Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 935430, 9 pages
http://dx.doi.org/10.1155/2013/935430

Research Article

A Kantorovich-Stancu Type Generalization of Szasz Operators including Brenke Type Polynomials

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Received 11 September 2013; Accepted 18 November 2013

Academic Editor: Anna Kaminska

We introduce a Kantorovich-Stancu type modification of a generalization of Szasz operators defined by means of the Brenke type polynomials and obtain approximation properties of these operators. Also, we give a Voronovskaya type theorem for Kantorovich-Stancu type operators including Gould-Hopper polynomials.

1. Introduction

For each positive n and $f \in C_B([0,\infty))$ or $C([0,\infty)) \cap E$, the Szasz-Mirakyan operators defined by

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{1}$$

have an important role in the approximation theory [1]. Their Korovkin type approximation properties and rates of convergence have been investigated by many researchers. Recently, there is a growing interest in defining linear positive operators via special functions (see [2–13]). In particular, many authors have studied various generalizations of Szasz operators via special functions. In [14], Jakimovski and Leviatan constructed a generalization of Szasz operators by means of the Appell polynomials. Then, Ismail [15] presented another generalization of Szasz operators by means of Sheffer polynomials, which involves the operators (1) defined by Jakimovski and Leviatan in [14]. In [11],Varma et al. considered the following generalization of Szasz operators by means of the Brenke type polynomials, which are motivated by the operators defined by Jakimovski and Leviatanand Ismail, for $x \geq 0$ and $n \in \mathbb{N}$:

$$L_n(f;x) := \frac{1}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{2}$$

under the following assumptions:

$$\begin{aligned} &\text{(i) } A(1) \neq 0, \quad \frac{a_{k-r}b_r}{A(1)} \geq 0, \quad 0 \leq r \leq k, k = 0,1,2,\ldots, \\ &\text{(ii) } B : [0,\infty) \longrightarrow (0,\infty), \\ &\text{(iii) (4) and (5) converge for } |t| < R \quad (R > 1), \end{aligned} \tag{3}$$

where

$$\begin{aligned} A(t) &= \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0, \\ B(t) &= \sum_{r=0}^{\infty} b_r t^r, \quad b_r \neq 0 \ (r \geq 0) \end{aligned} \tag{4}$$

are analytic functions and the Brenke type polynomials [16] have generating functions of the form

$$A(t)B(xt) = \sum_{k=0}^{\infty} p_k(x) t^k, \tag{5}$$

where

$$p_k(x) = \sum_{r=0}^{k} a_{k-r} b_r x^r, \quad k = 0,1,2,\ldots. \tag{6}$$

The Kantorovich type of Szasz-Mirakyan operators is defined by [17]

$$K_n(f;x) := ne^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_{k/n}^{(k+1)/n} f(t)\,dt. \quad (7)$$

The approximation properties of the Szasz-Mirakyan-Kantorovich operators and their various iterates were studied by many authors in [12, 18–23].

Recently, in [8], the Kantorovich type of the operators given by (2) under the assumptions (3) has been defined as

$$K_n(f;x) := \frac{n}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\int_{k/n}^{(k+1)/n} f(t)\,dt, \quad (8)$$

where $n \in \mathbb{N}$, $x \geq 0$ and $f \in C[0,\infty)$, and some of its properties have been investigated.

The purpose of this study is to introduce a Kantorovich-Stancu type modification of the operators given by (8) and to examine the approximation properties of these operators. We also present a Kantorovich-Stancu type of the operators including Gould-Hopper polynomials and then we prove a Voronovskaya type theorem for these operators including Gould-Hopper polynomials.

2. Construction of the Operators

For each positive integer n, $x \geq 0$ and $f \in C_B([0,\infty))$, or $C([0,\infty)) \cap E$, let us consider the following operators:

$$K_n^{(\alpha,\beta)}(f;x) := \frac{n+\beta}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} f(t)\,dt, \quad (9)$$

where α and β parameters satisfy the condition $0 \leq \alpha \leq \beta$. For the approximation properties of Stancu type operators, we refer to [24–27].

It is clear that for $\alpha = \beta = 0$, $K_n^{(\alpha,\beta)}(f;x)$ reduces to the operators defined by (8).

In the case of $B(t) = e^t$ and $A(t) = 1$, with the help of (5) it follows that $p_k(x) = x^k/k!$. So the operator $K_n^{(\alpha,\beta)}(f;x)$ gives the Kantorovich-Stancu type of Szasz-Mirakyan operators as follows:

$$K_n^{(\alpha,\beta)}(f;x) := (n+\beta)e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} f(t)\,dt, \quad (10)$$

where α and β parameters satisfy the condition $0 \leq \alpha \leq \beta$.

In the case of $\alpha = \beta = 0$, the operator (10) turns out to be the Szasz-Mirakyan-Kantorovich operators given by (7).

For $B(t) = e^t$, $K_n^{(\alpha,\beta)}(f;x)$ gives the Kantorovich-Stancu type of the operators $P_n(f;x)$ proposed by Jakimovski and Leviatan in [14].

Now, for the operators $K_n^{(\alpha,\beta)}$ given by (9), we give some results which are necessary to prove the main theorem.

Lemma 1. *Kantorovich-Stancu type operators, defined by* (9), *are linear and positive.*

Lemma 2. *For each $x \in [0,\infty)$, the Kantorovich-Stancu type operators* (9) *have the following properties:*

$$K_n^{(\alpha,\beta)}(1;x) = 1, \quad (11)$$

$$K_n^{(\alpha,\beta)}(s;x) = \frac{n}{n+\beta}\frac{B'(nx)}{B(nx)}x + \frac{A'(1)}{(n+\beta)A(1)} + \frac{2\alpha+1}{2(n+\beta)}, \quad (12)$$

$$\begin{aligned} K_n^{(\alpha,\beta)}(s^2;x) &= \left(\frac{n}{n+\beta}\right)^2\frac{B''(nx)}{B(nx)}x^2 \\ &+ \frac{nB'(nx)\left[2A'(1)+(2\alpha+2)A(1)\right]}{(n+\beta)^2A(1)B(nx)}x \\ &+ \frac{1}{(n+\beta)^2A(1)}\Big\{A''(1)+(2\alpha+2)A'(1) \\ &+ \left(\alpha^2+\alpha+\frac{1}{3}\right)A(1)\Big\}. \end{aligned} \quad (13)$$

Proof. From the generating function of the Brenke type polynomials given by (5), a few calculations reveal that

$$\begin{aligned} \sum_{k=0}^{\infty}p_k(nx) &= A(1)B(nx), \\ \sum_{k=0}^{\infty}kp_k(nx) &= A'(1)B(nx) + nxA(1)B'(nx), \\ \sum_{k=0}^{\infty}k^2p_k(nx) &= n^2x^2A(1)B''(nx) \\ &+ nxB'(nx)\left\{2A'(1)+A(1)\right\} \\ &+ B(nx)\left\{A''(1)+A'(1)\right\}. \end{aligned} \quad (14)$$

By using these equalities, we obtain the assertions of the lemma by simple calculation. □

Lemma 3. *For each $x \in [0,\infty)$, one has*

$$\begin{aligned} &K_n^{(\alpha,\beta)}((s-x)^2;x) \\ &= \left\{\left(\frac{n}{n+\beta}\right)^2\frac{B''(nx)}{B(nx)} - \frac{2nB'(nx)}{(n+\beta)B(nx)} + 1\right\}x^2 \\ &+ \Big\{\frac{nB'(nx)\left[2A'(1)+A(1)\right]}{(n+\beta)^2A(1)B(nx)} + \frac{(2\alpha+1)nB'(nx)}{(n+\beta)^2B(nx)} \\ &- \frac{2A'(1)}{(n+\beta)A(1)} - \frac{2\alpha+1}{n+\beta}\Big\}x + \frac{A''(1)+A'(1)}{(n+\beta)^2A(1)} \\ &+ \frac{(2\alpha+1)A'(1)}{(n+\beta)^2A(1)} + \frac{\alpha^2+\alpha+(1/3)}{(n+\beta)^2}. \end{aligned} \quad (15)$$

Theorem 4. *Let*

$$E := \left\{ f : x \in [0,\infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x \longrightarrow \infty \right\},$$
$$\lim_{y\to\infty} \frac{B'(y)}{B(y)} = 1, \qquad \lim_{y\to\infty} \frac{B''(y)}{B(y)} = 1. \tag{16}$$

If $f \in C[0,\infty) \cap E$, then

$$\lim_{n\to\infty} K_n^{(\alpha,\beta)}(f;x) = f(x), \tag{17}$$

and the operators $K_n^{(\alpha,\beta)}$ converge uniformly in each compact subset of $[0,\infty)$.

Proof. According to Lemma 2, by considering the equality (16), we get

$$\lim_{n\to\infty} K_n^{(\alpha,\beta)}(s^i;x) = x^i, \quad i = 0,1,2. \tag{18}$$

This convergence is satisfied uniformly in each compact subset of $[0,\infty)$. Then, the proof follows from the universal Korovkin-type property (vi) of Theorem 4.1.4 in [28]. □

3. Rates of Convergence

In this section, we compute the rates of convergence of the operators $K_n^{(\alpha,\beta)}(f)$ to f by means of a classical approach, the second modulus of continuity, and Peetre's K-functional.

Let $f \in \widetilde{C}[0,\infty)$. Then for $\delta > 0$, the modulus of continuity of f denoted by $w(f;\delta)$ is defined to be

$$w(f;\delta) := \sup_{\substack{x,y\in[0,\infty)\\ |x-y|\le\delta}} |f(x) - f(y)|, \tag{19}$$

where $\widetilde{C}[0,\infty)$ denotes the space of uniformly continuous functions on $[0,\infty)$. Then, for any $\delta > 0$ and each $x \in [0,\infty)$, it is well known that one can write

$$|f(x) - f(y)| \le w(f;\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{20}$$

The next result gives the rate of convergence of the sequence $K_n^{(\alpha,\beta)}(f)$ to f by means of the modulus of continuity.

Theorem 5. *For $f \in \widetilde{C}[0,\infty) \cap E$, one has*

$$\left|K_n^{(\alpha,\beta)}(f;x) - f(x)\right| \le 2w\left(f;\sqrt{\lambda_n(x)}\right), \tag{21}$$

where

$$\begin{aligned}
\lambda &= \lambda_n(x) \\
&= K_n^{(\alpha,\beta)}\left((s-x)^2;x\right) \\
&= \left\{\left(\frac{n}{n+\beta}\right)^2 \frac{B''(nx)}{B(nx)} - \frac{2nB'(nx)}{(n+\beta)B(nx)} + 1\right\} x^2 \\
&\quad + \left\{\frac{nB'(nx)\left[2A'(1) + A(1)\right]}{(n+\beta)^2 A(1) B(nx)} + \frac{(2\alpha+1)nB'(nx)}{(n+\beta)^2 B(nx)} \right. \\
&\quad \left. - \frac{2A'(1)}{(n+\beta)A(1)} - \frac{2\alpha+1}{n+\beta}\right\} x + \frac{A''(1) + A'(1)}{(n+\beta)^2 A(1)} \\
&\quad + \frac{(2\alpha+1)A'(1)}{(n+\beta)^2 A(1)} + \frac{\alpha^2 + \alpha + (1/3)}{(n+\beta)^2}.
\end{aligned} \tag{22}$$

Proof. Using linearity of the operators $K_n^{(\alpha,\beta)}$, (11) and (20), we get

$$\begin{aligned}
&\left|K_n^{(\alpha,\beta)}(f;x) - f(x)\right| \\
&\quad \le \frac{n+\beta}{A(1)B(nx)} \sum_{k=0}^{\infty} P_k(nx) \times \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |f(s) - f(x)|\, ds \\
&\quad \le \frac{n+\beta}{A(1)B(nx)} \sum_{k=0}^{\infty} P_k(nx) \times \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} \left(\frac{|s-x|}{\delta} + 1\right) w(f;\delta)\, ds \\
&\quad \le \left\{1 + \frac{n+\beta}{A(1)B(nx)\delta} \sum_{k=0}^{\infty} P_k(nx) \times \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s-x|\, ds\right\} w(f;\delta).
\end{aligned} \tag{23}$$

According to the Cauchy-Schwarz inequality for integration, we obtain that

$$\begin{aligned}
&\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s-x|\, ds \\
&\quad \le \frac{1}{\sqrt{n+\beta}} \left(\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s-x|^2 ds\right)^{1/2}
\end{aligned} \tag{24}$$

from which, it follows that

$$\begin{aligned}
&\sum_{k=0}^{\infty} P_k(nx) \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s-x|\, ds \\
&\quad \le \frac{1}{\sqrt{n+\beta}} \sum_{k=0}^{\infty} P_k(nx) \left(\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s-x|^2 ds\right)^{1/2}.
\end{aligned} \tag{25}$$

By using the Cauchy-Schwarz inequality for summation on the right hand side of (25), we may write

$$\begin{aligned}\sum_{k=0}^{\infty} P_k(nx)\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)}|s-x|\,ds &\le \frac{\sqrt{A(1)B(nx)}}{\sqrt{n+\beta}}\left(\frac{A(1)B(nx)}{n+\beta}K_n^{(\alpha,\beta)}\left((s-x)^2;x\right)\right)^{1/2}\\ &= \frac{A(1)B(nx)}{n+\beta}\left(K_n^{(\alpha,\beta)}\left((s-x)^2;x\right)\right)^{1/2}\\ &= \frac{A(1)B(nx)}{n+\beta}\left(\lambda_n(x)\right)^{1/2},\end{aligned} \tag{26}$$

where $\lambda_n(x)$ is given by (22). Considering this inequality in (23), we find that

$$\left|K_n^{(\alpha,\beta)}(f;x)-f(x)\right| \le \left\{1+\frac{1}{\delta}\sqrt{\lambda_n(x)}\right\}w(f;\delta). \tag{27}$$

If we set $\delta = \sqrt{\lambda_n(x)}$, the proof is completed. □

Now, we will study the rates of convergence of the operators $K_n^{(\alpha,\beta)}$ to f by means of the second modulus of continuity and Peetre's K-functional.

Recall that the second modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$w_2(f;\delta) := \sup_{0<t\le\delta}\left\|f(\cdot+2t)-2f(\cdot+t)+f(\cdot)\right\|_{C_B}, \tag{28}$$

where $C_B[0,\infty)$ is the class of real valued functions defined on $[0,\infty)$ which are bounded and uniformly continuous with the norm $\|f\|_{C_B} = \sup_{x\in[0,\infty)}|f(x)|$.

Peetre's K-functional of the function $f \in C_B[0,\infty)$ is defined by

$$K(f;\delta) := \inf_{g\in C_B^2[0,\infty)}\left\{\|f-g\|_{C_B}+\delta\|g\|_{C_B^2}\right\}, \tag{29}$$

where

$$C_B^2[0,\infty) := \left\{g\in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\right\} \tag{30}$$

and the norm $\|g\|_{C_B^2} := \|g\|_{C_B}+\|g'\|_{C_B}+\|g''\|_{C_B}$ (see [29]). It is clear that the following inequality:

$$K(f;\delta) \le M\left\{w_2\left(f;\sqrt{\delta}\right)+\min(1,\delta)\|f\|_{C_B}\right\} \tag{31}$$

holds for all $\delta > 0$. The constant M is independent of f and δ.

Theorem 6. *Let $f \in C_B^2[0,\infty)$. If $K_n^{(\alpha,\beta)}$ is defined by (9), then one has*

$$\left|K_n^{(\alpha,\beta)}(f;x)-f(x)\right| \le \zeta\|f\|_{C_B^2}, \tag{32}$$

where

$$\begin{aligned}\zeta = \zeta_n(x) &= \left\{\left(\frac{n}{n+\beta}\right)^2\frac{B''(nx)}{2B(nx)}-\frac{nB'(nx)}{(n+\beta)B(nx)}+\frac{1}{2}\right\}x^2\\ &\quad+\left\{\frac{nB'(nx)\left[2A'(1)+(2\alpha+2)A(1)\right]}{2(n+\beta)^2A(1)B(nx)}\right.\\ &\quad\left.-\frac{2A'(1)+(2\alpha+1)A(1)}{2(n+\beta)A(1)}+\frac{n}{n+\beta}\frac{B'(nx)}{B(nx)}-1\right\}x\\ &\quad+\frac{A''(1)+A'(1)}{2(n+\beta)^2A(1)}+\frac{(2\alpha+1)A'(1)}{2(n+\beta)^2A(1)}\\ &\quad+\frac{\alpha^2+\alpha+(1/3)}{2(n+\beta)^2}+\frac{2A'(1)+(2\alpha+1)A(1)}{2(n+\beta)A(1)}.\end{aligned} \tag{33}$$

Proof. We can write from the Taylor expansion of f, the linearity of the operators $K_n^{(\alpha,\beta)}$, and (11)

$$\begin{aligned}K_n^{(\alpha,\beta)}(f;x)-f(x) &= f'(x)K_n^{(\alpha,\beta)}(s-x;x)\\ &\quad+\frac{1}{2}f''(\eta)K_n^{(\alpha,\beta)}\left((s-x)^2;x\right), \quad \eta\in(x,s).\end{aligned} \tag{34}$$

From Lemma 2, it is obvious that

$$\begin{aligned}K_n^{(\alpha,\beta)}(s-x;x) &= \left\{\frac{n}{n+\beta}\frac{B'(nx)}{B(nx)}-1\right\}x+\frac{A'(1)}{(n+\beta)A(1)}+\frac{2\alpha+1}{2(n+\beta)} \ge 0\end{aligned} \tag{35}$$

for $s \ge x$. Thus, by considering Lemmas 2 and 3 in (34), one can write

$$\begin{aligned}\left|K_n^{(\alpha,\beta)}(f;x)-f(x)\right| &\le \left\{\left\{\frac{n}{n+\beta}\frac{B'(nx)}{B(nx)}-1\right\}x\right.\\ &\quad\left.+\frac{A'(1)}{(n+\beta)A(1)}+\frac{2\alpha+1}{2(n+\beta)}\right\}\left\|f'\right\|_{C_B}\\ &\quad+\frac{1}{2}\left[\left\{\left(\frac{n}{n+\beta}\right)^2\frac{B''(nx)}{B(nx)}\right.\right.\\ &\quad\left.-\frac{2nB'(nx)}{(n+\beta)B(nx)}+1\right\}x^2\end{aligned}$$

$$+\left\{\frac{nB'(nx)\left[2A'(1)+A(1)\right]}{(n+\beta)^2A(1)B(nx)}+\frac{(2\alpha+1)nB'(nx)}{(n+\beta)^2B(nx)}-\frac{2A'(1)}{(n+\beta)A(1)}-\frac{2\alpha+1}{n+\beta}\right\}x$$
$$+\frac{A''(1)+A'(1)}{(n+\beta)^2A(1)}+\frac{(2\alpha+1)A'(1)}{(n+\beta)^2A(1)}+\frac{\alpha^2+\alpha+(1/3)}{(n+\beta)^2}\Bigg]\left\|f''\right\|_{C_B}$$
$$\leq\Bigg[\left\{\left(\frac{n}{n+\beta}\right)^2\frac{B''(nx)}{2B(nx)}-\frac{nB'(nx)}{(n+\beta)B(nx)}+\frac{1}{2}\right\}x^2$$
$$+\left\{\frac{nB'(nx)\left[2A'(1)+(2\alpha+2)A(1)\right]}{2(n+\beta)^2A(1)B(nx)}-\frac{2A'(1)+(2\alpha+1)A(1)}{2(n+\beta)A(1)}+\frac{n}{n+\beta}\frac{B'(nx)}{B(nx)}-1\right\}x$$
$$+\frac{A''(1)+A'(1)}{2(n+\beta)^2A(1)}+\frac{(2\alpha+1)A'(1)}{2(n+\beta)^2A(1)}+\frac{\alpha^2+\alpha+(1/3)}{2(n+\beta)^2}+\frac{2A'(1)+(2\alpha+1)A(1)}{2(n+\beta)A(1)}\Bigg]\left\|f\right\|_{C_B^2}\quad(36)$$

which completes the proof. □

Theorem 7. *If $f\in C_B[0,\infty)$, then one has*

$$\left|K_n^{(\alpha,\beta)}(f;x)-f(x)\right|\leq 2M\left\{w_2\left(f;\sqrt{\delta}\right)+\min(1,\delta)\left\|f\right\|_{C_B}\right\},\quad(37)$$

where

$$\delta:=\delta_n(x)=\frac{1}{2}\zeta_n(x)\quad(38)$$

and $M>0$ is a constant which is independent of the function f and δ. Also, $\zeta_n(x)$ is the same as in Theorem 6.

Proof. Suppose that $g\in C_B^2[0,\infty)$. From Theorem 6, we have

$$\begin{aligned}\left|K_n^{(\alpha,\beta)}(f;x)-f(x)\right|&\leq\left|K_n^{(\alpha,\beta)}(f-g;x)\right|+\left|K_n^{(\alpha,\beta)}(g;x)-g(x)\right|\\&\quad+\left|g(x)-f(x)\right|\\&\leq 2\left\|f-g\right\|_{C_B}+\zeta\left\|g\right\|_{C_B^2}=2\left[\left\|f-g\right\|_{C_B}+\delta\left\|g\right\|_{C_B^2}\right].\end{aligned}\quad(39)$$

Since the left-hand side of inequality (39) does not depend on the function $g\in C_B^2[0,\infty)$, we get

$$\left|K_n^{(\alpha,\beta)}(f;x)-f(x)\right|\leq 2K(f;\delta),\quad(40)$$

where $K(f;\delta)$ is Peetre's K-functional defined by (29). By using the relation (31) in (39), the inequality

$$\left|K_n^{(\alpha,\beta)}(f;x)-f(x)\right|\leq 2M\left\{w_2\left(f;\sqrt{\delta}\right)+\min(1,\delta)\left\|f\right\|_{C_B}\right\}\quad(41)$$

holds. □

Remark 8. In Theorems 5–7, $\lambda_n,\zeta_n,\delta_n\to 0$ when $n\to\infty$ under the assumption (16).

4. Special Cases of the Operators $K_n^{(\alpha,\beta)}$ and Further Properties

Gould-Hopper polynomials $g_k^{d+1}(x,h)$ are defined through the identity

$$g_k^{d+1}(x,h)=\sum_{m=0}^{[k/(d+1)]}\frac{k!}{m!\,(k-(d+1)m)!}h^mx^{k-(d+1)m}\quad(42)$$

and satisfy the generating function

$$e^{ht^{d+1}}\exp(xt)=\sum_{k=0}^{\infty}g_k^{d+1}(x,h)\frac{t^k}{k!},\quad(43)$$

where, as usual, $[\cdot]$ denotes the integer part [30].

The Gould-Hopper polynomials are Brenke-type polynomials for the special case of $A(t)=e^{ht^{d+1}}$ and $B(t)=e^t$ in (5). From (2), the operators including the Gould-Hopper polynomials are as follows:

$$L_n^*(f;x):=e^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!}f\left(\frac{k}{n}\right),\quad(44)$$

where $x\in[0,\infty)$ and $h\geq 0$ (see [11]).

Similarly, the special case $A(t)=e^{ht^{d+1}}$ and $B(t)=e^t$ of (9) gives the following Kantorovich-Stancu type operators $K_n^{*(\alpha,\beta)}(f;x)$ including the Gould-Hopper polynomials:

$$K_n^{*(\alpha,\beta)}(f;x):=(n+\beta)e^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!}\times\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)}f(t)\,dt\quad(45)$$

under the assumption $h\geq 0$.

Remark 9. For $h = 0$, we have $g_k^{d+1}(nx,0) = (nx)^k$ and the operators given by (45) reduce to the Kantorovich-Stancu type of Szasz-Mirakyan operators given by (10).

Remark 10. For $\alpha = \beta = 0$, the operators (45) give the Kantorovich type operators including the Gould-Hopper polynomials given by

$$K_n^*(f;x) := ne^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!}\int_{k/n}^{(k+1)/n} f(t)\,dt \quad (46)$$

in [8].

Remark 11. For $h = 0$ in Remark 10, we get $g_k^{d+1}(nx,0) = (nx)^k$ and then the operators given by (46) reduce to the Szasz-Mirakyan-Kantorovich operators given by (7).

Now, in order to prove a Voronovskaya type theorem for the operators given by (45), let us prove the following lemmas.

Lemma 12. *For the operators $K_n^{*(\alpha,\beta)}$, one has*

$$\begin{aligned}
&K_n^{*(\alpha,\beta)}(1;x) = 1,\\
&K_n^{*(\alpha,\beta)}(s;x) = \frac{nx}{n+\beta} + \frac{h(d+1)}{n+\beta} + \frac{2\alpha+1}{2(n+\beta)},\\
&K_n^{*(\alpha,\beta)}(s^2;x)\\
&= \frac{n^2x^2}{(n+\beta)^2} + \frac{nx}{(n+\beta)^2}\\
&\quad\times\{2h(d+1)+(2\alpha+2)\} + \frac{1}{(n+\beta)^2}\\
&\quad\times\Big[h(h+1)(d+1)^2 + (2\alpha+1)h(d+1)\\
&\qquad+\Big(\alpha^2+\alpha+\frac{1}{3}\Big)\Big],\\
&K_n^{*(\alpha,\beta)}(s^3;x)\\
&= \frac{n^3x^3}{(n+\beta)^3} + \frac{3n^2x^2}{2(n+\beta)^3}\\
&\quad\times\{2\alpha+3+2(d+1)h\} + \frac{nx}{2(n+\beta)^3}\\
&\quad\times\Big\{6h^2(d+1)^2 + 6h(d+1)\\
&\qquad\times(3+d+2\alpha) + 12\alpha + 6\alpha^2 + 7\Big\}\\
&\quad+\frac{1}{4(n+\beta)^3}\Big\{4h^3(d+1)^3 + 6h^2(d+1)^2\\
&\qquad\times(2\alpha+2d+3) + 4\alpha^3 + 6\alpha^2\\
&\qquad+4\alpha+1+2h(d+1)\\
&\qquad\times\Big[2d^2+d(2\alpha+7)\\
&\qquad+12\alpha+6\alpha^2+7\Big]\Big\},\\
&K_n^{*(\alpha,\beta)}(s^4;x)\\
&= \frac{n^4x^4}{(n+\beta)^4} + \frac{4n^3x^3}{(n+\beta)^4}(h(d+1)+\alpha+2)\\
&\quad+\frac{3n^2x^2}{(n+\beta)^4}\Big\{2h^2(d+1)^2 + 2h(d+1)\\
&\qquad\times(2\alpha+d+4) + 2\alpha^2 + 6\alpha + 5\Big\}\\
&\quad+\frac{2nx}{(n+\beta)^4}\Big\{2h^3(d+1)^3 + 6h^2(d+1)^2(\alpha+d+2)\\
&\qquad+h(d+1)\\
&\qquad\times\Big(2d^2+10d+6(3+d)\alpha+6\alpha^2+15\Big)\\
&\qquad+(1+\alpha)(3+2\alpha(2+\alpha))\Big\} + \frac{1}{5(n+\beta)^4}\\
&\quad\times\Big\{5h(d+1)^4[1+h(7+h(6+h))]\\
&\qquad+10h(d+1)^3[1+h(3+h)](1+2\alpha)\\
&\qquad+10h(h+1)(d+1)^2(1+3\alpha(1+\alpha))\\
&\qquad+5h(d+1)(1+2\alpha[2+\alpha(3+2\alpha)])\\
&\qquad+5\alpha^4+10\alpha^3+10\alpha^2+5\alpha+1\Big\}.
\end{aligned} \quad (47)$$

Proof. The proof follows from the generating function (43) for the Gould-Hopper polynomials. □

Lemma 13. *For each $x \in [0,\infty)$, one has*

$$\begin{aligned}
&K_n^{*(\alpha,\beta)}\big((s-x)^2;x\big)\\
&= \frac{\beta^2x^2}{(n+\beta)^2}\\
&\quad+x\Big[\frac{n}{(n+\beta)^2}\{2h(d+1)+(2\alpha+2)\}\\
&\qquad-\frac{2h(d+1)+2\alpha+1}{n+\beta}\Big] + \frac{1}{(n+\beta)^2}\\
&\quad\times\Big[h(h+1)(d+1)^2\\
&\qquad+(2\alpha+1)h(d+1)+\Big(\alpha^2+\alpha+\frac{1}{3}\Big)\Big],\\
&K_n^{*(\alpha,\beta)}\big((s-x)^4;x\big) = \frac{\beta^4x^4}{(n+\beta)^4}\\
&\quad-x^3\left\{\frac{2\beta^2(-3n+(2h(d+1)+2\alpha+1)\beta)}{(n+\beta)^4}\right\}
\end{aligned}$$

$$+x^2\left\{\frac{1}{(n+\beta)^4}\left[3n^2-2n\beta(6h(d+1)+6\alpha+5)\right.\right.$$
$$+2\left\{3h^2(d+1)^2+3h(d+1)\right.$$
$$\left.\left.\left.\times(2\alpha+d+2)+3\alpha(1+\alpha)+1\right\}\beta^2\right]\right\}$$
$$+x\left\{\frac{2n}{(n+\beta)^4}\left\{2h^3(d+1)^3+6h^2(d+1)^2\right.\right.$$
$$\times(\alpha+d+2)+h(d+1)$$
$$\times\left(2d^2+10d+6(3+d)\alpha\right.$$
$$\left.+6\alpha^2+15\right)+(1+\alpha)$$
$$\left.\times(3+2\alpha(2+\alpha))\right\}$$
$$-\frac{1}{(n+\beta)^3}\left\{4h^3(d+1)^3+6h^2(d+1)^2\right.$$
$$\times(2\alpha+2d+3)+4\alpha^3+6\alpha^2$$
$$+4\alpha+1+2h(d+1)$$
$$\times\left[2d^2+d(2\alpha+7)+6\alpha^2\right.$$
$$\left.\left.\left.+12\alpha+7\right]\right\}\right\}$$
$$+\frac{1}{5(n+\beta)^4}\left\{5h(d+1)^4[1+h(7+h(6+h))]\right.$$
$$+10h(d+1)^3[1+h(3+h)](2\alpha+1)$$
$$+10(d+1)^2h(1+h)(1+3\alpha(1+\alpha))$$
$$+5h(d+1)(1+2\alpha[2+\alpha(3+2\alpha)])$$
$$\left.+5\alpha^4+10\alpha^3+10\alpha^2+5\alpha+1\right\}. \quad (48)$$

Proof. From Lemma 12, the proof is obvious. □

Theorem 14. *Let $f\in C^2[0,a]$. Then one has*

$$\lim_{n\to\infty}(n+\beta)\left[K_n^{*(\alpha,\beta)}(f;x)-f(x)\right]$$
$$=f'(x)\left\{\beta x+h(d+1)+\frac{2\alpha+1}{2}\right\}+\frac{xf''(x)}{2!}. \quad (49)$$

Proof. By Taylor's theorem for f, we have

$$f(s)=f(x)+(s-x)f'(x)$$
$$+\frac{(s-x)^2}{2!}f''(x)+(s-x)^2\eta(s;x), \quad (50)$$

where $\eta(s;x)\in C[0,a]$ and $\lim_{s\to x}\eta(s;x)=0$. By applying the operator $K_n^{*(\alpha,\beta)}$ to the both sides of (50), we have

$$K_n^{*(\alpha,\beta)}(f;x)=f(x)+f'(x)K_n^{*(\alpha,\beta)}(s-x;x)$$
$$+\frac{f''(x)}{2!}K_n^{*(\alpha,\beta)}\left((s-x)^2;x\right) \quad (51)$$
$$+K_n^{*(\alpha,\beta)}\left((s-x)^2\eta(s;x);x\right).$$

According to Lemmas 12 and 13, the equality (51) can be written as follows:

$$(n+\beta)\left[K_n^{*(\alpha,\beta)}(f;x)-f(x)\right]$$
$$=(n+\beta)\left\{\frac{\beta}{n+\beta}x+\frac{h(d+1)}{n+\beta}+\frac{2\alpha+1}{2(n+\beta)}\right\}f'(x)$$
$$+(n+\beta)\left\{x^2\left(\frac{\beta}{n+\beta}\right)^2\right.$$
$$+x\left[\frac{n}{(n+\beta)^2}\{2h(d+1)+(2\alpha+2)\}\right.$$
$$\left.-\frac{2h(d+1)+2\alpha+1}{n+\beta}\right]$$
$$+\frac{1}{(n+\beta)^2}\left[h(h+1)(d+1)^2+(2\alpha+1)\right.$$
$$\left.\left.\times h(d+1)+\left(\alpha^2+\alpha+\frac{1}{3}\right)\right]\right\}$$
$$\times\frac{f''(x)}{2!}+(n+\beta)K_n^{*(\alpha,\beta)}\left((s-x)^2\eta(s;x);x\right), \quad (52)$$

where

$$K_n^{*(\alpha,\beta)}\left((s-x)^2\eta(s;x);x\right)$$
$$=(n+\beta)e^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!} \quad (53)$$
$$\times\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)}(s-x)^2\eta(s;x)\,ds.$$

By applying Cauchy-Schwarz inequality, we can write

$$(n+\beta)K_n^{*(\alpha,\beta)}\left((s-x)^2\eta(s;x);x\right)$$
$$\leq(n+\beta)^2e^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!}$$
$$\times\left(\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)}(s-x)^4ds\right)^{1/2} \quad (54)$$
$$\times\left(\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)}\eta^2(s;x)\,ds\right)^{1/2}.$$

If we consider Cauchy-Schwarz inequality again on the right-hand side of inequality above, then we arrive at

$$\begin{aligned}
&(n+\beta) K_n^{*(\alpha,\beta)}\left((s-x)^2\eta(s;x);x\right)\\
&\quad\le\left((n+\beta)^3 e^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!}\right.\\
&\qquad\left.\times\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)}(s-x)^4 ds\right)^{1/2}\\
&\quad\times\left((n+\beta) e^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!}\right.\\
&\qquad\left.\times\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)}\eta^2(s;x)\,ds\right)^{1/2}\\
&\quad=\sqrt{(n+\beta)^2 K_n^{*(\alpha,\beta)}\left((s-x)^4;x\right)}\sqrt{K_n^{*(\alpha,\beta)}\left(\eta^2(s;x);x\right)}.
\end{aligned}\tag{55}$$

From Lemma 13, we have

$$\lim_{n\to\infty}(n+\beta)^2 K_n^{*(\alpha,\beta)}\left((s-x)^4;x\right)=3x^2.\tag{56}$$

On the other hand, since $\eta(s;x)\in C[0,a]$ and $\lim_{s\to x}\eta(s;x)=0$, then it follows from Theorem 4 that

$$\lim_{n\to\infty}K_n^{*(\alpha,\beta)}\left(\eta^2(s;x);x\right)=\eta^2(x;x)=0.\tag{57}$$

Therefore, we conclude from (55), (56), and (57) that

$$\lim_{n\to\infty}(n+\beta) K_n^{*(\alpha,\beta)}\left((s-x)^2\eta(s;x);x\right)=0\tag{58}$$

and then, by taking limit as $n\to\infty$ in (52) and using (58), we find

$$\begin{aligned}
&\lim_{n\to\infty}\left[K_n^{*(\alpha,\beta)}(f;x)-f(x)\right]\\
&\quad=f'(x)\left\{\beta x+h(d+1)+\frac{2\alpha+1}{2}\right\}+\frac{xf''(x)}{2!}
\end{aligned}\tag{59}$$

which completes the proof. □

Remark 15. For $\alpha=\beta=0$, Theorem 14 represents the Voronovskaya type theorem for the operators given by (46) (see [8]).

Remark 16. For $h=0$, it yields a Voronovskaya type theorem for the Kantorovich-Stancu type of Szasz-Mirakyan operators given by (10).

Remark 17. Getting $\alpha=\beta=h=0$ in Theorem 14 gives the Voronovskaya type result for the Szasz-Mirakyan-Kantorovich operators given by (7).

References

[1] O. Szasz, "Generalization of S. Bernstein's polynomials to the infinite interval," *Journal of Research of the National Bureau of Standards*, vol. 45, pp. 239–245, 1950.

[2] A. Altın, O. Doğru, and F. Taşdelen, "The generalization of Meyer-König and Zeller operators by generating functions," *Journal of Mathematical Analysis and Applications*, vol. 312, no. 1, pp. 181–194, 2005.

[3] O. Dogru, M. A. Özarslan, and F. Taşdelen, "On positive operators involving a certain class of generating functions," *Studia Scientiarum Mathematicarum Hungarica*, vol. 41, no. 4, pp. 415–429, 2004.

[4] G. İçöz, F. Taşdelen Yeşildal, and O. Doğru, "Kantrorovich process of linear positive operators via biorthogonal polynomials," *Journal of Inequalities and Special Functions*, vol. 3, pp. 77–84, 2012.

[5] G. İçöz, F. T. Yeşildal, and S. Varma, "On linear positive operators involving biorthogonal polynomial," *Ars Combinatoria*, vol. 105, pp. 319–331, 2012.

[6] M. A. Özarslan, "q-Laguerre type linear positive operators," *Studia Scientiarum Mathematicarum Hungarica*, vol. 44, no. 1, pp. 65–80, 2007.

[7] M. A. Özarslan and O. Duman, "Approximation properties of Poisson integrals for orthogonal expansions," *Taiwanese Journal of Mathematics*, vol. 12, no. 5, pp. 1147–1163, 2008.

[8] F. Taşdelen, R. Aktaş, and A. Altın, "A Kantorovich type of szasz operators including brenke-type polynomials," *Abstract and Applied Analysis*, vol. 2012, Article ID 867203, 13 pages, 2012.

[9] F. Taşdelen and A. Erençin, "The generalization of bivariate MKZ operators by multiple generating functions," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 727–735, 2007.

[10] S. Varma, G. İçöz, and S. Sucu, "On some extensions of Szasz operators including Boas-Buck-type polynomials," *Abstract and Applied Analysis*, vol. 2012, Article ID 680340, 15 pages, 2012.

[11] S. Varma, S. Sucu, and G. İçöz, "Generalization of Szasz operators involving Brenke type polynomials," *Computers & Mathematics with Applications*, vol. 64, no. 2, pp. 121–127, 2012.

[12] S. Varma and F. Taşdelen, "Szasz type operators involving Charlier polynomials," *Mathematical and Computer Modelling*, vol. 56, no. 5-6, pp. 118–122, 2012.

[13] S. Varma and F. Taşdelen, "On a generalization of Szasz-Durrmeyer operators with some orthogonal polynomials," *Studia Universitatis Babeş-Bolyai*, vol. 58, no. 2, pp. 225–232, 2013.

[14] A. Jakimovski and D. Leviatan, "Generalized Szasz operators for the approximation in the infinite interval," *Mathematica*, vol. 11, no. 34, pp. 97–103, 1969.

[15] M. E.-H. Ismail, "On a generalization of Szasz operators," *Mathematica*, vol. 39, no. 2, pp. 259–267, 1974.

[16] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, vol. 13, Gordon and Breach Science Publishers, New York, NY, USA, 1978.

[17] P. L. Butzer, "On the extensions of Bernstein polynomials to the infinite interval," *Proceedings of the American Mathematical Society*, vol. 5, pp. 547–553, 1954.

[18] O. Duman, M. A. Özarslan, and B. D. Vecchia, "Modified Szasz-Mirakjan-Kantorovich operators preserving linear functions," *Turkish Journal of Mathematics*, vol. 33, no. 2, pp. 151–158, 2009.

[19] D. Miclaus, "The Voronovskaja type theorem for the Szász-Mirakjan-Kantorovich operators," *Journal of Science and Arts*, vol. 2, no. 13, pp. 257–260, 2010.

[20] G. Nowak and A. Sikorska-Nowak, "Some approximation properties of modified Szasz-Mirakyan-Kantorovich operators," *Revue d'Analyse Numérique et de Théorie de l'Approximation*, vol. 38, no. 1, pp. 73–82, 2009.

[21] V. Totik, "Approximation by Szasz-Mirakjan-Kantorovich operators in L^p $(p > 1)$," *Analysis Mathematica*, vol. 9, no. 2, pp. 147–167, 1983.

[22] G. Başcanbaz-Tunca and A. Erençin, "A Voronovskaya type theorem for q-Szasz-Mirakyan-Kantorovich operators," *Revue d'Analyse Numérique et de Théorie de l'Approximation*, vol. 40, no. 1, pp. 14–23, 2011.

[23] Z. Walczak, "On approximation by modified Szasz-Mirakyan operators," *Glasnik Matematički*, vol. 37, no. 2, pp. 303–319, 2002.

[24] Ç. Atakut and İ. Büyükyazıcı, "Stancu type generalization of the Favard-Szàsz operators," *Applied Mathematics Letters*, vol. 23, no. 12, pp. 1479–1482, 2010.

[25] D. Bărbosu, "Kantorovich-Stancu type operators," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 3, article 53, 2004.

[26] İ. Büyükyazıcı, "Approximation by Stancu-Chlodowsky polynomials," *Computers & Mathematics with Applications*, vol. 59, no. 1, pp. 274–282, 2010.

[27] A. Erençin, G. Başcanbaz-Tunca, and F. Taşdelen, "Kantorovich type q-Bernstein-Stancu operators," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. 57, no. 1, pp. 89–105, 2012.

[28] F. Altomare and M. Campiti, *Korovkin-Type Approximation Theory and Its Applications*, vol. 17 of *de Gruyter Studies in Mathematics*, Walter de Gruyter, Berlin, Germany, 1994.

[29] Z. Ditzian and V. Totik, *Moduli of Smoothness*, vol. 9 of *Springer Series in Computational Mathematics*, Springer, New York, NY, USA, 1987.

[30] H. W. Gould and A. T. Hopper, "Operational formulas connected with two generalizations of Hermite polynomials," *Duke Mathematical Journal*, vol. 29, pp. 51–63, 1962.