mbx^*\right)\gamma mbx^* T_g\\
&\quad -2\left(p+q\gamma^2\right)\mu D_1\varepsilon-(1+q)\gamma mbx^*\varepsilon\Big]u_1^2\\
&-\Big[2q\gamma ma-(1+q)\sigma_2^2-(1+q)\\
&\quad \times\left(D+mbx^*+2ma+2\mu D_1\right)\gamma mbx^* T_g\\
&\quad -2q\gamma\mu D_1-\left(p+q\gamma^2\right)\mu D_1\varepsilon\\
&\quad -2(1+q)\gamma mbx^*\varepsilon\Big]u_2^2\\
&+\left(p+q\gamma^2\right)\mu D_1\left\|\varphi_2\right\|^2\int_t^\infty f(s)\,ds\\
&+(1+q)\gamma mbx^*\left\|\varphi_1\right\|^2\int_t^\infty g(s)\,ds.
\end{aligned}\tag{62}$$

For convenience, let

$$Q = \min\{2p(D+mbx^*) - p\sigma_1^2 + 2q\gamma^2(D+mbx^*) - q\gamma^2\sigma_1^2 - 2q\gamma^2 mbx^* - 2(p+q\gamma^2)\mu D_1\gamma mbx^* T_f - (1+q) \times (D+mbx^*)\gamma mbx^* T_g - 2(p+q\gamma^2)\mu D_1\varepsilon - (1+q)\gamma mbx^*\varepsilon, \quad 2q\gamma ma - (1+q)\sigma_2^2 - (1+q) \times (D+mbx^* + 2ma + 2\mu D_1)\gamma mbx^* T_g - 2q\gamma\mu D_1 - (p+q\gamma^2)\mu D_1\varepsilon - 2(1+q)\gamma mbx^*\varepsilon\}. \tag{63}$$

Integrating both sides of (62) from T to $t \geq T$, we have

$$\begin{aligned} E(V(t)) &+ Q\int_T^t E\left(u_1^2(s) + u_2^2(s)\right) ds \\ &\leq V(T) + \left(p+q\gamma^2\right)\mu D_1\|\varphi_2\|^2 \int_T^t \int_s^\infty f(u)\,du\,ds \\ &\quad + (1+q)\gamma mbx^*\|\varphi_1\|^2 \int_T^t \int_t^\infty g(u)\,du\,ds \\ &\leq V(T) + \left(p+q\gamma^2\right)\mu D_1\|\varphi_2\|^2 \int_0^\infty sf(s)\,ds \\ &\quad + (1+q)\gamma mbx^*\|\varphi_1\|^2 \int_0^\infty sg(s)\,ds \\ &= V(T) + \left(p+q\gamma^2\right)\mu D_1\|\varphi_2\|^2 T_f \\ &\quad + (1+q)\gamma mbx^*\|\varphi_1\|^2 T_g < \infty. \end{aligned} \tag{64}$$

Discussing as that in He et al. [18], by the Barbălat lemma, we conclude $E(u_1^2(t) + u_2^2(t)) \to 0$ as $t \to \infty$. Applying Definition 4, we obtain the conclusion. □

Now, we are in a position to prove the stability of the trivial solution $(0,0)$ of nonlinear system (11) using the Lyapunov functionals constructed above.

Theorem 8. *Let condition* (6) *hold. If conditions* (44) *are satisfied, then the trivial solution* $(0,0)$ *of the system* (11) *or the equilibrium* (S^*, x^*) *of system* (6) *is stochastically stable.*

Proof. Consider the Lyapunov function $V_1(u_1, u_2)$ defined in (33). It follows from (11) and Itô's formula that

$$\begin{aligned} dV_1(u_1,u_2) &= 2pu_1 du_1 + p(du_1)^2 + 2u_2 du_2 \\ &\quad + (du_2)^2 + 2q(\gamma u_1 + u_2)\,d(\gamma u_1 + u_2) \\ &\quad + q(d(\gamma u_1 + u_2))^2 \\ &= \Big\{2pu_1\Big[-(D+mbx^*)u_1 + \mu D_1\int_0^\infty f(s)u_2(t-s)\,ds - mau_2 + F_1\Big] + p\sigma_1^2 u_1^2 \\ &\quad + 2\gamma mbx^* u_2\int_0^\infty g(s)u_1(t-s)\,ds + 2u_2\widetilde{F}_2 + \sigma_2^2u_2^2 + 2q(\gamma u_1 + u_2) \\ &\quad \times\Big[-\gamma(D+mbx^*)u_1 + \gamma\mu D_1\int_0^\infty f(s)u_2(t-s)\,ds - \gamma mau_2 + \gamma F_1 \\ &\quad + \gamma mbx^*\int_0^\infty g(s)u_1(t-s)\,ds + \widetilde{F}_2\Big] + q\left(\gamma^2\sigma_1^2u_1^2 + \sigma_2^2u_2^2\right)\Big\}dt \\ &\quad + 2\sigma_2u_2^2dB_2 + 2q(\gamma u_1 + u_2)(\gamma\sigma_1u_1dB_1 + \sigma_2u_2dB_2) \\ &\quad + 2p\sigma_1u_1^2dB_1 = LV_1(u_1,u_2)\,dt + 2\sigma_2u_2^2dB_2 \\ &\quad + 2q(\gamma u_1 + u_2)(\gamma\sigma_1u_1dB_1 + \sigma_2u_2dB_2) \\ &\quad + 2p\sigma_1u_1^2dB_1, \end{aligned} \tag{65}$$

where

$$\begin{aligned} LV_1(u_1,u_2) = &-\left[2p(D+mbx^*) - p\sigma_1^2 + 2q\gamma^2(D+mbx^*) - q\gamma^2\sigma_1^2\right]u_1^2 \\ &- \left[2q\gamma ma - (1+q)\sigma_2^2\right]u_2^2 \\ &- 2\left[pma + q\gamma^2ma + q\gamma(D+mbx^*)\right]u_1u_2 \\ &+ 2(1+q)\gamma mbx^*u_2\int_0^\infty g(s)u_1(t-s)\,ds \\ &+ 2q\gamma^2mbx^*u_1\int_0^\infty g(s)u_1(t-s)\,ds \\ &+ 2\left(p+q\gamma^2\right)\mu D_1u_1\int_0^\infty f(s)u_2(t-s)\,ds \\ &+ 2q\gamma\mu D_1u_2\int_0^\infty f(s)u_2(t-s)\,ds \end{aligned}$$

$$\begin{aligned}&+2pu_1F_1+2u_2\widetilde{F}_2\\&+2q\left(\gamma u_1+u_2\right)\left(\gamma F_1+\widetilde{F}_2\right).\end{aligned}\tag{66}$$

From the terms of the right-hand side of (66), we observe that

$$\begin{aligned}&u_1\int_0^\infty f(s)\,u_2(t-s)\,ds\\&=u_1u_2-u_1\int_0^t f(s)\int_{t-s}^t du_2(\tau)\,ds+h_1(t)\\&=u_1u_2-\gamma mbx^*H_1(u_1,u_2)-u_1\int_0^t f(s)\int_{t-s}^t \widetilde{F}_2d\tau\,ds\\&\quad-u_1\int_0^t f(s)\int_{t-s}^t \sigma_2u_2(\tau)\,dB_2(\tau)\,ds+h_1(t),\end{aligned}\tag{67}$$

where $h_1(t)$ is defined in (49), and

$$\begin{aligned}&u_2\int_0^\infty g(s)\,u_1(t-s)\,ds\\&=u_1u_2-u_2\int_0^t g(s)\int_{t-s}^t du_1(\tau)\,ds+h_2(t)\\&=u_1u_2+(D+mbx^*)\,H_2(u_1,u_2)\\&\quad+maH_3(u_1,u_2)-\mu D_1H_4(u_1,u_2)\\&\quad-u_2\int_0^t g(s)\int_{t-s}^t F_1d\tau\,ds\\&\quad+u_2\int_0^t g(s)\int_{t-s}^t \sigma_2u_2(\tau)\,dB_2(\tau)\,ds+h_2(t),\end{aligned}\tag{68}$$

where $h_2(t)$ is defined in (52). Substituting (67) and (68) into (46), we get

$$\begin{aligned}&LV_1(u_1,u_2)\\&\le-\Big[2p(D+mbx^*)-p\sigma_1^2\\&\qquad+2q\gamma^2(D+mbx^*)-q\gamma^2\sigma_1^2-q\gamma^2mbx^*\\&\qquad-\left(p+q\gamma^2\right)\mu D_1\gamma mbx^*T_f\Big]u_1^2\\&\quad-\Big[2q\gamma ma-(1+q)\,\sigma_2^2-(1+q)\\&\qquad\times(D+mbx^*+ma+\mu D_1)\,\gamma mbx^*T_g\\&\qquad-q\gamma\mu D_1\Big]u_2^2\\&\quad-2\Big[pma+q\gamma^2ma\\&\qquad+q\gamma(D+mbx^*)-(1+q)\,\gamma mbx^*\\&\qquad-\left(p+q\gamma^2\right)\mu D_1\Big]u_1u_2+(1+q)\,\gamma mbx^*\\&\qquad\times\Big[\mu D_1\int_0^\infty g(s)\int_{t-s}^t\int_0^\infty f(v)\,u_2^2(\tau-v)\,dv\,d\tau\,ds\\&\qquad+(D+mbx^*)\int_0^\infty g(s)\int_{t-s}^t u_1^2(\tau)\,d\tau\,ds\\&\qquad+ma\int_0^\infty g(s)\int_{t-s}^t u_2^2(\tau)\,d\tau\,ds\Big]\\&\quad+q\gamma^2mbx^*\int_0^\infty g(s)\,u_1^2(t-s)\,ds\\&\quad+q\gamma\mu D_1\int_0^\infty f(s)\,u_2^2(t-s)\,ds\\&\quad+\left(p+q\gamma^2\right)\mu D_1\gamma mbx^*\\&\quad\times\int_0^\infty f(s)\int_{t-s}^t\int_0^\infty g(v)\,u_1^2(\tau-v)\,dv\,d\tau\,ds\\&\quad+2pu_1F_1+2u_2\widetilde{F}_2+2q(\gamma u_1+u_2)\left(\gamma F_1+\widetilde{F}_2\right)\\&\quad-2\left(p+q\gamma^2\right)\mu D_1u_1\int_0^\infty f(s)\int_{t-s}^t\widetilde{F}_2d\tau\,ds\\&\quad+2\left(p+q\gamma^2\right)\mu D_1h_1(t)\\&\quad-2(1+q)\,\gamma mbx^*u_2\int_0^\infty g(s)\int_{t-s}^t F_1d\tau\,ds\\&\quad+2(1+q)\,\gamma mbx^*h_2(t).\end{aligned}\tag{69}$$

For the functions $V_2(u_1,u_2)$ and $V_3(u_1,u_2)$ defined in (55) and (57), one has

$$\begin{aligned}&L(V_1+V_2+V_3)\\&\le-\Big[2p(D+mbx^*)-p\sigma_1^2+2q\gamma^2(D+mbx^*)\\&\qquad-q\gamma^2\sigma_1^2-2q\gamma^2mbx^*\\&\qquad-2\left(p+q\gamma^2\right)\mu D_1\gamma mbx^*T_f-(1+q)\\&\qquad\times(D+mbx^*)\,\gamma mbx^*T_g\Big]u_1^2\\&\quad-\Big[2q\gamma ma-(1+q)\,\sigma_2^2-(1+q)\\&\qquad\times(D+mbx^*+2ma+2\mu D_1)\,\gamma mbx^*T_g\\&\qquad-2q\gamma\mu D_1\Big]u_2^2\\&\quad-2\Big[pma+q\gamma^2ma+q\gamma(D+mbx^*)\\&\qquad-(1+q)\,\gamma mbx^*-\left(p+q\gamma^2\right)\mu D_1\Big]u_1u_2\\&\quad+2pu_1F_1+2u_2\widetilde{F}_2+2q(\gamma u_1+u_2)\left(\gamma F_1+\widetilde{F}_2\right)\\&\quad-2\left(p+q\gamma^2\right)\mu D_1u_1\int_0^\infty f(s)\int_{t-s}^t\widetilde{F}_2d\tau\,ds\\&\quad+2\left(p+q\gamma^2\right)\mu D_1h_1(t)\end{aligned}$$

$$\begin{aligned}&- 2(1+q)\gamma m b x^* u_2 \int_0^\infty g(s)\int_{t-s}^t F_1 d\tau\, ds\\&+ 2(1+q)\gamma m b x^* h_2(t).\end{aligned} \tag{70}$$

It follows from the expression of $h_1(t)$ and $h_2(t)$ that

$$\begin{aligned}h_1(t) &\le 2u_1^2\int_t^\infty f(s)\,ds + \left(u_2^2 + \|\varphi_2\|^2\right)\int_t^\infty f(s)\,ds,\\h_2(t) &\le 2u_2^2\int_t^\infty g(s)\,ds + \left(u_1^2 + \|\varphi_1\|^2\right)\int_t^\infty g(s)\,ds.\end{aligned} \tag{71}$$

For $V(u_1,u_2) = V_1(u_1,u_2) + V_2(u_1,u_2) + V_3(u_1,u_2)$, one has

$$\begin{aligned}&LV(u_1,u_2)\\&\quad\le -\Big[2p(D+mbx^*) - p\sigma_1^2\\&\qquad + 2q\gamma^2(D+mbx^*) - q\gamma^2\sigma_1^2 - 2q\gamma^2 mbx^*\\&\qquad - 2\left(p+q\gamma^2\right)\mu D_1\gamma mbx^* T_f - (1+q)\\&\qquad \times (D+mbx^*)\gamma mbx^* T_g\\&\qquad - 2\left(p+q\gamma^2\right)\mu D_1\int_t^\infty f(s)\,ds\\&\qquad - (1+q)\gamma mbx^*\int_t^\infty g(s)\,ds\Big]u_1^2\\&\quad - \Big[2q\gamma ma - (1+q)\sigma_2^2 - (1+q)\\&\qquad \times (D+mbx^* + 2ma + 2\mu D_1)\gamma mbx^* T_g\\&\qquad - 2q\gamma\mu D_1 - \left(p+q\gamma^2\right)\mu D_1\int_t^\infty f(s)\,ds\\&\qquad - 2(1+q)\gamma mbx^*\int_t^\infty g(s)\,ds\Big]u_2^2\\&\quad + \left(p+q\gamma^2\right)\mu D_1\|\varphi_2\|^2\int_t^\infty f(s)\,ds\\&\quad + (1+q)\gamma mbx^*\|\varphi_1\|^2\int_t^\infty g(s)\,ds\\&\quad + 2pu_1F_1 + 2u_2\widetilde{F}_2 + 2q(\gamma u_1 + u_2)\left(\gamma F_1 + \widetilde{F}_2\right)\\&\quad - 2(1+q)\gamma mbx^* u_2\int_0^\infty g(s)\int_{t-s}^t F_1\,d\tau\,ds\\&\quad - 2\left(p+q\gamma^2\right)\mu D_1 u_1\int_0^\infty f(s)\int_{t-s}^t \widetilde{F}_2\,d\tau\,ds.\end{aligned} \tag{72}$$

Since F_1 and $\widetilde{F}_2$ are terms of order ≥ 2 in u_1, u_2, then we have

$$\lim_{u_1,u_2\to 0}\frac{F_1(u_1,u_2)}{\sqrt{u_1^2+u_2^2}} = \lim_{u_1,u_2\to 0}\frac{\widetilde{F}_2(u_1,u_2)}{\sqrt{u_1^2+u_2^2}} = 0. \tag{73}$$

For $\varepsilon > 0$, we can find a constant $\zeta \in (0,1)$ such that

$$F_1(u_1,u_2) \le \frac{\varepsilon}{\sqrt{2}}\sqrt{u_1^2+u_2^2}, \qquad \widetilde{F}_2(u_1,u_2) \le \frac{\varepsilon}{\sqrt{2}}\sqrt{u_1^2+u_2^2} \tag{74}$$

provided that $u_1^2 + u_2^2 \le 2\zeta^2$. Now consider the class of processes

$$\Psi = \left\{\varphi \in \mathscr{H} \mid P\left\{\sup_{-\infty\le s\le 0}|\varphi(s)| < \zeta\right\} = 1\right\}. \tag{75}$$

Notice that for $u_t \in \Psi$,

$$\begin{aligned}\left|\int_0^\infty g(s)\int_{t-s}^t F_1(\tau)\,d\tau\,ds\right| &\le \varepsilon T_g\zeta,\\\left|\int_0^\infty f(s)\int_{t-s}^t \widetilde{F}_2(\tau)\,d\tau\,ds\right| &\le \varepsilon T_f\zeta\end{aligned} \tag{76}$$

are valid. Substituting (74)-(76) into (72), we obtain

$$\begin{aligned}&LV(u_1,u_2)\\&\quad\le -\Big[2p(D+mbx^*) - p\sigma_1^2\\&\qquad + 2q\gamma^2(D+mbx^*) - q\gamma^2\sigma_1^2 - 2q\gamma^2 mbx^*\\&\qquad - 2\left(p+q\gamma^2\right)\mu D_1\gamma mbx^* T_f\\&\qquad - (1+q)(D+mbx^*)\gamma mbx^* T_g\\&\qquad - 2\left(p+q\gamma^2\right)\mu D_1\int_t^\infty f(s)\,ds\\&\qquad - (1+q)\gamma mbx^*\int_t^\infty g(s)\,ds\Big]u_1^2\\&\quad - \Big[2q\gamma ma - (1+q)\sigma_2^2 - (1+q)\\&\qquad \times (D+mbx^* + 2ma + 2\mu D_1)\gamma mbx^* T_g\\&\qquad - 2q\gamma\mu D_1 - \left(p+q\gamma^2\right)\mu D_1\int_t^\infty f(s)\,ds\\&\qquad - 2(1+q)\gamma mbx^*\int_t^\infty g(s)\,ds\Big]u_2^2\\&\quad + \left(p+q\gamma^2\right)\mu D_1\|\varphi_2\|^2\int_t^\infty f(s)\,ds\\&\quad + (1+q)\gamma mbx^*\|\varphi_1\|^2\int_t^\infty g(s)\,ds\\&\quad + 2\varepsilon\Big[p+1+q(\gamma+1)^2 + (1+q)\gamma mbx^* T_g\\&\qquad + \left(p+q\gamma^2\right)\mu D_1 T_f\Big]\zeta^2.\end{aligned} \tag{77}$$

Integrating both sides of the above formula from T to $t \wedge T_{\varepsilon_1}$ yields

$$\begin{aligned} E\left(V\left(t \wedge T_{\varepsilon_1}\right)\right) \leq{} & V(T) + \left(p + q\gamma^2\right)\mu D_1 \|\varphi_2\|^2 \\ & \times \int_0^{t\wedge T_{\varepsilon_1}} \int_s^{\infty} f(\tau)\, d\tau\, ds \\ & + (1+q)\gamma m b x^* \|\varphi_1\|^2 \\ & \times \int_0^{t\wedge T_{\varepsilon_1}} \int_s^{\infty} g(\tau)\, d\tau\, ds + 2\varepsilon k_1 \zeta^2 \\ \leq{} & V(T) + \left(p + q\gamma^2\right)\mu D_1 \|\varphi_2\|^2 \int_0^{\infty} s f(s)\, ds \\ & + (1+q)\gamma m b x^* \|\varphi_1\|^2 \\ & \times \int_0^{\infty} s g(s)\, ds + 2\varepsilon k_1 \zeta^2, \end{aligned} \tag{78}$$

where

$$\begin{aligned} k_1 = {} & p + 1 + q(\gamma+1)^2 + (1+q)\gamma m b x^* T_g \\ & + \left(p + q\gamma^2\right)\mu D_1 T_f. \end{aligned} \tag{79}$$

By the definition of function $V(u_1, u_2)$, we can find a constant $k_2 > 0$ such that

$$V(T) \leq k_2 \left(\|\varphi_1\|^2 + \|\varphi_2\|^2\right). \tag{80}$$

Obviously,

$$E\left(V\left(t \wedge T_{\varepsilon_1}\right)\right) \leq k_3 \left(\|\varphi_1\|^2 + \|\varphi_2\|^2\right) + 2\varepsilon k_1 \zeta^2, \tag{81}$$

where $k_3 = \max\{k_2 + (1+q)\gamma m b x^*, k_2 + (p + q\gamma^2)\mu D_1\}$. Now for $\varepsilon_1, \varepsilon_2 \in (0,1)$, let

$$\delta = \min\left\{\left(\frac{1 \wedge p}{2\varepsilon k_1 + k_3}\varepsilon_2\right)^{1/2} \varepsilon_1, \frac{\varepsilon_1}{2}, \frac{\zeta}{2}\right\} \tag{82}$$

and $\|\varphi_1\|^2 + \|\varphi_2\|^2 < \delta^2$. Then it follows that

$$E\left(V\left(t \wedge T_{\varepsilon_1}\right)\right) \leq (2\varepsilon k_1 + k_3)\delta^2 \leq (1 \wedge p)\varepsilon_1^2 \varepsilon_2. \tag{83}$$

On the other hand, we have

$$\begin{aligned} E\left(V\left(t \wedge T_{\varepsilon_1}\right)\right) & \geq E\left[1_{\{T_{\varepsilon_1} \leq t\}} V\left(t \wedge T_{\varepsilon_1}\right)\right] \\ & = E\left[1_{\{T_{\varepsilon_1} \leq t\}} V\left(T_{\varepsilon_1}\right)\right] \\ & = P\left\{T_{\varepsilon_1} \leq t\right\} V\left(T_{\varepsilon_1}\right) \\ & \geq (1 \wedge p)\varepsilon_1^2 P\left\{T_{\varepsilon_1} \leq t\right\}. \end{aligned} \tag{84}$$

Hence, we have $P\{T_{\varepsilon_1} \leq t\} \leq \varepsilon_2$. Let $t \to \infty$; then

$$P\left\{T_{\varepsilon_1} < \infty\right\} \leq \varepsilon_2. \tag{85}$$

Equivalently,

$$P\left\{u_1^2 + u_2^2 < \varepsilon_1^2\right\} \geq 1 - \varepsilon_2. \tag{86}$$

Applying Definition 4, we obtain the conclusion. □

5. Simulations and Discussions

In this paper, we have considered a stochastic chemostat model simulating the process of wastewater treatment. The model incorporates a general nutrient uptake function and two distributed delays. The first delay models the fact that nutrient is partially recycled after the death of the biomass by bacterial decomposition and the second indicates that the growth of the species depends on the past concentration of the nutrient. Furthermore, we consider the stochastic perturbations which are of white noise type and are proportional to the distances of $S(t)$, $x(t)$ from the values of the positive equilibrium S^*, x^*. By constructing appropriate Liapunov-like functionals, some sufficient conditions for the stochastic stability of the positive equilibrium have been obtained.

For model (3), we have first analyzed the stochastic stability of the positive equilibrium E^* in the case when the delays are ignored, that is, the average delays $T_f = T_g = 0$. Our findings in Theorem 6 reveal that E^* is stochastically stable provided that the intensities of noises are small. When at least one of the average delays T_f and T_g is not equal to zero, our results in Theorem 8 reveal that E^* is stochastically stable provided that the average delays T_f and T_g are both small. Obviously, Theorem 8 reduces to Theorem 6 when $T_f = T_g = 0$, which indicates that if the average delays are sufficiently small, E^* is still stochastically stable; and in the case of $\sigma_i = 0$ $(i = 1, 2)$, Theorem 8 reduces to He et al. [18, Theorem 3.1]; that is to say, the equilibrium E^* of model (3) is still stable if σ_1 and σ_2 are sufficient small, which preserves the dynamics of its corresponding deterministic counterpart (5).

To illustrate the results obtained above, some numerical simulations are carried out by using Milstein scheme [50]. Here we assume that the specific growth function $U(S)$ is of Michaelis-Menten type

$$U(S) = \frac{S}{a_1 + S}, \tag{87}$$

where a_1 is the half-saturation constant. For the kernel functions $f(s)$ and $g(s)$, we consider two special cases: (1) $f(s) = g(s) = \delta(0)$; (2) $f(s) = \alpha e^{-\alpha s}$ and $g(s) = \beta e^{-\beta s}$. For case (1), the discretization of model (3) for $t = 0, \Delta t, 2\Delta t, \ldots, n\Delta t$ takes the form

$$\begin{aligned} S_{i+1} = {} & S_i + \left[D\left(S^0 - S_i\right) - mU\left(S_i\right)x_i + \mu D_1 x_i\right]\Delta t \\ & + \sigma_1\left(S_i - S^*\right)\sqrt{\Delta t}\,\xi_i, \\ x_{i+1} = {} & x_i + x_i\left[-\left(D_w + D_1\right) + \gamma m U\left(S_i\right)\right]\Delta t \\ & + \sigma_2\left(x_i - x^*\right)\sqrt{\Delta t}\,\xi_i, \end{aligned} \tag{88}$$

where time increment $\Delta t > 0$ and ξ_i is $N(0,1)$-distributed independent random variables which can be generated

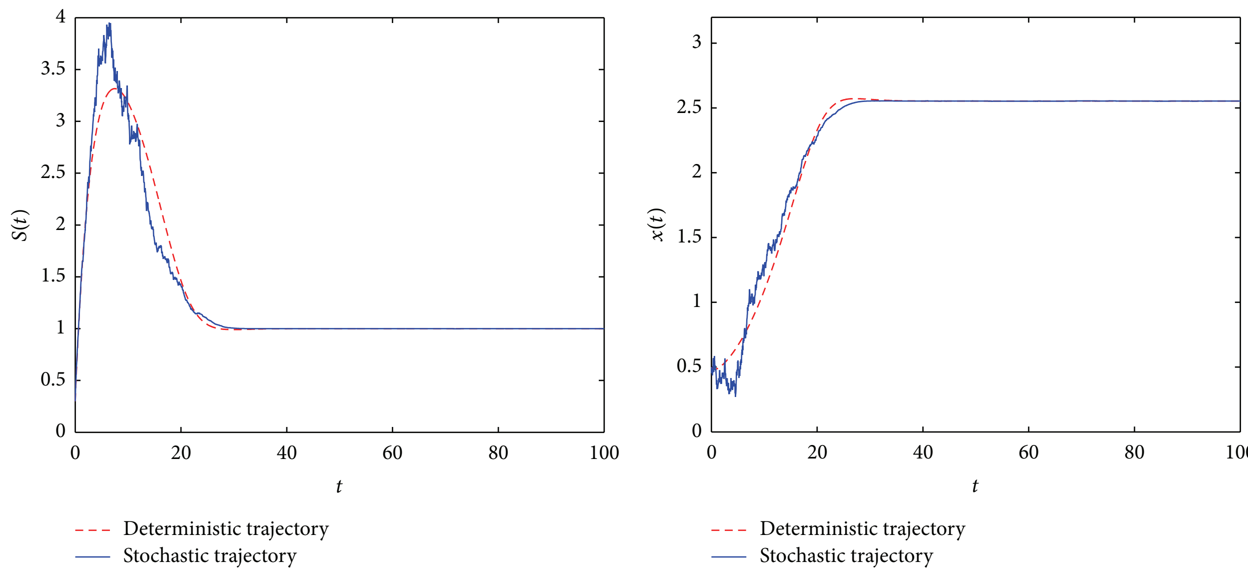

FIGURE 1: The dynamics of stochastic model compared with deterministic model with $\sigma_1 = 0.1$ and $\sigma_2 = 0.08$. Here $S(0) = 0.3$, $x(0) = 0.5$.

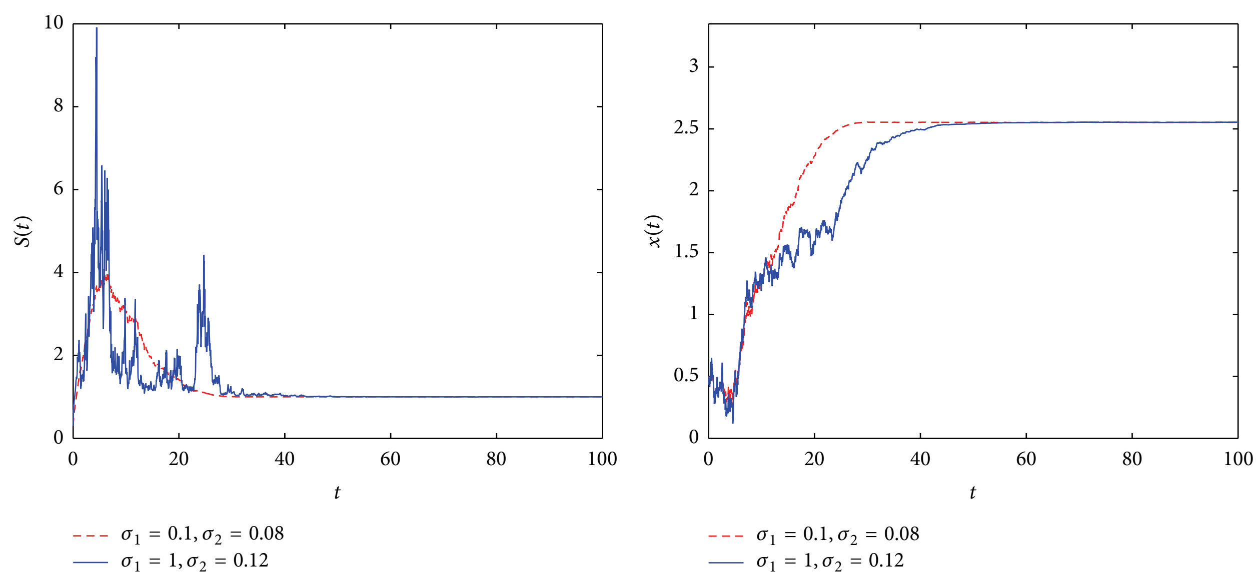

FIGURE 2: The dynamics of stochastic model with different values of σ_1 and σ_2. Here $S(0) = 0.3$, $x(0) = 0.5$.

numerically by pseudorandom number generators. For case (2), define

$$
\begin{aligned}
y(t) &= \int_0^{\infty} \alpha e^{-\alpha s} x(t-s)\, ds, \\
z(t) &= \int_0^{\infty} \beta e^{-\beta s} U(S(t-s))\, ds,
\end{aligned} \tag{89}
$$

then the discretization of model (3) for $t = 0, \Delta t, 2\Delta t, \ldots, n\Delta t$ takes the form

$$
\begin{aligned}
S_{i+1} &= S_i + \left[D\left(S^0 - S_i\right) - mU\left(S_i\right) x_i + \mu D_1 y_i\right] \Delta t \\
&\quad + \sigma_1 \left(S_i - S^*\right) \sqrt{\Delta t}\xi_i, \\
x_{i+1} &= x_i + x_i\left[-\left(D_w + D_1\right) + \gamma m z_i\right] \Delta t \\
&\quad + \sigma_2 \left(x_i - x^*\right) \sqrt{\Delta t}\xi_i, \\
y_{i+1} &= y_i + \left(-\alpha y_i + \alpha x_i\right) \Delta t, \\
z_{i+1} &= z_i + \left(-\beta z_i + \beta U\left(S_i\right)\right) \Delta t.
\end{aligned} \tag{90}
$$

Let in model (3) $D = D_w = 0.3$, $D_1 = 0.1$, $S^0 = 5$, $m = 0.7$, $a_1 = 0.4$, $\mu = 0.3$, $\gamma = 0.8$. It is easy to compute that $a \doteq 0.7143$, $b \doteq 0.2041$, $p \doteq 0.3100$, $q \doteq 0.2694$, and $E^* = (1, 2.55)$.

The first two examples given below concern case (1) when the delays are ignored; that is to say, it is assumed that the

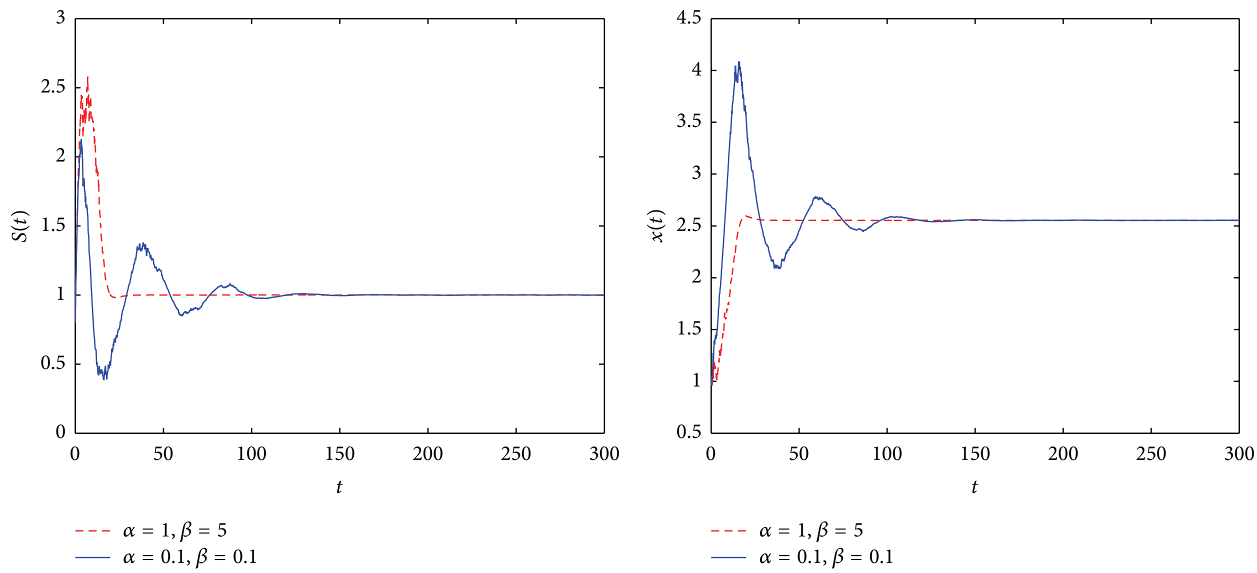

FIGURE 3: The dynamics of stochastic functional model with different α, β. Here $S(0) = 0.3$, $x(0) = 0.5$, $y(0) = 0.3$, $z(0) = 0.5$.

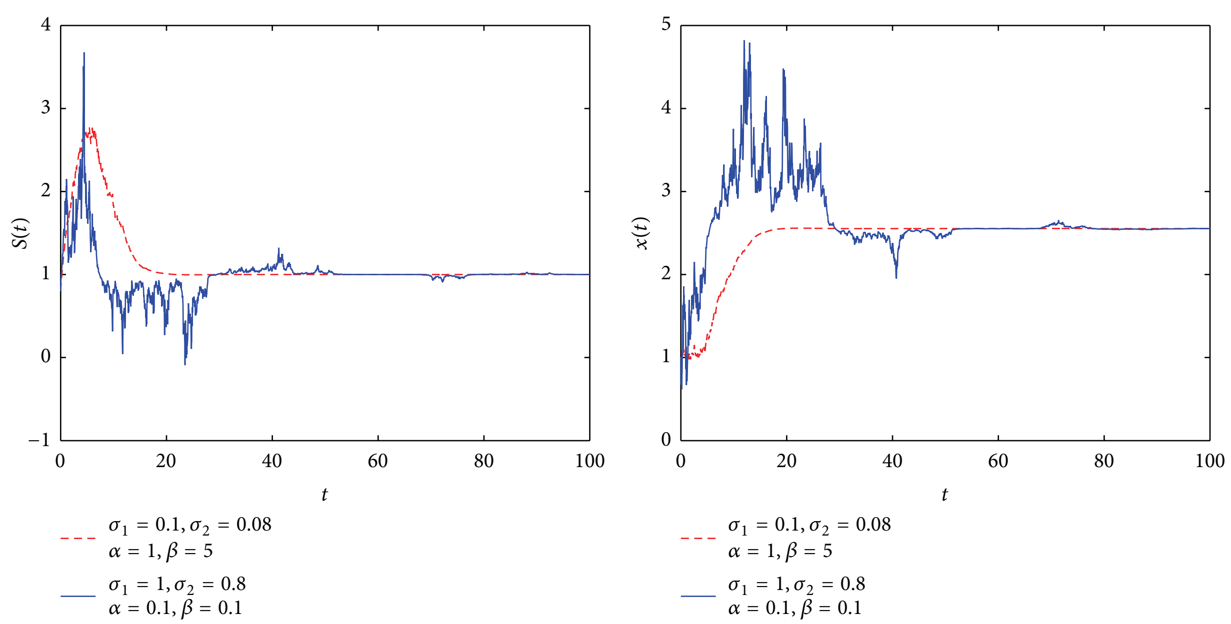

FIGURE 4: The dynamics of stochastic functional model with different σ_1, σ_2 and α, β. Here $S(0) = 0.3$, $x(0) = 0.5$, $y(0) = 0.3$, $z(0) = 0.5$.

process of nutrient recycling and the growth response of the species are immediate and, therefore, $T_f = T_g = 0$. Example 1 verifies the results obtained in Theorem 6.

Example 1. Let $\sigma_1 = 0.1$ and $\sigma_2 = 0.08$, then by straightforward computations, we have that $0.01 = \sigma_1^2 < 2D + 2mbx^* \doteq 1.3285$, $0.0064 = \sigma_2^2 < (2q/(1+q))\gamma(ma - \mu D_1) \doteq 0.1596$. In view of Theorem 6, the equilibrium E^* of (3) is stochastically asymptotically stable, which is consistent with the simulation results as shown in Figure 1.

To further study the combined effects of σ_i, $i = 1, 2$ when $T_f = T_g = 0$, we need to consider four situations: (a) σ_1 increases, σ_2 increases; (b) σ_1 increases, σ_2 decreases; (c) σ_1 decreases, σ_2 increases; (d) σ_1 decreases, σ_2 decreases. Here we only give one example about situation (a); other situations can be considered similarly.

Example 2. Let the intensities σ_i, $i = 1, 2$ increase from $\sigma_1 = 0.1$, $\sigma_2 = 0.08$ to $\sigma_1 = 1$, $\sigma_2 = 0.12$, respectively. Simulations show that the trajectories of model (3) still approach ultimately to the positive equilibrium E^*, but they need to go through more oscillations and more time to return to E^* (see Figure 2).

The next two examples concern case (2) when $f(s)$ and $g(s)$ take weak kernels; that is, $f(s) = \alpha e^{-\alpha s}$ and $g(s) = \beta e^{-\beta s}$,

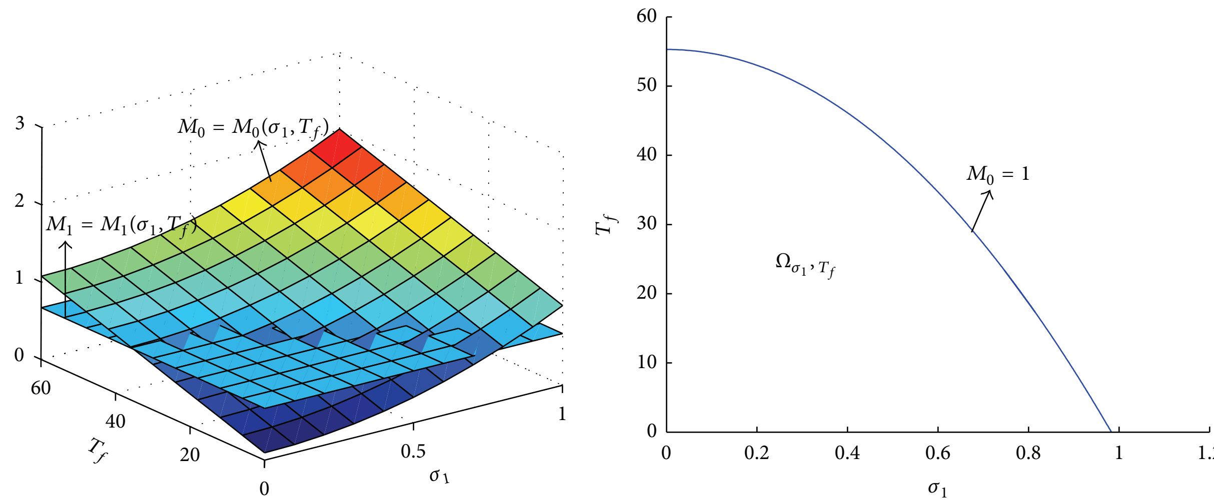

FIGURE 5: The positive equilibrium E^* is stochastically stable provided that $(\sigma_1, T_f) \in \Omega_{\sigma_1, T_f}$. Here $\sigma_2 = 0.08$ and $T_g = 0.2$.

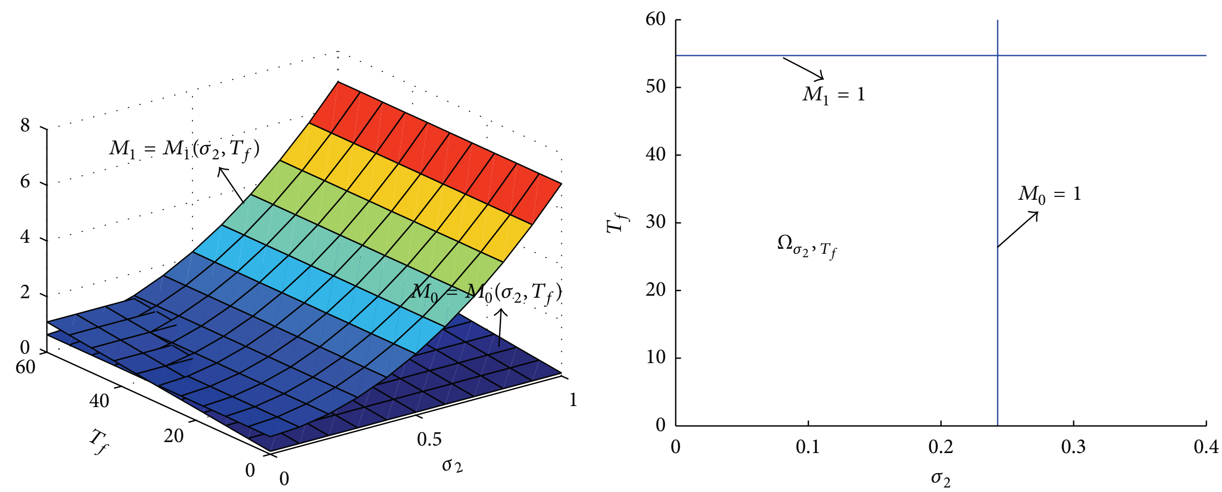

FIGURE 6: The positive equilibrium E^* is stochastically stable provided that $(\sigma_2, T_f) \in \Omega_{\sigma_2, T_f}$. Here $\sigma_1 = 0.1$ and $T_g = 0.2$.

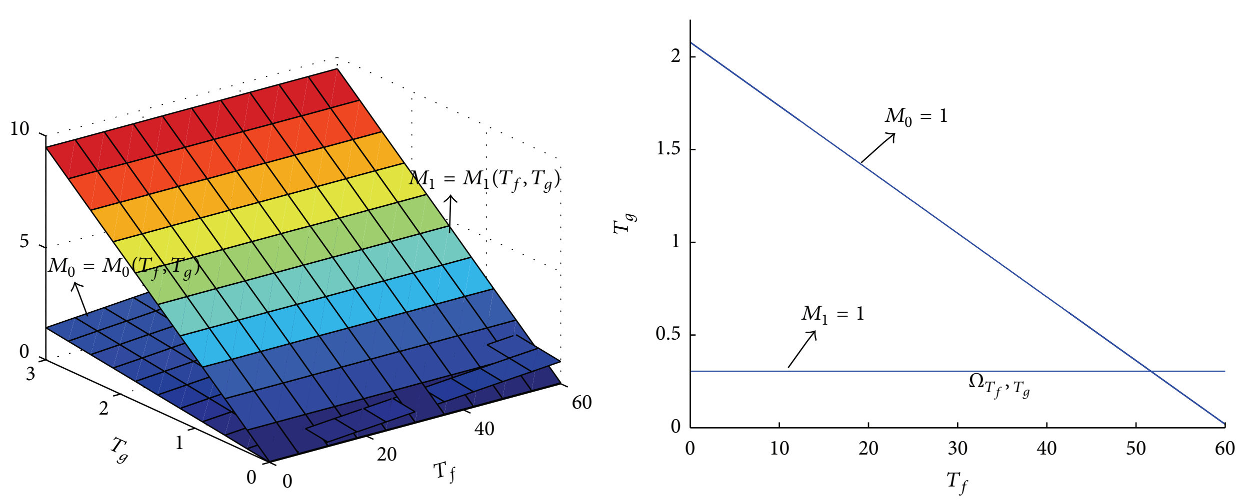

FIGURE 7: The positive equilibrium E^* is stochastically stable provided that $(T_f, T_g) \in \Omega_{T_f, T_g}$. Here $\sigma_1 = 0.1$ and $\sigma_2 = 0.08$.

which means that $T_f = 1/\alpha$ and $T_g = 1/\beta$. Example 3 verifies the results obtained in Theorem 8.

Example 3. Let $\sigma_1 = 0.1, \sigma_2 = 0.08, \alpha = 1$ and $\beta = 5$. It is easy to compute that $(p+q\gamma^2)\sigma_1^2 + 2(p+q\gamma^2)\mu D_1 \gamma mbx^* T_f + (1+q)(D+mbx^*)\gamma mbx^* T_g \doteq 0.0624$, $2p(D+mbx^*)+2q\gamma^2 D \doteq 0.5154$ and $\sigma_2^2 + (D+mbx^*+2ma+2\mu D_1)\gamma mbx^* T_g \doteq 0.1069$, $(2q/(1+q))\gamma(ma-\mu D_1) \doteq 0.1596$; thus conditions (44) are satisfied. By Theorem 8, the equilibrium E^* of model (3) is stochastically stable. Our simulation supports this result as shown in Figure 3.

To examine the combined effects of the noise intensities and the delays on the dynamics of model (3), we first consider the case when the values of $\sigma_i, i = 1, 2$ in Example 3 are fixed and the values of α and β are reduced from 1 and 5 to 0.1 and 0.1, respectively. That is to say, the average delays T_f and T_g increase from 1 and 0.2 to 10 and 10, respectively. Simulation results show that the solution of (3) will suffer more oscillations and more time to approach the equilibrium E^* when delays increase (see Figure 3). When both the values of the noise intensities and the delays vary, the dynamics of model (3) may become more complicated. Here we only consider the case when σ_i $(i = 1, 2)$, T_f and T_g (i.e., $1/\alpha$ and $1/\beta$) all increase. See the following Example.

Example 4. Let σ_i $(i = 1, 2)$, T_f and T_g (i.e., $1/\alpha$ and $1/\beta$) increase from 0.1, 0.08, 1, and 0.2 (i.e., $\alpha = 1$ and $\beta = 5$) to 1, 0.8, 10, and 10 (i.e., $\alpha = 0.1$ and $\beta = 0.1$), respectively. It is found that the trajectories of model (3) fluctuate wildly and suffer more oscillations and need more time to approach the equilibrium E^*; please see Figure 4.

Notice also that conditions (44) in Theorem 8 are only sufficient conditions to insure the stochastic stability of E^*, which are dependent on parameters σ_1, σ_2, T_f, and T_g. Define

$$\begin{aligned} M_0 = {} & \left(\left(p+q\gamma^2\right)\sigma_1^2 + 2\left(p+q\gamma^2\right)\mu D_1 \gamma mbx^* T_f \right. \\ & \left. + (1+q)\left(D+mbx^*\right)\gamma mbx^* T_g\right) \\ & \times \left(2p\left(D+mbx^*\right) + 2q\gamma^2 D\right)^{-1}, \\ M_1 = {} & \frac{\sigma_2^2 + \left(D+mbx^*+2ma+2\mu D_1\right)\gamma mbx^* T_g}{\left(2q/\left(1+q\right)\right)\gamma\left(ma-\mu D_1\right)}. \end{aligned} \tag{91}$$

Thus, conditions (44) are equivalent to those when parameters σ_1, σ_2, T_f, and T_g are seated in the following parameter set:

$$\begin{aligned} \Omega = {} & \left\{\left(\sigma_1, \sigma_2, T_f, T_g\right) \mid \max\left\{M_0, M_1\right\} < 1, \right. \\ & \left. \sigma_i \geq 0, T_f \geq 0, T_g \geq 0\right\}, \end{aligned} \tag{92}$$

from which we can further perform some approximate sensitivity analysis of the stochastic stability of E^* with respect to these parameters. To do this, we can let two of the parameters (e.g., σ_1 and T_f) vary and the other two (σ_2 and T_g) be fixed, which have six cases in all.

Let us first consider the case when $\sigma_2 = 0.08$ and $T_g = 0.2$; then M_0 and M_1 are both functions of σ_1 and T_f. Then Ω defined in (92) is equivalent to

$$\Omega_{\sigma_1, T_f} = \left\{\left(\sigma_1, T_f\right) \mid \left(\sigma_1, 0.08, T_f, 0.2\right) \in \Omega\right\}, \tag{93}$$

which is the projection of surfaces $M_0 = M_0(\sigma_1, T_f)$ and $M_1 = M_1(\sigma_1, T_f)$ in the first octant such that $\max\{M_0, M_1\} < 1$ (see Figure 5). The positive equilibrium E^* is stochastically stable provided that $(\sigma_1, T_f) \in \Omega_{\sigma_1, T_f}$.

To better observe the dependence of the stochastic stability of E^* on all parameters, we further consider another two cases when $\sigma_1 = 0.1$ and $T_g = 0.2$ are fixed and $\sigma_1 = 0.1$ and $\sigma_2 = 0.08$ are fixed. Accordingly, Ω defined in (92) is equivalent, respectively, to

$$\begin{aligned} \Omega_{\sigma_2, T_f} &= \left\{\left(\sigma_2, T_f\right) \mid \left(0.1, \sigma_2, T_f, 0.2\right) \in \Omega\right\}, \\ \Omega_{T_f, T_g} &= \left\{\left(T_f, T_g\right) \mid \left(0.1, 0.08, T_f, T_g\right) \in \Omega\right\}, \end{aligned} \tag{94}$$

which are plotted, respectively, in Figures 6 and 7 (other three cases can be considered similarly). From Figures 5–7, we find that the stochastic stability of E^* is greatly affected by σ_1, σ_2, and T_g and less affected by T_f (which is consistent with the results observed in [13, 17]). We would like to point out here that E^* may also be stable when the parameters are seated outside of the set Ω, since (44) are only sufficient conditions ensuring the stochastic stability of E^*.

In conclusion, this paper presents an investigation on the combined effect of the noises and delays on a bottom-microbe model. Our findings are useful for better understanding of the dynamics of microbial population in the activated sludge process. We should point out that there are still some other interesting topics about the wastewater treatment deserving further investigation, for example, membrane reactor, and so forth. We leave these for future considerations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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