Research Article

Existence and Uniqueness of Generalized Solutions to a Telegraph Equation with an Integral Boundary Condition via Galerkin's Method

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We consider a telegraph equation with nonlocal boundary conditions, and using the application of Galerkin's method we established the existence and uniqueness of a generalized solution.

1. Introduction

The initial work devoted to the second-order partial differential equations with nonlocal integral conditions goes back to Cannon [1]. Subsequent investigations include [1–10] and many other references therein.

Nonlocal problems for some classes of partial differential equations have attracted much interest in the last few years, and several papers have been devoted to this subject. The problem of existence and uniqueness of solutions is of growing interest, as well as methods of obtaining explicit or approximate solutions for this kind of problems. The main reason of such a wide interest in this type of problems is that the non-local conditions occur in various equations of mathematical physics and in mathematical biology.

Nonlocal conditions come up when values of the function on the boundary are connected to values inside the domain. There are various types of non-local integral conditions for hyperbolic, parabolic, or elliptic equations; we point out an important non-local

problem (see [11, 12]), studied for a parabolic equation with the following non local conditions.

$$u(0,t) = \int_0^1 \alpha_1(x)u(x,t)dx + d_1(t),$$
$$u(1,t) = \int_0^1 \alpha_2(x)u(x,t)dx + d_2(t),$$

(1.1)

connected with quasistationary thermoelasticity theory. The non-local boundary conditions of type

$$\alpha(t) = \int_0^1 u(x,t)dx, \qquad u(1,t) = 0,$$

(1.2)

arise in modeling the technology of integral circuits, studied in [13].

Problems with nonlocal integral conditions for multidimensional hyperbolic equations have been intensively studied in recent years. However, the nonlocal integral conditions for equations were in the form of second kind as

$$\left.\frac{\partial u}{\partial \eta}\right|_{(x,t)\in\partial\Omega\times I} + \int_0^T \int_\Omega K(x,\xi,\tau)u(\xi,\tau)d\xi d\tau = 0, \quad \forall x \in \partial\Omega,$$

(1.3)

or

$$u(x,t)|_{(x,t)\in\partial\Omega\times I} + \int_\Omega K(x,\xi)u(\xi,t)d\xi = 0, \quad \forall x \in \partial\Omega,$$

(1.4)

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary, $\partial u/\partial \eta$ is the inward normal vector to $\partial\Omega$; see [14, 15]. Non-local integral conditions of first kind as

$$\int_\Omega K(x,\xi)u(\xi,t)d\xi = 0, \quad \forall x \in \partial\Omega,$$

(1.5)

are studied in [15]. Under some assumptions on the kernel K, existence and uniqueness of the generalized solution are established.

The present work deals with the application of the Galerkin method to determine a function $u = u(x,t)$, $x \in \Omega \in \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$, $t \in I = (0,T)$, which satisfies, in weak sense, the telegraph equation

$$lu = \frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu - c\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad (x,t) \in Q = \Omega \times I,$$

(1.6)

subject to the initial conditions

$$u(x,0) = \varphi(x),$$
$$\frac{\partial u}{\partial t}(x,0) = \psi(x) \tag{1.7}$$

and the integral condition

$$\left.\frac{\partial u}{\partial \eta}\right|_{(x,t)\in\partial\Omega\times I} + \int_{\Omega} K(x,\xi)u(\xi,t)d\xi = 0, \quad \forall x \in \partial\Omega, \tag{1.8}$$

where $\partial u / \partial \eta$ is the inward normal vector to $\partial\Omega$. This equation arises in the analysis of propagation of electrical signals in a cable transmission line. Both the current and voltage satisfy an equation of this form. This equation arises also in the propagation of pressure waves in the study of pulsate blood flow in arteries.

Cannon et al. [5] have developed and analyzed finite-difference and finite-element Galerkin's methods for the solution of one-dimensional heat equation with the Neumann and integral conditions. Cannon and van der Hoek [4] presented numerical schemes based on the finite difference method. In [16], the authors have applied the Galerkin method to a parabolic problem with an integral boundary condition and established the existence, uniqueness, and continuous dependence upon the data of a weak solution. In [17–21], the authors have used the method of semidiscretization in time and have established the existence and uniqueness of a weak solution. In [22], the authors have discussed some existence uniqueness results for a neutral functional differential equation with a nonlocal initial condition via the Galerkin approximation.

The paper is organized as follows. In Section 2, we specify notations, state some inequalities and make the sense of the desired solution precise. In Section 3, we establish the uniqueness of the solution. Finally, Section 4, is devoted to the construction of the approximate solution and its existence via the Galerkin method.

2. Notation and Definition

Let $L^2(Q)$ be the usual space of Lebesgue square integrable real functions on Q whose inner product and norm will be denoted by $(\,,)$ and $\|\cdot\|$, respectively.

$W^{1,2}(Q)$ is the Sobolev space consisting of functions such that all derivatives lower than one belong to $L^2(Q)$ equipped with the norm

$$\|u\|^2_{W^{1,2}(Q)} = \|u\|^2 + \|\nabla u\|^2 + \|u_t\|^2. \tag{2.1}$$

We define the space $W^{1,2}_T(Q) = \{v(x,t) \in W^{1,2}(Q), v(x,T) = 0\}$.

2.1. Some Useful Inequalities: Gronwall Inequalities [23]

Let $h(t)$ and $y(t)$ be two integrable nonnegative functions on the interval I with $h(t)$ nondecreasing. If

$$y(t) \leq h(t) + c \int_0^t y(s)ds, \quad \forall t \in I, \tag{2.2}$$

where c is a positive constant, then $y(t) \leq h(t)e^{ct}, \forall t \in I$.

Cauchy Schwarz inequality:

$$\left(\int f(t)g(t)dt \right)^2 \leq \left(\int |f(t)|^2 dt \right) \left(\int |g(t)|^2 dt \right). \tag{2.3}$$

ε-Cauchy inequality:

$$|\alpha\beta| \leq \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2, \quad \forall \alpha, \beta \in \mathbb{R}, \ \forall \varepsilon \in \mathbb{R}_+^*. \tag{2.4}$$

Trace inequality:

$$\int_{\partial\Omega} |v|^2 ds dt \leq \int_{\Omega} \left(\varepsilon |\nabla v|^2 + c(\varepsilon)|v|^2 \right) dx dt, \tag{2.5}$$

where $c(\varepsilon)$ is a positive constant that depends only on ε and on the domain Ω.

2.2. Definition of Generalized Solution

We define the generalized solution of the problem (1.6)–(1.8) as a weak solution that satisfies for all $v \in W_T^{1,2}(Q)$

$$(lu, v) = (f, v). \tag{2.6}$$

Integrating by parts, we get

$$\int_Q u_{tt}(x,t)v(x,t)dxdt = -\int_\Omega \psi(x)v(x,0)dx - \int_Q u_t(x,t)v_t(x,t)dxdt,$$

$$\int_Q u_t(x,t)v(x,t)dxdt = -\int_\Omega \varphi(x)v(x,0)dx - \int_Q u(x,t)v_t(x,t)dxdt,$$

$$\int_Q u_{xx}(x,t)v(x,t)dxdt = -\int_0^T \int_{\partial\Omega} v\left(\int_\Omega K(x,\xi)u(\xi,t)d\xi\right)dsdt$$

$$- \int_Q \nabla u(x,t)\nabla v(x,t)dxdt. \tag{2.7}$$

Regrouping the identities (2.7), then (2.6) becomes

$$\int_Q (c\nabla u\nabla v - u_t v_t - auv_t + buv)dxdt + c\int_0^T \int_{\partial\Omega} v\left(\int_\Omega K(x,\xi)u(\xi,t)d\xi\right)dsdt$$

$$\tag{2.8}$$

$$= \int_Q fvdxdt + \int_\Omega \psi(x)v(x,0)dx + a\int_\Omega \varphi(x)v(x,0)dx.$$

Definition 2.1. By a generalized solution of problem (1.6)–(1.8) one means a function $u \in W^{1,2}(Q)$ such that the identity (2.8) holds for all function $v \in W_T^{1,2}(Q)$.

3. Uniqueness of the Generalized Solution

Now we will show that the generalized solution of problem (1.6)–(1.8), if it exists, is unique.

Theorem 3.1. *Assume that $\varphi \in W^{1,2}(\Omega)$, $\psi \in L_2(\Omega)$, $f \in L_2(Q)$, $K(x,y) \in C(\Omega \times \Omega)$, and*

$$\max_{\overline{Q}}|K(x,y)| \le k_1, \tag{3.1}$$

then the generalized solution of problem (1.6)–(1.8), if it exists, is unique.

Proof. Suppose that there exist two different generalized solutions u_1 and u_2 for the problem (1.6)–(1.8); then obviously their difference $u = u_1 - u_2$ is a generalized solution of the problem (1.6)–(1.8) with homogeneous equation and homogeneous initial and non local conditions, that is, $f = \varphi = \psi = 0$. We will prove that $u = 0$ in Q. Let $v \in W_T^{1,2}(Q)$, and denote $Q^\tau = \{(x,t); 0 < x < 1, 0 < t \le \tau \le T\}$. Consider the function

$$v(x,t) = \begin{cases} \int_t^\tau u(x,\theta)d\theta, & 0 \le t \le \tau, \\ 0, & \tau \le t \le T. \end{cases} \tag{3.2}$$

The identity (2.8) becomes

$$\int_Q (c\nabla u \nabla v - u_t v_t - auv_t + buv)\,dxdt + c\int_0^T \int_{\partial\Omega} v\left(\int_\Omega K(x,\xi)u(\xi,t)\,d\xi\right)dsdt = 0. \qquad (3.3)$$

Substituting v into (3.3), integrating by parts, and then using the fact that $v_t(x,t) = -u(x,t)$, it follows that

$$\int_\Omega \left(c(\nabla v(x,0))^2 + u^2(x,\tau) + bv^2(x,0)\right)dx = -2c\int_0^\tau \int_{\partial\Omega} v\left(\int_\Omega K(s,\xi)u(\xi,t)\,d\xi\right)dsdt$$
$$+ 2a\int_0^\tau \int_\Omega u^2(x,t)\,dxdt. \qquad (3.4)$$

Using the assumption on the function K, we get

$$\int_\Omega \left((\nabla v(x,0))^2 + u^2(x,\tau) + v^2(x,0)\right)dx$$
$$\leq C_1\left(\int_0^\tau \int_{\partial\Omega} |v|\left(\int_\Omega |u(\xi,t)|\,dx\right)dsdt + \int_0^\tau \int_\Omega u^2(x,t)\,dxdt\right), \qquad (3.5)$$

where $C_1 = \max(2ck_1, 2a)/\min(c,1,b)$. Now, applying Cauchy Schwarz inequality to the first term in the right-hand side of (3.5), then ε-Cauchy inequality with $\varepsilon = 1$, yields

$$\int_\Omega \left((\nabla v(x,0))^2 + u^2(x,\tau) + v^2(x,0)\right)dx \leq C_1\left(\int_0^\tau \int_{\partial\Omega} |v|^2\,dsdt + 2\int_0^\tau \int_\Omega u^2(x,t)\,dxdt\right). \qquad (3.6)$$

Using the trace inequality, we obtain

$$\int_\Omega \left((\nabla v(x,0))^2 + u^2(x,\tau) + v^2(x,0)\right)dx$$
$$\leq C_1\left(\int_0^\tau \int_\Omega \left(\epsilon(\nabla v)^2 + c(\varepsilon)v^2\right)dxdt + 2\int_0^\tau \int_\Omega u^2(x,t)\,dxdt\right). \qquad (3.7)$$

Considering the function

$$w(x,t) = \int_0^t u(x,\theta)\,d\theta, \qquad (3.8)$$

it is easy to see that $v(x,t) = w(x,\tau) - w(x,t)$, $\nabla v(x,0) = \nabla w(x,\tau)$, and $(\nabla v(x,t))^2 \leq 2(\nabla w(x,\tau))^2 + 2(\nabla w(x,t))^2$. Consequently, substituting w in (3.7), we get

$$\int_\Omega \left((\nabla w(x,\tau))^2 + w^2(x,\tau) + u^2(x,\tau)\right) dx \leq 2\tau C_1 \int_\Omega \left(\epsilon(\nabla w(x,\tau))^2 + c(\epsilon)w^2(x,\tau)\right) dx$$
$$+ 2C_1 \int_0^\tau \int_\Omega \left(\epsilon(\nabla w)^2 + c(\epsilon)w^2 + u^2\right) dx dt. \tag{3.9}$$

Seting

$$C_2 = 2C_1 \max(\epsilon, c(\epsilon)), \quad C_3 = 2C_1 \max(\epsilon, c(\epsilon), 1), \tag{3.10}$$

it follows that

$$\int_\Omega \left((\nabla w(x,\tau))^2 + w^2(x,\tau) + u^2(x,\tau)\right) dx \leq \tau C_2 \int_\Omega \left((\nabla w(x,\tau))^2 + w^2(x,\tau)\right) dx$$
$$+ C_3 \int_0^\tau \int_\Omega \left((\nabla w)^2 + w^2 + u^2\right) dx dt. \tag{3.11}$$

Since τ is arbitrary chosen, let $1 - \tau C_2 > 0$; then (3.11) becomes

$$(1 - \tau C_2) \int_\Omega \left((\nabla w(x,\tau))^2 + w^2(x,\tau) + u^2(x,\tau)\right) dx \leq C_3 \int_0^\tau \int_\Omega \left((\nabla w)^2 + w^2 + u^2\right) dx dt. \tag{3.12}$$

Applying Gronwall Lemma, we get

$$\int_\Omega \left((\nabla w(x,\tau))^2 + w^2(x,\tau) + u^2(x,\tau)\right) dx \leq 0, \quad \forall \tau \in \left]0, \frac{1}{C_2}\right[. \tag{3.13}$$

Consequently, we obtain $u(x,\tau) = 0$, for all $x \in \Omega$ and $\tau \in]0, 1/C_2[$.

If $T \leq 1/C_2$, then $u = 0$ in Q. In the case where $T \geq 1/C_3$, we see that $]0, T[\subset \cup_{n=1}^{n=n_0}](n-1)/C_2, n/C_2[$, where $n_0 = [C_2 T] + 1, [C_2 T]$ is the entire part of $C_2 T$; then repeating the preceding reasoning for $\tau \in](n-1)/C_2, n/C_2[$, we get $u(x,\tau) = 0$, for all $\tau \in](n-1)/C_2, n/C_2[$, and then $u(x,t) = 0$ in Q. Thus, the uniqueness is proved. $\square$

Remark 3.2. It should be noted that we can prove the uniqueness theorem by using a priori estimates, so we establish that the generalized solution, if it exists, satisfies the inequality

$$\|u\|_{W^{1,2}(Q)}^2 \leq C\left[\|f\|^2 + \|\psi\|_{L_2(\Omega)}^2 + \|\varphi\|_{W^{1,2}(\Omega)}^2\right]. \tag{3.14}$$

4. Existence of Generalized Solution

Theorem 4.1. *Assume that the assumptions of Theorem 4.1 hold; then the non-local problem* (1.6)–(1.8) *has a unique solution* $u \in W^{1,2}(Q)$.

Proof. In order to prove the existence of the generalized solution we apply Galerkin's method. Let $\{w_k(x)\}$ be a fundamental system in $W^{1,2}(\Omega)$, such that $(w_k, w_i)_{L_2(Q)} = \delta_{k,i}$. Now we will try to find an approximate solution of the problem (1.6)–(1.8) in the form

$$u^{(n)} = \sum_{k=1}^{n} \alpha_k(t) w_k(x). \tag{4.1}$$

The approximates of the functions $\varphi(x)$ and $\psi(x)$ are denoted, respectively, by

$$\varphi^{(n)}(x) = \sum_{k=1}^{n} \varphi_k w_k(x), \qquad \psi^{(n)}(x) = \sum_{\Sigma k=1}^{n} \psi_k w_k(x),$$

$$\alpha_k(0) = \varphi_k, \quad \alpha'_k(0) = \psi_k. \tag{4.2}$$

Substituting the approximation solution in (1.6), multiplying both sides by $w_i(x)$, and then integrating according to x on Ω, we get

$$\int_{\Omega} \left(c \nabla u^{(n)} \nabla w_i + u_{tt}^{(n)} w_i + a u_t^{(n)} w_i + b u^{(n)} w_i \right) dx$$

$$+ c \int_{\partial\Omega} w_i \left(\int_{\Omega} K(x,\xi) u^{(n)}(\xi,t) d\xi \right) ds = \int_{\Omega} f w_i dx. \tag{4.3}$$

Substituting (4.1) in (4.3), we obtain

$$\int_{\Omega} \sum_{k=1}^{n} \left(c \alpha_k \nabla w_k \nabla w_i + \alpha''_k w_k w_i + a \alpha'_k w_k w_i + b \alpha_k w_k w_i \right) dx$$

$$+ c \int_{\partial\Omega} \sum_{k=1}^{n} w_i \left(\alpha_k \int_{\Omega} K(x,\xi) w_k(\xi) d\xi \right) ds = \int_{\Omega} f w_i dx. \tag{4.4}$$

This implies that

$$\sum_{k=1}^{n} \left[\alpha''_k (w_k, w_i)_{L_2(\Omega)} + \alpha'_k (a w_k, w_i)_{L_2(\Omega)} + \alpha_k \left((c \nabla w_k, \nabla w_i)_{L_2(\Omega)} + (b w_k, w_i)_{L_2(\Omega)} \right) \right]$$

$$+ \sum_{k=1}^{n} \alpha_k \int_{\partial\Omega} c w_i \left(\int_{\Omega} K(x,\xi) w_k(\xi) d\xi \right) ds = \int_{\Omega} f w_i dx. \tag{4.5}$$

Denote

$$\beta_{k,i} = (c\nabla w_k, \nabla w_i)_{L_2(\Omega)} + (bw_k, w_i)_{L_2(\Omega)} + \int_{\partial\Omega} cw_i \left(\int_\Omega K(x,\xi)w_k(\xi)d\xi \right) ds,$$

$$f_i = \int_\Omega f w_i dx.$$

(4.6)

Then (4.5) becomes

$$\sum_{k=1}^n \left[\alpha_k'' \delta_{k,i} + a\alpha_k' \delta_{k,i} + \alpha_k \beta_{k,i} \right] = f_i. \tag{4.7}$$

We obtain a system of differential equations of second order according to the variable t with smooth coefficients and the initial conditions

$$\alpha_k(0) = \varphi_k, \qquad \alpha_k'(0) = \psi_k. \tag{4.8}$$

Consequently, we get a Cauchy problem of linear differential equations with smooth coefficients that is uniquely solvable. So it has a unique solution $u^{(n)}$ satisfying (4.3). □

Lemma 4.2. *The sequence $(u^{(n)})$ is bounded.*

Proof. Multiplying (4.4) by α_i' and summing over i from 1 to n, we get

$$\int_\Omega \left(u_t^{(n)} u_{tt}^{(n)} + c\nabla u^{(n)} \nabla u_t^{(n)} + a u_t^{(n)} u_t^{(n)} + b u^{(n)} u_t^{(n)} \right) dx$$

$$+ c\int_{\partial\Omega} u_t^{(n)} \left(\int_\Omega K(x,\xi) u^{(n)}(\xi,t) d\xi \right) ds = \int_\Omega f u_t^{(n)} dx.$$

(4.9)

Integrating by parts the left-hand side of (4.9) over t on $(0,\tau)$ yields

$$\int_0^\tau \int_\Omega u_t^{(n)} u_{tt}^{(n)} dx dt = \frac{1}{2} \int_\Omega \left[\left(u_t^{(n)}(x,\tau) \right)^2 - \left(u_t^{(n)}(x,0) \right)^2 \right] dx,$$

$$\int_0^\tau \int_\Omega \nabla u^{(n)} \nabla u_t^{(n)} dx dt = \frac{1}{2} \int_\Omega \left[\left(\nabla u^{(n)}(x,\tau) \right)^2 - \left(\nabla u^{(n)}(x,0) \right)^2 \right] dx,$$

$$\int_0^\tau \int_\Omega u^{(n)} u_t^{(n)} dx dt = \frac{1}{2} \int_\Omega \left[\left(u^{(n)}(x,\tau) \right)^2 - \left(u^{(n)}(x,0) \right)^2 \right] dx,$$

$$\int_0^\tau \int_{\partial\Omega} u_t^{(n)} \left(\int_\Omega K(x,\xi) u^{(n)}(\xi) d\xi \right) ds dt = \int_{\partial\Omega} u^{(n)}(x,\tau) \left(\int_\Omega K(x,\xi) u^{(n)}(\xi,\tau) d\xi \right) ds$$

$$- \int_{\partial\Omega} u^{(n)}(x,0) \left(\int_\Omega K(x,\xi) u^{(n)}(\xi,0) d\xi \right) ds$$

$$- \int_0^\tau \int_{\partial\Omega} u^{(n)} \left(\int_\Omega K(x,\xi) u_t^{(n)}(\xi) d\xi \right) ds dt.$$

(4.10)

Substituting the four last identities into (4.9), we get

$$\int_\Omega \left[\left(u_t^{(n)}(x,\tau) \right)^2 + c\left(\nabla u^{(n)}(x,\tau) \right)^2 + b\left(u^{(n)}(x,\tau) \right)^2 \right] dx$$

$$= \int_\Omega \left[\left(u_t^{(n)}(x,0) \right)^2 + c\left(\nabla u^{(n)}(x,0) \right)^2 + b\left(u^{(n)}(x,0) \right)^2 \right] dx$$

$$- 2c \int_{\partial\Omega} u^{(n)}(x,\tau) \left(\int_\Omega K(x,\xi) u^{(n)}(\xi,\tau) d\xi \right) ds$$

$$+ 2c \int_{\partial\Omega} u^{(n)}(x,0) \left(\int_\Omega K(x,\xi) u^{(n)}(\xi,0) d\xi \right) ds \qquad (4.11)$$

$$+ 2c \int_0^\tau \int_{\partial\Omega} u^{(n)}(x,t) \left(\int_\Omega K(x,\xi) u_t^{(n)}(\xi,t) d\xi \right) ds\, dt$$

$$- 2a \int_0^\tau \int_\Omega \left(u_t^{(n)} \right)^2 dx\, dt + 2 \int_0^\tau \int_\Omega f u_t^{(n)} dx\, dt.$$

Applying Cauchy Schwarz inequality, ϵ-Cauchy inequality, the hypothesis on the operator K, and the fact that $-2a \int_0^\tau \int_\Omega (u_t^{(n)})^2 dx\, dt \leq 0$ to the second term in the right hand side of (4.11), we get

$$\left| \int_{\partial\Omega} u^{(n)}(x,\tau) \left(\int_\Omega K(x,\xi) u^{(n)}(\xi,\tau) d\xi \right) ds \right|$$

$$\leq \frac{1}{2} \left(c(\varepsilon) + k_1^2 |\partial\Omega| \right) \int_\Omega \left(u^{(n)}(x,\tau) \right)^2 dx + \frac{\varepsilon}{2} \int_\Omega \left(\nabla u^{(n)}(x,\tau) \right)^2 dx. \qquad (4.12)$$

We may obtain similar inequalities for the third and the fourth terms in the right-hand side of (4.11). Now if we apply Cauchy Schwarz inequality to the last term in (4.11) we get

$$\left| \int_0^\tau \int_\Omega f u_t^{(n)} dx\, dt \right| \leq \frac{1}{2} \int_0^\tau \int_\Omega |f(x,t)|^2 dx\, dt + \frac{1}{2} \int_0^\tau \int_\Omega \left| u_t^{(n)}(x,t) \right|^2 dx\, dt. \qquad (4.13)$$

Using the result inequalities in (4.11) and then regrouping the same terms yield

$$
\int_{\Omega} \left[\left(u_t^{(n)}(x,\tau) \right)^2 + c(1-\varepsilon) \left(\nabla u^{(n)}(x,\tau) \right)^2 + \left(b - c \left(c(\varepsilon) + k_1^2 |\partial\Omega| \right) \right) \left(u^{(n)}(x,\tau) \right)^2 \right] dx
$$

$$
\leq \int_{\Omega} \left[\left(u_t^{(n)}(x,0) \right)^2 + (c+c\varepsilon) \left(\nabla u^{(n)}(x,0) \right)^2 + \left(b + c \left(c(\varepsilon) + k_1^2 |\partial\Omega| \right) \right) \left(u^{(n)}(x,0) \right)^2 \right] dx
$$

$$
+ c \left(c(\varepsilon) + k_1^2 |\partial\Omega| \right) \int_0^\tau \int_{\Omega} \left(u^{(n)}(x,t) \right)^2 dxdt
$$

$$
+ c\varepsilon \int_0^\tau \int_{\Omega} \left(\nabla u^{(n)}(x,t) \right)^2 dxdt
$$

$$
+ \int_0^\tau \int_{\Omega} |f(x,t)|^2 dxdt + \int_0^\tau \int_{\Omega} \left| u_t^{(n)}(x,t) \right|^2 dxdt.
$$

(4.14)

Choose ϵ such that $(1-\varepsilon) > 0$ and $b - c(c(\varepsilon) + k_1^2|\partial\Omega|) > 0$ and let

$$
M = \max \left\{ 1, (c+c\varepsilon), \left(b + c \left(c(\varepsilon) + k_1^2 |\partial\Omega| \right) \right) \right\},
$$

$$
m = \min \left\{ 1 - \varepsilon, b - c(c(\varepsilon) + k_1^2|\partial\Omega|) \right\}, \qquad K_2 = \frac{M}{m},
$$

(4.15)

then (4.14) becomes

$$
\int_{\Omega} \left[\left(u_t^{(n)}(x,\tau) \right)^2 + \left(\nabla u^{(n)}(x,\tau) \right)^2 + \left(u^{(n)}(x,\tau) \right)^2 \right] dx
$$

$$
\leq K_2 \left[\int_{\Omega} \left(\left(\psi^{(n)}(x) \right)^2 + \left(\nabla \varphi^{(n)}(x) \right)^2 + \left(\varphi^{(n)}(x) \right)^2 \right) dx \right.
$$

$$
+ \int_0^\tau \int_{\Omega} \left(\left| u_t^{(n)}(x,t) \right|^2 + \left(u^{(n)}(x,t) \right)^2 \left(\nabla u^{(n)}(x,t) \right)^2 \right) dxdt
$$

$$
\left. + \int_0^\tau \int_{\Omega} |f(x,t)|^2 dxdt \right].
$$

(4.16)

Now, we apply Gronwall Lemma to get

$$
\int_{\Omega} \left[\left(u_t^{(n)}(x,\tau) \right)^2 + \left(\nabla u^{(n)}(x,\tau) \right)^2 + \left(u^{(n)}(x,\tau) \right)^2 \right] dx
$$

$$
\leq e^{K_2 T} \left(\int_{\Omega} \left[\left(\psi^{(n)}(x) \right)^2 + \left(\nabla \varphi^{(n)}(x) \right)^2 + \left(\varphi^{(n)}(x) \right)^2 \right] dx + \|f\|^2 \right).
$$

(4.17)

Since $\varphi^{(n)}$ and $\psi^{(n)}$ are the orthogonal projections of φ and ψ, respectively, from the Pythagorean relation we have that $\|\varphi^{(n)}\|_{W^{1,2}(\Omega)}^2 \leq \|\varphi\|_{W^{1,2}(\Omega)}^2$ and $\|\psi^{(n)}\|_{L_2(\Omega)}^2 \leq \|\psi\|_{L_2(\Omega)}^2$. Integrating (4.17) according to τ on $[0, T]$ yields

$$\left\|u^{(n)}\right\|_{W^{1,2}(Q)}^2 \leq Te^{K_2 T}\left[\|f\|^2 + \|\psi\|_{L_2(\Omega)}^2 + \|\varphi\|_{W^{1,2}(\Omega)}^2\right]. \tag{4.18}$$

Consequently, the sequence $\{u^{(n)}\}$ is bounded.   □

Remark 4.3. We have proved that the sequence $\{u^{(n)}\}$ is bounded, so we can extract a subsequence which we denote by $\{u^{(n_k)}\}$ that is weakly convergent; then we prove that its limit is the desired solution of the problem (1.6)–(1.8).

Lemma 4.4. *The limit of the subsequence $\{u^{(n_k)}\}$ is the solution of the problem* (1.6)–(1.8).

Proof. For this we prove that the limit of the subsequence $\{u^{(n_k)}\}$ satisfies the identity (2.8) for all functions $v = \sum_{i=1}^n v_i(t)w_i(x) \in W_T^{1,2}(Q)$. Since the set $S_n = \{v(x,t) = \sum_{k=1}^n v_k(t)w_k(x), v_k(t) \in C^2(0,T), v_k(T) = 0\}$ is such that $\cup_{n=1}^\infty S_n$ is dense in $W_T^{1,2}(Q)$, it suffices to prove (2.8) for $v \in S_n$. Multiplying (4.3) by the function $v_i(t) \in W^{1,2}(0,T)$, $v_i(T) = 0$, and then taking the sum from $i = 0$ to n, we obtain

$$\int_\Omega \left(c\nabla u^{(n_k)}\nabla v + u_{tt}^{(n_k)}v + au_t^{(n_k)}v + bu^{(n_k)}v\right)dx$$
$$+ c\int_{\partial\Omega} v\left(\int_\Omega K(x,\xi)u^{(n_k)}(\xi,t)d\xi\right)ds = \int_\Omega fvdx. \tag{4.19}$$

Integrating by parts on $[0, T]$, we get

$$\int_Q \left(c\nabla u^{(n_k)}\nabla v - u_t^{(n_k)}v - au^{(n_k)}v + bu^{(n_k)}v\right)dxdt + c\int_0^T\int_{\partial\Omega}v\left(\int_\Omega K(x,\xi)u^{(n_k)}(\xi,t)d\xi\right)dsdt$$
$$= \int_\Omega \psi^{(n_k)}(x)v(x,0)dx + a\int_\Omega \varphi^{(n_k n_k)}(x)v(x,0)dx + \int_Q fvdxdt. \tag{4.20}$$

Denote the weak limit of the subsequence $\{u^{(n_k)}\}$ by u. When k tends to infinity, we get

$$\int_0^T\int_{\partial\Omega}v\left(\int_\Omega K(x,\xi)u^{(n_k)}(\xi,t)d\xi\right)dsdt \longrightarrow \int_0^T\int_{\partial\Omega}v\left(\int_\Omega K(x,\xi)u(\xi,t)d\xi\right)dsdt. \tag{4.21}$$

Indeed, using Cauchy Schwarz inequality then ε-Cauchy inequality, we obtain

$$
\begin{aligned}
&\left| \int_0^T \int_{\partial\Omega} v \left(\int_\Omega K(x,\xi) u^{(n_k)}(\xi,t) d\xi \right) ds dt - \int_0^T \int_{\partial\Omega} v \left(\int_\Omega K(x,\xi) u(\xi,t) d\xi \right) ds dt \right| \\
&\quad \leq k_1 |\partial\Omega| \left(\int_0^T \int_{\partial\Omega} |v|^2 ds dt \right)^{1/2} \\
&\quad \times \left(\int_0^T \int_\Omega \left| u^{(n_k)}(\xi,t) - u(\xi,t) \right|^2 d\xi ds dt \right)^{1/2} \longrightarrow 0,
\end{aligned}
\tag{4.22}
$$

and then by passing to the limit in (4.20) we get that the limit u satisfies (2.8). $\qquad\square$

Example 4.5. Now we present an example to demonstrate the applications of the results established in the earlier sections. We consider the following:

$$
\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u - \frac{\partial^2 u}{\partial x^2} = \sin x (\sin t + \cos t), \quad 0 < t \leq T, \ x \in (0,\pi),
\tag{4.23}
$$

with the initial conditions

$$
u(x,0) = 0, \quad \frac{\partial u}{\partial t(x,0)} = \sin x, \quad x \in (0,\pi),
\tag{4.24}
$$

and the nonlocal boundary condition

$$
\frac{\partial u}{\partial x}(\pi,t) - \frac{\partial u}{\partial x}(0,t) - \int_0^\pi u(s,t) ds = 0.
\tag{4.25}
$$

The results of the earlier sections guarantee the existence and uniqueness of a solution. It may be noted that $u(x,t) = \sin x \sin t$ is the unique solution of (4.23)–(4.25).

Acknowledgment

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