Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 962963, 14 pages
doi:10.1155/2012/962963

Research Article

Free-Boundary Seepage from Asymmetric Soil Channels

Adrian Carabineanu[1, 2]

[1] *Institute of Mathematical Statistics and Applied Mathematics, Romanian Academy, Calea 13 Septembrie 13, 010702 Bucharest, Romania*
[2] *Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest, Romania*

Correspondence should be addressed to Adrian Carabineanu, acara@fmi.unibuc.ro

Received 30 March 2012; Accepted 28 May 2012

Academic Editor: Petru Jebelean

We present an inverse method for the study of the seepage from soil channels without lining. We give integral representations of the complex potential, velocity field, stream lines, free phreatic lines, and contour of the channel by means of Levi-Civitá's function ω. For different values of the Taylor coefficients of ω, we calculate numerically the contour of the channel, the phreatic lines, the seepage loss, the velocity field, the stream lines, and the equipotential lines. Examples are given for various symmetric or asymmetric channels, with smooth contours or with angular points.

1. Introduction

The study of the seepage from soil channels is important for drainage, irrigation, or water transportation. In this paper, we present a new inverse method for investigating the seepage from soil channels (watercourses) with no lining.

The inverse methods do not solve the *direct seepage problem: given the contour of the channel, calculate the corresponding seepage loss*, but there is a reason to pay attention to this kind of methods: the possibility to obtain exact analytical results.

A valuable tool for studying the direct problem by means of the inverse method is Kacimov's comparison theorem [1] which states that for two arbitrary channels having the cross sections S_1 and S_2, the relation $S_1 \subseteq S_2$ implies the relation $Q_1 \leq Q_2$ between the corresponding seepage discharges. Therefore, it is important to have a great number of channel contours obtained by means of the inverse method. We shall review some papers where various alternatives of the inverse method for the seepage problem from soil channels have been employed. Kozeny (see [2, 3]) studied the seepage from a curved channel using

Zhukovskii's function and found that the resultant channel has a trochoid form. In [4], Anakhaev obtained a solution for curvilinear watercourses by representing the watercourse profiles in the Zhukovskii plane by means of the equation of a family of lemniscates. Other types of watercourses with different relative widths were studied by Anakhaev in [5]. For the particular case of a circular base of the watercourse profile, the solution of Anakhaev coincides with the known exact solutions derived by Vedernikov [6] and Pavlovskii [7]. Chahar utilized in [8] the inverse method to obtain an exact solution for seepage from a curved channel whose boundary is mapped on a circle from the complex velocity plane. The channel shape is an approximate semiellipse with the top width as the major axis and twice the water depth as the minor axis and vice versa. The average of the corresponding ellipse and parabola gives almost the exact shape of the channel. In a subsequent paper dedicated to the same class of curvilinear bottomed channels Chahar [9] discusses the optimal section properties from the least area and minimum seepage loss points of view. In [10], Chahar extends his method to the case of seepage from curved channels with a drainage layer at shallow depth. Kacimov and Obsonov [11] used the inverse method to find the shape of a soil channel of constant hydraulic gradient. In [11, 12] the authors utilized an inverse method where the shape of the unknown channel is searched as part of the solution.

In most of the above cited papers, the profiles of the channels are considered to be symmetric. In reality the great majority of watercourses do not have symmetric profiles. Even the channels which are symmetric by construction become asymmetric because of erosion or sediments.

There are some papers dedicated to the study of the seepage from asymmetric watercourses. For example in [13], Anakhaev and Temukuev conceived a semi-inverse method based on successive conformal mappings of the domain from the Zhukovskii plane onto the complex potential domain.

In the present paper, we present a new variant of the complex velocity-complex potential pair method for studying the seepage from asymmetric soil channels. The symmetric case, which herein is considered as a particular case, was already investigated in [14, 15]. We consider the conformal mapping $f(\zeta)$ of the unit half-disk onto the half-strip from the complex potential plane. We shall use Levi-Civitá's function $\omega(\zeta)$ in order to construct the conformal mapping $z(\zeta)$ of the unit half-disk onto the flow domain. The radii $[-1, 0)$ and $(0, 1]$ of the unit half-disk correspond through the conformal mapping $z(\zeta)$ to the free (phreatic) lines of the flow domain. On these radii, the imaginary part of $\omega(\zeta)$ vanishes by virtue of the conditions imposed on the free lines. According to Schwarz's principle of symmetry, we may extend the domain of definition of $\omega(\zeta)$ to the whole unit disk. The analytic function ω is afterwards expanded into a Taylor series. In comparison to the above mentioned inverse method, our method is more general; it is not restricted to special classes of contours of the channel. We have to give only the expression of the function $\omega(\zeta)$ (in fact we shall give the coefficients of the Taylor series of $\omega(\zeta)$ and some additional terms for the case of profiles with angular points) in order to construct the channel profile and solve the corresponding free boundary seepage problem. In fact, an inverse method has the maximum efficiency if it can be employed to solve the direct problem. Our method satisfies this requirement; by successive attempts, for every a priori given contour, we may endeavor to guess the corresponding coefficients of the Taylor series and so, to use the inverse method in order to solve the direct problem.

In Section 6, we present some calculated channel profiles and the corresponding phreatic lines, stream lines, equipotential lines and seepage losses. The integrals occurring in the corresponding integral representations are numerically calculated. In fact, we have conceived a Matlab code. The input data consist of the Taylor coefficients of (ζ). The output consists of the contour of the channel, seepage loss, phreatic lines, stream lines, and equipotential lines calculated in the nodes of a mesh from the flow domain.

2. The Free Boundary Value Problem

From the equation of continuity we have

$$\operatorname{div} \mathbf{v} = \mathbf{0} \tag{2.1}$$

and from Darcy's law for a homogeneous isotropic porous medium

$$\mathbf{v} = \operatorname{grad} \varphi, \quad \varphi = -k\left(\frac{p}{\rho g} + y\right) + \text{const}, \tag{2.2}$$

we deduce that

$$\Delta\varphi = 0. \tag{2.3}$$

Here φ is the potential of the velocity, $\mathbf{v} = (u, v)$ is the velocity, p is the pressure and ρ is the density of the fluid, k is the constant filtration coefficient (hydraulic conductivity), g is the gravity constant, and (x, y) are the Cartesian coordinates.

Let ψ (the stream function) be the harmonic conjugate of φ. For $z = x + iy$ the analytic function $f(z) = \varphi(x, y) + i\psi(x, y)$ is the complex potential and

$$\frac{df}{dz} = \frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} = w, \tag{2.4}$$

where

$$w = u - iv \tag{2.5}$$

is the complex velocity.

Now we are going to establish the boundary conditions. We consider a soil channel whose profile is a curve which has the following equation:

$$y = y(x), \quad x \in [-L, L], \qquad y(L) = y(-L) = 0. \tag{2.6}$$

Let $y = 0$ be the level of the water in the channel (Figure 1(a)). Assuming that there is no lining of the bottom AB of the channel, the pressure on AB is

$$p = p_{\text{atm}} - \rho g y \tag{2.7}$$

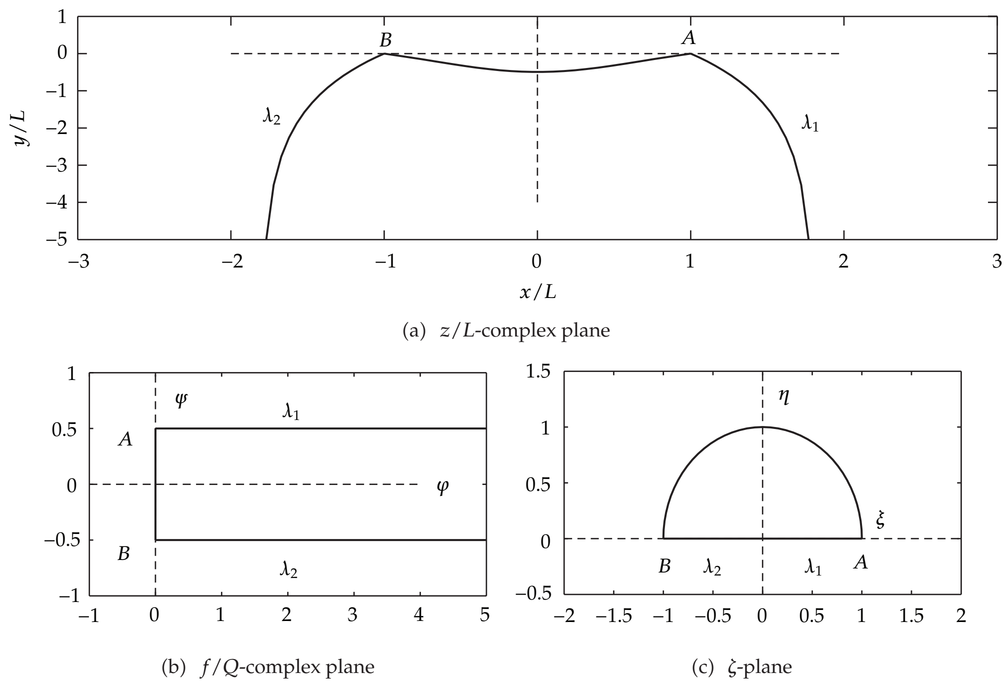


Figure 1: (a) Flow domain in the porous medium. (b) Half-strip in the plane of the complex potential. (c) Half disk.

(p_{atm} is the atmospheric pressure), whence we deduce that

$$\varphi_{AB} = 0. \tag{2.8}$$

On AB the tangential velocity $\partial\varphi/\partial s$ vanishes, hence we have

$$\arg w(z)_{AB} = \arg(u - iv) = -\arctan\frac{dy}{dx} + \frac{\pi}{2}. \tag{2.9}$$

The free boundaries (phreatic lines) λ_1 and λ_2 are streamlines, whence

$$\psi_{\lambda_1} = \frac{Q}{2}, \qquad \psi_{\lambda_2} = -\frac{Q}{2}, \tag{2.10}$$

where Q is the seepage loss. We study the seepage flow without capillarity, evaporation, or infiltration. Hence on the free phreatic lines the pressure has the constant value p_{atm}. We have therefore

$$\varphi + ky_{\lambda_1\cup\lambda_2} = 0. \tag{2.11}$$

Deriving along the tangential direction we get

$$\frac{\partial\varphi}{\partial s} + k\frac{\partial y}{\partial s}_{\lambda_1\cup\lambda_2} = 0 \Longrightarrow u^2 + v^2 + kv_{\lambda_1\cup\lambda_2} = 0. \tag{2.12}$$

The subscripts in (2.8)–(2.12) indicate that the relations we have in view are valid on the corresponding boundaries AB, λ_1, λ_2, $\lambda_1 \cup \lambda_2$.

3. Levi-Civitá's Function

From (2.8) and (2.10) it follows that the image of the domain of motion in the plane of the complex potential is a half-strip (Figure 1(b)). The function

$$f = -\frac{Q}{\pi}\ln\zeta + \frac{Qi}{2}, \quad \zeta = \xi + i\eta, \qquad \ln(-1+0i) = \pi i, \quad \ln(-1-0i) = -\pi i, \tag{3.1}$$

is the conformal mapping of the unit half-disk from the ζ-plane (Figure 1(c)) onto the half-strip from the f-plane. From (2.4) and (3.1) we deduce that f, z, and w may be regarded as functions of ζ, ($z(\zeta)$ is the conformal mapping of the unit half-disk from the ζ-plane onto the flow domain from the z-plane). The free lines λ_1 and λ_2 represent the image of the real diameter $\zeta = \xi + i\eta$, $\eta = 0$, $\xi \in [-1, 0) \cup (0, 1]$ by the conformal mapping $z(\zeta)$ and the contour of the channel is the imagine of the half-circle $\zeta = \exp(is)$, $s \in [0, \pi]$ by the same mapping.

We introduce the auxiliary function $w^*(\zeta)$ by means of the relation

$$w^*(\zeta) = w(\zeta) - \frac{ik}{2} = u - i\left(v + \frac{k}{2}\right). \tag{3.2}$$

From (2.12) and (3.2) it results

$$V^*(\xi) = |w(\xi)| = \frac{k}{2}, \quad \xi \in [-1, 0) \cup (0, 1]. \tag{3.3}$$

In the sequel we introduce Levi-Civitá's function $\omega(\zeta) = \sigma(\xi, \eta) + i\tau(\xi, \eta)$ by means of the relation

$$w^*(\zeta) = \frac{k}{2}\exp(-i\omega(\zeta)). \tag{3.4}$$

Since $w^* = V^*\exp(i\arg w^*)$, from (3.4) we deduce that

$$\sigma = -\arg w^*, \qquad \tau = \ln\frac{2V^*}{k}. \tag{3.5}$$

At infinity, under the channel, the direction of the velocity is assumed to be vertical. Hence

$$w(0) = ik, \qquad w^*(0) = \frac{ik}{2}, \qquad \omega(0) = -\frac{\pi}{2}. \tag{3.6}$$

From (3.3) and (3.5) it follows that

$$\tau(\xi, 0) = 0, \quad \xi \in [-1, 0) \cup (0, 1]. \tag{3.7}$$

According to Schwarz's principle of symmetry, the function $\omega(\zeta)$ can be extended to the whole unit disk by means of the relation

$$\omega(\zeta) = \overline{\omega\left(\bar{\zeta}\right)}. \tag{3.8}$$

From (3.6) and (3.8) it follows that the Taylor's series of the analytic complex function $\omega(\zeta)$ is

$$\omega(\zeta) = -\frac{\pi}{2} + \sum_{j=1}^{\infty} a_j \zeta^j, \quad a_j \in \mathbb{R},\ |\zeta| < 1. \tag{3.9}$$

Denoting $\sigma(s) = \sigma(\cos s, \sin s)$, and $\tau(s) = \tau(\cos s, \sin s)$, we deduce from (3.9) on the unit circle the Fourier expansions

$$\begin{aligned} \sigma(s) &= -\frac{\pi}{2} + \sum_{j=1}^{\infty} a_j \cos js, \quad s \in [0, 2\pi],\ a_j \in \mathbb{R}, \\ \tau(s) &= \sum_{j=1}^{\infty} a_j \sin js, \quad s \in [0, 2\pi],\ a_j \in \mathbb{R}. \end{aligned} \tag{3.10}$$

For the symmetric channels we have

$$\tau(\xi, \eta) = \tau(-\xi, \eta), \qquad \sigma(\xi, \eta) = -\pi - \sigma(-\xi, \eta), \tag{3.11}$$

and taking into account (3.9) and (3.11) we get the following Taylor and Fourier expansions

$$\begin{aligned} \omega(\zeta) &= -\frac{\pi}{2} + \sum_{j=1}^{\infty} a_{2j+1} \zeta^{2j+1}, \quad a_{2j+1} \in \mathbb{R},\ |\zeta| < 1, \\ \sigma(s) &= -\frac{\pi}{2} + \sum_{j=1}^{\infty} a_{2j+1} \cos(2j+1)s, \quad s \in [0, 2\pi],\ a_{2j+1} \in \mathbb{R}, \\ \tau(s) &= \sum_{j=1}^{\infty} a_{2j+1} \sin(2j+1)s, \quad s \in [0, 2\pi],\ a_{2j+1} \in \mathbb{R}. \end{aligned} \tag{3.12}$$

4. Channel Profiles with Angular Points

We assume that the points $z(\exp is_k)$, $0 < s_k < \pi$, $s = 1, 2, \ldots, n$, of the channel profile are angular points. In this case the function $\sigma(s)$ is discontinuous at s_k. We denote by $\mu_k\pi$ the jump of σ in s_k, that is,

$$\mu_k\pi = \lim_{s\searrow s_k}\sigma(s) - \lim_{s\nearrow s_k}\sigma(s). \tag{4.1}$$

Let $\omega_k(\zeta)$ be the analytic function in the unit disk such that

$$\operatorname{Re}\omega_k(\exp(is)) = \begin{cases} \alpha, & s \in [0, s_k) \cup (2\pi - s_k, 2\pi], \\ \alpha + \mu_k\pi, & s \in (s_k, 2\pi - s_k). \end{cases} \tag{4.2}$$

We observe that for $\zeta = \exp(is)$ we have

$$\begin{aligned} &\operatorname{Re}\left(-i\ln\frac{\exp(2i\pi - is_k) - \zeta}{\exp(is_k) - \zeta}\right) \\ &\quad = \arg\frac{\exp(2i\pi - is_k) - \zeta}{\exp(is_k) - \zeta} \\ &\quad = \begin{cases} \operatorname{meas}\triangleleft(\exp(2\pi i - is_k), \zeta, \exp(is_k)), & \zeta = \zeta_1 = \exp(is),\ s \in [0, s_k) \cup (2\pi - s_k, 2\pi], \\ \operatorname{meas}\triangleleft(\exp(2\pi i - is_k), \zeta, \exp(is_k)), & \zeta = \zeta_2 = \exp(is),\ s \in (s_k, 2\pi - s_k), \end{cases} \\ &\quad = \begin{cases} \pi - s_k, & s \in [0, s_k) \cup (2\pi - s_k, 2\pi], \\ 2\pi - s_k, & s \in (s_k, 2\pi - s_k), \end{cases} \end{aligned} \tag{4.3}$$

where meas means the measure of the corresponding angle.

Assigning to ω_k the expression $\omega_k(\zeta) = -ia\ln(\exp(2i\pi - is_k) - \zeta)/(\exp(is_k) - \zeta) + b$, and determining a and b from the boundary conditions (4.2), it follows that

$$\omega_k(\zeta) = -i\mu_k\ln\frac{\exp(2i\pi - is_k) - \zeta}{\exp(is_k) - \zeta} - \mu_k\pi + \mu_k s_k + \alpha. \tag{4.4}$$

Imposing $\omega_k(0) = 0$, we get the final expression

$$\omega_k(\zeta) = -i\mu_k\ln\frac{\exp(2i\pi - is_k) - \zeta}{\exp(is_k) - \zeta} - 2\mu_k\pi + 2\mu_k s_k. \tag{4.5}$$

The expression of Levi-Civitá's function will be

$$\omega(\zeta) = -\frac{\pi}{2} + \sum_{k=1}^{n}\omega_k(\zeta) + \sum_{j=1}^{\infty}a_j\zeta^j, \quad a_j \in \mathbb{R},\ |\zeta| < 1. \tag{4.6}$$

Separating in (4.6) the real and the imaginary parts, we have for $\zeta = \exp(is)$

$$\sigma(s) = -\frac{\pi}{2} + \sum_{k=1}^{n}\sigma_k(s) + \sum_{j=1}^{\infty}a_j \cos js, \quad s \in [0, 2\pi],\ a_j \in \mathbb{R}, \tag{4.7}$$

with

$$\sigma_k(s) = \begin{cases} -\mu\pi + \mu s_k, & s \in [0, s_k) \cup (2\pi - s_k, 2\pi], \\ \mu s_k, & s \in (s_k, 2\pi - s_k), \end{cases} \tag{4.8}$$

$$\tau(s) = \sum_{k=1}^{n}\tau_k(s) + \sum_{j=1}^{\infty}a_j \sin js, \quad s \in [0, 2\pi],\ a_j \in \mathbb{R}, \tag{4.9}$$

with

$$\tau_k(s) = -\mu \ln\left|\frac{\sin(s/2 + s_k/2)}{\sin(s/2 - s_k/2)}\right|. \tag{4.10}$$

5. Integral Representations and Conformal Mappings

From the relation

$$\frac{df}{dz} = w(\zeta) = w^*(\zeta) + \frac{ik}{2}, \tag{5.1}$$

it follows that

$$dz = \frac{df}{w^*(\zeta) + ik/2}. \tag{5.2}$$

Taking into account (3.1) and (3.4) we get

$$dz = \frac{-2Q}{k\pi(i + \exp(-i\omega(\zeta)))}\frac{d\zeta}{\zeta}, \tag{5.3}$$

whence one obtains the following integral representation of the conformal mapping $z(\zeta)$:

$$z(\zeta) = z(\zeta_0) - \int_{\zeta_0}^{\zeta}\frac{2Q}{k\pi(i + \exp(i + \exp(-i\omega(\zeta))))}\frac{d\zeta}{\zeta}. \tag{5.4}$$

Considering $\zeta = \exp(is)$ in (5.3) it results in

$$dx + idy = \frac{-2Qids}{k\pi(i + \exp(-i\sigma(s) + \tau(s)))} \tag{5.5}$$

on the profile of the channel. Separating in (5.5) the real parts and the imaginary ones, we obtain

$$\begin{aligned} \frac{dx}{ds} &= -\frac{2Q}{k\pi}\frac{1-\sin\sigma(s)\exp(\tau(s))}{1-2\sin\sigma(s)\exp(\tau(s))+\exp(2\tau(s))}, \\ \frac{dy}{ds} &= -\frac{2Q}{k\pi}\frac{\cos\sigma(s)\exp(\tau(s))}{1-2\sin\sigma(s)\exp(\tau(s))+\exp(2\tau(s))}, \end{aligned} \tag{5.6}$$

whence it follows

$$x(s) = L - \frac{2Q}{k\pi}\int_0^s \frac{1-\sin\sigma(s)\exp(\tau(s))}{1-2\sin\sigma(s)\exp(\tau(s))+\exp(2\tau(s))}ds, \tag{5.7}$$

$$y(s) = -\frac{2Q}{k\pi}\int_0^s \frac{\cos\sigma(s)\exp(\tau(s))}{1-2\sin\sigma(s)\exp(\tau(s))+\exp(2\tau(s))}ds. \tag{5.8}$$

6. The Inverse Method: Numerical and Analytical Results

6.1. Smooth Profiles

Assigning in (3.9) various values to the Taylor coefficients a_j and using the formulas (3.2) and (3.4) and the integral representations (5.4)–(5.8) we calculate numerically the contour of the channel, the phreatic lines, the seepage loss, the velocity field, the stream lines, and the equipotential lines.

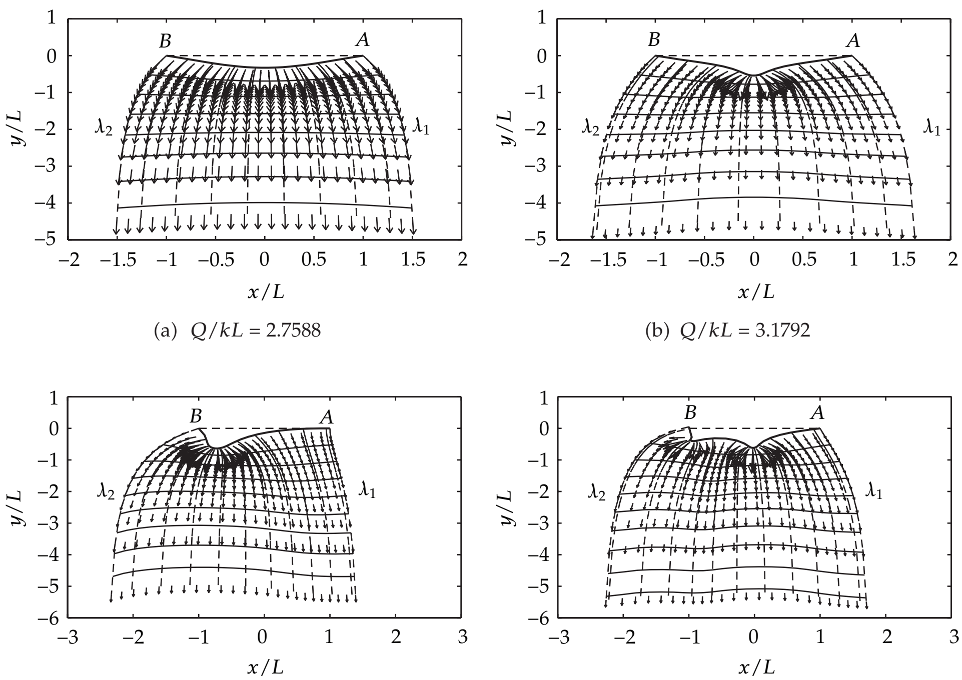
The integrals are numerically computed (we utilized the trapezium formula). The conformal mapping $z(\zeta)$ is:

$$z(\zeta) = z(\zeta_0) - \frac{2Q}{k\pi i}\int_{\zeta_0}^{\zeta} \frac{1}{1+\exp\left(-i\sum_{j=1}^{\infty} a_j\zeta^j\right)}\frac{d\zeta}{\zeta}. \tag{6.1}$$

If $\zeta = r\exp(is)$, $0 < r < 1$, we choose $\zeta_0 = \exp(is)$, $z(\zeta_0) = x(s) + iy(s)$. The path of integration is the segment $[\zeta_0, \zeta]$. For the numerical computations we used in the ζ-complex plane the mesh points $\{\zeta_{jl} = (j/n)\exp(l\pi/m),\ j = 1,2,\ldots,n,\ l = 0,1,\ldots,m\}$. In the flow domain we considered the mesh points $z(\zeta_{jl})$ obtained by means of the conformal mapping (6.1).

The parametric equation of the phreatic lines λ_1 and λ_2 are

$$\begin{aligned} z(\xi) &= L - \frac{2Q}{k\pi i}\int_1^{\xi} \frac{1}{1+\exp\left(-i\sum_{j=1}^{\infty} a_j\xi^j\right)}\frac{d\xi}{\xi}, \quad \xi \in (0,1], \\ z(\xi) &= -L - \frac{2Q}{k\pi i}\int_{-1}^{\xi} \frac{1}{1+\exp\left(-i\sum_{j=1}^{\infty} a_j\xi^j\right)}\frac{d\xi}{\xi}, \quad \xi \in [-1,0). \end{aligned} \tag{6.2}$$



(a) $Q/kL = 2.7588$

(b) $Q/kL = 3.1792$

(c) $Q/kL = 2.8640$

(d) $Q/kL = 3.3038$

Figure 2: Seepage from channels with smooth contours.

From (3.2), (3.4), and (6.1) we obtain the complex velocity in the points $z(\zeta)$ as follows:

$$w(z(\zeta)) = w(\zeta) = \frac{ik}{2}\left[1 + \exp\left(-i\sum_{j=1}^{\infty} a_j \zeta^j\right)\right]. \tag{6.3}$$

Imposing $x(\pi) = -L$ in (5.7), we get the seepage loss

$$Q = \frac{kL\pi}{2\int_0^{\pi/2}\left(1 - \sin\sigma(s)\exp(\tau(s))\right)/\left(1 - 2\sin\sigma(s)\exp(\tau(s)) + \exp(2\tau(s))\right)\mathrm{d}s}. \tag{6.4}$$

In Figure 2, we present the seepage from channels having various smooth profiles. We use dimensionless variables ($x/L, y/L, w/k$, and f/kL instead of x, y, w, and f). We employ solid lines for the contour of the channel and the equipotential lines, dashed lines for the stream lines (including the phreatic lines λ_1 and λ_2), and arrows for the velocity field. We also indicate the numerical values of the dimensionless seepage loss $Q^* = Q/kL$.

We considered the following expressions of ω: $\omega(\zeta) = -\pi/2 + (\pi/4)\zeta$ in Figure 2(a), $\omega(\zeta) = -\pi/2 + (\pi/3)\zeta - (\pi/6)\zeta^3 + (\pi/12)\zeta^5$ in Figure 2(b), $\omega(\zeta) = -\pi/2 + (\pi/3)\zeta - (\pi/3)\zeta^2$ in Figure 2(c), and $\omega(\zeta) = -\pi/2 + (\pi/3)\zeta - (\pi/3)\zeta^4 + (\pi/6)\zeta^5$ in Figure 2(d).

6.2. Profiles with Angular Points

In this case, we employ for $\omega(\zeta)$, $\sigma(s)$, and $\tau(s)$ the formulas (4.6)–(4.8). The conformal mapping $z(\zeta)$ is

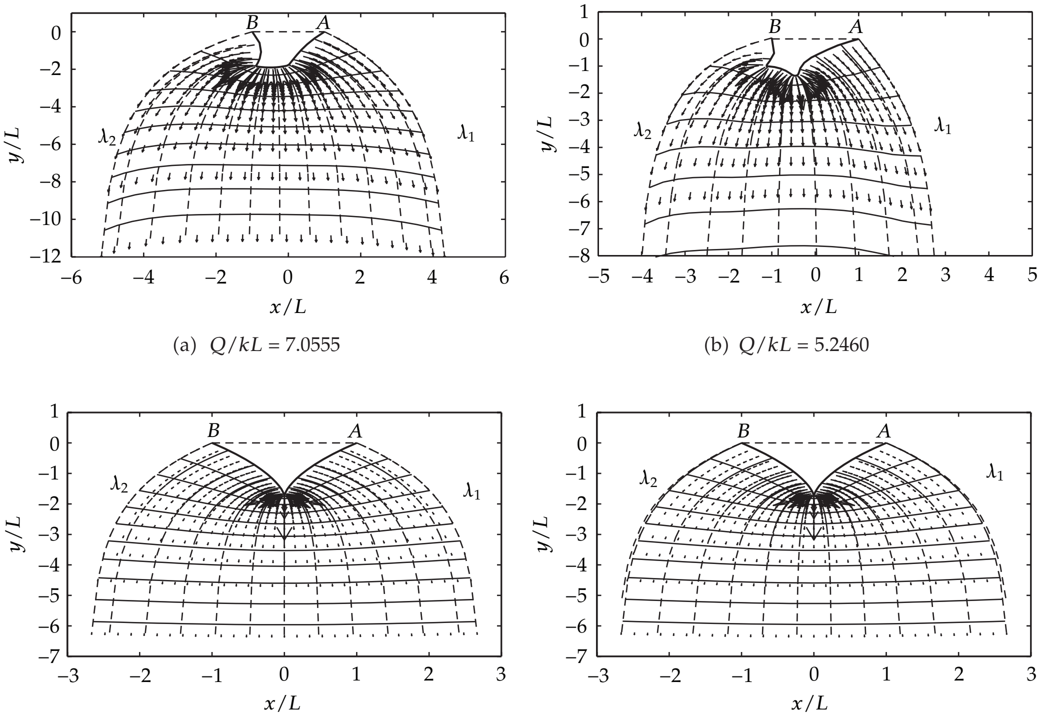
$$z(\zeta) = z(\zeta_0) - \frac{2Q}{k\pi i} \times \int_{\zeta_0}^{\zeta} \frac{1}{1+\exp\left(-i\sum_{j=1}^{\infty} a_j\zeta^j\right)\prod_{k=1}^{n}\left[\exp(2i\mu_k(\pi-s_k))\left((\exp(is_k)-\zeta)/(\exp(-is_k)-\zeta)\right)^{\mu_k}\right]}\frac{d\zeta}{\zeta}. \tag{6.5}$$

The parametric equation of the phreatic lines λ_1 and λ_2 are

$$\begin{aligned} z(\xi) &= L - \frac{2Q}{k\pi i} \times \int_{1}^{\xi} \frac{1}{1+\exp\left(-i\sum_{j=1}^{\infty} a_j\xi^j\right)\prod_{k=1}^{n}\left[\exp(2i\mu_k(\pi-s_k))\left((\exp(is_k)-\xi)/(\exp(-is_k)-\xi)\right)^{\mu_k}\right]}\frac{d\xi}{\xi}, \quad \xi \in (0,1], \\ z(\xi) &= -L - \frac{2Q}{k\pi i} \times \int_{-1}^{\xi} \frac{1}{1+\exp\left(-i\sum_{j=1}^{\infty} a_j\xi^j\right)\prod_{k=1}^{n}\left[\exp(2i\mu_k(\pi-s_k))\left((\exp(is_k)-\xi)/(\exp(-is_k)-\xi)\right)^{\mu_k}\right]}\frac{d\xi}{\xi}, \quad \xi \in [-1,0). \end{aligned} \tag{6.6}$$

The complex velocity at the points $z(\zeta)$ is as follows:

$$w(\zeta) = \frac{ik}{2}\left[1+\exp\left(-i\sum_{j=1}^{\infty}a_j\zeta^j\right)\prod_{k=1}^{n}\left[\exp(2i\mu_k(\pi-s_k))\left(\frac{\exp(is_k)-\zeta}{\exp(-is_k)-\zeta}\right)^{\mu_k}\right]\right]. \tag{6.7}$$



(a) $Q/kL = 7.0555$

(b) $Q/kL = 5.2460$

(c) $Q/kL = 5.4951$

(d) $Q/kL = 5.5039$

Figure 3: Seepage from channels with angular points.

In Figure 3, we present the seepage from various channels whose profiles have angular points. In Figures 3(a) and 3(b) we considered, respectively,

$$\omega(\zeta) = \frac{\pi}{12} + \frac{i}{4}\ln\frac{\zeta - \exp(-i(\pi/3))}{\zeta - \exp(i(\pi/3))} + \frac{i}{2}\ln\frac{\zeta - \exp(-i(3\pi/4))}{\zeta - \exp(i(3\pi/4))} + \frac{\pi}{4}\zeta, \tag{6.8}$$

$$\omega(\zeta) = \frac{5\pi}{36} + \frac{i}{4}\ln\frac{\zeta - \exp(-i(\pi/6))}{\zeta - \exp(i(\pi/6))} + \frac{2i}{3}\ln\frac{\zeta - \exp(-i(5\pi/6))}{\zeta - \exp(i(5\pi/6))} + \frac{\pi}{4}\zeta - \frac{\pi}{4}\zeta^3 - \frac{\pi}{8}\zeta^4 + \frac{\pi}{12}\zeta^5, \tag{6.9}$$

and we calculated numerically for each case the contour of the channel, the equipotential lines, the stream lines (including the phreatic lines), the velocity field, and seepage loss.

In Figures 3(c) and 3(d) we considered Levi-Civitá's function $\omega(\zeta) = i\ln(\zeta+i)/(\zeta-i) + \pi/2$.

In this case, we can perform some analytical calculations, and we get the complex velocity

$$w(\zeta) = \frac{k}{\zeta - i}. \tag{6.10}$$

We also may obtain the conformal mapping of the unit half-disk onto the flow domain in the porous medium

$$z(\zeta) = -\frac{2L}{\pi - 2}\left(\zeta - i \ln \zeta - \frac{\pi}{2}\right), \tag{6.11}$$

as well as the parametric equations of the free lines

$$z(\xi) = -\frac{2L}{\pi - 2}\left(\xi - i \ln \xi - \frac{\pi}{2}\right), \quad \xi \in [-1, 0) \cup [0, 1). \tag{6.12}$$

On the contour of the channel we have

$$\sigma(s) = \begin{cases} 0, & s \in \left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{3\pi}{2}, 2\pi\right], \\ \pi, & s \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right), \end{cases} \tag{6.13}$$

$$\tau(s) = \ln \tan\left(\frac{s}{2} + \frac{\pi}{4}\right), \tag{6.14}$$

$$x(s) = L - \frac{2L}{\pi - 2}(s + \cos s - 1), \tag{6.15}$$

$$y(s) = -\frac{2L}{\pi - 2} \sin s. \tag{6.16}$$

The seepage loss is

$$Q = \frac{2kL\pi}{\pi - 2}. \tag{6.17}$$

Equations (6.15) and (6.16) are the parametric equations of an arc of cycloid with an angular point. In Figure 3(c) we present the contour of the channel, the equipotential lines, the stream lines (including the phreatic lines), the velocity field, and seepage loss calculated numerically and in Figure 3(d) we present the same things calculated analytically. We notice a very good agreement.

From the mathematical point of view, we obtained in this paper the following result: to any sequence $(-\pi/2, a_1, a_2, a_3, \ldots)$ of real coefficients of the Taylor expansion of the function $\omega(\zeta)$ there corresponds a channel contour for which one may calculate the phreatic lines, the velocity field, and the seepage discharge. We have to mention that for some values of a_k, $k = 1, 2, \ldots$. one may obtain results which are unacceptable from a physical point of view: self-intersecting channel profiles, self-intersecting phreatic lines, and unreasonable values of the coordinates of the velocity. For obtaining acceptable results we have to impose some restrictions on a_k, $k = 1, 2, \ldots$. These coefficients also represent the Fourier coefficients of the function $\sigma(s)$ which satisfies the inequality $-\pi \le \sigma(s) \le \pi$. We have therefore

$$|a_k| = \frac{1}{\pi}\left|\int_{-\pi}^{\pi} \sigma(s) \cos ks \, ds\right| \le \int_{-\pi}^{\pi} |\cos ks| ds = 4, \quad k = 1, 2, \ldots. \tag{6.18}$$

These restrictions are not sufficient to ensure the physical correctness of the results. In order to decide if the results are acceptable or not we have to examine the graphic representations obtained by the aid of a numeric code which calculates the values of the integral representations of the functions we have in view.

Acknowledgment

The paper was supported by the ECOSCID project of the University of Bucharest.

References

[1] A. R. Kacimov, "Discussion of 'Design of minimum seepage loss nonpolygonal canal sections' by Prabhata K. Swamee and Deepak Kashyap," *Journal of Irrigation and Drainage Engineering*, vol. 129, no. 1, pp. 68–70, 2003.
[2] M. E. Harr, *Groundwater and Seepage*, McGraw-Hill, New York, NY, USA, 1962.
[3] P. Ya. Polubarinova-Kochina, *Theory of Ground Water Movement*, Princeton University Press, Princeton, NJ, USA, 1962.
[4] K. N. Anakhaev, "Free percolation and seepage flows from watercourses," *Fluid Dynamics*, vol. 39, no. 5, pp. 756–761, 2004.
[5] K. N. Anakhaev, "Calculation of free seepage from watercourses with curvilinear profiles," *Water Resources*, vol. 34, no. 3, pp. 295–300, 2007.
[6] V. V. Vedernikov, "Seepage theory and its application in the field of irrigation and drainage," *Gosstroiizdat*, vol. 248, 1939 (Russian).
[7] N. N. Pavlovskii, "Groundwater Flow," *Akademii Nauk SSSR*, vol. 36, 1956 (Russian).
[8] B. R. Chahar, "Analytical solution to seepage problem from a soil channel with a curvilinear bottom," *Water Resources Research*, vol. 42, no. 1, Article ID W01403, 2006.
[9] B. R. Chahar, "Optimal design of a special class of curvilinear bottomed channel section," *Journal of Hydraulic Engineering*, vol. 133, no. 5, pp. 571–576, 2007.
[10] B. R. Chahar, "Seepage from a special class of a curved channel with drainage layer at shallow depth," *Water Resources Research*, vol. 45, no. 9, Article ID W09423, 2009.
[11] A. R. Kacimov and Yu. V. Obnosov, "Analytical determination of seeping soil slopes of a constant exit gradient," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 82, no. 6, pp. 363–376, 2002.
[12] N. B. Ilyinsky and A. R. Kasimov, "An inverse problem of filtration from a channel in the presence of backwater," *Trudy Seminara po Kraevym Zadacham*, no. 20, pp. 104–115, 1983 (Russian).
[13] K. N. Anakhaev and K. M. Temukuev, "Filtration from water-currents of an asymmetrical structure," *Matematicheskoe Modelirovanie*, vol. 21, no. 2, pp. 73–78, 2009.
[14] A. Carabineanu, "A free boundary problem in porous media," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 37, no. 7, pp. 563–570, 1992.
[15] A. Carabineanu, "Free boundary seepage from open earthen channels," *Annals of the University of Bucharest*, vol. 2, no. 60, pp. 3–17, 2011.