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Research Article

Adaptive Sliding-Mode Tracking Control for a Class of Nonholonomic Mechanical Systems

Wei Sun

School of Automation, Southeast University, Nanjing 210096, China

Correspondence should be addressed to Wei Sun; tellsunwei@sina.com

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This paper investigates the problem of finite-time tracking control for nonholonomic mechanical systems with affine constraints. The control scheme is provided by flexibly incorporating terminal sliding-mode control with the method of relay switching control and related adaptive technique. The proposed relay switching controller ensures that the output tracking error converges to zero in a finite time. As an application, a boat on a running river is given to show the effectiveness of the control scheme.

1. Introduction

In recent decades, sliding-mode control (SMC) has received considerable attention for it is less sensitive to the parameter variations and noise disturbances, and a great number of results have been acquired [1–4]. To get more faster error convergence on the sliding mode, a finite time mechanism, terminal sliding-mode (TSM) control scheme, was presented in [5–7]. Based on it, a series of tracking control problems have been solved in [8–11] and the references therein.

On the other hand, nonholonomic constraints arise in many mechanical systems when there is a rolling or sliding contact, such as wheeled mobile robots, n-trailer systems, space robots, underwater vehicles, and multifingered robotic hands. Although considerable effort [11–15] has been made for nonholonomic systems during the last decades, controller design for these systems is still a challenging problem owing to the existence of nonintegrable geometry constraints.

It is worth pointing out that the existing results [12–15] were mainly aimed at the classic nonholonomic linear constraints (i.e., $J(q)\dot{q} = 0$). However, there are rare results on affine constraints [16, 17] (i.e., $J(q)\dot{q} = A(q)$), which are frequently encountered in some mechanical systems, such as a running river with the varying stream, ball on rotating table with invariable angular velocity, and space robot with initial angular momentum. Therefore, researching the tracking problem for such systems is an innovatory and significative work.

This paper, using the terminal sliding-mode technique, investigates the tracking control problem for a class of uncertainty nonholonomic mechanical systems with affine constraints. To achieve the tracking objective, by flexibly using the algebra processing technique, we triumphantly reduce the number of state variables which provide a motion complying with affine constraints. In order to do so with uncertainties, an appropriate adaptive law is established to identify uncertainty parameter. The main contributions of the paper are briefly characterized by the following features.

(i) Because of the introduction of affine constraints to mechanical systems, it is difficult to find linearly independent vector fields to cancel the constraint forces $J(q)\lambda$ in dynamic equation. Hence, a new diffeomorphism transform is presented to deal with it.

(ii) Based on the asymptotic tracking idea for uncertain multi-input nonlinear systems, the strategy of terminal sliding-mode control, and related adaptive theory, an adaptive relay switching tracking controller is designed which ensures that the output tracking error converges to zero in a finite time.

(iii) As a practical application, a boat on a running river with varying stream is given to illustrate the reasonability of the assumptions and the effectiveness of the control strategy.

The rest of this paper is organized as follows. System description is given in Section 2. The design scheme of the adaptive relay switching controller is addressed in Section 3. Section 4 gives the main results. As for the application, a practical example is considered in Section 5. Section 6 provides some concluding remarks.

2. System Description

2.1. Dynamics Model. In this paper, we consider a class of nonholonomic mechanical systems described by Euler-Lagrangian formulation:

$$M(q)\ddot{q} + V(q,\dot{q})\dot{q} + G(q) + \Phi_u(\sigma,q,\dot{q}) = f + B(q)\tau, \tag{1}$$

$$J^T(q)\dot{q} = A(q), \tag{2}$$

where $q = [q_1,\ldots,q_n]^T \in \mathbb{R}^n$ is the generalized coordinates and $\dot{q},\ddot{q} \in \mathbb{R}^n$ represent the generalized velocity vector and acceleration vector, respectively; $M(q) \in \mathbb{R}^{n\times n}$ is inertia matrix; $V(q,\dot{q})\dot{q} \in \mathbb{R}^n$ represents the vector of centripetal, Coriolis forces; $G(q) \in \mathbb{R}^n$ represents the vector of gravitational forces; $\Phi_u(\sigma,q,\dot{q})$ represents uncertainty of system; $\sigma \in \mathbb{R}^l$ denotes the unknown parameter vector; $B(q) \in \mathbb{R}^{n\times r}$ is an input transformation matrix; $f \in \mathbb{R}^n$ denotes the vector of constraint forces; $J(q) = [j_1(q),\ldots,j_m(q)] \in \mathbb{R}^{n\times m}(m < n)$ is a constraint matrix with full rank; τ represents the r-vector of the generalized control input with $r > n-m$; $A(q) = [a_1(q),\ldots,a_m(q)]^T \in \mathbb{R}^m$ is a known vector function.

Constraint equation (2) is regarded as affine constraints. When it is imposed on the mechanical system (1), the constraint (generalized reaction) forces are given by

$$f = J(q)\lambda, \tag{3}$$

where $\lambda \in \mathbb{R}^m$ is a Lagrangian multiplier corresponding to m nonholonomic affine constraints.

Remark 1. It is worth emphasizing that the system studied in this paper is more general than that in some existing literatures such as [12–15], where dynamic equation satisfies the classical linear constraints. In fact, by taking $A(q) = 0$, (2) transforms to linear constraints, whose tracking problem has been extensively studied in [15, 18–20].

2.2. Reduced Dynamics and State Transformation. This part mainly focuses on reducing the number of state variables which provide motion complying with the affine constraints.

It is easy to find a full-rank matrix $S \in \mathbb{R}^{n\times(n-m)}$ satisfying

$$J^T(q)S(q) = 0. \tag{4}$$

Define $\varsigma(t) = [q,-t]^T$; then (2) can be expressed concisely as

$$\begin{bmatrix} J^T(q) & A(q) \end{bmatrix}\dot{\varsigma}(t) = 0. \tag{5}$$

Let

$$E(q) = \begin{bmatrix} S(q) & \eta(q) \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n-m+1)}, \tag{6}$$

where $\eta(q) \in \mathbb{R}^n$ satisfies $J^T(q)\eta(q) = A(q)$. One can deduce that $E(q)$ is a full rank and satisfies

$$\begin{bmatrix} J^T(q) & A(q) \end{bmatrix} E(q) = 0. \tag{7}$$

According to (5) and (7), we know that there exists an $(n-m+1)$-dimensional vector $\dot{\bar{z}} = [\dot{\bar{z}}_{n-m}^T, \dot{\bar{z}}_{n-m+1}]^T$ such that $\dot{\varsigma} = E\dot{\bar{z}}$, that is,

$$\begin{bmatrix} \dot{q} \\ -1 \end{bmatrix} = \begin{bmatrix} S(q) & \eta(q) \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{\bar{z}}_{n-m}^T \\ \dot{\bar{z}}_{n-m+1} \end{bmatrix}, \tag{8}$$

which implies that $\dot{\bar{z}}_{n-m+1} = 1$. For convenience, define $z \triangleq \bar{z}_{n-m} \in \mathbb{R}^{n-m}$. In view of the relationship (8), the generalized velocity vectors can be written as

$$\dot{q} = S(q)\dot{z} + \eta(q). \tag{9}$$

It is clear that z corresponds to the internal state variable.

Substituting (9) into (1), premultiplying $S^T(q)$ on both sides of it, and using $J^T(q)S(q) = 0$, the dynamics of the mechanical system made up by (1) and (2) can be described clearly as

$$M_1(q)\ddot{z} + V_1(q,\dot{q})\dot{z} + G_1(q,\dot{q}) + \Phi_1(\sigma,q,\dot{q}) = u, \tag{10}$$

where $M_1(q) = S^T(q)M(q)S(q)$, $V_1(q,\dot{q}) = S^T(q)(M(q)\dot{S}(q) + V(q,\dot{q})S(q))$, $G_1(q,\dot{q}) = S^T(q)(M(q)\dot{\eta}(q) + V(q,\dot{q})\eta(q) + G(q))$, $\Phi_1(\sigma,q,\dot{q}) = S^T(q)\Phi_u(\sigma,q,\dot{q})$, and $u = S^T(q)B(q)\tau$. In order to guarantee that all $(n-m)$ degrees of freedom are actuated independently, we suppose that $S^T(q)B(q)$ is full rank.

Remark 2. The aforementioned diffeomorphism transform method differs from the traditional ones in [13, 18–22]. More specifically, when the affine constraints are imposed on the mechanical system, it is difficult to find linearly independent vector fields to proceed with a simple diffeomorphism transformation for canceling the constraint forces in dynamic equations. Hence, we present the forementioned diffeomorphism transform to achieve this goal.

Remark 3. The diffeomorphism transformations consist of (4) and (9), ensure that the transformed system (10) still satisfies constraint equation (2), and possess the practical physical meaning. This can also be confirmed by the practical example in Section 5.

The control objective of this paper can be specified as follows. Given the desired trajectories $z_r(t)$ and $\dot{z}_r(t)$, which are assumed to be bounded and should satisfy constraint equation (2), we determine a control law such that all states of the closed-loop system are globally bounded and the output tracking error $\varepsilon(t) \triangleq z(t) - z_r(t)$ and its time derivative $\dot{\varepsilon}(t)$ converge to zero in a finite time.

In order to solve the above tracking problem, we make the following reasonable assumption.

Assumption 1. The matrix M_1 is symmetric, positive definite and there exist two known scalars $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $\lambda_{\min}[M_1^{-1}(q)] \geq \alpha_1$, $\|M_1^{-1}(q)\| \leq \alpha_2$.

3. Control Design

This subsection will construct a relay switching controller composed of an adaptive TSM controller and an adaptive Pre-TSM controller. For simplicity, sometimes the arguments of functions are dropped in the remainder of the paper.

Step 1 (Adaptive TSM Controller Design). Define $\xi = [z^T, \dot{z}^T]^T \in \mathbb{R}^{2(n-m)}$; then system (10) can be expressed as

$$\dot{\xi} = \begin{bmatrix} \dot{z} \\ -M_1^{-1}(q)\left[V_1(q,\dot{q})\dot{z} + G_1(q,\dot{q}) + \Phi_1(\sigma,q,\dot{q})\right] \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} M_1^{-1}(q)\,u. \tag{11}$$

The following reference model is chosen as that in [5]:

$$\begin{bmatrix} \dot{z}_r \\ \ddot{z}_r \end{bmatrix} = \begin{bmatrix} 0 & I \\ R & Q \end{bmatrix} \begin{bmatrix} z_r \\ \dot{z}_r \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} r(t) \triangleq A\xi_r + \overline{B}r(t), \tag{12}$$

where R, Q, and B_1 are constant matrices such that system (12) is stable; I is an identity matrix; $z_r(t)$, $\dot{z}_r(t)$, and $r(t)$ are measurable and bounded signals. For convenience, define $\xi_r = [z_r^T, \dot{z}_r^T]^T \in \mathbb{R}^{2(n-m)}$.

Now, we develop the following tracking error system which will be used in the subsequent controller design and stability analysis:

$$e(t) = \xi(t) - \xi_r(t) = \left[\varepsilon^T(t), \dot{\varepsilon}^T(t)\right]^T. \tag{13}$$

We directly get the following equations from (11)~(13):

$$\begin{aligned} \dot{e}(t) &= \dot{\xi}(t) - \dot{\xi}_r(t) \\ &= Ae(t) + B\left(M_1^{-1}(q)\,u(t) + L(q,\dot{q},z,\dot{z},r)\right), \end{aligned} \tag{14}$$

$$\begin{aligned} L(q,\dot{q},z,\dot{z},r) = {} & -M_1^{-1}(q)(V_1\dot{z} + G_1) - M_1^{-1}(q)\Phi_1 \\ & - Rz - Q\dot{z} - B_1 r(t), \end{aligned} \tag{15}$$

where $B = [0, I]^T$. To ensure that $e(t)$ reaches zero in finite time, one defines a fast switching surface as

$$s_i = \dot{\varepsilon}_i + c_i\varepsilon_i^{b/a}, \quad \varepsilon = [\varepsilon_1, \ldots, \varepsilon_{n-m}]^T, \tag{16}$$

where $c_i, i = 1\cdots n{-}m$, are positive constants and both a and b are odd and satisfy $b < a < 2b$. Define $S = [s_1, \ldots, s_{n-m}]^T$, $C = \operatorname{diag}[c_1, \ldots, c_{n-m}]$. Dynamic equation (16) can be rewritten as

$$S = \dot{\varepsilon} + C\varepsilon^{b/a}. \tag{17}$$

According to the parameter separation technique (Lemma 2.1 in [23]), for uncertain term $\Phi_1(\sigma, q, \dot{q})$ in system (1), there exist an unknown constant $\Theta \geq 1$ and a known smooth function $\Phi(q,\dot{q}) \geq 1$, such that $\|\Phi_1(\sigma,q,\dot{q})\| \leq \Theta\Phi(q,\dot{q})$. Then, choose a continuously differentiable, positive definite and radially unbounded function V_1 as

$$V_1 = \frac{1}{2}S^TS + \frac{\gamma}{2}\widetilde{\Theta}^2, \tag{18}$$

where γ is a design parameter and $\widetilde{\Theta}(t) = \Theta - \widehat{\Theta}(t)$ represents parameter estimation error.

According to (9) and the definition of $V_1(q,\dot{q})$, $G_1(q,\dot{q})$, we have $\|V_1(q,\dot{q})\dot{z} + G_1(q,\dot{q})\| < \Psi(q,\dot{q},\dot{z})$, where $\Psi(q,\dot{q},\dot{z})$ is a known positive smooth function. Taking the time derivative of V_1, using Assumption 1 and substituting (14)–(17) into it result in the following:

$$\begin{aligned} \dot{V}_1 = {} & S^T\left[\frac{b}{a}C\operatorname{diag}\left(\varepsilon_i^{(b/a)-1}\right)\dot{\varepsilon}\right] + S^T[0,I] \\ & \times\left[Ae(t) + B\left(M^{-1}(q)\,u + L(q,\dot{q},z,\dot{z},r)\right)\right] - \gamma\widetilde{\Theta}\dot{\widehat{\Theta}} \\ \leq {} & \left\|S^T\right\|\left\|\frac{b}{a}C\operatorname{diag}\left(\varepsilon_i^{(b/a)-1}\right)\dot{\varepsilon}\right\| + \left\|S^T\right\| \\ & \times\left\|[0,I]\,Ae(t) - (Rz + Q\dot{z} + B_1 r)\right\| \\ & + \alpha_2\left\|S^T\right\|\Psi(\cdot) + S^TM_1^{-1}u + \alpha_2\Theta\left\|S^T\right\|\Phi(\cdot) - \gamma\widetilde{\Theta}\dot{\widehat{\Theta}} \\ \leq {} & \left\|S^T\right\|\left\|\frac{b}{a}C\operatorname{diag}\left(\varepsilon_i^{(b/a)-1}\right)\dot{\varepsilon}\right\| + \left\|S^T\right\| \\ & \times\left\|[0,I]\,Ae(t) - (Rz + Q\dot{z} + B_1 r)\right\| + \alpha_2\left\|S^T\right\|\Psi(\cdot) \\ & + S^TM_1^{-1}u + \alpha_2\widehat{\Theta}\left\|S^T\right\|\Phi(\cdot) + \alpha_2\widetilde{\Theta}\left\|S^T\right\|\Phi(\cdot) \\ & - \gamma\widetilde{\Theta}\dot{\widehat{\Theta}} \triangleq \left\|S^T\right\|\Upsilon(\cdot) + S^TM_1^{-1}u + \alpha_2\widetilde{\Theta}\left\|S^T\right\|\Phi(\cdot) \\ & - \gamma\widetilde{\Theta}\dot{\widehat{\Theta}}, \end{aligned} \tag{19}$$

where $\Upsilon = \|(b/a)C\operatorname{diag}(\varepsilon_i^{(b/a)-1})\dot{\varepsilon}\| + \|[0,I]Ae(t) - (Rz + Q\dot{z} + B_1 r)\| + \alpha_2\Psi(\cdot) + \alpha_2\widehat{\Theta}\Phi(\cdot)$.

Now, if an FTC control law $u_{FT}(t)$ is taken as

$$u_{FT}(t) = -\frac{S}{\alpha_1\|S\|}(\Upsilon + 1) \tag{20}$$

with the following adaptive law:

$$\dot{\widehat{\Theta}}(t) = \frac{\alpha_2}{\gamma}\Phi(q,\dot{q})\|S\|, \tag{21}$$

then the following expression can be obtained from (19):

$$\dot{V}_1 \leq -\|S\|. \tag{22}$$

Remark 4. The FTC controller composed of (20) and (21) may cause the singularity of closed-loop (14)~(21) due to the existence of the term $\operatorname{diag}(\varepsilon_i^{(b/a)-1})$, since $\operatorname{diag}(\varepsilon_i^{(b/a)-1})\dot{\varepsilon}$ may be sufficiently large if $\varepsilon_i(t)$ is sufficiently small. However, on the sliding surfaces, the singularity does not occur. Since, on the sliding mode, $S = 0$ implies $\dot{\varepsilon}_i = -c_i\varepsilon_i^{b/a}$, then one can further get

$$\operatorname{diag}\left(\varepsilon_i^{(b/a)-1}\right)\dot{\varepsilon} = -\left[c_1\varepsilon_1^{(2b/a)-1}, \ldots, c_{n-m}\varepsilon_{n-m}^{(2b/a)-1}\right]. \tag{23}$$

This shows that each component of $\operatorname{diag}(\varepsilon_i^{(b/a)-1})\dot{\varepsilon}$ is bounded as $\varepsilon(t)$ is sufficiently small for $2b > a$. Consequently, once the trajectory of $e(t)$ arrives on the sliding surfaces, the control law is bounded and does not cause the singularity. However, when $e(t)$ moves to the switching surface, singularity may occur. To avoid this phenomenon, we introduce the following controller.

Step 2 (adaptive pre-TSM controller design). Firstly, define Ω as

$$\Omega \triangleq \{e : \|e - e^*\| < \varepsilon_0 \ll 1\}, \tag{24}$$

where e^* is a fixed point on the switching surface $S = 0$ and ε_0 is a sufficiently small constant.

Let us construct an augmented linear system as

$$\dot{x}(t) = Ax(t) + Bv(t), \tag{25}$$

where

$$v(t) = \begin{cases} B^T e^{-A^T t} G_c^T(0, t_f) \\ \quad \times \left[e^{-A^T t_f} x(t_f) - x(0) \right], & 0 \le t \le t_f, \\ 0, & t > t_f, \end{cases} \tag{26}$$

where $G_c^T(0, t_f)$ is the controllable gramian matrix with the following form:

$$G_c^T(0, t_f) = \int_0^{t_f} e^{-At} BB^T e^{-A^T t} \mathrm{d}t. \tag{27}$$

Based on linear system theory, under the control law $v(t)$, $x(t)$ starting from any initial state vector $x(0)$ can be transferred to any given final state $x(t_f)$ at time t_f. Here, let the final state $x(t_f)$ be on the nonlinear switching surface $S = 0$ and $x(t_f) = e^*$.

The remaining task is to design the preterminal sliding-mode controller which guarantees that an arbitrary point in the space $\mathbb{R}^{2n-m}$ arrives at Ω in a finite time.

Define

$$\varrho(t) = e(t) - x(t), \qquad \varrho(0) = e(0) = \varrho_0. \tag{28}$$

Equations (14) and (25) give rise to

$$\dot{\varrho}(t) = A\varrho(t) + B\left[M_1^{-1}(q) u + L(\cdot) - v(t) \right], \tag{29}$$

where $L(\cdot)$ is given in (15). Because of the stabilization of A, there exists a positive definite matrix P such that

$$A^T P + PA = -I. \tag{30}$$

Choose a candidate Lyapunov function V_2 as

$$V_2 = \frac{1}{2}\varrho^T P \varrho + \frac{\kappa}{2}\widetilde{\Theta}^2, \tag{31}$$

where κ is a design parameter. In view of Assumption 1, the time derivative of V_2 satisfies

$$\begin{aligned} \dot{V}_2 &= -\frac{1}{2}\varrho^T \varrho + \varrho^T PB \left[M_1^{-1}(q) u + L(q, \dot{q}, z, \dot{z}, r) - v(t) \right] \\ &\quad - \kappa \widetilde{\Theta} \dot{\widehat{\Theta}} \\ &\le -\frac{1}{2}\varrho^T \varrho + \varrho^T PBM_1^{-1}(q) u \\ &\quad + \left\| \varrho^T PB \right\| \left[\alpha_2 \Psi(\cdot) + \|Rz + Q\dot{z} + B_1 r\| + \alpha_2 \widehat{\Theta}\Phi(\cdot) \right. \\ &\quad \left. + \|v(t)\| \right] + \alpha_2 \widetilde{\Theta}\Phi(\cdot) \left\| \varrho^T PB \right\| - \kappa \widetilde{\Theta} \dot{\widehat{\Theta}}. \end{aligned} \tag{32}$$

If we take a pre-terminal controller as

$$\begin{aligned} u_{\text{pr}}(t) &= -\frac{\left(\varrho^T PB\right)^T}{\alpha_1 \left\|\varrho^T PB\right\|} \\ &\quad \times \left[\|v(t)\| + \alpha_2 \Psi(\cdot) + \|Rz + Q\dot{z} + B_1 r\| + \alpha_2 \widehat{\Theta}\Phi(\cdot) \right] \end{aligned} \tag{33}$$

with adaptive law

$$\dot{\widehat{\Theta}} = \frac{\alpha_2}{\kappa} \Phi(q, \dot{q}) \left\| \varrho^T PB \right\|, \tag{34}$$

then (32) reduces to

$$\dot{V}_2 \le -\frac{1}{2}\varrho^T \varrho. \tag{35}$$

4. Main Results

Firstly, let us recapitulate how to manipulate the fore-mentioned two adaptive controllers to realize the control objective.

(i) According to (29), design a preterminal controller made up by (33) and (34) such that the trajectory of $e(t)$ in (14) enters Ω in a given finite time.

(ii) Design the TSM controller formed by (20) and (21) such that the trajectory of $e(t)$ starting from Ω first reaches switching surface $S = 0$ and then moves to zero in a finite time along this surface.

Next, we present the following theorem, which summarizes the main results of this paper.

Theorem 5. *Suppose that Assumption 1 holds for the nonholonomic mechanical system described by* (1) *and* (2)*, then for a desired trajectory $z_r(t)$ satisfying the constraint equation* (2)*, according to the above manipulations (i) and (ii), the following are guaranteed:*

(a) *all states of the closed-loop system are globally bounded;*

(b) *the output tracking errors $\varepsilon(t)$ and $\dot{\varepsilon}(t)$ converge to zero in a finite time.*

Proof. From (35), one knows that V_2 is monotonically decreasing, that is, $V_2(t) \leq V_2(0)$, for all $t \geq 0$, which results in $\|\varrho(t)\| \leq \sqrt{2V_2(0)/\|P\|} < \infty$ and $|\widetilde{\Theta}(t)| \leq \sqrt{2V_2(0)/\kappa} < \infty$. Therefore, it follows that $e(t)$ is bounded from (25)~(28). Moreover, the boundedness of $z_r(t), \dot{z}_r(t)$ shows that $z(t)$ and $\dot{z}(t)$ are bounded, so are $u(t)$ in (33) and $\dot{\varrho}(t)$. In addition, integrating on (35) from 0 to ∞ and using the boundedness of V_2, we have $\int_0^\infty \varrho^T \varrho d\tau < \infty$. The boundedness of $\varrho^T(t)\dot{\varrho}(t)$ means that $\varrho^T(t)\varrho(t)$ is uniformly continuous. According to the Barbalat lemma [24], one has $\lim_{t\to\infty}\varrho(t)^T\varrho(t) = 0$ and $\lim_{t\to\infty}\varrho(t) = 0$. Therefore, there exists a finite time $T_0 = T_0(\varrho_0)$ such that $\|\varrho(\varrho_0, t)\| \leq \varepsilon_0$, $t \geq T_0$. If one selects $t_f = T_0$, then

$$\varrho\left(\varrho_0, t_f\right) = \varrho\left(\varrho_0, T_0\right) = e\left(t_f\right) - x\left(t_f\right) = e\left(t_f\right) - e^*, \quad e\left(t_f\right) \in \Omega. \tag{36}$$

This shows that under the action of the pre-terminal control law (33), $e(t)$ starting from any initial state vector $e(0)$ enters the small neighborhood Ω of e^* after time t_f. On the other hand, (18) and (22) imply that the trajectory of the system globally asymptotically converges to $S = 0$. In the following, we will talk about the convergence rate of dynamic surface S. (18) and (22) result in

$$\dot{V}_1 = S^T\dot{S} - \gamma\widetilde{\Theta}\dot{\widehat{\Theta}} = S^T\dot{S} - a_2\widetilde{\Theta}\Phi\left(q,\dot{q}\right)\|S\| \leq -\|S\|. \tag{37}$$

It is equivalent to

$$\left\|S^T\right\|\left\|\dot{S}\right\| + a_2\widetilde{\Theta}\Phi\left(q,\dot{q}\right)\|S\| \geq \|S\|, \tag{38}$$

or

$$\left\|\dot{S}\right\| \geq 1 - a_2\widetilde{\Theta}\Phi\left(q,\dot{q}\right) \triangleq L, \quad \text{when } \|S\| \neq 0. \tag{39}$$

Integrating on both sides of (21) and according to the definition of $\widetilde{\Theta}$, one has

$$\widetilde{\Theta} = -\int_0^t \frac{a_2}{\gamma}\Phi\left(q,\dot{q}\right)\|S\| d\tau - \widehat{\Theta}(0) + \Theta. \tag{40}$$

This ensures $\widetilde{\Theta} < 0$ by appropriately choosing $\widehat{\Theta}(0)$. Hence, one further gets $L > 0$. $\|\dot{S}\| > L$ means that $\sum_1^{n-m}\dot{s}_i^2 > L^2$; without loss of generality, suppose $\dot{s}_i^2 > L^2/(n-m)$. $\dot{s}_i > L/\sqrt{n-m}$ and $\dot{s}_i < -L/\sqrt{n-m}$ can conclude $(L/\sqrt{n-m})t + s_i(0) < s_i(t)$ and $(L/\sqrt{n-m})t - s_i(0) < -s_i(t)$, respectively. In both cases, one can deduce that $s_i(t)$ tends to zero in a finite time on account of the convergence of S. In view of the above analysis, one knows that $S(t)$ tends to zero in finite time; that is, the error dynamic $e(t)$ reaches the nonlinear switching surface $S = 0$ in a finite time.

Next, we prove that the singularity cannot occur by using the above proposed relay switching controller.

Specifically, given a specified initial value $e(0)$, the trajectory of error dynamics $e(t)$ can be divided into two stages in which it firstly moves to the switching surface in a certain time interval $[0, t_f]$ and then slides along $S = 0$ to the origin at a finite time. According to Remark 4, once error $e(t)$ arrives to the sliding surface, the singularity may not occur. In the following, one can prove that under control law composed of (20) and (21), the trajectory $e(t)$ initiating from the open set Ω cannot escape to infinity. For this purpose, we assume that at some time instant t_0^*, $e(t_0^*) = e_0^* \in \Omega$. By (18) and (22), $S(e(t))$ can be close to zero in a finite time. Since $S(e^*) = 0$, $e_0^* \in \Omega$ and ε_0 is sufficiently small; $\|S(e_0^*)\|$ is very small, which guarantees that $\|S(e(t))\|$ is sufficiently small. Therefore, by (17), one has

$$\dot{\varepsilon}(t) = -C\varepsilon^{b/a}(t) + S(t) \approx -C\varepsilon^{b/a}(t). \tag{41}$$

Consequently, e_0^* is far away from the origin when e^* is away from the origin. This shows that the initial value $\varepsilon(t_0^*)$ is away from zero. By solving the inequalities (22) and (41) in time interval $[t_0^*, \infty)$ and by considering the initial values $\varepsilon(t_0^*)$ and $\|S(e(t_0^*))\|$, we conclude that $\|S(e(t))\|$ reaches zero firstly before $\varepsilon_i(t)$ becomes very small. As shown previously, on the switching surface $S = 0$, the control signal is bounded. This illustrates that the control law $u(t)$ given in (20) and (21) is bounded if the starting state of the trajectories $e(t)$ is in a sufficiently small neighborhood of Ω, and in this sense, the singularity is avoided.

Altogether, we consider the trajectories of (14) in time intervals $[0, t_f]$ and $[t_f, \infty)$. In view of the above analysis, as t_f is fixed, $[0, t_f]$ is a finite one. Equations (31) and (32) guarantee that $e(t)$ arrives at Ω and does not escape to infinite in $[0, t_f]$. When $t \geq t_f$, the controller is switched to the TSM controller under which the trajectory arrives at the switching surface $S = 0$ first and then moves along this surface to the origin in a finite time. The TSM controller has been proved to be bounded in time interval $[0, t_f]$ and all the signals in the closed loop are bounded on the switching surface $S = 0$. Till now, the theorem is proved completely. □

Remark 6. If one adopts a general finite time controller in [5], the control signals may tend to infinity before the state of the error system reaches the switching surface. For instance, the term $\|\operatorname{diag}(\varepsilon_1^{(b/a)-1}, \ldots, \varepsilon_{n-m}^{(b/a)-1})\dot{\varepsilon}\|$ in controller (20) is infinity, if $\varepsilon_i (i = 1, \ldots, n-m)$ are all sufficiently small for $b < a < 2b$. Therefore, one proposed a relay switching control scheme to avoid this phenomenon.

5. Simulation

Consider a boat on a running river (see Figure 1). The x-axis and y-axis denote the transverse direction and the downstream direction of the river, respectively. According to the motion of the boat on the river, one can get the following kinematic equations:

$$\begin{aligned}\dot{x} &= V\cos\theta - C(x)\cos\theta\sin\theta, \\ \dot{y} &= V\sin\theta + C(x)\cos^2\theta,\end{aligned} \tag{42}$$

where $C(x)$ and V denote the stream of the river and the speed of the boat, respectively. After some simple calculations, the affine constraints can be obtained as follows:

$$\dot{y}\cos\theta - \dot{x}\sin\theta = C(x)\cos\theta. \tag{43}$$

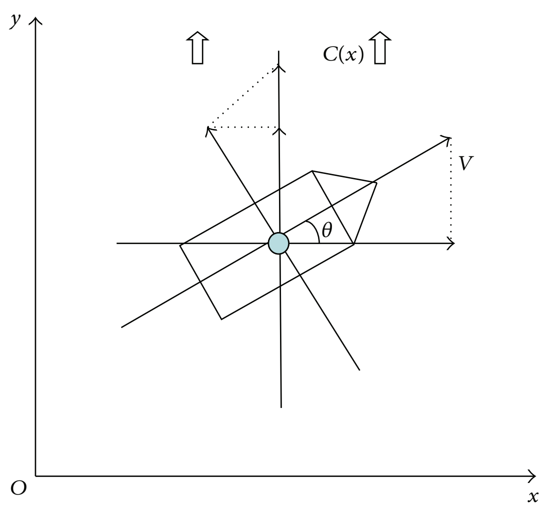

Figure 1: Boat on a running river.

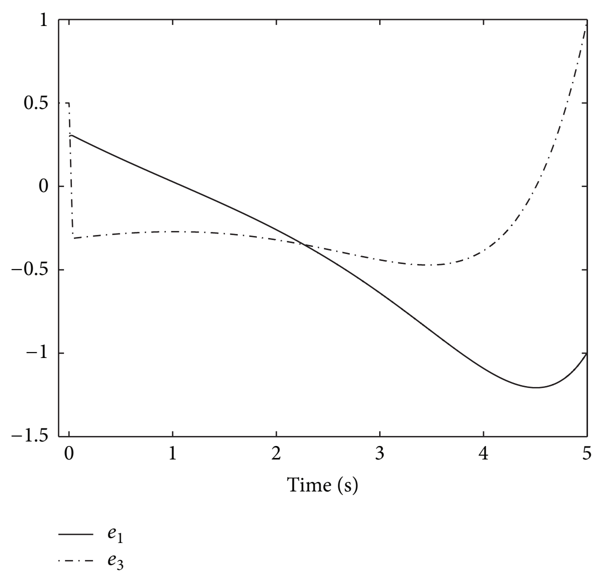

Figure 2: The trajectories of $e_1(t)$ and $e_3(t)$.

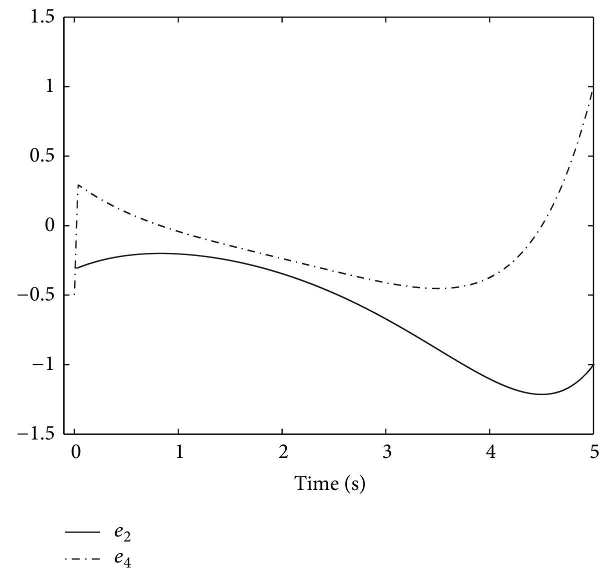

Figure 3: The trajectories of $e_2(t)$ and $e_4(t)$.

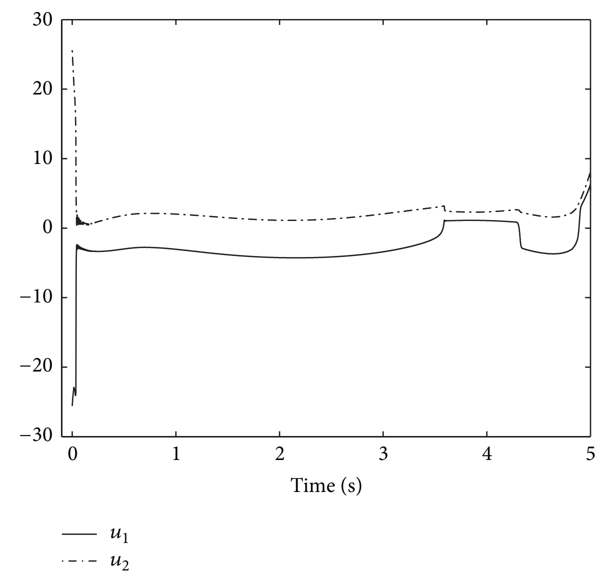

Figure 4: The trajectories of $u_1(t)$ and $u_2(t)$.

From the above equation, the following can be obtained:

$$q = [y, x, \theta]^T,$$
$$J(q) = [\cos q_3, -\sin q_3, 0]^T, \tag{44}$$
$$A(q) = C(q_2)\cos q_3.$$

The dynamics model of the boat on a running river can be expressed as

$$\begin{aligned} m_1\ddot{y} + \sigma_1\dot{y} &= \lambda\cos\theta + \tau_1\sin\theta, \\ m_1\ddot{x} + \sigma_2\dot{x} &= -\lambda\sin\theta + \tau_1\cos\theta, \\ I_1\ddot{\theta} &= \tau_2, \end{aligned} \tag{45}$$

where m_1 is the mass of the boat and I_1 is the inertia of the boat; $\sigma_1\dot{y}$ and $\sigma_2\dot{x}$ denote the external resistance, where σ_1 and σ_2 are unknown. In this simulation, let $C(q_2) = q_2$, and choose

$$S(q) = \begin{bmatrix} \sin q_3 & 0 \\ \cos q_3 & 0 \\ 0 & 1 \end{bmatrix}, \tag{46}$$
$$\eta(q) = \left[q_2,\ 0,\ 2\sqrt{2}q_2\right]^T.$$

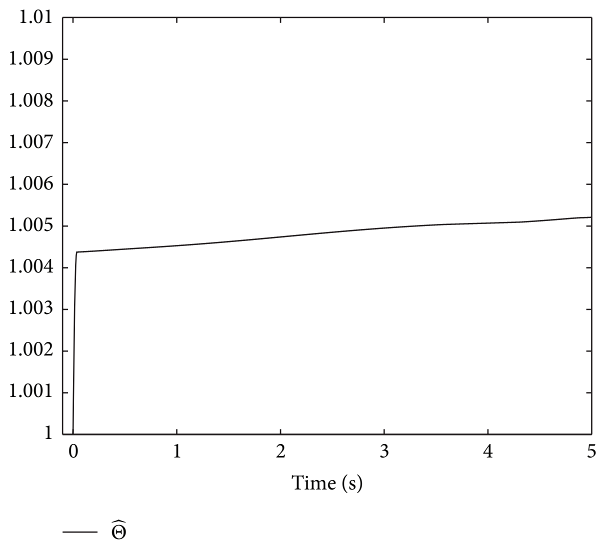

FIGURE 5: The trajectory of $\widehat{\Theta}(t)$.

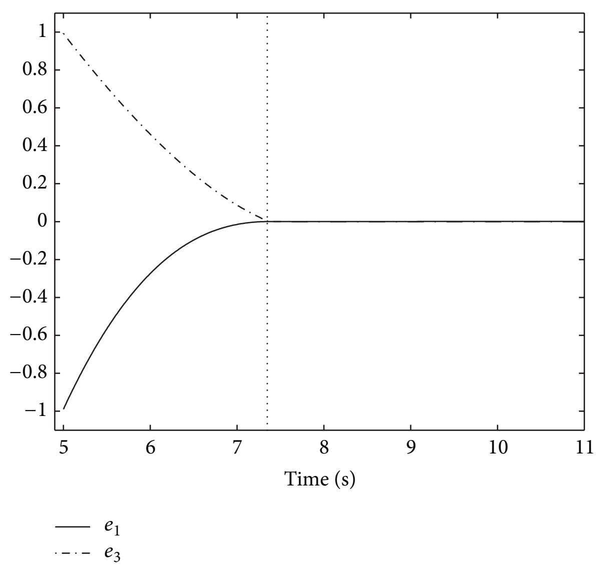

FIGURE 6: The trajectories of $e_1(t)$ and $e_3(t)$.

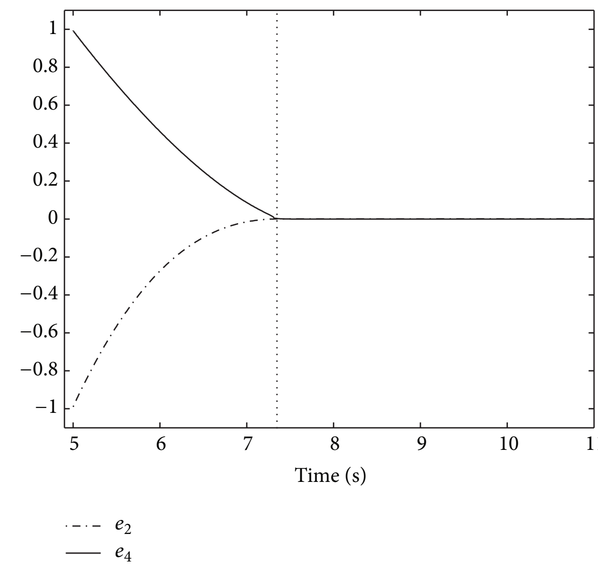

FIGURE 7: The trajectories of $e_2(t)$ and $e_4(t)$.

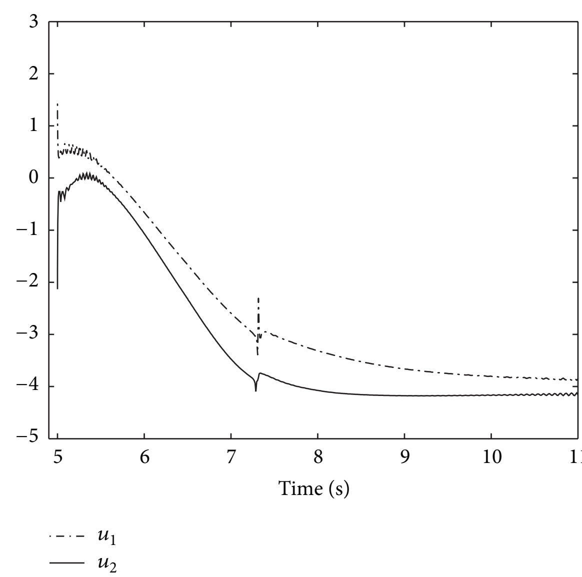

FIGURE 8: The trajectories of $u_1(t)$ and $u_2(t)$.

It follows from the procedure of the aforementioned diffeomorphism transformation that

$$\begin{aligned} \dot{q}_1 &= \dot{z}_1 \sin q_3 + q_2, \\ \dot{q}_2 &= \dot{z}_1 \cos q_3, \\ \dot{q}_3 &= \dot{z}_2 + 2\sqrt{2} q_2. \end{aligned} \tag{47}$$

Then, the original dynamics system can be converted into the following form:

$$\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} \dot{q}_2 \sin q_3 \\ 2\sqrt{2}\dot{q}_2 \end{bmatrix} + \begin{bmatrix} \sigma_1 \dot{q}_1 \sin q_3 + \sigma_2 \dot{q}_2 \cos q_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}. \tag{48}$$

For the given $J(q)$, $S(q)$, and $\eta(q)$, the desired trajectory $q_d = [\pi/3, e^{-t}, \pi/4]^T$ satisfies kinematic constraint $J^T(q_d)\dot{q}_d = A(q_d)$ and diffeomorphism transform $\dot{q}_d = S(q_d)\dot{z}_d + \eta(q_d)$ with $z_r = [\sqrt{2}e^{-t}, 2\sqrt{2}e^{-t}]^T$. The control objective, based on the proposed scheme, is to determine an adaptive relay switching control law such that the trajectory z follows z_r. According to $[\dot{z}_r^T, \ddot{z}_r^T]^T = A[z_r^T, \dot{z}_r^T]^T$, the reference model is chosen as

$$\dot{\xi}_r = A\xi + \overline{B}r, \tag{49}$$

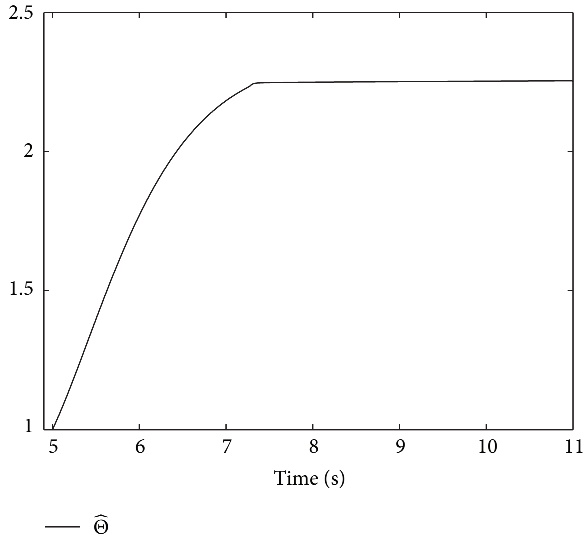

FIGURE 9: The trajectory of $\widehat{\Theta}(t)$.

where

$$\xi_r = \begin{bmatrix} z_{1r} \\ z_{2r} \\ \dot{z}_{1r} \\ \dot{z}_{2r} \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix},$$
$$\overline{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad r(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \tag{50}$$

In this simulation, the parameters in the system are selected as $m = 1\,\text{kg}$, $I = 1\,\text{kg}\cdot\text{m}^2$, $\sigma_1 = \sigma_2 = 1$, $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\gamma = \kappa = 15$, $a = 3$, $b = 5$, $c_1 = c_2 = 1$, $\varepsilon_0 = 0.01$, $x(t_f) = [-1, -1, 1, 1]^T$, and $t_f = 5$. In order to eliminate the effects of the chattering, we replace $s_i/\|S\|$ by $s_i/(\|S\| + 0.005)$. The results of the simulation are shown in Figures 2–9. Specifically, Figures 2 and 3 show that output tracking errors go to Ω at time instant $t = 5$ under the action of adaptive Pre-TSM controller. Figures 4 and 5 show that both the adaptive pre-TSM controller $u(t)$ and state $\widehat{\Theta}(t)$ are bounded. On the other hand, the satisfactory results of TSM design scheme after $t = 5$ are presented in Figures 6–9. Figures 6 and 7 show that the errors $e(t)$ become zero at approximate time instant $t = 7.4$. Figures 8 and 9 show that the adaptive TSM controller $u(t)$ and $\widehat{\Theta}(t)$ tend to the steady state and both of them are bounded. Thereby, the practical simulation example confirms the validity of the proposed algorithm.

6. Conclusions

This paper studies tracking problem for a class of uncertain nonholonomic mechanical systems based on the idea of terminal sliding-mode control. The adaptive relay switching tracking controller guarantees that output tracking error converges to zero in a finite time and avoids the singularity problem. A practical mechanical model is constructed to confirm the reasonability of the assumption and the effectiveness of the control scheme.

Acknowledgments

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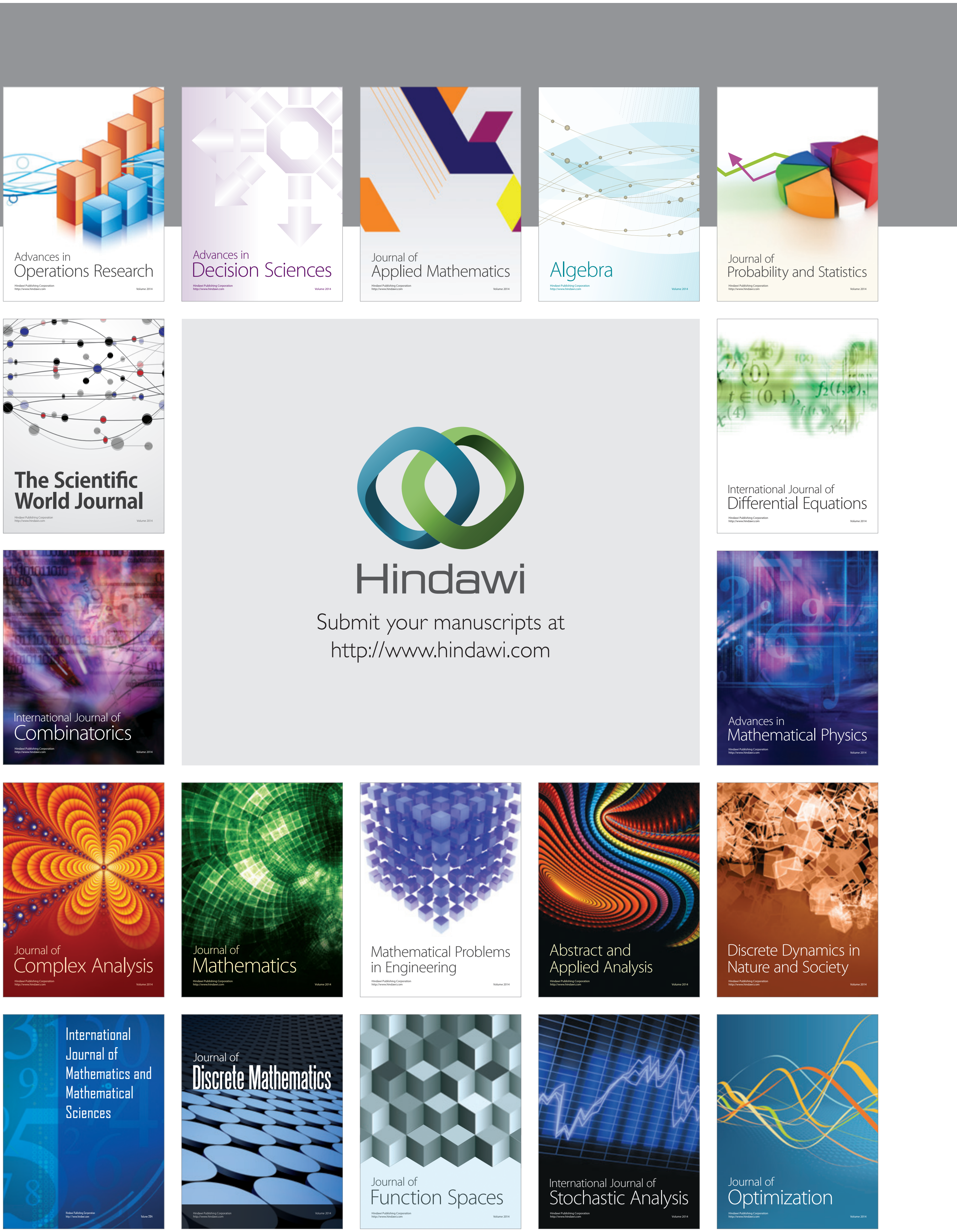
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