Spatial Reasoning with $\mathcal{RCC}8$ and Connectedness Constraints in Euclidean Spaces

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Abstract

The language $\mathcal{RCC}8$ is a widely-studied formalism for describing topological arrangements of spatial regions. The variables of this language range over the collection of non-empty, regular closed sets of n-dimensional Euclidean space, here denoted $\mathsf{RC}^+(\mathbb{R}^n)$, and its non-logical primitives allow us to specify how the interiors, exteriors and boundaries of these sets intersect. The key question is the *satisfiability problem*: given a finite set of atomic $\mathcal{RCC}8$-constraints in m variables, determine whether there exists an m-tuple of elements of $\mathsf{RC}^+(\mathbb{R}^n)$ satisfying them. These problems are known to coincide for all $n \geq 1$, so that $\mathcal{RCC}8$-satisfiability is independent of dimension. This common satisfiability problem is NLogSpace-complete. Unfortunately, $\mathcal{RCC}8$ lacks the means to say that a spatial region comprises a 'single piece', and the present article investigates what happens when this facility is added. We consider two extensions of $\mathcal{RCC}8$: $\mathcal{RCC}8c$, in which we can state that a region is *connected*, and $\mathcal{RCC}8c^\circ$, in which we can instead state that a region has a *connected interior*. The satisfiability problems for both these languages are easily seen to depend on the dimension n, for $n \leq 3$. Furthermore, in the case of $\mathcal{RCC}8c^\circ$, we show that there exist finite sets of constraints that are satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, but only by 'wild' regions having no possible physical meaning. This prompts us to consider interpretations over the more restrictive domain of non-empty, regular closed, *polyhedral* sets, $\mathsf{RCP}^+(\mathbb{R}^n)$. We show that (a) the satisfiability problems for $\mathcal{RCC}8c$ (equivalently, $\mathcal{RCC}8c^\circ$) over $\mathsf{RC}^+(\mathbb{R})$ and $\mathsf{RCP}^+(\mathbb{R})$ are distinct and both NP-complete; (b) the satisfiability problems for $\mathcal{RCC}8c$ over $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$ are identical and NP-complete; (c) the satisfiability problems for $\mathcal{RCC}8c^\circ$ over $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$ are distinct, and the latter is NP-complete. Decidability of the satisfiability problem for $\mathcal{RCC}8c^\circ$ over $\mathsf{RC}^+(\mathbb{R}^2)$ is open. For $n \geq 3$, $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ are not interestingly different from $\mathcal{RCC}8$. We finish by answering the following question: given that a set of $\mathcal{RCC}8c$- or $\mathcal{RCC}8c^\circ$-constraints is satisfiable over $\mathsf{RC}^+(\mathbb{R}^n)$ or $\mathsf{RCP}^+(\mathbb{R}^n)$, how complex is the simplest satisfying assignment? In particular, we exhibit, for both languages, a sequence of constraints Φ_n, satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$, such that the size of Φ_n grows polynomially in n, while the smallest configuration of polygons satisfying Φ_n cuts the plane into a number of pieces that grows exponentially. We further show that, over $\mathsf{RC}^+(\mathbb{R}^2)$, $\mathcal{RCC}8c$ again requires exponentially large satisfying diagrams, while $\mathcal{RCC}8c^\circ$ can force regions in satisfying configurations to have infinitely many components.

Keywords: qualitative spatial reasoning; spatial logic; Euclidean space; connectedness; satisfiability; complexity.

1. Introduction

Spatial reasoning in everyday life possesses two distinctive—and related—characteristics: it is primarily concerned with *extended*, as opposed to point-like entities, and it typically invokes *qualitative*, as opposed to quantitative, concepts [1, 2, 3]. This observation has prompted consideration, within the Artificial Intelligence community, of representation languages whose variables range over some specified collection of extended spatial objects, and whose non-logical primitives are interpreted as qualitative spatial properties and relations involving those objects. As might

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be expected, the logical properties of such languages depend on the geometry of the spaces over which they are interpreted—in most applications, two- and three-dimensional Euclidean space. The present article draws attention to some hitherto overlooked subtleties regarding this dependency.

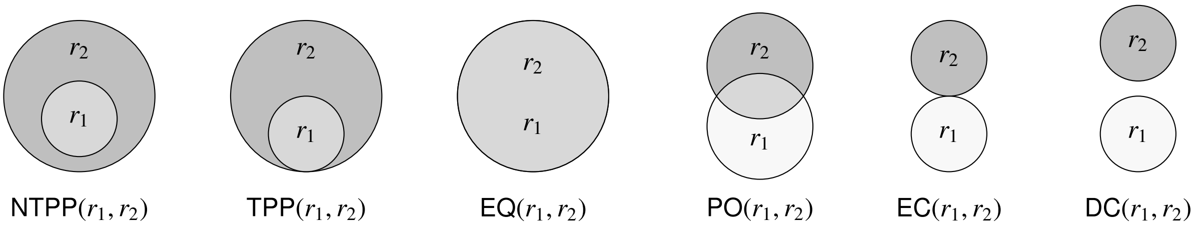


Figure 1: $\mathcal{RCC}8$-relations over discs in $\mathbb{R}^2$.

By far the best-known language for Qualitative Spatial Reasoning is $\mathcal{RCC}8$, originally proposed—in essentially equivalent formulations—by Egenhofer and Franzosa [4], Egenhofer and Herring [5], Randell *et al.* [6] and Smith and Park [7]. This quantifier-free language allows us to specify how regions and their interiors are related to each other. It employs an infinite collection of variables $r_1, r_2, \ldots$, ranging over spatial regions, together with six binary predicates: NTPP (*non-tangential proper part*), TPP (*tangential proper part*), EQ (*equality*), PO (*partial overlap*), EC (*external contact*) and DC (*disjointness*). The relations denoted by these predicates are illustrated, for closed discs in the plane, in Fig. 1. More formally: NTPP(r_1, r_2) if r_1 is included in the interior of r_2; TPP(r_1, r_2) if r_1 is included in r_2 but not in its interior; PO(r_1, r_2) if the interiors of r_1 and r_2 intersect, but neither is included in the other; and EC(r_1, r_2) if r_1 and r_2 intersect, but their interiors do not. A *constraint* is a statement $R(r_i, r_j)$, where R is one of these six predicates. For example, the constraints

$$\mathsf{EC}(r_1, r_2), \quad \mathsf{TPP}(r_1, r_3), \quad \mathsf{NTPP}(r_2, r_4) \tag{1}$$

state that regions r_1 and r_2 are in external contact, with the former a tangential proper part of r_3 and the latter a non-tangential proper part of r_4. Fig. 2 shows two arrangements of regions satisfying these constraints.

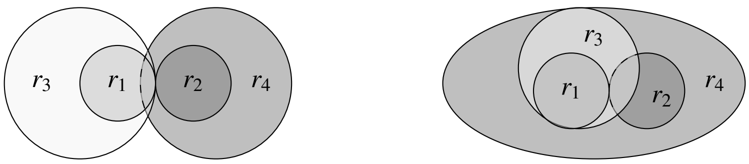


Figure 2: Two arrangements of regions in the plane satisfying (1).

The $\mathcal{RCC}8$-relations mentioned above were defined in [6] by means of a formalism referred to there as the *Region Connection Calculus*. Of these, the relations NTPP and TPP are asymmetric: counting their converses, NTPP^{-1} and TPP^{-1}, we obtain eight relations in all, hence the name $\mathcal{RCC}8$. Syntactically, the original Region Connection Calculus is the language of first-order logic (with equality) over the signature consisting of a single binary predicate C, variously referred to as 'contact', or (confusingly) 'connection.' The origin of this predicate can be traced back, via Clarke [8, 9], to the philosophical work of Whitehead [10] and de Laguna [11]. Semantically, one is supposed to think of C as holding between two regions just in case they share at least one point, though matters are somewhat muddied by the recurrent suggestion that this notion should be regarded as an (undefined) primitive. However, the etymology of the term $\mathcal{RCC}8$ need not concern us further: it is now standardly used for the quantifer-free language featuring the six primitives illustrated in Fig. 1, and we simply follow suit. The motivation for focussing on this particular collection of primitives is not always clear. Egenhofer and Franzosa [4] and Egenhofer and Herring [5] classify relationships between regions in terms of intersections of their interiors, exteriors and boundaries and show, in particular, that only the $\mathcal{RCC}8$ relations are possible between closed disc-homeomorphs in the Euclidean plane. Düntsch, Wang and McCloskey [12] observe that NTPP, TPP, EQ, PO, EC, DC, NTPP^{-1} and TPP^{-1} are exactly the atoms of the smallest relation algebra defined on the set of closed discs in the Euclidean plane that contains the contact relation, C. In fact, Li and Ying [13] show that the set of closed disc-homeomorphs in the Euclidean plane realizes the

a) b)

Figure 3: Non-regular and regular closed subsets of a) $\mathbb{R}^2$, and b) $\mathbb{R}^3$.

same relation algebra. (See also Li and Li [14] for an interesting extension of this result.) In any case, the language $\mathcal{RCC}8$ is by now firmly established as a basic formalism in the field of qualitative spatial reasoning. In particular, the standard geographic query language for RDF data GeoSPARQL[1], suggested by the Open Geospatial Consortium, is based on the $\mathcal{RCC}8$ relations.

How are we to understand spatial *reasoning* in $\mathcal{RCC}8$? Observe that, in (1), nothing is said about the relation between r_3 and r_4. What are the possibilities? A little thought suffices to convince us that DC and EC are both impossible. As we might say, the two sets of constraints

$$\mathsf{EC}(r_1, r_2), \quad \mathsf{TPP}(r_1, r_3), \quad \mathsf{NTPP}(r_2, r_4), \quad \mathsf{DC}(r_3, r_4), \tag{2}$$

$$\mathsf{EC}(r_1, r_2), \quad \mathsf{TPP}(r_1, r_3), \quad \mathsf{NTPP}(r_2, r_4), \quad \mathsf{EC}(r_3, r_4) \tag{3}$$

are *unsatisfiable*. Allowing ourselves to combine $\mathcal{RCC}8$-constraints using arbitrary sentential connectives, we could express this knowledge as a formula

$$(\mathsf{EC}(r_1, r_2) \wedge \mathsf{TPP}(r_1, r_3) \wedge \mathsf{NTPP}(r_2, r_4)) \rightarrow \neg(\mathsf{DC}(r_3, r_4) \vee \mathsf{EC}(r_3, r_4)),$$

which we take to be true of *all* tuples of regions $r_1, \ldots, r_4$. The validity of this formula thus represents a geometrical fact to which an agent employing $\mathcal{RCC}8$ as a spatial representation language should have access. Satisfiability and validity being dual notions, it suffices, from a computational perspective, to consider only the former. And since, in this context, nothing essential is added by sentential connectives, we may confine attention in the sequel to finite sets of constraints, interpreted conjunctively. Following common practice, we refer to such a set as an $\mathcal{RCC}8$-*constraint network* (or, simply, $\mathcal{RCC}8$-*network*).

When introducing the notion of satisfiability of $\mathcal{RCC}8$-networks, we employed the term *spatial region* as if it needed no clarification, giving as examples the regions in Figs. 1 and 2. But what is a *spatial region*, exactly? In formal terms, when we ask whether an $\mathcal{RCC}8$-network is satisfiable, over what domain are we taking its variables to range? Fix some topological space T. A subset $X \subseteq T$ is said to be *regular closed* if X is the topological closure of some open set in T (equivalently, if it is equal to the closure of its interior). We denote the non-empty, regular closed subsets of T by $\mathsf{RC}^+(T)$. Most recent literature on Qualitative Spatial Reasoning takes regions to be elements of $\mathsf{RC}^+(\mathbb{R}^n)$ for some fixed n (usually 2 or 3). Roughly speaking, the regular closed subsets of $\mathbb{R}^2$ are those closed sets with no 'filaments' or 'isolated points' (Fig. 3a); similarly, the regular closed subsets of $\mathbb{R}^3$ are those closed sets with no 'flanges,' 'filaments' or 'isolated points' (Fig. 3b). Determining the satisfiability of collections of $\mathcal{RCC}8$ constraints over $\mathsf{RC}^+(\mathbb{R}^n)$ has been the subject of intensive research; see, e.g. [15, 16, 17, 18]. It is known that satisfiability does not depend on dimension: that is, if an $\mathcal{RCC}8$-network is satisfiable over $\mathsf{RC}^+(\mathbb{R}^n)$ for some $n \geq 1$, then it is satisfiable over $\mathsf{RC}^+(\mathbb{R}^n)$ for every such n [19]. This (common) satisfiability problem is known to be NLogSpace-complete [15, 17, 20, 21]. We mention in this connection that some authors allow $\mathcal{RCC}8$-constraints to feature *sets* of the basic $\mathcal{RCC}8$-relations of Fig. 1, interpreted disjunctively. Thus, for example, the constraint $\{\mathsf{NTPP} \cup \mathsf{TPP} \cup \mathsf{EQ}\}(r_1, r_2)$ states that r_1 is a (proper or improper) part of r_2. This extension entails a computational cost: the satisfiability problem for sets of disjunctive $\mathcal{RCC}8$ constraints is NP-complete [22], though optimized algorithms have been developed to attack it [23]. More generally, one could allow the unrestricted use of Boolean connectives in $\mathcal{RCC}8$-constraints, in which case it is easy to see that the problem of determining satisfiability over $\mathsf{RC}^+(\mathbb{R}^n)$ is also NP-complete. However, from the point of

[1] `www.opengeospatial.org/standards/geosparql`

view of this article, nothing is gained by considering such extensions of the $\mathcal{RCC}8$-formalism, and we do not consider them in the sequel.

At first glance, the regular closed sets in Euclidean space serve as an attractive mathematical model of our pre-theoretic notion of a spatial region. No two regular closed subsets of $\mathbb{R}^n$ differ only with respect to boundary points; at the same time, the regular closed sets of any topological space form a Boolean algebra under the natural operations of 'fusion,' 'complementation' and 'taking common parts.' Thus, confining attention to regular closed sets allows us to finesse the apparently senseless question of whether the regions occupied by physical objects include their boundaries, while at the same time retaining a simple algebra for manipulating regions. Closer inspection, however, reveals a more complicated picture. Most obviously, regular closed sets may consist of more than one 'piece,' and may contain 'holes'; and such sets may not necessarily be what we have in mind when we speak of *regions*. In 3-dimensional space (or more), this matter may safely be ignored: any collection of $\mathcal{RCC}8$-constraints satisfiable by elements of $\mathsf{RC}^+(\mathbb{R}^n)$ $(n \geq 3)$ is satisfiable by regions homeomorphic to n-dimensional balls. In dimensions 1 and 2, by contrast, we are not free to assume that regions are so simple, and in particular are not free to assume that they are connected. This is most easily seen in 1-dimensional space, where the connected, non-empty, regular closed subsets are the intervals of the forms $(-\infty, b]$, $[a, b]$ or $[a, \infty)$ (where $a < b$). Thus, for example, the $\mathcal{RCC}8$-network

$$\mathsf{EC}(r_1, r_2), \qquad \mathsf{EC}(r_2, r_3), \qquad \mathsf{EC}(r_3, r_1). \tag{4}$$

is satisfiable over $\mathsf{RC}^+(\mathbb{R})$, for example by the assignment $r_1 = [0, 1]$, $r_2 = [1, 2]$, $r_3 = [-1, 0] \cup [2, 3]$; however, (4) is obviously not satisfiable by connected elements of $\mathsf{RC}^+(\mathbb{R})$. Likewise, there exist $\mathcal{RCC}8$-networks that are satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, but not by connected elements of $\mathsf{RC}^+(\mathbb{R}^2)$. (We shall encounter an example in Sec. 4.) The moral of these observations is that the notion of regionhood provided by the non-empty regular closed sets may be too liberal for many applications.

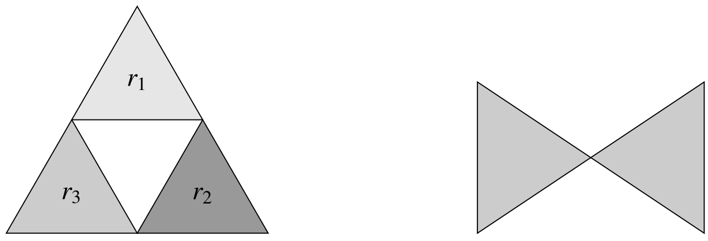
What, then, would a more conservative approach look like? We have two options. The first is to restrict, by fiat, the domain over which our variables can range to those regular closed sets satisfying certain additional properties; the second is to expand our language with additional primitives able to express those properties. An example of the first approach is provided by Schaefer, Sedgwick and Štefankovič [24, 25], who considered the $\mathcal{RCC}8$-satisfiability problem over sets of *closed disc-homeomorphs* in $\mathbb{R}^2$. They showed that this problem is NP-complete rather than NLogSpace-complete over $\mathsf{RC}^+(\mathbb{R}^2)$. (However, it is membership in NP rather than NP-hardness that is remarkable.) An example of the second approach is provided by Davis, Gotts and Cohn [26], who investigated the extension of $\mathcal{RCC}8$ with constraints of the form $conv(r)$ ('r is convex'). They showed that the satisfiability problem for this language, interpreted over $\mathsf{RC}^+(\mathbb{R}^2)$, is again decidable, though with the same complexity as the satisfiability problem for quantifier-free real arithmetic. The present article also takes this latter approach. In particular, we investigate the language $\mathcal{RCC}8c$, which extends $\mathcal{RCC}8$ with constraints of the form $c(r)$ ('r is connected'). Unlike $\mathcal{RCC}8$, $\mathcal{RCC}8c$ can discriminate between low-dimensional Euclidean spaces. Indeed, as we saw in our discussion of (4), the $\mathcal{RCC}8c$-network

$$\mathsf{EC}(r_1, r_2), \qquad \mathsf{EC}(r_2, r_3), \qquad \mathsf{EC}(r_3, r_1), \qquad c(r_1), \qquad c(r_2), \qquad c(r_3) \tag{5}$$

is not satisfiable over $\mathsf{RC}^+(\mathbb{R})$; on the other hand, (5) *is* satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, for example, as in Fig. 4a. In Sec. 4, we present an $\mathcal{RCC}8c$-network that is not satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, but is satisfiable over $\mathsf{RC}^+(\mathbb{R}^3)$. However, it is easily seen that the satisfiability of $\mathcal{RCC}8c$-networks over $\mathsf{RC}(\mathbb{R}^n)$ coincides for all $n \geq 3$. As we might put it, $\mathcal{RCC}8c$ can tell the difference between dimensions 1, 2 and 3; however, it cannot tell the difference between dimensions greater than or equal to 3.

Actually, the topological notion of connectedness is perhaps not *quite* what we might have in mind for certain applications of qualitative spatial reasoning. Consider, for example, the region formed by two closed triangles touching externally at a common vertex, as in Fig. 4b. This set is—according to the usual definition—connected, though its *interior* is not. And indeed, we are loath to take such a figure to represent, say, a contiguous plot of land on a map, since no extended object, however small, could squeeze from one part of this region to another without crossing its boundary.[2] Accordingly, we additionally consider the language $\mathcal{RCC}8c^\circ$, which extends $\mathcal{RCC}8$ with constraints of

[2] One famous exception (proving the rule) is the town Jungholz, an exclave of Austria connected to the rest of that country by a single point (`en.wikipedia.org/wiki/Jungholz`). This configuration caused various problems before Austria joined the EU. See also `jungholz.enclaves.org`.

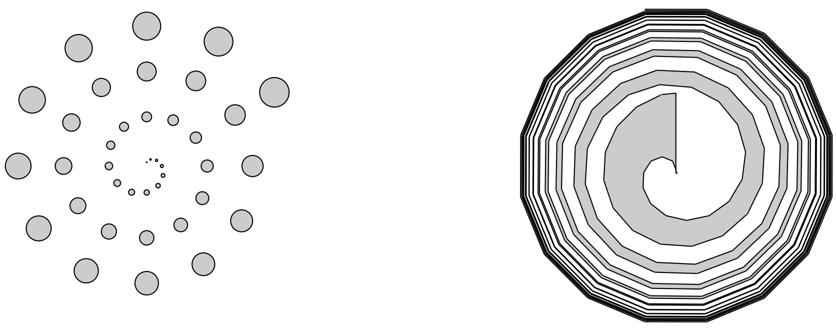


a) b)

Figure 4: a) satisfying (5) over $\mathsf{RC}^+(\mathbb{R}^2)$ and b) a connected but not interior-connected region.

the form $c^\circ(r)$ ('r has a connected interior'). For regular closed sets in Euclidean space of dimension greater than 1, the property of having a connected interior is strictly stronger than the property of being connected. (For $\mathsf{RC}^+(\mathbb{R})$, connectedness and the property of having a connected interior coincide.) Again, it is routine to show that $\mathcal{RCC}8c^\circ$ can tell the difference between dimensions 1, 2 and 3, but no more.

The property of having a connected interior, however, brings into relief a further important issue regarding the sets of points which we are prepared to countenance as regions—one that has gone largely unnoticed in the literature on Qualitative Spatial Reasoning. Consider the two closed subsets of $\mathbb{R}^2$ depicted in Fig. 5. The left-hand region is disconnected—indeed it has infinitely many components spiralling endlessly inwards towards a limit point. (Note that, since the set in question is closed, it must contain this limit point.) By contrast, the right-hand region is connected—indeed it has a connected interior—but spirals endlessly outwards in ever more compressed cycles towards a limiting hexadecagon. (Again, since the set in question is closed, it includes this limiting hexadecagon.) Both sets are easily seen to be regular closed. Nevertheless, both are in some sense *illegitimate*: they could never correspond to the parts of surfaces occupied (or left unoccupied) by physical objects. The question thus arises as to whether such 'regions' make a difference to the satisfiability of spatial constraints. After all, it might be of little comfort to know that a collection of constraints is satisfiable if the *only* satisfying assignments involve regions which make no physical sense. The answer



a) b)

Figure 5: Non-tame regular closed subsets of $\mathbb{R}^2$.

depends, of course, on where we draw the distinction between legitimate and illegitimate regions; but fortunately, we have a natural answer at hand. Let us say that a region is *polyhedral* if it is a finite union of finite intersections of closed half-planes. Equivalently, a polyhedral region is a regular closed region that is *semi-linear*—definable (using standard Cartesian coordinates) by means of a Boolean combination of linear inequalities. Polyhedral sets, in this sense, are not required to be connected, or bounded, and are not required to have connected complements; however, they never exhibit the strangeness of the regular closed sets depicted in Fig. 5. In particular, all polyhedra consist of finitely many 'pieces' (contrast Fig. 5a), and have boundaries that are 'reachable' from their interiors (contrast Fig. 5b). (We make the relevant notions precise below.) The regular closed polyhedral sets in $\mathbb{R}^n$ form a Boolean sub-algebra of the regular closed algebra, and we denote its non-empty elements by $\mathsf{RCP}^+(\mathbb{R}^n)$.

Arguably, the really important problem regarding any language for Qualitative Spatial Reasoning is that of determining the satisfiability of networks over $\mathsf{RCP}^+(\mathbb{R}^n)$, rather than over $\mathsf{RC}^+(\mathbb{R}^n)$. For any bounded subset of $\mathbb{R}^n$ can be approximated with arbitrary precision by an element of $\mathsf{RCP}^+(\mathbb{R}^n)$, a fact which underlies the almost universal adoption of semi-linear sets as a spatial data model for Geographical Information Systems (GIS). Does restricting attention to polyhedral regions make a difference to satisfiability? For $\mathcal{RCC}8$ (which lacks connectedness constraints), the answer is no: any collection of $\mathcal{RCC}8$-constraints satisfiable over $\mathsf{RC}^+(\mathbb{R}^n)$ ($n \geq 1$) is easily seen to be satisfiable over $\mathsf{RCP}^+(\mathbb{R}^n)$; as we might put it, $\mathcal{RCC}8$ is insensitive to the difference between $\mathsf{RC}^+(\mathbb{R}^n)$ and $\mathsf{RCP}^+(\mathbb{R}^n)$. In the sequel, we show that $\mathcal{RCC}8c$ (equivalently, $\mathcal{RCC}8c^\circ$) *is* sensitive to the difference between $\mathsf{RC}^+(\mathbb{R})$ and $\mathsf{RCP}^+(\mathbb{R})$; we further show that $\mathcal{RCC}8c^\circ$ is sensitive to the difference between $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$, but that $\mathcal{RCC}8c$ is not. Both languages are easily seen to be insensitive to the difference between $\mathsf{RC}^+(\mathbb{R}^n)$ and $\mathsf{RCP}^+(\mathbb{R}^n)$ for $n \geq 3$. Thus, when working with $\mathcal{RCC}8$ or $\mathcal{RCC}8c$ in the Euclidean plane, we can ignore the issue of 'wild' regions; when working with $\mathcal{RCC}8c^\circ$, we cannot. That is, when interpreting $\mathcal{RCC}8c^\circ$ over the Euclidean plane, we have two problems to consider: satisfiability over $\mathsf{RCP}^+(\mathbb{R}^2)$ and satisfiability over $\mathsf{RC}^+(\mathbb{R}^2)$.

The plan of the article is as follows. Sec. 2 provides formal definitions of the languages $\mathcal{RCC}8$, $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$. Sec. 3 establishes various topological facts about their principal domains of interpretation, $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$. Sec. 4 provides a systematic treatment of various results on $\mathcal{RCC}8$ that have appeared in the literature. Specifically, we show that, for $n \geq 1$, the satisfiability problems for $\mathcal{RCC}8$ over both $\mathsf{RC}^+(\mathbb{R}^n)$ and $\mathsf{RCP}^+(\mathbb{R}^n)$ all coincide, and are NLogSpace-complete. We also obtain, as a simple corollary, corresponding results for the languages $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ where $n \geq 3$. In Sec. 5, we show that $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ are sensitive to dimension n in the range $1 \leq n \leq 3$; we further show that these languages are sensitive to the difference between $\mathsf{RC}^+(\mathbb{R})$ and $\mathsf{RCP}^+(\mathbb{R})$, that $\mathcal{RCC}8c^\circ$ is sensitive to the difference between $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$, but that $\mathcal{RCC}8c$ is insensitive to the difference between $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$. Having mapped out the landscape of satisfiability problems, we then turn to the question of computational complexity. Sec. 6 deals with the 1-dimensional case: we show that the satisfiability problems for $\mathcal{RCC}8c$ (equivalently, $\mathcal{RCC}8c^\circ$) over $\mathsf{RC}^+(\mathbb{R})$ and $\mathsf{RCP}^+(\mathbb{R})$ are both NP-complete. Sec. 7 considers the more interesting, and challenging, two-dimensional case. We show that the satisfiability problem for $\mathcal{RCC}8c$ over $\mathsf{RC}^+(\mathbb{R}^2)$—equivalently, over $\mathsf{RCP}^+(\mathbb{R}^2)$—is NP-complete; and we show that the satisfiability problem for $\mathcal{RCC}8c^\circ$ over $\mathsf{RCP}^+(\mathbb{R}^2)$ is also NP-complete. The decidability of the satisfiability problem for $\mathcal{RCC}8c^\circ$ over $\mathsf{RC}^+(\mathbb{R}^2)$ is left open. In the final section, we investigate the ability of the languages considered above to enforce 'fragmented' satisfying arrangements, showing that $\mathcal{RCC}8$, $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ all exhibit different behaviour. For any $\mathcal{RCC}8$-network satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, we are guaranteed a satisfying tuple whose diagram is bounded in size by a polynomial function of the number of constraints. We show that, when interpreted over $\mathsf{RCP}^+(\mathbb{R}^2)$, this result fails for $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$: specifically, we construct a sequence of satisfiable constraints Φ_n (in either language) such that the size of Φ_n grows polynomially as a function of n, and such that the smallest configuration of polygons satisfying Φ_n cuts the plane into a number of pieces that grows exponentially in n. We further show that, over $\mathsf{RC}^+(\mathbb{R}^2)$, $\mathcal{RCC}8c$ again requires exponentially large satisfying diagrams, while $\mathcal{RCC}8c^\circ$ can force regions in satisfying configurations to have infinitely many components. The differences that emerge between the superficially similar languages $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ in the Euclidean plane provide a striking illustration of the subtleties that need to be confronted when working with even relatively simple languages extending $\mathcal{RCC}8$.

2. Preliminaries

Let T be a topological space. We denote the closure of any $X \subseteq T$ by X^-, the interior of X by X°, and the boundary of X by $\delta X = X^- \setminus X^\circ$. We call X *regular closed* if $X = X^{\circ -}$, and denote by $\mathsf{RC}(T)$ the set of all regular closed subsets of T. We preferentially use the (possibly decorated) letters p, q, r, s, t to range over regular closed sets. It is a standard result that $\mathsf{RC}(T)$ is a Boolean algebra under the operations $r + s = r \cup s$, $r \cdot s = (r^\circ \cap s^\circ)^-$, $-r = (T \setminus r)^-$, $1 = T$ and $0 = \emptyset$; in fact, this Boolean algebra is complete, with $\sum X = ((\bigcup X)^\circ)^-$ and $\prod X = (\bigcap X^\circ)^-$ for any $X \subseteq T$ [27, pp. 25–27]. We write $\mathsf{RC}^+(T)$ as an abbreviation for $\mathsf{RC}(T) \setminus \{\emptyset\}$. A subset $X \subseteq T$ is *compact* if any cover of X by open sets has a finite subcover; when $T = \mathbb{R}^n$, this property coincides with that of being closed and bounded. Any subset $X \subseteq T$ has a *subspace topology*, whose open sets are those of the form $O \cap X$, where O is open in T; X is *connected* if it cannot be covered by the union of two non-empty and disjoint subsets which are open in the subspace topology. A maximal connected subset of X is called a *component* of X. We say that X is *interior-connected* if X° is connected. For elements of $\mathsf{RC}(T)$, interior-connectedness implies connectedness (indeed, if X° is connected then its closure, X, is connected as well), but not *vice versa*.

2.1. Constraint networks

If T is any topological space, we define eight binary relations on $\mathsf{RC}^+(T)$ as follows:

$$\begin{array}{rll}
\mathsf{DC}(r_1, r_2) & \text{iff} & r_1 \cap r_2 = \emptyset, \\
\mathsf{EC}(r_1, r_2) & \text{iff} & r_1 \cap r_2 \neq \emptyset \text{ but } r_1^\circ \cap r_2^\circ = \emptyset, \\
\mathsf{PO}(r_1, r_2) & \text{iff} & r_1^\circ \cap r_2^\circ \neq \emptyset,\ r_1 \not\subseteq r_2 \text{ and } r_2 \not\subseteq r_1, \\
\mathsf{EQ}(r_1, r_2) & \text{iff} & r_1 = r_2, \\
\mathsf{TPP}(r_1, r_2) & \text{iff} & r_1 \subseteq r_2 \text{ but } r_1 \not\subseteq r_2^\circ \text{ and } r_2 \not\subseteq r_1, \\
\mathsf{NTPP}(r_1, r_2) & \text{iff} & r_1 \subseteq r_2^\circ \text{ but } r_2 \not\subseteq r_1, \\
\mathsf{TPP}^{-1}(r_1, r_2) & \text{iff} & \mathsf{TPP}(r_2, r_1), \\
\mathsf{NTPP}^{-1}(r_1, r_2) & \text{iff} & \mathsf{NTPP}(r_2, r_1).
\end{array}$$

Together, these are known as the *$\mathcal{RCC}8$-relations*. The first six are illustrated, for closed discs in the plane, in Fig. 1. It is routine to show that the $\mathcal{RCC}8$-relations are jointly exhaustive and pairwise disjoint (JEPD) over $\mathsf{RC}^+(T)$ for any topological space T: given $r, s \in \mathsf{RC}^+(T)$, the ordered pair (r, s) lies in exactly one of the $\mathcal{RCC}8$-relations.

Fix some countably infinite set of *variables*, $\mathcal{V}$. An *$\mathcal{RCC}8$-constraint* is an expression $R(r, s)$, where $r, s \in \mathcal{V}$ and R is one of the symbols DC, EC, PO, EQ, TPP, NTPP, TPP^{-1} or NTPP^{-1}; an *$\mathcal{RCC}8c$-constraint* is either an $\mathcal{RCC}8$-constraint or an expression of the form $c(r)$, where $r \in \mathcal{V}$; an *$\mathcal{RCC}8c^\circ$-constraint* is either an $\mathcal{RCC}8$-constraint or an expression of the form $c^\circ(r)$, where $r \in \mathcal{V}$. An *$\mathcal{RCC}8$-constraint network* is a finite set of $\mathcal{RCC}8$-constraints; and similarly for *$\mathcal{RCC}8c$-constraint network* and *$\mathcal{RCC}8c^\circ$-constraint network*. When writing constraint networks, the relations TPP^{-1} and NTPP^{-1} can always be eliminated by transposing variables. In the sequel, therefore, we typically employ only the six relations DC, EC, PO, EQ, TPP and NTPP.

If T is a topological space, a *frame over T* is a subset $\mathfrak{F} \subseteq \mathsf{RC}^+(T)$. Typical examples of frames are the set of closed disc-homeomorphs in $\mathbb{R}^2$ or the set of regular closed polyhedra in $\mathbb{R}^3$. Where $\mathfrak{F}$ is clear from context, we refer to its elements as *regions*. An *assignment* over a frame $\mathfrak{F}$ is a function $\mathfrak{a}\colon \mathcal{V} \to \mathfrak{F}$. We say that $\mathfrak{a}$ *satisfies* an $\mathcal{RCC}8$-constraint $R(r, s)$ if $\mathfrak{a}(r)$ stands in the relation R to $\mathfrak{a}(s)$; we say that $\mathfrak{a}$ *satisfies* $c(r)$ if $\mathfrak{a}(r)$ is connected, and $\mathfrak{a}$ *satisfies* $c^\circ(r)$ if $\mathfrak{a}(r)$ is interior-connected. A constraint network Φ is *satisfied* by $\mathfrak{a}$ if $\mathfrak{a}$ satisfies all the constraints in Φ; and Φ is *satisfiable* over $\mathfrak{F}$ if some $\mathfrak{a}\colon \mathcal{V} \to \mathfrak{F}$ satisfies Φ. To aid readability, we use the (possibly decorated) letters $p, q, r, s, t, \ldots$ *both* for variables *and* for the regions they are mapped to by some (putative) assignment. We remark that, since, for regular closed sets, interior-connectedness implies connectedness, any assignment satisfying an $\mathcal{RCC}8c^\circ$-constraint network Φ automatically satisfies the $\mathcal{RCC}8c$-constraint network obtained by replacing every occurrence of c° in Φ by c. We remark in passing that, in place of regular closed sets, we could just as easily have worked with regular *open* sets. (A set X is regular open if $X = (X^-)^\circ$.) The regular open subsets of a topological space T form a Boolean algebra isomorphic to $\mathsf{RC}(T)$. Of course, in that case, we would have to work with the predicates c^- (denoting the property of having a connected closure) and c, in place of c and c°; but otherwise, there would be no difference. Our choice of regular closed sets is purely a matter of convention.

We employ the convention that, if $\bar{r} = r_1, \ldots, r_n$ is some tuple of variables, $\Phi(\bar{r})$ is the constraint network Φ with variables taken in the indicated order. (Thus, for instance, we can meaningfully say that the two constraint networks $\Phi(\bar{r})$ and $\Psi(\bar{s})$ are satisfied by the *same* tuples of regions.) We take the notation $\Phi(\bar{r})$ to indicate that Φ contains no variables other than $r_1, \ldots, r_n$; however, we do not insist that each r_i actually occurs in Φ. If $\mathcal{L}$ is any of the languages $\mathcal{RCC}8$, $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$, we denote by $Sat(\mathcal{L}, \mathfrak{F})$ the *satisfiability problem* for $\mathcal{L}$-constraint networks over $\mathfrak{F}$, namely:

Given: an $\mathcal{L}$-constraint network Φ.
Return: yes, if Φ is satisfiable over $\mathfrak{F}$; no, otherwise.

2.2. Type-certificates and point-certificates

For a set $\mathcal{V}$ of variables, we take the set $\mathcal{V}^\circ = \{r^\circ \mid r \in \mathcal{V}\}$ and call its elements *interior terms*. Interior terms will be used in some of the proofs below; however, we stress that they are not part of the syntax of the languages $\mathcal{RCC}8$, $\mathcal{RCC}8c$ or $\mathcal{RCC}8c^\circ$. Variables and interior terms will be referred to as *terms*.

Let $\Phi(\bar{r})$ be an $\mathcal{RCC}8$-constraint network. Observe that the semantics of $\mathcal{RCC}8$-constraints was defined above using *universal* conditions such as $r_1 \subseteq r_2$ (that is, $\forall x\,(x \in r_1 \to x \in r_2)$) and *existential* conditions such as $r_2 \not\subseteq r_1$ (that is, $\exists x\,(x \in r_2 \land x \notin r_1)$). Thus, to satisfy Φ in $\mathsf{RC}^+(T)$, we have to find an assignment $\mathfrak{a}$ over $\mathsf{RC}^+(T)$ providing witnesses for all the existential conditions given by Φ (including non-emptiness of regions) and complying with the universal ones. The following notion will be used to characterize membership of points in regions assigned to variables of Φ. A Φ-*type* (or simply *type* when Φ is clear) is any set $\boldsymbol{\tau}$ of terms with variables from Φ such that

(type) for each variable r in Φ, if $r^\circ \in \boldsymbol{\tau}$ then $r \in \boldsymbol{\tau}$.

A set of Φ-types $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_m$ is called a *type-certificate for* Φ if it satisfies the following *existential conditions*:

(reg-e) for each variable r in Φ, there is k ($1 \leq k \leq m$) for which $r^\circ \in \boldsymbol{\tau}_k$;

(ec-e) for each $\mathsf{EC}(r_i, r_j) \in \Phi$, there is k ($1 \leq k \leq m$) for which $r_i, r_j \in \boldsymbol{\tau}_k$;

(po-e) for each $\mathsf{PO}(r_i, r_j) \in \Phi$, there is k ($1 \leq k \leq m$) for which $r_i^\circ, r_j^\circ \in \boldsymbol{\tau}_k$;

(diff-e) for each $\mathsf{PO}(r_i, r_j) \in \Phi$, each $\mathsf{PO}(r_j, r_i) \in \Phi$, each $\mathsf{TPP}(r_i, r_j) \in \Phi$ and each $\mathsf{NTPP}(r_i, r_j) \in \Phi$, there is k ($1 \leq k \leq m$) for which $r_j^\circ \in \boldsymbol{\tau}_k$ and $r_i \notin \boldsymbol{\tau}_k$ (note that $r_j \not\subseteq r_i$ if and only if $r_j^\circ \setminus r_i \neq \emptyset$ for regular closed r_i and r_j);

(tpp-e) for each $\mathsf{TPP}(r_i, r_j) \in \Phi$, there is k ($1 \leq k \leq m$) for which $r_i \in \boldsymbol{\tau}_k$ and $r_j^\circ \notin \boldsymbol{\tau}_k$,

as well as the following *universal conditions*, for all k ($1 \leq k \leq m$):

(dc-u) for each $\mathsf{DC}(r_i, r_j) \in \Phi$, either $r_i \notin \boldsymbol{\tau}_k$ or $r_j \notin \boldsymbol{\tau}_k$;

(ec-u) for each $\mathsf{EC}(r_i, r_j) \in \Phi$, either $r_i^\circ \notin \boldsymbol{\tau}_k$ or $r_j \notin \boldsymbol{\tau}_k$ and either $r_j^\circ \notin \boldsymbol{\tau}_k$ or $r_i \notin \boldsymbol{\tau}_k$ (note that $r_i^\circ \cap r_j^\circ = \emptyset$ is equivalent to $r_i^\circ \cap r_j = \emptyset$ and $r_i \cap r_j^\circ = \emptyset$ for regular closed r_i and r_j);

(eq-u) for each $\mathsf{EQ}(r_i, r_j) \in \Phi$, $r_i \in \boldsymbol{\tau}_k$ if and only if $r_j \in \boldsymbol{\tau}_k$, and $r_i^\circ \in \boldsymbol{\tau}_k$ if and only if $r_j^\circ \in \boldsymbol{\tau}_k$;

(tpp-u) for each $\mathsf{TPP}(r_i, r_j) \in \Phi$, if $r_i \in \boldsymbol{\tau}_k$ then $r_j \in \boldsymbol{\tau}_k$, and if $r_i^\circ \in \boldsymbol{\tau}_k$ then $r_j^\circ \in \boldsymbol{\tau}_k$;

(ntpp-u) for each $\mathsf{NTPP}(r_i, r_j) \in \Phi$, if $r_i \in \boldsymbol{\tau}_k$ then $r_j^\circ \in \boldsymbol{\tau}_k$.

Given an assignment $\mathfrak{a}$ over $\mathsf{RC}^+(T)$, we associate with every point $x \in T$ the Φ-type

$$\boldsymbol{\tau}(x, \mathfrak{a}) \;=\; \{\, r \in \bar{r} \mid x \in \mathfrak{a}(r) \,\} \;\cup\; \{\, r^\circ \mid x \in (\mathfrak{a}(r))^\circ,\ r \in \bar{r} \,\}$$

and call it the Φ-*type of* x *under* $\mathfrak{a}$. By the semantics of $\mathcal{RCC}8$, if $\mathfrak{a}$ satisfies $\Phi(\bar{r})$ then there are points x_i, for $1 \leq i \leq 3|\Phi|$, whose Φ-types $\boldsymbol{\tau}(x_i, \mathfrak{a})$ form a type-certificate for Φ; we call any such set a *point-certificate* Φ (*under* $\mathfrak{a}$). Thus, every satisfiable constraint network Φ has a type-certificate of cardinality at most $3|\Phi|$. In Theorem 11, we establish a converse: if a set of $\mathcal{RCC}8$-constraints Φ has a type-certificate, then Φ is satisfiable over $\mathsf{RCP}^+(\mathbb{R})$. Lemma 24 extends this result to the language $\mathcal{RCC}8c$: if there is a type-certificate for Φ that satisfies certain additional 'planarity' conditions, then Φ is satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$ with some chosen regions being connected. Lemma 26 gives a (partial) analogue for the language $\mathcal{RCC}8c^\circ$.

2.3. Important frames

Various frames over $\mathbb{R}^n$ present themselves for consideration. Obviously, $\mathsf{RC}^+(\mathbb{R}^n)$ itself is a frame; however, as mentioned in Sec. 1, we may wish to consider more restricted frames of 'well-behaved' regions.

A function $T \to S$ from one topological space to another is *continuous* if the inverse image of any open set is an open set. A *homeomorphism* is a continuous function having a continuous inverse (on its range). We refer to a homeomorphism $\alpha\colon [0,1] \to \mathbb{R}^n$ as a *Jordan arc*. The points $\alpha(0)$ and $\alpha(1)$ are called the *endpoints* of α; all other points of α are called *internal points*. Where no confusion results, we identify any Jordan arc α with its locus $\alpha([0,1])$. Two Jordan arcs α and β *properly intersect* if there is a point x which is an internal point of both α and β. If α, β are Jordan arcs having the unique common point $\alpha(1) = \beta(0)$, we write $\alpha\beta$ to denote, ambiguously, any Jordan arc from

$\alpha(0)$ to $\beta(1)$ with locus $\alpha \cup \beta$. If $x_1 = \alpha(a_1)$ and $x_2 = \alpha(a_2)$ are points on α with $a_2 > a_1$, then $\alpha[x_1, x_2]$ denotes a Jordan arc whose locus is the segment of α between x_1 and x_2, and $\alpha[x_2, x_1]$ denotes the same Jordan arc but with reversed orientation. A subset X of $\mathbb{R}^n$ is *arc-connected* if any two points of X are joined by a Jordan arc lying in X. Arc-connectedness implies connectedness, and, for open sets, the converse implication holds. If $X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, an *end-cut* (*to* x) *in* X is a Jordan arc $\alpha \subseteq X \cup \{x\}$ with endpoint x. If such an end-cut exists, then x is *accessible from* X. Denote by $\mathbb{S}^1$ the unit circle (in $\mathbb{R}^2$) and by $\mathbb{S}^2$ the unit sphere (in $\mathbb{R}^3$), with both sets having the subspace topology. Topologically, we may think of $\mathbb{S}^n$ ($n = 1, 2$) as the result of adding a 'point at infinity' to $\mathbb{R}^n$. We refer to a homeomorphism $\gamma\colon \mathbb{S}^1 \to T$, where T is either $\mathbb{R}^2$ or $\mathbb{S}^2$, as a *Jordan curve*. Again, we identify Jordan curves with their loci, where convenient. The *Jordan curve theorem* states that, for any Jordan curve γ in $\mathbb{S}^2$, its complement $\mathbb{S}^2 \setminus \gamma$ has exactly two components. This theorem has a converse: if X is a closed set such that $\mathbb{S}^2 \setminus X$ has two components, with each $x \in X$ accessible from both of them, then X is (the locus of) a Jordan curve (see, e.g., [28]). The Schönflies theorem states that, if γ_1 and γ_2 are Jordan curves in $\mathbb{S}^2$ and f is a homeomorphism from (the locus of) γ_1 onto (the locus of) γ_2, then f can be extended to a homeomorphism from $\mathbb{S}^2$ onto itself.

A subset $X \subseteq \mathbb{R}^2$ (or $X \subseteq \mathbb{S}^2$) is a *closed disc-homeomorph* if it is the homeomorphic image of the unit disc $\{(a, b) \mid a^2 + b^2 \leq 1\}$. (The Schönflies theorem tells us that, topologically speaking, closed disc-homeomorphs in $\mathbb{S}^2$—or indeed in $\mathbb{R}^2$—are all the same.) Let $\mathfrak{D}$ denote the set of all closed disc-homeomorphs in $\mathbb{R}^2$. Every element of $\mathfrak{D}$ is a non-empty, regular closed subset of $\mathbb{R}^2$; thus, $\mathfrak{D}$ is a frame over $\mathbb{R}^2$.

A *half-space* (in $\mathbb{R}^n$) is the set of points satisfying any non-degenerate linear inequality $a_1x_1 + \cdots + a_nx_n \geq c$. Thus, every half-space is regular closed. Let the Boolean subalgebra of $\mathsf{RC}(\mathbb{R}^n)$ (finitely) generated by the half-spaces be denoted by $\mathsf{RCP}(\mathbb{R}^n)$, and write $\mathsf{RCP}^+(\mathbb{R}^n)$ as an abbreviation for $\mathsf{RCP}(\mathbb{R}^n) \setminus \{\emptyset\}$. We refer to the elements of $\mathsf{RCP}^+(\mathbb{R}^n)$ as *polyhedra* in $\mathbb{R}^n$. (If $n = 2$, we speak of poly*gons* rather than poly*hedra*.) Thus, $\mathsf{RCP}^+(\mathbb{R}^n)$ is a frame over $\mathbb{R}^n$. Note that polyhedra, in our sense, need not be connected, and need not have connected complements.

A subset of $\mathbb{R}^n$ is said to be *semi-algebraic* if it is the solution set of a Boolean combination of polynomial inequalities with n variables. (Equivalently, a subset of $\mathbb{R}^n$ is semi-algebraic if it is the set of points of $\mathbb{R}^n$ satisfying a first-order formula in the language of arithmetic with n free variables.) The regular closed semi-algebraic subsets of $\mathbb{R}^n$ can be shown to form a Boolean subalgebra of $\mathsf{RC}(\mathbb{R}^n)$; hence, the non-empty regular closed semi-algebraic subsets of $\mathbb{R}^n$ also form a frame.

Let $\mathbf{X}$ be any collection of subsets of a topological space. We say that $\mathbf{X}$ has *finite decomposition* if every $X \in \mathbf{X}$ is the union of finitely many connected elements of $\mathbf{X}$. It is simple to show that $\mathsf{RCP}^+(\mathbb{R}^n)$ has this property. On the other hand, $\mathsf{RC}^+(\mathbb{R}^n)$ lacks finite decomposition, since it evidently contains regions with infinitely many components: the region in Fig. 5a is an example in the case $n = 2$. A set $r \in \mathsf{RC}^+(\mathbb{R}^n)$ has the *curve selection property* if every $x \in \delta r$ is accessible from r°. It is immediate that all polyhedra have curve selection. On the other hand, curve selection fails for elements of $\mathsf{RC}^+(\mathbb{R}^n)$. Consider, for example, the spiral region illustrated in Fig. 5b: there are no end-cuts in the interior of that region to any points lying on the limiting hexadecagon. We call a frame $\mathfrak{F} \subseteq \mathsf{RC}^+(\mathbb{R}^n)$ *tame* if it has finite decomposition and each element of it has curve selection. Thus, the frame $\mathsf{RCP}^+(\mathbb{R}^n)$ is tame for all $n \geq 1$, but $\mathsf{RC}^+(\mathbb{R}^n)$ is not.

In fact, the frame of non-empty, regular closed, semi-algebraic subsets of $\mathbb{R}^n$ can also be shown to be tame, for all $n \geq 1$ (see [29, Theorems 2.4.4 and 2.5.5] for proofs). However, in $\mathbb{R}$, the regular closed, semi-algebraic subsets coincide with the regular closed, semi-linear subsets; and it is well-known that we may transform any finite collection of regular closed semi-algebraic subsets of $\mathbb{R}^2$ into a collection of regular closed polygons by means of a homeomorphism of $\mathbb{R}^2$ onto itself; hence satisfiability of any topological constraint network over the non-empty, regular closed semi-algebraic subsets of $\mathbb{R}^2$ is equivalent to satisfiability over $\mathsf{RCP}^+(\mathbb{R}^2)$. Since the languages considered in this article cannot distinguish between regular closed sets $\mathsf{RC}^+(\mathbb{R}^n)$ and polyhedra $\mathsf{RCP}^+(\mathbb{R}^3)$, for $n \geq 3$, it follows that they cannot distinguish between polyhedra and regular closed, semi-algebraic sets in *any* dimension; in the sequel, therefore, we concentrate on the former for simplicity. The issue of tameness will play a large part in the ensuing discussion.

2.4. Graphs and designs

We employ the usual concepts and terminology of graph theory, with one or two non-standard extensions. A *graph* is a pair $G = (V, E)$, where V is a finite set and E a set of 2-element subsets of V. We call the elements of V *vertices* and the elements of E *edges*. If $|V| = n$ and E is the set of *all* 2-element subsets of V, we denote G by K_n, the *complete* *graph* on n vertices. A graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. If $e = (u, v)$ is an edge in some graph, we consider the operation of 'shrinking' that edge: e itself disappears, and the vertices u and v are replaced by a single new vertex, joined by edges to all and only those other vertices to which either u or v were joined in G. A graph H is a *minor* of G if H is the result of first taking a subgraph of G and then successively shrinking some number of edges.

If $G = (V, E)$ is a graph, a *realization* of G (in $\mathbb{R}^2$) is an injective mapping f from V to points of $\mathbb{R}^2$ and from E to Jordan arcs in $\mathbb{R}^2$ such that, for each $e = (u, v) \in E$: (*i*) the endpoints of $f(e)$ are $f(u)$ and $f(v)$; and (*ii*) no internal points of $f(e)$ are in $f(V)$. It is sometimes more convenient to replace $\mathbb{R}^2$ in this context with the space $\mathbb{S}^2$, in which case we speak of a *realization in* $\mathbb{S}^2$. Of particular interest are realizations in which no two arcs have any internal points in common. (By the definition of realization, no internal point of one arc can be an end-point of another.) We call such a realization a *drawing* of G. It is well-known that a graph G has a drawing in $\mathbb{R}^2$ if and only if it has a drawing in $\mathbb{S}^2$. If G has a drawing (in $\mathbb{R}^2$ or $\mathbb{S}^2$), G is said to be *planar*, and any drawing of G is called a *plane graph*. It is easy to see that any minor of a planar graph is a planar graph. Given any drawing of a (planar) graph G, we can recover G up to isomorphism. It is therefore customary to treat plane graphs as graphs, without comment. A graph G is *k-connected* if it has at least k vertices, and the removal of up to $k-1$ of those vertices leaves G connected. Whitney's theorem (see [30, p. 79]) states that a 3-connected, planar graph has a unique drawing in $\mathbb{S}^2$, up to a homeomorphism of $\mathbb{S}^2$ onto itself.

The following generalization of the notion of planar graph will be used in this article. A *design* is a pair $D = (G, S)$, where $G = (V, E)$ is a graph and S a set of 2-element subsets of E. If $D = (G, S)$ is a design, a *drawing* of D (in either $\mathbb{R}^2$ or $\mathbb{S}^2$) is a realization f of $G = (V, E)$ with the additional property that, for all distinct $e, e' \in E$, $f(e)$ and $f(e')$ have an internal point in common if and only if $(e, e') \in S$. We take the *size* of D to be $|D| = |V| + |E| + |S|$. Again, a design has a drawing in $\mathbb{R}^2$ if and only if it has a drawing in $\mathbb{S}^2$. By the *drawing problem for designs*, we understand the following:

> **Given:** a design D.
> **Return:** yes, if D has a drawing; no, otherwise.

It is immediate from the above definitions that a graph G is planar if and only if the design $(G, \emptyset)$ has a drawing. In fact, we can without loss of generality assume that all drawings have the following, rather particular form. Let us say that a drawing is *rectified* if all arcs are piecewise-linear (i.e., consist of finitely many straight line segments), and no two arcs share any (non-punctual) line segment. In that case, two arcs share an internal point if and only if they cross at that point (in the obvious sense); we call any such point an *arc-crossing* of the drawing. The following upper bound on the complexity of drawings of designs was obtained by Schaefer and Štefankovič:

Proposition 1 ([25]). *Any drawable design D has a rectified drawing with $2^{O(|D|)}$ arc-crossings.*

The corresponding lower bound is due to Kratochvíl and Matoušek:

Proposition 2 ([31]). *There exists a sequence of drawable designs $\{D_n\}_{n\geq 1}$, with D_n of size $O(n)$, such that any rectified drawing of D_n has at least 2^n arc-crossings.*

This lower bound notwithstanding, we have the following remarkable complexity result due to Schaefer, Sedgwick and Štefankovič:

Proposition 3 ([24]). *The drawing problem for designs is* NP*-complete.*

These authors then derive the following corollary, mentioned in Sec. 1:

Proposition 4 ([24, 25]). *Sat*$(\mathcal{RCC}8, \mathfrak{D})$ *is in* NP.

We note in passing that Proposition 1 places an immediate upper bound on the total size of drawings for designs, via the following result of Schnyder:

Proposition 5 ([32]). *Any plane graph with n vertices has a drawing on a $2n \times 2n$ grid in which all vertices have integer coordinates and all edges are straight lines.*

Closely related to the drawing problem for designs is the so-called string graph problem. Let $\alpha_1, \ldots, \alpha_n$ be Jordan arcs. Form the graph (V, E) where $V = \{\alpha_1, \ldots, \alpha_n\}$ and $(\alpha_i, \alpha_j) \in E$ just in case $i \neq j$ and α_i and α_j have at least one point in common. Any graph G isomorphic to such a (V, E) is called a *string graph*, and the sequence of Jordan arcs $\alpha_1, \ldots, \alpha_n$ is called a *string representation* of G. By the *string graph problem*, we understand the following:

> **Given:** a graph G.
> **Return:** yes, if G is a string graph; no, otherwise.

Kratochvíl obtained the following lower bound:

Proposition 6 ([33])**.** *The string graph problem is* NP*-hard.*

A matching upper bound can be obtained from Proposition 3, as shown in [24].

3. $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ in the Euclidean plane: technical lemmas

This section presents some technical lemmas concerning the frames $\mathsf{RCP}^+(\mathbb{R}^2)$ and $\mathsf{RC}^+(\mathbb{R}^2)$; the reader may prefer to omit the proofs, which are straightforward, on first reading.

3.1. Components, i-components and thickenings

We make use of the following simple facts about $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$. Any connected $r \in \mathsf{RCP}^+(\mathbb{R}^2)$ is arc-connected. This is not true for $\mathsf{RC}^+(\mathbb{R}^2)$: the region in Fig. 5b is connected, but not arc-connected (there is no arc connecting a point on the limiting hexadecagon to a point in the interior of the region). On the other hand, for any interior-connected $r \in \mathsf{RC}^+(\mathbb{R}^2)$, the (connected) open set r° is arc-connected. The collection of sets $\mathsf{RCP}^+(\mathbb{R}^n)$ is closed under taking components: if $r \in \mathsf{RCP}^+(\mathbb{R}^n)$ and s is a component of r, then $s \in \mathsf{RCP}^+(\mathbb{R}^n)$. This is not true for $\mathsf{RC}^+(\mathbb{R}^2)$: the region in Fig. 5a has, as a component, the singleton set containing the central point; but this set is not regular closed.

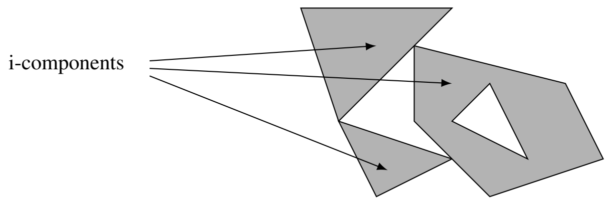


Figure 6: A bounded region r in $\mathsf{RCP}^+(\mathbb{R}^2)$ (dark grey) with 3 i-components.

The following terminology will be useful. If $r \in \mathsf{RC}^+(\mathbb{R}^n)$ and X is a component of r°, then we call X^- an *i-component* of r (Fig. 6). Thus, r is interior-connected just in case it has exactly one i-component. It is easily verified that both $\mathsf{RC}^+(\mathbb{R}^n)$ and $\mathsf{RCP}^+(\mathbb{R}^n)$ are closed under taking i-components. Any $r \in \mathsf{RCP}^+(\mathbb{R}^2)$ is the sum (in the Boolean algebra sense) of its components, and any $r \in \mathsf{RC}^+(\mathbb{R}^2)$ is the sum of its i-components. If $r \in \mathsf{RCP}^+(\mathbb{R}^2)$, then r has only finitely many i-components (hence only finitely many components), so that the relevant sums are finite. The proofs in this section make occasional use of i-components; however, i-components of a region are so visually salient that it is natural to ask what the languages $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ can say about them: we address this question in Sec. 8.

The following lemma establishes a technique that will be used throughout this article.

Lemma 7. *Let X and Y be closed subsets of $\mathbb{R}^n$. Then there exist elements r, s of $\mathsf{RC}^+(\mathbb{R}^n)$ with $X \subseteq r^\circ$ and $Y \subseteq s^\circ$, such that* (*i*) $X \cap Y = \emptyset \Rightarrow r \cap s = \emptyset$; (*ii*) $X \nsubseteq Y \Rightarrow r \nsubseteq s$; (*iii*) $Y \nsubseteq X \Rightarrow s \nsubseteq r$; (*iv*) *$X$ is connected $\Rightarrow$ r is interior-connected*; (*v*) *Y is connected $\Rightarrow$ s is interior-connected. In addition, if X and Y are both bounded, then r and s may be assumed to be bounded elements of* $\mathsf{RCP}^+(\mathbb{R}^n)$.

Proof. If x, y are distinct points of $\mathbb{R}^n$, let $r_{x,y}$, $s_{x,y}$ be non-intersecting closed n-dimensional hyper-cubes containing x and y in their respective interiors. For any point $x \in \mathbb{R}^n \setminus Y$, the collection $\{s^\circ_{x,y}\}_{y \in Y}$ forms an (open) cover of Y. Recalling that $\mathsf{RC}(\mathbb{R}^n)$ is a complete Boolean algebra, set $r_x = \prod_{y \in Y} r_{x,y}$ and $s_x = \sum_{y \in Y} s_{x,y}$. Notice that these sums are infinite, so that r_x and s_x are in $\mathsf{RC}^+(\mathbb{R}^n)$, but not necessarily in $\mathsf{RCP}^+(\mathbb{R}^n)$. Evidently, $x \in r^\circ_x$ and $Y \subseteq s^\circ_x$, and r_x is interior-connected. Moreover, we easily see that, if Y is connected, then s_x is interior-connected. Now, the collection $\{r^\circ_x\}_{x \in X}$ forms an (open) cover of X. Set $r = \sum_{x \in X} r_x$ and $s = \prod_{x \in X} s_x$. (Again, r and s are in $\mathsf{RC}^+(\mathbb{R}^n)$ but not necessarily in $\mathsf{RCP}^+(\mathbb{R}^n)$.) Implications (*i*)–(*iii*) are easy. Implication (*iv*) follows from the fact that each r_x is interior-connected. For implication (*v*), we note that, if Y is connected, then some i-component of s includes Y, and so we can simply replace s by that i-component. This secures implication (*v*) without disturbing implications (*i*)–(*iii*).

For the final statement of the lemma, suppose in addition that X and Y are bounded (hence compact). Then the open covers $\{s^\circ_{x,y}\}_{y \in Y}$ and $\{r^\circ_x\}_{x \in X}$ have finite subcovers, say $\{s^\circ_{x,y}\}_{y \in Y_0}$ and $\{r^\circ_x\}_{x \in X_0}$. We then define the regions r_x, s_x, r and s by taking the sums and products only over those elements of the finite subcovers. In that case, these regions will be bounded elements of $\mathsf{RCP}^+(\mathbb{R}^n)$, as required. □

Thus, we can in effect 'thicken' arbitrary closed subsets of $\mathbb{R}^n$ into interior-connected elements of $\mathsf{RC}^+(\mathbb{R}^n)$ without spoiling the properties of disjointness and non-inclusion. Indeed, *bounded* closed sets can be thickened into regions which are not merely bounded, but also polyhedral. For unbounded closed subsets of $\mathbb{R}^n$, we cannot in general insist that their thickened versions are polyhedral.

Lemma 7 can be generalized: given a collection of sets $X_1, \dots, X_m$, we may thicken them all simultaneously, maintaining pairwise satisfaction of the relevant spatial relations.

Lemma 8. *Let $\bar{X} = X_1, \dots, X_m$ be a tuple of closed subsets of $\mathbb{R}^n$. Then there exist elements $\bar{r} = r_1, \dots, r_m$ of* $\mathsf{RC}^+(\mathbb{R}^n)$ *such that, for all i and j ($1 \le i, j \le m$), (i) $X_i \subseteq r^\circ_i$; (ii) X_i is connected $\Rightarrow r_i$ is interior-connected*; (*iii*) *$X_i \cap X_j = \emptyset \Rightarrow r_i \cap r_j = \emptyset$; (iv) $X_i \not\subseteq X_j \Rightarrow r_i \not\subseteq r_j$. In addition, if $X_1, \dots, X_m$ are all bounded, then $r_1, \dots, r_m$ may be assumed to be bounded elements of* $\mathsf{RCP}^+(\mathbb{R}^n)$.

Proof. We first observe that the proof of Lemma 7 yields slightly more than advertised. Conditions (*ii*) and (*iii*) can be strengthened as follows: (*ii′*) $X \not\subseteq Y \Rightarrow X \not\subseteq s$; and (*iii′*) $Y \not\subseteq X \Rightarrow Y \not\subseteq r$. Now, for all i, j with $1 \le i < j \le m$, apply Lemma 7 to $X = X_i$ and $Y = X_j$ to obtain the sets $r_{i,j} = r$ and $s_{i,j} = s$. For all $1 \le i \le m$, let

$$r'_i = \left(\prod_{j=1}^{i-1} s_{j,i} \right) \cdot \left(\prod_{j=i+1}^{m} r_{i,j} \right).$$

If X_i is connected, let r_i be the i-component of r'_i including X_i; otherwise, let $r_i = r'_i$. Using Conditions (*ii′*) and (*iii′*), it is easy to see that the r_i are as required. □

Lemmas 7 and 8 are required in the sequel only in the case $n = 2$.

3.2. Encoding graphs and designs in $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$

Let $D = (G, S)$ be a design, and let the vertices $v_1, \dots, v_n$ and edges $e_1, \dots, e_m$ of G be ordered in some fixed way. We denote by $u(j)$ and $v(j)$ the indices of the vertices incident on e_j (with $u(j) < v(j)$). Now let $\bar{r} = r_1, \dots, r_n$ and $\bar{s} = s_1, \dots, s_m$ be tuples of elements of $\mathsf{RC}^+(\mathbb{R}^2)$. We say that a drawing f of D is *strongly embedded in* the tuple $\bar{r}\bar{s}$ if

$$f(v_i) \in r^\circ_i \text{ for all } i \;\; (1 \le i \le n) \qquad \text{and} \qquad f(e_j) \subseteq r^\circ_{u(j)} \cup s^\circ_j \cup r^\circ_{v(j)} \text{ for all } j \;\; (1 \le j \le m). \tag{6}$$

If D has a drawing that is strongly embedded in the tuple $\bar{r}\bar{s}$, then D is said to be *strongly embeddable in $\bar{r}\bar{s}$*. If G is a planar graph such that the design $(G, \emptyset)$ is strongly embeddable in $\bar{r}\bar{s}$, then G is said to be *strongly embeddable in $\bar{r}\bar{s}$*. Fig. 7 shows that the planar graph K_4 is strongly embeddable in the tuple $r_1, \dots, r_4, s_1, \dots, s_6$.

We employ $\mathcal{RCC}8c^\circ$-constraint networks to capture strong embeddability of planar graphs. Let G be a graph with vertices $v_1, \dots, v_n$ and edges $e_1, \dots, e_m$ ordered in some fixed way. Define the $\mathcal{RCC}8c^\circ$-constraint network $\Omega^\circ_G(\bar{r}, \bar{s})$, for $\bar{r} = r_1, \dots, r_n$ and $\bar{s} = s_1, \dots, s_m$, by taking, for $1 \le i \ne i' \le n$ and $1 \le j \ne j' \le m$,

$$\begin{array}{llll} c^\circ(r_i), & \mathsf{DC}(r_i, r_{i'}), & \mathsf{PO}(r_i, s_j) & \text{if } v_i \text{ is incident on } e_j, \\ c^\circ(s_j), & \mathsf{DC}(s_j, s_{j'}), & \mathsf{DC}(r_i, s_j) & \text{if } v_i \text{ is not incident on } e_j. \end{array} \tag{7}$$

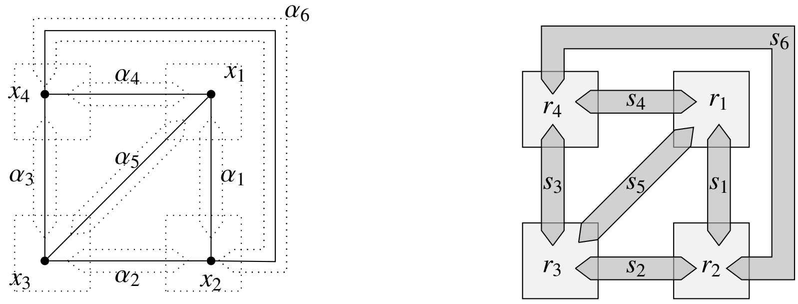


Figure 7: A drawing f of K_4 (with $x_i = f(v_i)$ and $\alpha_j = f(e_j)$) and its strong embedding in the tuple $r_1, \ldots, r_4, s_1, \ldots, s_6$.

Lemma 9. *Let G be a graph. (a) If G is planar then Ω_G° is satisfiable over* $\mathsf{RCP}^+(\mathbb{R}^2)$. *(b) Conversely, let $\hat{\Omega}_G^\circ$ be the result of replacing any number (possibly zero) of the* DC*-constraints in Ω_G° by the corresponding* EC*-constraints. If $\hat{\Omega}_G^\circ$ is satisfiable over* $\mathsf{RC}^+(\mathbb{R}^2)$ *then G is planar, and any satisfying tuple has a rectified (= piecewise-linear) drawing of G strongly embedded in it.*

Proof. (*a*) If G is planar, consider any drawing f of G with points $x_i = f(v_i)$, for $1 \leq i \leq n$, and arcs $\alpha_j = f(e_j)$, for $1 \leq j \leq m$. For all i, $1 \leq i \leq n$, let d_i be a disc centred on x_i, with the d_i pairwise disjoint. For all j, $1 \leq j \leq m$, let α'_j be some segment of α_j which meets $d_{u(j)}$ and $d_{v(j)}$ only at its endpoints. Applying Lemma 8 to the sets $d_1, \ldots, d_n$, $\alpha'_1, \ldots, \alpha'_m$, we obtain $r_1, \ldots, r_n, s_1, \ldots, s_m$ in $\mathsf{RCP}^+(\mathbb{R}^2)$ satisfying Ω_G°.

(*b*) Conversely, if Ω_G° is satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, let $r_1, \ldots, r_n, s_1, \ldots, s_m$ be a satisfying assignment. From the PO-constraints, any s_j partially overlaps the two regions $r_{u(j)}$ and $r_{v(j)}$. For each j, $1 \leq j \leq m$, pick points $y_j \in s_j^\circ \cap r_{u(j)}^\circ$ and $z_j \in s_j^\circ \cap r_{v(j)}^\circ$, and a piecewise-linear Jordan arc $\alpha'_j \subseteq s_j^\circ$ joining y_j to z_j. Now, for each i, $1 \leq i \leq n$, pick a point $x_i \in r_i^\circ$ not lying on any of the α'_j, and for each j with $u(j) = i$, draw a piecewise-linear arc $\beta_{i,j} \subseteq r_i^\circ$ from x_i to y_j, and for each j with $v(j) = i$, draw a piecewise-linear arc $\gamma_{i,j} \subseteq r_i^\circ$ from z_j to x_i. It is easy to see that this can be done in such a way that, for each i, the arcs $\beta_{i,j}$ and $\gamma_{i,j}$ (with j varying), lying in r_i°, intersect precisely at their common endpoint x_i, and that they intersect any arc $\alpha_{j'}$ only when $j = j'$, and then precisely at the other end-point, y_j or z_j. For each j, let $\alpha_j = \beta_{u(j),j}\alpha'_j\gamma_{v(j),j}$; thus, α_j joins $x_{u(j)}$ to $x_{v(j)}$. But either $\mathsf{DC}(r_i, r_{i'})$ or $\mathsf{EC}(r_i, r_{i'})$ imply $r_i^\circ \cap r_j^\circ = \emptyset$, and similarly for the constraints $\mathsf{DC}(r_i, s_j)/\mathsf{EC}(r_i, s_j)$ and $\mathsf{DC}(s_j, s_{j'})/\mathsf{EC}(s_j, s_{j'})$. It follows that the various piecewise-linear arcs α_j intersect only at shared endpoints, and thus constitute a rectified drawing of G, which is strongly embedded in $r_1, \ldots, r_n, s_1, \ldots, s_m$. □

A weaker form of embeddings will also be required in the sequel. As before, let $D = (G, S)$ be a design; let the vertices $v_1, \ldots, v_n$ and edges $e_1, \ldots, e_m$ of G be ordered in some fixed way; and let $\bar{r} = r_1, \ldots, r_n$ and $\bar{s} = s_1, \ldots, s_m$ be tuples of elements of $\mathsf{RC}^+(\mathbb{R}^2)$. We say that a drawing f of D is *weakly embedded in* the tuple $\bar{r}\bar{s}$ if

$$f(v_i) \in r_i^\circ \text{ for all } i \ (1 \leq i \leq n) \qquad \text{and} \qquad f(e_j) \subseteq r_{u(j)}^\circ \cup s_j \cup r_{v(j)}^\circ \text{ for all } j \ (1 \leq j \leq m).$$

(Thus, we have replaced s_j° in (6) by s_j.) Now define the $\mathcal{RCC}8c$-constraint network $\Theta_D(\bar{r}, \bar{s})$, for $\bar{r} = r_1, \ldots, r_n$ and $\bar{s} = s_1, \ldots, s_m$, by taking, for $1 \leq i \neq i' \leq n$ and $1 \leq j \neq j' \leq m$,

$c(r_i)$,	$\mathsf{DC}(r_i, r_{i'})$,		$\mathsf{PO}(r_i, s_j)$	if v_i is incident on e_j,
$c(s_j)$,	$\mathsf{DC}(s_j, s_{j'})$	if $(e_j, e_{j'}) \notin S$,	$\mathsf{DC}(r_i, s_j)$	if v_i is not incident on e_j,
	$\mathsf{EC}(s_j, s_{j'})$	if $(e_j, e_{j'}) \in S$.		

(Thus, we have replaced the constraints $\mathsf{DC}(s_j, s_{j'})$ in (7) by $\mathsf{EC}(s_j, s_{j'})$ whenever $(e_j, e_{j'}) \in S$, and we have replaced c° by c.)

Lemma 10. *Let D be a design. If D has a drawing, then Θ_D is satisfiable over* $\mathsf{RCP}^+(\mathbb{R}^2)$. *Conversely, if Θ_D is satisfiable over* $\mathsf{RCP}^+(\mathbb{R}^2)$ *then D has a drawing; furthermore, any satisfying tuple has a rectified drawing of D weakly embedded in it.*

Proof. The first statement of the lemma is proved in an almost identical way to that of Lemma 9. If D has a drawing, then it has a rectified drawing. The regions r_i are defined as before; however, the regions s_j have the 'string-of-sausages' form shown in Fig. 8, allowing any crossing arcs α_i to lie in a region s_i such that $\mathsf{EC}(s_j, s_i)$. For the

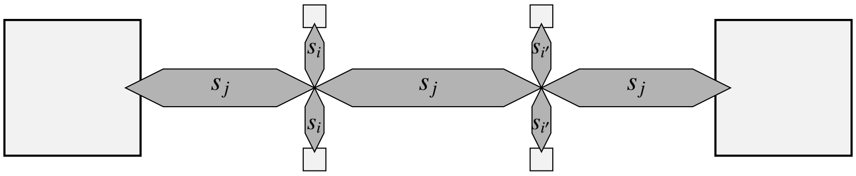


Figure 8: Encoding designs using $\mathcal{RCC}8c$-networks. The crossing of an edge e_j by edges e_i and $e_{i'}$ corresponds to the 'strings-of-sausages,' s_j, s_i and $s_{i'}$, arranged as shown.

converse, we use the fact that connectedness entails arc-connectedness in $\mathsf{RCP}^+(\mathbb{R}^2)$ to construct, for any tuple $\bar{r}\bar{s}$ satisfying Θ_D, a rectified drawing f of D weakly embedded in $\bar{r}\bar{s}$. □

We remark that Lemma 10, in contrast to Lemma 9, makes reference to satisfiability over $\mathsf{RCP}^+(\mathbb{R}^2)$ only. However, we will show later (Theorem 25) that D has a drawing if Θ_D is satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$.

4. Complexity and separation results for $\mathcal{RCC}8$

In this section, we summarize what is known about the satisfiability problem for $\mathcal{RCC}8$-constraint networks over various frames. Although most of the results are not new, we give them with proofs which will serve as a starting point for constructions later on in the article.

Nebel [17] considered the problem of determining whether, for a given $\mathcal{RCC}8$-constraint network Φ, there exists a topological space T such that Φ is satisfiable over $\mathsf{RC}^+(T)$. He showed this problem to be in NLogSpace by reducing it—via Bennett's [15] encoding of $\mathcal{RCC}8$ in intuitionistic propositional logic—to 2SAT, which is known to be in NLogSpace (a different proof was given in [21]). Renz [20] observed further that any $\mathcal{RCC}8$-constraint network satisfiable over $\mathsf{RC}^+(T)$, for some topological space T, can also be satisfied over $\mathsf{RCP}^+(\mathbb{R}^n)$, for any $n \geq 1$; moreover, for $n \geq 3$, it can be satisfied over $\mathsf{RCP}^+(\mathbb{R}^n)$ by a tuple of interior-connected regions. The next two theorems give a direct and concise proof of these results.

Theorem 11. *Let $\Phi(\bar{r})$ be an $\mathcal{RCC}8$-constraint network. Then the following are equivalent*:

(a) *there exists a type-certificate for* Φ;

(b) Φ *is satisfiable over* $\mathsf{RC}^+(T)$ *for some topological space* T;

(c) Φ *is satisfiable over* $\mathsf{RCP}^+(\mathbb{R}^n)$ *for any* $n \geq 3$; *moreover, the satisfying assignment may be chosen so that all the regions in $\bar{p}$ are interior-connected and all the regions in $\bar{q}$ are not connected, for any partitioning $\bar{p}\bar{q}$ of $\bar{r}$*;

(d) Φ *is satisfiable over* $\mathsf{RCP}^+(\mathbb{R})$ *and* $\mathsf{RCP}^+(\mathbb{R}^2)$.

Proof. The implication (d) $\Rightarrow$ (c) is easy. Suppose Φ is satisfiable over $\mathsf{RCP}^+(\mathbb{R})$. Take any satisfying assignment under which every region in $\bar{r}$ is disconnected (this can always be achieved by taking a disjoint copy of a given satisfying assignment). Then we cylindrify this assignment in $\mathbb{R}^3$ and use the third dimension to make the regions in $\bar{p}$ interior connected (in the same way as in [20]) while keeping the regions in $\bar{q}$ disconnected.

The implication (c) $\Rightarrow$ (b) is trivial. The implication (b) $\Rightarrow$ (a) follows from the observation in Sec. 2.2 that, for every satisfiable Φ, there is a type-certificate (of at most $3|\Phi|$ types).

Thus, it remains to prove (a) $\Rightarrow$ (d). Suppose there exists a type-certificate for $\Phi(\bar{r})$. Without loss of generality, we may assume that Φ contains no $\mathsf{EQ}(r_i, r_j)$—if this is not the case, we replace all occurrences of r_i with r_j and remove $\mathsf{EQ}(r_j, r_j)$. We write $r_j <_\Phi r_i$ if there is a sequence

$$j = k_1 < \cdots < k_l = i \quad \text{such that} \quad \mathsf{(N)TPP}(r_{k_1}, r_{k_2}), \ldots, \mathsf{(N)TPP}(r_{k_{l-1}}, r_{k_l}) \in \Phi.$$

By **(type)**, **(reg-e)**, **(tpp-u)**, **(ntpp-u)** and **(diff-e)**, $<_\Phi$ is a strict partial order on $\bar{r}$. We also write $r_j \leq_\Phi r_i$ if either $r_j = r_i$ or $r_j <_\Phi r_i$, and write $r_j \ll_\Phi r_i$ if $r_j <_\Phi r_i$ and at least one of the relations in the sequence above is NTPP. In view of **(type)**, **(tpp-u)** and **(ntpp-u)**, we then have the following, for each type $\boldsymbol{\tau}$ in the type-certificate for Φ:

$$\text{if} \quad r_j \leq_\Phi r_i \quad \text{and} \quad r_j \in \boldsymbol{\tau} \quad \text{then} \quad r_i \in \boldsymbol{\tau}, \tag{8}$$

$$\text{if} \quad r_j \leq_\Phi r_i \quad \text{and} \quad r_j^\circ \in \tau \quad \text{then} \quad r_i^\circ, r_i \in \boldsymbol{\tau}, \tag{9}$$

$$\text{if} \quad r_j \ll_\Phi r_i \quad \text{and} \quad r_j \in \boldsymbol{\tau} \quad \text{then} \quad r_i^\circ, r_i \in \boldsymbol{\tau}. \tag{10}$$

As $<_\Phi$ is a strict partial order, we can assume the variables $\bar{r} = r_1, \dots, r_n$ to be ordered in such a way that

$$j < i \quad \text{whenever} \quad r_j <_\Phi r_i. \tag{11}$$

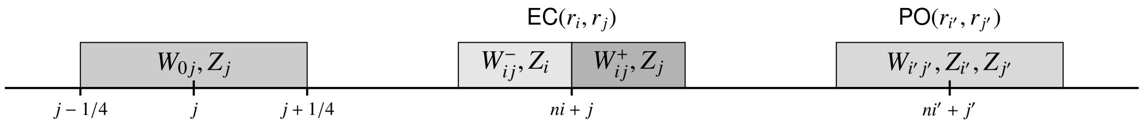


Figure 9: Three types of intervals in the proof of Theorem 11.

We begin by considering $\mathsf{RCP}^+(\mathbb{R})$. First we define regions $Z_1, \dots, Z_n$ to satisfy the existential conditions of the constraints in Φ and then, by taking account of the TPP and NTPP constraints, extend $Z_1, \dots, Z_n$ to regions $X_1, \dots, X_n$ satisfying Φ. For $0 \leq i \leq n$ and $1 \leq j \leq n$, let W_{ij}^- and W_{ij}^+ be the non-empty adjacent intervals $[ni + j - 1/4, ni + j]$ and $[ni + j, ni + j + 1/4]$, respectively, and let W_{ij} be the union of W_{ij}^- and W_{ij}^+. Each Z_j is defined to be the union of the following intervals (see Fig. 9):

$$\begin{array}{ll} W_{0j}, & \\ W_{ij}^-, & \text{if } \mathsf{EC}(r_i, r_j) \in \Phi, \\ W_{ij}^+, & \text{if } \mathsf{EC}(r_j, r_i) \in \Phi, \\ W_{ij}, & \text{if } \mathsf{PO}(r_i, r_j) \in \Phi \text{ or } \mathsf{PO}(r_j, r_i) \in \Phi. \end{array}$$

The centre of each W_{0j} is a point whose Φ-type is guaranteed by **(reg-e)** and **(diff-e)**, while the centre of each W_{ij} with $i > 0$ is a point whose Φ-type is guaranteed by the respective **(ec-e)** or **(po-e)**. In addition, the construction we are about to present will ensure that the two boundary points, $j - 1/4$ and $j + 1/4$, of each W_{ij} provide a Φ-type required by **(tpp-e)**. By construction, **(dc-e)**, **(po-e)** and **(type)**, the following conditions are respected by the Z_i:

$$Z_j \cap Z_{j'} \neq \emptyset \quad \Longrightarrow \quad \text{any type-certificate for } \Phi \text{ has a type } \boldsymbol{\tau} \text{ such that } r_j, r_{j'} \in \boldsymbol{\tau}; \tag{12}$$

$$Z_j^\circ \cap Z_{j'}^\circ \neq \emptyset \quad \Longrightarrow \quad \text{any type-certificate for } \Phi \text{ has a type } \boldsymbol{\tau} \text{ such that } r_j^\circ, r_{j'}^\circ \in \boldsymbol{\tau}. \tag{13}$$

Note that all the intervals W_{ij} are of length $1/2$. We fix some ε such that $0 < \varepsilon \leq 1/(6n)$. Given a union s of disjoint intervals $[a_i, b_i]$, we denote by $s^{+\varepsilon}$ the union of the intervals $[a_i - \varepsilon, b_i + \varepsilon]$. We shall define a sequence of regions $X_1, \dots, X_n$ by repeated application of the $\cdot^{+\varepsilon}$-operator to unions of the W_{ij}. The choice of ε guarantees that distinct intervals W_{ij} and $W_{i'j'}$ remain disjoint even after n applications of $\cdot^{+\varepsilon}$.

We proceed step-by-step in the order $X_1, \dots, X_n$ and set, for $1 \leq i \leq n$,

$$X_i \quad = \quad Z_i \quad + \sum\nolimits_{\mathsf{TPP}(r_j, r_i) \in \Phi} X_j \quad + \sum\nolimits_{\mathsf{NTPP}(r_j, r_i) \in \Phi} X_j^{+\varepsilon}. \tag{14}$$

This definition is legitimate by (11). The constructed regions for the constraint network (1) are illustrated in Fig. 10.

We now prove that $X_1, \dots, X_n$ satisfy Φ. First, we show, by induction on i, that, for all $j, j' \leq i$,

$$X_j \cap X_{j'} \neq \emptyset \quad \Longrightarrow \quad \text{any type-certificate for } \Phi \text{ has a type } \boldsymbol{\tau} \text{ such that } r_j, r_{j'} \in \boldsymbol{\tau};$$

$$X_j^\circ \cap X_{j'}^\circ \neq \emptyset \quad \Longrightarrow \quad \text{any type-certificate for } \Phi \text{ has a type } \boldsymbol{\tau} \text{ such that } r_j^\circ, r_{j'} \in \boldsymbol{\tau} \text{ or } r_j, r_{j'}^\circ \in \boldsymbol{\tau}.$$

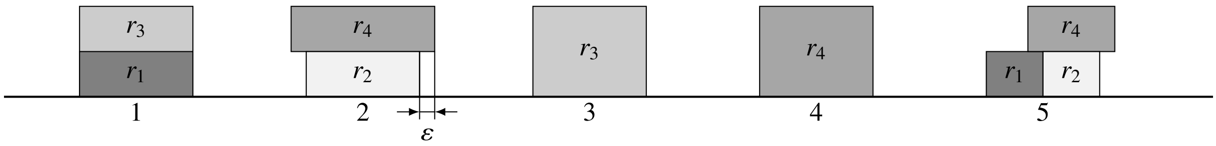


Figure 10: An assignment satisfying the constraint network (1).

The basis of induction, $i = 1$, is trivial. For the inductive step, it is enough to consider only $j < j' = i$ (because the case with $j, j' < i$ trivially follows from the induction hypothesis).

Suppose $X_j \cap X_i \neq \emptyset$. Two cases are possible. If $X_j \cap Z_i = \emptyset$ then, by (14) and the choice of ε, there is $j' < i$ such that $X_j \cap X_{j'} \neq \emptyset$ and either $\mathsf{TPP}(r_{j'}, r_i) \in \Phi$ or $\mathsf{NTPP}(r_{j'}, r_i) \in \Phi$, that is, $r_{j'} \leq_\Phi r_i$; the required type exists by the induction hypothesis and (8). Otherwise, we have $X_j \cap Z_i \neq \emptyset$ and so, by (14), either $Z_j \cap Z_i \neq \emptyset$ or there is $j' < j$ such that $\mathsf{TPP}(r_{j'}, r_j) \in \Phi$ and $X_{j'} \cap Z_i \neq \emptyset$ or there is $j' < j$ such that $\mathsf{NTPP}(r_{j'}, r_j) \in \Phi$ and $(X_{j'})^{+\varepsilon} \cap Z_i \neq \emptyset$. Thus, by the choice of ε and (14), there is $j' \leq j$ such that $Z_{j'} \cap Z_i \neq \emptyset$ and $r_{j'} \leq_\Phi r_j$, and so the required type exists by (12) and (8).

Suppose $X_j^\circ \cap X_i^\circ \neq \emptyset$. Two cases are possible. If $X_j^\circ \cap Z_i^\circ = \emptyset$ then, by (14) and the choice of ε, there is $j' < i$ such that either $X_j^\circ \cap X_{j'}^\circ \neq \emptyset$ and $r_{j'} \leq_\Phi r_i$, or $X_j \cap X_{j'} \neq \emptyset$ and $r_{j'} \ll_\Phi r_i$. In the former case, by the induction hypothesis, any type-certificate for Φ has a type $\boldsymbol{\tau}$ such that $r_j^\circ, r_{j'} \in \boldsymbol{\tau}$ or $r_j, r_{j'}^\circ \in \boldsymbol{\tau}$, from which, by (8) and (9), we obtain $r_j^\circ, r_i \in \boldsymbol{\tau}$ or $r_j, r_i^\circ \in \boldsymbol{\tau}$. In the latter case, by the induction hypothesis, any type-certificate for Φ has a type $\boldsymbol{\tau}$ such that $r_j, r_{j'} \in \boldsymbol{\tau}$, which gives $r_j, r_i^\circ \in \boldsymbol{\tau}$ by (10). If $X_j^\circ \cap Z_i^\circ \neq \emptyset$ then, by (14) and the choice of ε, there is $j' \leq j$ such that either $Z_{j'}^\circ \cap Z_i^\circ \neq \emptyset$ and $r_{j'} \leq_\Phi r_j$ or $Z_{j'} \cap Z_i \neq \emptyset$ and $r_{j'} \ll_\Phi r_j$. The required type exists by (13) and (9) in the former case and by (12) and (10) in the latter case.

Next, we show that $R(X_j, X_i)$, for all $R(r_j, r_i) \in \Phi$.

- If $\mathsf{DC}(r_j, r_i) \in \Phi$ then $X_j \cap X_i = \emptyset$, for otherwise the type-certificate for Φ would contain a type with both r_j and r_i, contrary to **(dc-u)**.
- If $\mathsf{EC}(r_j, r_i) \in \Phi$ then, by construction, $Z_j \cap Z_i \neq \emptyset$, whence, by (14), $X_j \cap X_i \neq \emptyset$. We also have $X_j^\circ \cap X_i^\circ \neq \emptyset$, for otherwise the type-certificate for Φ would contain a type with either r_j°, r_i or r_j, r_i°, contrary to **(ec-u)**.
- If $\mathsf{PO}(r_j, r_i) \in \Phi$ then, by construction, $Z_j^\circ \cap Z_i^\circ \neq \emptyset$, whence, by (14), $X_j^\circ \cap X_i^\circ \neq \emptyset$. By **(diff-e)** and **(type)**, there is a type τ in the type-certificate for Φ with $r_j^\circ, r_j \in \tau$ and $r_i \notin \tau$. If we assume $X_j \subseteq X_i$ then, by (14), $r_j \leq_\Phi r_i$ and, by (8), $r_i \in \tau$, which is impossible. Symmetrically, we cannot have $X_i \subseteq X_j$.
- If $\mathsf{TPP}(r_j, r_i) \in \Phi$ then, by (14), $X_j \subseteq X_i$. By **(diff-e)** and **(type)**, there is a type τ in the type-certificate for Φ with $r_j^\circ, r_j \in \tau$ and $r_i \notin \tau$. If we assume $X_j \subseteq X_i^\circ$ then, by (14), $r_j \ll_\Phi r_i$, and, by (10), $r_i \in \tau$, which is impossible. We cannot also have $X_i \subseteq X_j$ because $j < i$.
- If $\mathsf{NTPP}(r_j, r_i) \in \Phi$ then, by (14), we have $X_j \subseteq X_i^\circ$.

The construction for $\mathsf{RCP}^+(\mathbb{R}^2)$ uses rectangles instead of intervals; the argument is identical. This completes the proof of Theorem 11. ☐

As a consequence of Theorem 11, we obtain the following:

Theorem 12. *All of the problems Sat*($\mathcal{RCC}8, \mathsf{RC}^+(\mathbb{R}^n)$) *and Sat*($\mathcal{RCC}8, \mathsf{RCP}^+(\mathbb{R}^n)$)*, where* $n \geq 1$*, are identical and* NLogSpace*-complete.*

Proof. We can encode the satisfiability problem for $\mathcal{RCC}8$-constraint networks as satisfiability of binary clauses with the quantifier prefix of the form $\exists^*\forall$. Indeed, for each variable r in Φ, we take two unary predicates, $P_r(x)$ and $P_{r^\circ}(x)$, to represent membership of the respective terms, r and r°, in Φ-types. Recall that Φ has a type-certificate in case it has a type-certificate of at most $3|\Phi|$ types. Then the existential and universal conditions for the type-certificate can easily be encoded as binary clauses: for example,

$$P_{r_1}(x_i) \wedge P_{r_2}(x_i), \tag{ec-e}$$

$$P_{r_1}(y) \rightarrow P_{r_2^\circ}(y), \tag{nttp-u}$$

$$\neg P_{r_1}(y) \vee \neg P_{r_2^\circ}(y), \quad \neg P_{r_2}(y) \vee \neg P_{r_1^\circ}(y), \tag{ec-u}$$

where the x_i are $3|\Phi|$-many existentially quantified variables and y is a single universally quantified variable. The upper complexity bound follows then from the fact that satisfiability of binary clauses of this form is NLogSpace-complete [34, Exercise 8.3.7].

NLogSpace-hardness can be shown by reduction of the reachability problem for directed acyclic graphs (DAGs), which is known to be NLogSpace-complete. Let $G = (V, E)$ be a DAG with $V = \{v_1, \dots, v_k\}$. We take variables $v_1, \dots, v_k$ and consider the set Φ_G of constraints containing $\mathsf{NTPP}(v_i, v_j)$ just in case $(v_i, v_j) \in E$. It should be clear that v_k is reachable from v_0 if and only if $\mathsf{DC}(v_0, v_k)$ is inconsistent with Φ_G. □

Note that if instead of atomic $\mathcal{RCC}8$-constrains we consider *unions* of $\mathcal{RCC}8$-predicates, then the satisfiability problem becomes NP-complete; for this and related tractability results consult [22, 19, 20, 35].

Let Φ be a set of constraints in either $\mathcal{RCC}8c$ or $\mathcal{RCC}8c^\circ$, and let Ψ be the result of removing from Φ all constraints of the form $c(r)$ or $c^\circ(r)$. Then, as a consequence of Theorem 11, we see that the following are equivalent: Ψ is satisfiable over $\mathsf{RC}(T)$ for some topological space T; Φ is satisfiable over $\mathsf{RC}^+(\mathbb{R}^n)$, for $n \geq 3$; Φ is satisfiable over $\mathsf{RCP}^+(\mathbb{R}^n)$, for $n \geq 3$. Thus, $\mathcal{RCC}8$ is not sensitive to the dimension of the Euclidean space or to tameness of regions; and in dimension 3 or above, $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ are not interestingly different from $\mathcal{RCC}8$.

We end this section with some separation results for $\mathcal{RCC}8$ over various salient sub-frames of $\mathsf{RC}^+(\mathbb{R}^2)$. To make the presentation more compact, we help ourselves with a result to be proved below (Theorem 25), which establishes the insensitivity of $\mathcal{RCC}8c$ to tameness in the plane. Recall that $\mathfrak{D}$ denotes the frame of all closed disc-homeomorphs in $\mathbb{R}^2$. Define $\mathfrak{C}$ to be the frame of all *connected* elements of $\mathsf{RC}^+(\mathbb{R}^2)$, and $\mathfrak{I}$ the frame of all *interior-connected* elements of $\mathsf{RC}^+(\mathbb{R}^2)$. Thus, $\mathfrak{D} \subseteq \mathfrak{I} \subseteq \mathfrak{C} \subseteq \mathsf{RC}^+(\mathbb{R}^2)$. It was observed [16] that $\mathit{Sat}(\mathcal{RCC}8, \mathfrak{D}) \subsetneqq \mathit{Sat}(\mathcal{RCC}8, \mathsf{RC}^+(\mathbb{R}^2))$. In fact, we have the following:

Theorem 13. *$\mathit{Sat}(\mathcal{RCC}8, \mathfrak{D}) \subsetneqq \mathit{Sat}(\mathcal{RCC}8, \mathfrak{I}) \subsetneqq \mathit{Sat}(\mathcal{RCC}8, \mathfrak{C}) \subsetneqq \mathit{Sat}(\mathcal{RCC}8, \mathsf{RC}^+(\mathbb{R}^2))$.*

Proof. The inclusions are trivial. To show that they are proper, we employ the graph K_5, with vertices and edges numbered identically to the points and arcs in Fig. 11a.

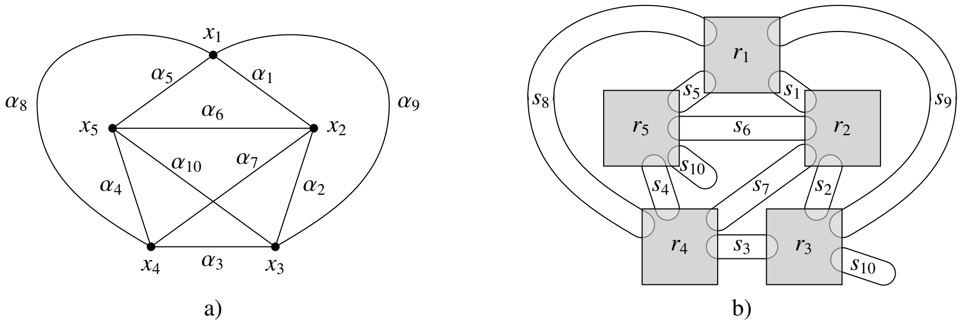


Figure 11: Graph K_5 and satisfying Φ over $\mathsf{RC}^+(\mathbb{R}^2)$.

To show that $\mathit{Sat}(\mathcal{RCC}8, \mathfrak{C}) \subsetneqq \mathit{Sat}(\mathcal{RCC}8, \mathsf{RC}^+(\mathbb{R}^2))$, let Φ be the result of removing all c-constraints from $\Theta_{(K_5,\emptyset)}$. Since K_5 is non-planar, Lemma 10 ensures that $\Theta_{(K_5,\emptyset)}$ is not satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$, whence, by Theorem 25, $\Theta_{(K_5,\emptyset)}$ is not satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, and thus Φ is unsatisfiable over $\mathfrak{C}$. On the other hand, Φ is trivially satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, as we see from Fig 11b. Note that s_{10} is not connected in this assignment.

To show that $\mathit{Sat}(\mathcal{RCC}8, \mathfrak{I}) \subsetneqq \mathit{Sat}(\mathcal{RCC}8, \mathfrak{C})$, let $\hat{\Omega}^\circ_{K_5}$ be the result of replacing the constraint $\mathsf{DC}(s_7, s_{10})$ in $\Omega^\circ_{K_5}$ by $\mathsf{EC}(s_7, s_{10})$, and let Ψ be the result of removing all c°-constraints from $\hat{\Omega}^\circ_{K_5}$. By Lemma 9 (b), $\hat{\Omega}^\circ_{K_5}$ is unsatisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, whence Ψ is not satisfiable over $\mathfrak{I}$. On the other hand, Ψ is easily satisfiable over $\mathfrak{C}$ by taking s_7 and s_{10} in Fig. 11b to be crossing strings-of-sausages as in Fig 8.

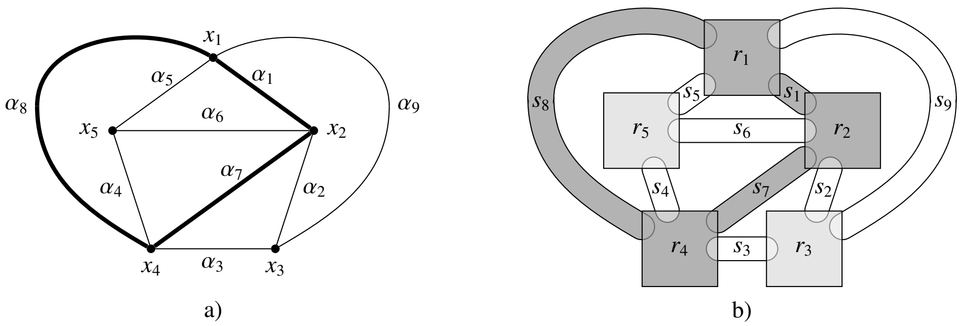


Figure 12: A drawing of the graph K_5^- and satisfying Ξ over $\mathfrak{I}$.

To show that $Sat(\mathcal{RCC}8, \mathfrak{D}) \subsetneqq Sat(\mathcal{RCC}8, \mathfrak{I})$, we consider the graph K_5 with vertices and edges numbered identically to the points and arcs in Fig. 11a and let K_5^- be the graph K_5 without the edge α_{10}. Since K_5^- is 3-connected, by Whitney's theorem (see [30, p. 79]), Fig. 12a presents its only drawing in $\mathbb{S}^2$ up to a homeomorphism of $\mathbb{S}^2$ onto itself. In particular, in all drawings (in $\mathbb{R}^2$), the points x_3 and x_5 are separated by the arcs α_1, α_7 and α_8. Let Ξ be the result of removing all c°-constraints from $\Omega^\circ_{K_5^-}$. As we see from Fig. 12b, $\Omega^\circ_{K_5^-}$ is satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$—and hence Ξ is satisfiable over $\mathfrak{I}$. However, by Lemma 9 (b), in any such satisfying assignment, the regular closed set $r_1 \cup s_1 \cup r_2 \cup s_7 \cup r_4 \cup s_8$ separates r_3 from r_5. Now let t be a new variable, and Ξ' the result of adding to Ξ the constraints

$$\mathsf{TPP}(r_1,t),\quad \mathsf{TPP}(s_1,t),\quad \mathsf{TPP}(r_2,t),\quad \mathsf{TPP}(s_7,t),\quad \mathsf{TPP}(r_4,t),\quad \mathsf{TPP}(s_8,t),\qquad \mathsf{DC}(r_3,t),\quad \mathsf{DC}(r_5,t).$$

Setting $t = r_1 \cup s_1 \cup r_2 \cup s_7 \cup r_4 \cup s_8$ yields a satisfying assignment over $\mathfrak{I}$. On the other hand, we have shown that t separates r_3 and r_5, and so must have a non-connected complement. Therefore, Ξ' is not satisfiable over $\mathfrak{D}$. □

5. Separation results with connectedness constraints

We now turn to the main subject of the present article: the languages $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ in one- and two-dimensional space. Our first task is to show that these languages are indeed sensitive to dimension up to 3. Again, to obtain a compact presentation, we anticipate the result of Theorem 25 stating that $\mathcal{RCC}8c$ is insensitive to tameness in the Euclidean plane.

Theorem 14. *Let $\mathcal{L}$ be either of the languages $\mathcal{RCC}8c$ or $\mathcal{RCC}8c^\circ$. Then*

$$Sat(\mathcal{L}, \mathsf{RC}^+(\mathbb{R})) \subsetneqq Sat(\mathcal{L}, \mathsf{RC}^+(\mathbb{R}^2)) \subsetneqq Sat(\mathcal{L}, \mathsf{RC}^+(\mathbb{R}^3)),$$
$$Sat(\mathcal{L}, \mathsf{RCP}^+(\mathbb{R})) \subsetneqq Sat(\mathcal{L}, \mathsf{RCP}^+(\mathbb{R}^2)) \subsetneqq Sat(\mathcal{L}, \mathsf{RCP}^+(\mathbb{R}^3)).$$

Proof. The inclusions hold because any tuple in $\mathsf{RC}^+(\mathbb{R}^n)$ can easily be cylindrified to form a tuple in $\mathsf{RC}^+(\mathbb{R}^m)$, for $m > n$, satisfying the same $\mathcal{L}$-networks. To show that the leftmost inclusions are proper, consider the network (5), saying that three connected regions r_1, r_2 and r_3 are in external contact with each other. This network cannot be satisfied over $\mathsf{RC}^+(\mathbb{R})$ because the r_i must be non-empty, closed, non-punctual intervals. However, we can easily satisfy (5) over any $\mathsf{RCP}^+(\mathbb{R}^n)$ for $n \geq 2$ (see Fig. 4a). The same argument holds for the $\mathcal{RCC}8c^\circ$-network obtained from (5) by replacing c with c°.

To show that the rightmost inclusions are proper for $\mathcal{RCC}8c^\circ$, consider the constraint network $\Omega^\circ_{K_5}$ for the non-planar graph K_5. By Lemma 9 (b), it is not satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, and thus over $\mathsf{RCP}^+(\mathbb{R}^2)$; on the other hand, it is clearly satisfiable over $\mathsf{RCP}^+(\mathbb{R}^3)$, and so over $\mathsf{RC}^+(\mathbb{R}^3)$. To show that the inclusions are proper for $\mathcal{RCC}8c$, consider the constraint network $\Theta_{(K_5,\emptyset)}$ for the same graph K_5. By Lemma 10, it is not satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$, whence, by Theorem 25, it is not satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$. On the other hand, again, it is clearly satisfiable over both $\mathsf{RCP}^+(\mathbb{R}^3)$ and $\mathsf{RC}^+(\mathbb{R}^3)$. □

What of sensitivity of $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ to tameness in low dimensions? Here a more complicated picture emerges. In one-dimensional space (where connectedness and interior-connectedness coincide) we indeed observe sensitivity to tameness:

Theorem 15. *Sat*($\mathcal{RCC}8c, \mathsf{RCP}^+(\mathbb{R})$) $\subsetneq$ *Sat*($\mathcal{RCC}8c, \mathsf{RC}^+(\mathbb{R})$). *Indeed, there exists an $\mathcal{RCC}8c$-formula that is satisfiable over $\mathsf{RC}^+(\mathbb{R})$, but only by tuples some of whose elements have infinitely many components.*

Proof. The inclusion is trivial. Consider the $\mathcal{RCC}8c$-network

$$\begin{aligned} &c(r_1), \quad c(r_2), \quad \mathsf{EC}(r_1,r_2), \quad \mathsf{EC}(r_1,r_3), \quad \mathsf{EC}(r_1,r_4), \\ &\qquad\qquad\qquad\qquad\qquad \mathsf{DC}(r_2,r_3), \quad \mathsf{DC}(r_2,r_4), \quad \mathsf{EC}(r_3,r_4). \end{aligned} \tag{15}$$

Fig. 13, where 0 is the accumulation point for r_3 and r_4, shows how this network can be satisfied over $\mathsf{RC}^+(\mathbb{R})$. To see that it cannot be satisfied over $\mathsf{RCP}^+(\mathbb{R})$, let $r_1 = [c,d]$ and $r_2 = [a,b]$. Since $\mathsf{EC}(r_1,r_2)$, and by applying a reflection if necessary, $b = c$. Since $\mathsf{DC}(r_2,r_3)$ and $\mathsf{EC}(r_1,r_3)$, r_3 includes some interval $[d,e]$ (with $e > d$). Since $\mathsf{EC}(r_3,r_4)$ and $\mathsf{DC}(r_2,r_4)$, it follows that r_4 has no points in $(a,b) \cup (d,e)$, contrary to $\mathsf{EC}(r_1,r_4)$. □

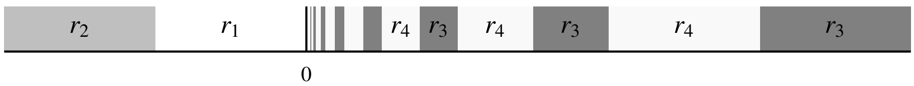


Figure 13: Subsets of $\mathbb{R}$ used in the proof of Theorem 15.

In two dimensions, we find that $\mathcal{RCC}8c^\circ$ is sensitive to tameness.

Theorem 16. *Sat*($\mathcal{RCC}8c^\circ, \mathsf{RCP}^+(\mathbb{R}^2)$) $\subsetneq$ *Sat*($\mathcal{RCC}8c^\circ, \mathsf{RC}^+(\mathbb{R}^2)$).

Proof. Again, the inclusion is trivial. To show it is proper, we exhibit an $\mathcal{RCC}8c^\circ$-network Φ satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, but not over $\mathsf{RCP}^+(\mathbb{R}^2)$. Let $G = (V,E)$ be the cyclic graph C_8 (see Fig. 14a), i.e.,

$$V = \{v_1,\ldots,v_8\} \quad \text{and} \quad E = \{(v_i,v_{i+1}) \mid 1 \le i \le 7\} \cup \{(v_8,v_1)\}.$$

Let $\bar{r} = r_1,\ldots,r_8$ and $\bar{s} = s_1,\ldots,s_8$ be tuples of variables and consider the network $\Omega^\circ_G(\bar{r},\bar{s})$. Let $\bar{t} = t_1,t_2$ be a pair of fresh variables. We form the network $\Phi(\bar{r},\bar{s},\bar{t})$ by adding to Ω°_G the constraints

$$c^\circ(t_k), \qquad \mathsf{EC}(t_k,r_i), \qquad \mathsf{EC}(t_k,s_i), \qquad 1 \le k \le 2, \;\; 1 \le i \le 8, \tag{16}$$

$$\mathsf{EC}(t_1,t_2). \tag{17}$$

We first show that Φ is not satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$. For suppose to the contrary that there exist regular closed polygons $\bar{r}$, $\bar{s}$, $\bar{t}$ making Φ true. By Lemma 9 (b), there exists a drawing of the cyclic graph C_8, with points $x_1,\ldots,x_8$ and arcs $\alpha_1,\ldots,\alpha_8$, such that $x_i \subseteq r^\circ_i$ and $\alpha_i \subseteq r^\circ_i \cup s^\circ_i \cup r^\circ_{i+1}$, for $1 \le i \le 7$, and $\alpha_8 \subseteq r^\circ_8 \cup s^\circ_8 \cup r^\circ_1$. Let γ denote the Jordan curve $\alpha_1 \ldots \alpha_8$. Note that the four points x_2, x_4, x_6 and x_8 divide γ into the four segments $\alpha_2\alpha_3$, $\alpha_4\alpha_5$, $\alpha_6\alpha_7$ and $\alpha_8\alpha_1$, as shown in Fig. 14b.

Setting $k = 1$ in (16), we have $\mathsf{EC}(t_1,r_1)$ and $\mathsf{EC}(t_1,r_5)$, so let $y_2 \in t_1 \cap r_1$ and $y_3 \in t_1 \cap r_5$. From Ω°_G, r_1 is interior-connected; and using the curve selection property, we can construct a Jordan arc $\beta^0_1 \subseteq r_1$ from x_1 to y_2. Similarly, we can construct a Jordan arc $\beta^0_3 \subseteq r_5$ from y_3 to x_5. By (16), t_1 is also interior-connected, whence the curve selection property again ensures the existence of a Jordan arc $\beta_2 \subseteq \{y_2\} \cup t^\circ_1 \cup \{y_3\}$ from y_2 to y_3. Let y_1 be the last point of β^0_1 lying on γ, and let $\beta_1 = \beta^0_1[y_1,y_2]$ (that is, β_1 is the segment of β^0_1 between y_1 and y_2). Let y_4 be the first point of β^0_3 lying on γ, and let $\beta_3 = \beta^0_3[y_3,y_4]$. From the DC-constraints in Ω°_G, y_1 must be an internal point of $\alpha_8\alpha_1$, and y_4 an

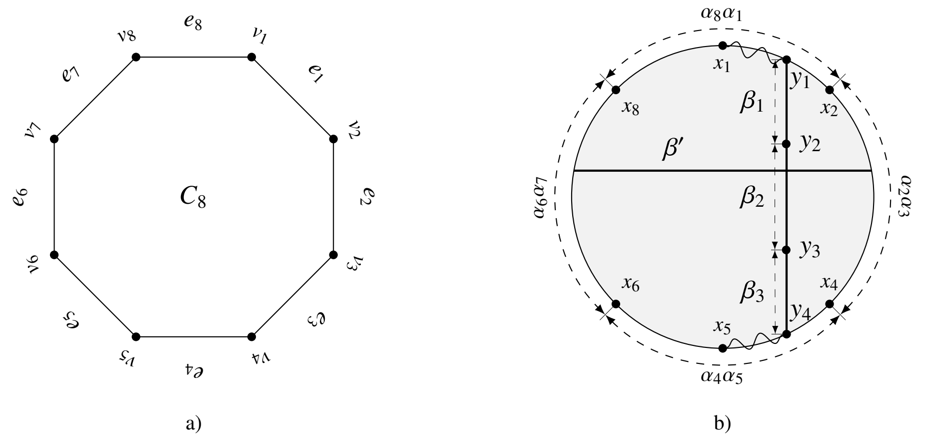


Figure 14: The network Φ given in the proof of Theorem 16 is not satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$.

internal point of $\alpha_4\alpha_5$. Furthermore, $\beta = \beta_1\beta_2\beta_3 \subseteq r_1 \cup t_1^\circ \cup r_5$ is a chord of γ. By setting $k = 2$ in (16), we can find another chord $\beta' \subseteq r_3 \cup t_2^\circ \cup r_7$ joining an internal point of $\alpha_2\alpha_3$ to an internal point of $\alpha_6\alpha_7$. Since $\gamma \subseteq \bigcup_{i=1}^{8}(r_i^\circ \cup s_i^\circ)$, (16) guarantees that t_1 and t_2 are disjoint from γ. In addition, (17) ensures that t_1 and t_2—and hence also the chords β and β'—lie on the same side of γ. By inspection of Fig. 14, β and β'—and hence $r_1 \cup t_1^\circ \cup r_5$ and $r_3 \cup t_2^\circ \cup r_7$—intersect. On the other hand, Ω_G° ensures that the r_i are pairwise disjoint; moreover (16) prevents t_1° or t_2° from intersecting any of the r_j. Therefore, t_1° intersects t_2°, contradicting (17). This completes the proof that Φ is not satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$.

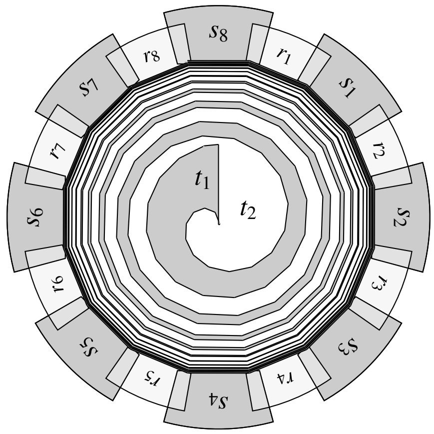


Figure 15: A satisfying assignment for Φ over $\mathsf{RC}^+(\mathbb{R}^2)$ in Theorem 16.

We next show that Φ is satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$. Let $r_1, \ldots, r_8$ and $s_1, \ldots, s_8$ be arranged as a (hexadecagonal) annulus as in Fig. 15. Denote the inner disc of the annulus by $d \in \mathsf{RC}^+(\mathbb{R}^2)$. Now let t_1 be an infinitely long spiral strip lying in d, and converging to the frontier of the annulus as shown; and let $t_2 = d \cdot (-t_1)$. It is obvious that Ω_G° and (16)–(17) hold. $\square$

We remark that it was falsely claimed [36] that $Sat(\mathcal{RCC}8c^\circ, \mathsf{RCP}^+(\mathbb{R}^2)) = Sat(\mathcal{RCC}8c^\circ, \mathsf{RC}^+(\mathbb{R}^2))$.[3] Observe that the set $t_1 \in \mathsf{RC}^+(\mathbb{R}^2)$ illustrated in Fig. 15 lacks the curve selection property: we can find points $y_2 \in t_1 \cap r_1$ and $y_3 \in t_1 \cap r_5$; but there is no arc $\beta_2 \subseteq \{y_2\} \cup t_1^\circ \cup \{y_3\}$ connecting them. Observe also that the proof that Φ is unsatisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$ collapses if we replace the interior-connectedness predicate c° with the connectedness predicate c: if

[3] This is the second statement of Theorem 6, p. 537; the first statement of that theorem, repeated here as Theorem 25, is correct.

β and β' are arcs lying in t_1 and t_2, respectively, but not necessarily in t_1° and t_2°, then the condition $\mathsf{EC}(t_1, t_2)$ cannot be used to force their disjointness. Indeed, as we shall see below (Theorem 25), $\mathcal{RCC}8c$ is *in*sensitive to tameness in two dimensions.

6. NP-completeness over $\mathsf{RCP}^+(\mathbb{R})$ and $\mathsf{RC}^+(\mathbb{R})$

In the one-dimensional space $\mathbb{R}$, the notions of connectedness and interior-connectedness coincide: a non-empty regular closed subset of $\mathbb{R}$ is (interior-) connected if and only if it has one of the forms $[a, b]$, $(-\infty, b]$, $[a, +\infty)$ or $(-\infty, +\infty)$, for some $a < b$ in $\mathbb{R}$. Hence, we only consider the language $\mathcal{RCC}8c$. Note also that polyhedra in $\mathbb{R}$ are finite unions of regular closed, connected subsets—i.e., finite unions of non-punctual, closed intervals. Arbitrary regular closed subsets of $\mathbb{R}$ may be infinite unions of such intervals together with their accumulation points (see Figs. 13 and 19).

We show first that the satisfiability problem for $\mathcal{RCC}8c$-networks is harder over $\mathbb{R}$ than over $\mathbb{R}^3$ (cf. Corollary 12) unless NLogSpace = NP:

Theorem 17. *Both Sat*($\mathcal{RCC}8c$, $\mathsf{RCP}^+(\mathbb{R})$) *and Sat*($\mathcal{RCC}8c$, $\mathsf{RC}^+(\mathbb{R})$) *are* NP*-hard.*[4]

Proof. The proof is by reduction of the 3-colourability problem, which is known to be NP-complete: given a graph, decide whether its vertices can be painted in 3 different colours in such a way that no two vertices connected by an edge of the graph are of the same colour. So suppose we are given a graph $G = (V, E)$. We are going to construct an $\mathcal{RCC}8c$-network Φ_G such that Φ_G is satisfiable over $\mathsf{RCP}^+(\mathbb{R})$ or $\mathsf{RC}^+(\mathbb{R})$ if and only if G is 3-colourable. (The idea of the construction is similar to the one used to prove NP-hardness of Allen's interval calculus [38].) We represent the colours by the connected regions p_1, p_2, p_3: the constraints

$$\mathsf{TPP}(p_1, p), \quad \mathsf{NTPP}(p_2, p), \quad \mathsf{TPP}(p_3, p), \quad c(p), \tag{18}$$

$$\mathsf{EC}(p_1, p_2), \quad \mathsf{EC}(p_2, p_3), \quad \mathsf{DC}(p_1, p_3), \quad c(p_i), \quad \text{for } i = 1, 2, 3, \tag{19}$$

say that p is a closed interval divided into three subintervals, p_1, p_2 and p_3, with p_2 being in the middle. The constraints

$$\mathsf{PO}(q_i, p_i), \quad \mathsf{PO}(q_i, p_{i+1}), \quad \mathsf{NTPP}(q_i, p), \quad c(q_i), \quad \text{for } i = 1, 2, \qquad \text{and} \qquad \mathsf{DC}(q_1, q_2) \tag{20}$$

say that the meeting point of p_1 and p_2 is in the interior of some interval q_1, the meeting point of p_2 and p_3 is in the interior of some interval q_2, and that the q_i are disjoint and lie inside p, as shown in Fig. 16.

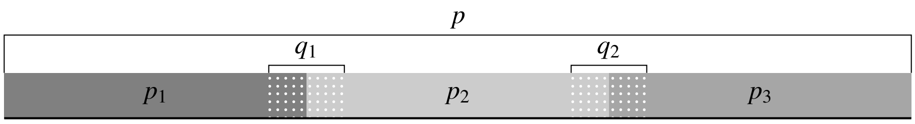


Figure 16: Satisfying (18)–(20) over $\mathsf{RC}^+(\mathbb{R})$.

The constraints

$$\mathsf{DC}(v, q_1), \quad \mathsf{DC}(v, q_2), \quad \mathsf{NTPP}(v, p), \quad c(v), \quad \text{for } v \in V, \tag{21}$$

ensure then that each interval v, for $v \in V$, lies entirely inside one of the p_i (here we deliberately overload notation and use v to denote both vertices of G and the region variables representing those vertices). Thus, it remains to ensure that the intervals corresponding to adjacent vertices in G are of different colours. This can be done using the following constraints, for all $(u, v) \in E$:

$$\mathsf{NTPP}(t_{uv}, p), \quad \mathsf{EC}(u, t_{uv}), \quad \mathsf{EC}(v, t_{uv}), \quad \mathsf{DC}(u, v), \quad c(t_{uv}), \tag{22}$$

$$\mathsf{NTPP}(s_{uv}, t_{uv}), \quad \mathsf{PO}(s_{uv}, p_2), \quad c(s_{uv}). \tag{23}$$

[4] As observed by one of the anonymous reviewers, this theorem follows from the classification of tractable subalgebras of Allen's interval algebra given in [37].

By (23), one of the end points of the interval s_{uv} belongs to p_2, while the other either to p_1 or p_3. It follows that the end points of t_{uv} are of different colours, and so u and v must be of different colours, too. Thus, if the union Φ_G of (18)–(23) is satisfiable, then G is 3-colourable.

To prove the converse, suppose that G is 3-colourable with the colours p_1, p_2 and p_3. First we satisfy (18)–(20) as shown in Fig. 16. Now, take any edge $(u, v) \in E$. If u is of colour p_1 and v of colour p_2, then we define s_{uv} and t_{uv} such that their left ends are in p_1, right ends are in p_2, $\mathsf{NTPP}(q_1, s_{uv})$ and $\mathsf{NTPP}(s_{uv}, t_{uv})$, and take u to be some interval inside p_1 for which $\mathsf{EC}(u, t_{uv})$ holds, and v to be some interval inside p_2 for which $\mathsf{EC}(v, t_{uv})$ holds. If v is of colour p_3, then we define s_{uv} as before, but extend t_{uv} to some point inside p_3 after q_2, and take v to be in p_3 as well. The remaining cases are mirror-images, and left to the reader. It should be clear that we can define such an assignment simultaneously for all edges in G. ☐

Now, we establish a matching upper bound. Let us assume first that an $\mathcal{RCC}8c$-network $\Phi(\bar{r})$ is satisfied over $\mathsf{RCP}^+(\mathbb{R})$ by some assignment $\mathfrak{a}$. For any open interval I of the form (a, b), $(-\infty, b)$ or (a, ∞), we take the set

$$\chi_{\mathfrak{a}}(I) = \{\, r \in \bar{r} \mid \mathfrak{a}(r) \cap I \neq \emptyset \,\}$$

and call it the Φ-*character of I under* $\mathfrak{a}$. We say that $\mathfrak{a}$ is *uniform over I* if each of its points belongs to *each* of the regions of $\chi_{\mathfrak{a}}(I)$; in other words, if $\chi_{\mathfrak{a}}(I) = \chi_{\mathfrak{a}}(I')$, for each open subinterval $I' \subseteq I$. Since every element of $\mathsf{RCP}^+(\mathbb{R})$ is a finite union of closed intervals, we can find finitely many points $z_1 < \cdots < z_k$ in $\mathbb{R}$ such that $\mathfrak{a}$ is uniform over each of the open intervals (z_i, z_{i+1}), for $0 \leq i \leq k$, where $z_0 = -\infty$ and $z_{k+1} = +\infty$. For all i ($0 \leq i \leq k$), let χ_i be the Φ-character of (z_i, z_{i+1}) under $\mathfrak{a}$. Notice that $z_i \in \mathfrak{a}(r)$ if and only if $r \in \chi_{i-1} \cup \chi_i$. As the exact values of $z_1, \ldots, z_k$ are irrelevant as far as satisfiability is concerned, we can think of $\mathfrak{a}$ as given by the tuple $(\chi_0, \ldots, \chi_k)$. We illustrate this representation by an example.

Example 18. Consider the following $\mathcal{RCC}8c$-network Φ:

$$\mathsf{DC}(p, q), \quad \mathsf{EC}(p, r), \quad \mathsf{EC}(q, r), \quad c(r),$$
$$\mathsf{NTPP}(p, p_1), \quad \mathsf{NTPP}(p_1, p_2), \quad \mathsf{NTPP}(q, q_1), \quad \mathsf{NTPP}(q_1, q_2), \quad \mathsf{DC}(p_2, q_2).$$

This network is satisfied over $\mathsf{RCP}^+(\mathbb{R})$ by the assignment $(\chi_0, \ldots, \chi_6)$, where each χ_i is the Φ-character of (z_i, z_{i+1}), $i = 0, \ldots, 6$, in Fig. 17 (for instance, $\chi_0 = \{p, p_1, p_2\}$).

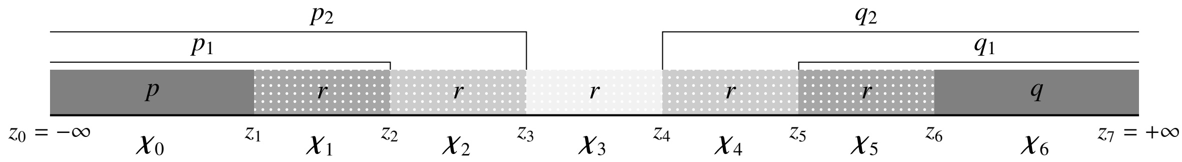


Figure 17: Satisfying Φ over $\mathsf{RCP}^+(\mathbb{R})$.

Our plan is to show that, given an assignment $\mathfrak{a}$ satisfying Φ, we can transform it into an assignment that has a small (polynomial in $|\Phi|$) number of Φ-characters.

Lemma 19. *Let $\Phi(r_1, \ldots, r_n)$ be an $\mathcal{RCC}8c$-network satisfied by an assignment $\mathfrak{a}$ over $\mathsf{RCP}^+(\mathbb{R})$ and let $x_1, \ldots, x_m$ be a point-certificate for Φ under $\mathfrak{a}$. Let $[z_1, z_2]$ be a closed non-punctual interval containing no x_i and such that $\mathfrak{a}$ is uniform over $(z_1 - \varepsilon, z_1)$ and $(z_2, z_2 + \varepsilon)$, for some $\varepsilon > 0$. Then Φ is satisfied by an assignment $\mathfrak{a}'$ that differs from $\mathfrak{a}$ only on $[z_1, z_2]$ in such a way that $[z_1, z_2]$ is partitioned into $2n - 1$ intervals with $\mathfrak{a}'$ uniform over the interior of each of them.*

Proof. Without loss of generality, we may assume that Φ contains no $\mathsf{EQ}(r_i, r_j)$—if this is not the case, we replace all occurrences of r_i with r_j and remove $\mathsf{EQ}(r_j, r_j)$. Denote the Φ-character of $(z_1 - \varepsilon, z_1)$ by χ_- and the Φ-character of $(z_2, z_2 + \varepsilon)$ by χ_+. Starting with χ_- and χ_+, we construct a tuple of Φ-characters that gives the required assignment between z_1 and z_2.

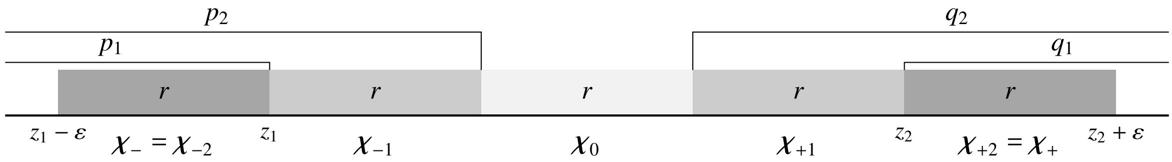


Figure 18: Proof of Lemma 19: construction of the assignment $\mathfrak{a}'$ for Example 18.

It may be impossible (due to connectedness constraints in Φ) to take the empty set for the character of (z_1, z_2): cf. r in Example 18. On the other hand, due to DC-constraints in Φ, it may also be impossible to split $[z_1, z_2]$ in half to obtain a pair of adjacent open intervals with the characters χ_- and χ_+, respectively: in Example 18 (see Fig. 18), χ_- contains p_1 and χ_+ contains q_1, which cannot share boundary points because they are included in disjoint p_2 and q_2, respectively. The following notation will be useful. If χ is a set of variables of Φ, denote by $\chi\!\uparrow$ the smallest extension of χ that contains each s with $r \in \chi\!\uparrow$ and either $\mathsf{TPP}(r, s) \in \Phi$ or $\mathsf{NTPP}(r, s) \in \Phi$. Our strategy, then, will be to ensure that (z_1, z_2) has character

$$\chi_0 = \{\, r \in \chi_- \cap \chi_+ \mid c(r) \in \Phi \,\}\!\uparrow$$

(although the assignment $\mathfrak{a}'$ we are constructing will not necessarily be uniform over this interval). By definition, $\chi_0 \subseteq \chi_- \cap \chi_+$. Moreover, χ_0 respects all the universal conditions of the constraints in Φ. (In Fig. 18, we have $\chi_0 = \{r\}$.) However, χ_0 cannot be simply assigned to (z_1, z_2) and thus made adjacent to χ_- and χ_+ because of the NTPP constraints in Φ, as illustrated by Example 18, where $\chi_- = \{r, p_1, p_2\}$ and $\chi_0 = \{r\}$: since $\mathsf{NTPP}(p_1, p_2) \in \Phi$, the regions p_1 and p_2 cannot share boundary points, and so we have to insert an intermediate character $\chi_{-1} = \{r, p_2\}$ between χ_- and χ_0; see Fig. 18. In general, we define two sequences of characters

$$\chi_- = \chi_{-n} \supseteq \cdots \supseteq \chi_{-0} = \chi_0 = \chi_{+0} \subseteq \cdots \subseteq \chi_{+n} = \chi_+. \tag{24}$$

We begin with $\chi_{-n} = \chi_-$ and, if $\chi_{-n}, \dots, \chi_{-(j+1)}$ have already been defined, then we set

$$\chi_{-j} \quad = \quad (\chi_0 \cup \{\, s \in \chi_{-(j+1)} \mid r \in \chi_{-(j+1)} \text{ and } \mathsf{NTPP}(r, s) \in \Phi \,\})\!\uparrow.$$

Intuitively, χ_{-j} eliminates from $\chi_{-(j+1)}$ the elements that are minimal with respect to NTPP: in Example 18, χ_{-1} eliminates p_1 from χ_{-2} and χ_{-0} further eliminates p_2 from χ_{-1}. First, observe that $\chi_0 \subseteq \chi_{-j} \subseteq \chi_{-(j+1)}$, for all j, $0 \leq j < n$. Moreover, we claim that $\chi_{-0} = \chi_0$. Indeed, assume that $\chi_{-0} \supsetneq \chi_0$. Let $s_0 \in \chi_{-0} \setminus \chi_0$. Then there is some $s_1 \in \chi_{-1}$ such that $\mathsf{NTPP}(s_1, s_0) \in \Phi$. We have $s_1 \notin \chi_0$ for otherwise $s_0 \in \chi_0$, which is impossible. Thus, there is some $s_2 \in \chi_{-2}$ such that $\mathsf{NTPP}(s_2, s_1) \in \Phi$. As Φ is satisfied (by $\mathfrak{a}$), we cannot repeat this argument more than n times, which proves our claim. The other sequence, $\chi_{+n}, \dots, \chi_{-0}$, is defined symmetrically.

We can redefine the assignment $\mathfrak{a}$ over $[z_1, z_2]$ by taking the tuple of characters (24). We claim that the resulting assignment $\mathfrak{a}'$ satisfies Φ. Indeed, by construction, $\mathfrak{a}'(r)$ is connected, for all $c(r) \in \Phi$. The existential conditions of the constraints in Φ are satisfied by $\mathfrak{a}'$ because their adjacent Φ-characters are the same as under $\mathfrak{a}$. The new characters are subsets of χ_- and χ_+, and so respect all the DC constraints in Φ; the universal conditions of the EC constraints in Φ are respected because the characters (24) are subsets of χ_- and χ_+ and touch only them; the universal conditions of the TPP and NTPP constraints in Φ are satisfied by construction. ☐

As an almost immediate consequence, we obtain:

Theorem 20. *Sat*($\mathcal{RCC}8c$, $\mathsf{RCP}^+(\mathbb{R})$) *is* NP*-complete.*

Proof. NP-hardness is by Theorem 17. The matching upper bound follows from a polynomial model property of the kind established by Lemma 19: if Φ is satisfied by $\mathfrak{a}$ then we can fix at most $3|\Phi|$ points of a point-certificate for Φ under $\mathfrak{a}$ and, by Lemma 19, find a tuple of at most $3|\Phi| \times 2|\Phi|$ Φ-characters that induce a satisfying assignment for Φ. Clearly, we can guess such a tuple of $6|\Phi|^2$ Φ-characters and check in polynomial time that: (*i*) the set of Φ-types obtained by taking $\{r^\circ \mid r \in \chi_1 \cap \chi_2\} \cup \chi_1 \cup \chi_2$, for each pair of adjacent Φ-characters χ_1 and χ_2, is a type-certificate for Φ; and (*ii*) any region r with $c(r) \in \Phi$ is connected in the sense that the set of Φ-characters containing r is contiguous in the tuple. ☐

We now show how the construction above can be modified to the case of arbitrary regular closed regions in $\mathbb{R}$. Recall (cf. Fig. 13) that such regions can be *infinite* unions of closed non-punctual intervals with their accumulation points. The first step in the proof of Theorem 20 was to pick some point-certificate for $\Phi(\bar{r})$. Over $\mathsf{RC}^+(\mathbb{R})$, however, a satisfying assignment may not be uniform in any of the intervals to the left or to the right of a point z in the point-certificate. Nevertheless, since the regions are regular closed sets, there is an interval $(z-\varepsilon, z)$ such that each of its subintervals of the form $(z-\varepsilon', z)$, $0 < \varepsilon' < \varepsilon$, contains points not only from the same collection of $\mathfrak{a}(r_i)$ as $(z-\varepsilon, z)$ but also points from the same collection of $-\mathfrak{a}(r_i)$; recall that $-X = (\mathbb{R} \setminus X)^-$. For example, in the context of Fig. 13, the point 0 belongs to any point-certificate because its type is required for $\mathsf{EC}(r_1, r_3)$, $\mathsf{EC}(r_1, r_4)$ and $\mathsf{EC}(r_3, r_4)$. Every sufficiently small neighbourhood $(-\varepsilon, 0)$ contains points in $\mathfrak{a}(r_1)$ only (these points are also in the complements of $\mathfrak{a}(r_i)$ for $i = 2, 3, 4$); but every sufficiently small neighbourhood $(0, \varepsilon)$ contains not only points in $\mathfrak{a}(r_3)$ and points in $\mathfrak{a}(r_4)$ but also points in the complements of $\mathfrak{a}(r_3)$ and $\mathfrak{a}(r_4)$.

We say that $\mathfrak{a}$ is *left-uniform in* z_2 *over* (z_1, z_2) if every subinterval $(z_2 - \varepsilon, z_2) \subseteq (z_1, z_2)$ satisfies the following:

$$X \cap (z_1, z_2) \neq \emptyset \quad \text{iff} \quad X \cap (z_2 - \varepsilon, z_2) \neq \emptyset, \qquad \text{for all } X \text{ of the form } \mathfrak{a}(r) \text{ and } -\mathfrak{a}(r).$$

We say that $\mathfrak{a}$ is *right-uniform in* z_1 *over* (z_1, z_2) if the mirror image condition is satisfied. For example, the assignment in Fig. 13 is right-uniform in 0 over $(0, +\infty)$ but it is not uniform over any $(0, \varepsilon)$. On the other hand, if $\mathfrak{a}$ is uniform over (a, b) then it is both left-uniform in b over (a, b) and right-uniform in a over (a, b). Note that an interval with a left- or right-uniform assignment can contain points of exponentially many distinct Φ-types (this is in contrast to an interval with a uniform assignment, all points of which are of the same Φ-type). Nevertheless, an assignment left-uniform in z_2 over (z_1, z_2) can be redefined on that interval so that it is uniform over each of the intervals forming an infinite sequence converging to z_2; moreover, the converging intervals can be made to have a small (linear in $|\Phi|$) number of Φ-characters (the case of right-uniform assignments is symmetrical). This will provide us with polynomial representations of assignments.

Lemma 21. *Let $\Phi(r_1, \dots, r_n)$ be an $\mathcal{RCC}8c$-network satisfied by an assignment $\mathfrak{a}$ over $\mathsf{RC}^+(\mathbb{R})$. Fix a point-certificate for Φ under $\mathfrak{a}$. Let z be one of them and $\varepsilon > 0$ be such that the distance between any two points in the certificate is at least 3ε. If $\mathfrak{a}$ is left-uniform in z over $(z-\varepsilon, z)$ (right-uniform in z over $(z, z+\varepsilon)$) then Φ has a satisfying assignment $\mathfrak{a}'$ that differs from $\mathfrak{a}$ only on $[z-\varepsilon, z)$ (respectively, on $(z, z+\varepsilon]$) and such that this interval is a union of intervals converging to z and having at most $2n$ distinct Φ-characters under $\mathfrak{a}'$, with $\mathfrak{a}'$ uniform over the interior of each of these intervals.*

Proof. Let Θ be the set of all r with $\mathfrak{a}(r) \cap (z-\varepsilon, z) \neq \emptyset$ and $(-\mathfrak{a}(r)) \cap (z-\varepsilon, z) \neq \emptyset$. If $\Theta = \emptyset$ then $\mathfrak{a}$ is uniform over $(z-\varepsilon, z)$ and we are done. Otherwise, for each $r \in \Theta$, z is a boundary point of $\mathfrak{a}(r)$, and so, since $\mathfrak{a}(r)$ is regular closed, there is a pair of open intervals $I_r, I'_r \subseteq (z-\varepsilon, z)$ such that $I_r \subseteq \mathfrak{a}(r)$ and $I'_r \cap \mathfrak{a}(r) = \emptyset$ with $\mathfrak{a}$ uniform over both I_r and I'_r (that is, each of the two intervals is either entirely in $\mathfrak{a}(r_i)$ or does not intersect $\mathfrak{a}(r_i)$, for any r_i, $1 \leq i \leq n$). Let $\chi_1, \dots, \chi_{2k}$ be a tuple of Φ-characters of the intervals I_r and I'_r, for $r \in \Theta$. Consider an infinite sequence of intervals in $(z-\varepsilon, z)$ converging to z and assign the characters $\chi_1, \dots, \chi_{2k}$ to the members of this sequence in a cyclical way (see Fig. 19). Define $\mathfrak{a}'$ by taking the Φ-types of points induced by this sequence of Φ-characters over $[z-\varepsilon, z)$; $\mathfrak{a}'$ coincides with $\mathfrak{a}$ elsewhere. To see that $\mathfrak{a}'$ satisfies Φ, observe that $\mathfrak{a}'(r)$ is connected, for each connected $\mathfrak{a}(r)$. The Φ-type of z under $\mathfrak{a}'$ coincides with the type of z under $\mathfrak{a}$, and so the existential conditions of all constraints are satisfied because the types of the points in the point-certificate have not changed. The Φ-characters under $\mathfrak{a}'$ are all taken from $\mathfrak{a}$, and thus satisfy the universal conditions of the DC, TPP and NTPP constraints in Φ. Finally, the Φ-type of $z-\varepsilon$ is also consistent with Φ (for otherwise, $\mathfrak{a}$ would not be left-uniform in z over $(z-\varepsilon, z)$). □

Figure 19: Subsets of $(z-\varepsilon, z)$ to represent the tuple χ_1, χ_2, χ_3.

Theorem 22. *Sat($\mathcal{RCC}8c$, $\mathsf{RC}^+(\mathbb{R})$) is* NP*-complete.*

Proof. NP-hardness was shown by Theorem 17. The matching upper bound follows from the polynomial model property established by Lemmas 19 and 21. Indeed, suppose an $\mathcal{RCC}8c$-netweork $\Phi(r_1, \ldots, r_n)$ is satisfied by an assignment $\mathfrak{a}$ over $\mathsf{RC}^+(\mathbb{R})$. Let $z_1 < \cdots < z_m$, for $m \leq 3|\Phi|$, be a point-certificate for Φ under $\mathfrak{a}$. Take some ε such that $0 < \varepsilon < \min_{1 \leq i < m}(z_{i+1} - z_i)/3$, $\mathfrak{a}$ is left-uniform in z_i over each $(z_i - \varepsilon, z_i)$ and right-uniform in z_i over each $(z_i, z_i + \varepsilon)$. By Lemma 21, we can construct a satisfying assignment $\mathfrak{a}'$ for Φ such that $\mathfrak{a}'$ differs from $\mathfrak{a}$ only on $[z_i - \varepsilon, z_i)$ and $(z_i, z_i + \varepsilon]$, $1 \leq i \leq m$, and on each of these intervals $\mathfrak{a}'$ is induced by a converging sequence of $2n$ Φ-characters. Next, by Lemma 19, each closed interval $[z_i + \varepsilon, z_{i+1} - \varepsilon]$, $1 \leq i < m$, is subdivided into a sequence of $2n - 1$ subintervals and $\mathfrak{a}'$ is redefined so that the result is uniform over each of the subintervals; the two infinite intervals, $(-\infty, z_1 - \varepsilon]$ and $[z_m + \varepsilon, \infty)$, are dealt with in a similar way. This gives us a polynomial representation of a satisfying assignment. More precisely, we obtain a sequence of $6mn$ Φ-characters that are arranged in the following structure

$$\cdots = \underbrace{\chi^j_{2n,L}, \ldots, \chi^j_{1,L}}_{2n\ \Phi\text{-characters}}, \overset{z_j}{\downarrow} \underbrace{\chi^j_{1,R}, \ldots, \chi^j_{2n,R}}_{2n\ \Phi\text{-characters}} = \underbrace{\chi^j_{-n}, \ldots, \chi^j_{-0} = \chi^{j+1}_{+0}, \ldots, \chi^{j+1}_{+n}}_{2n-1\ \Phi\text{-characters}} = \underbrace{\chi^{j+1}_{2n,L}, \ldots, \chi^{j+1}_{1,L}}_{2n\ \Phi\text{-characters}}, \overset{z_{j+1}}{\downarrow} \underbrace{\chi^{j+1}_{1,R}, \ldots, \chi^{j+1}_{2n,R}}_{2n\ \Phi\text{-characters}} = \cdots$$

and such that the following Φ-types form a type-certificate for Φ:

- $\boldsymbol{\tau}^j_{+i}$ and $\boldsymbol{\tau}^j_{-i}$, for i, $0 \leq i \leq n$, and j, $1 \leq j \leq m$;
- $\left\{r^\circ \mid r \in \chi^j_{+i} \cap \chi^j_{+(i+1)}\right\} \cup \chi^j_{+i} \cup \chi^j_{+(i+1)}$ and $\left\{r^\circ \mid r \in \chi^j_{-i} \cap \chi^j_{-(i+1)}\right\} \cup \chi^j_{-i} \cup \chi^j_{-(i+1)}$, for i, $0 \leq i < n$, and j, $1 \leq j \leq m$;
- $\boldsymbol{\tau}^j_{i,L}$ and $\boldsymbol{\tau}^j_{i,R}$, for i, $1 \leq i \leq 2n$, and j, $1 \leq j \leq m$;
- $\left\{r^\circ \mid r \in \bigcap_{i=1}^{2n}(\chi^j_{i,L} \cap \chi^j_{i,R})\right\} \cup \bigcup_{i=1}^{2n}(\chi^j_{i,L} \cup \chi^j_{i,R})$, for j, $1 \leq j \leq m$,

where $\boldsymbol{\tau}^j_\sigma$ is the Φ-type for χ^j_σ, that is, $\boldsymbol{\tau}^j_\sigma = \{r^\circ \mid r \in \chi^j_\sigma\} \cup \chi^j_\sigma$. Informally, the Φ-types of the first and the third sort correspond to the internal points of the respective Φ-characters; the Φ-types of the second sort correspond to the boundary points shared by pairs of adjacent intervals having characters either χ^j_{+i} and $\chi^j_{+(i+1)}$, or χ^{j+1}_{-i} and $\chi^{j+1}_{-(i+1)}$; finally, the Φ-types of the fourth sort correspond to the points z_j, which are on the boundary of all the intervals converging to z_j from both the left and the right.

Our NP satisfiability checking algorithm guesses such a structure of $O(|\Phi|)$ Φ-characters and then checks whether its induced Φ-types form a type-certificate for Φ and every region r_i with $c(r_i) \in \Phi$ is connected in the obvious sense. □

7. NP-completeness of $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ in the Euclidean plane

In this section, we show that the problems $Sat(\mathcal{RCC}8c, \mathsf{RCP}^+(\mathbb{R}^2))$ and $Sat(\mathcal{RCC}8c, \mathsf{RC}^+(\mathbb{R}^2))$ are identical and NP-complete. In addition, we show that the problem $Sat(\mathcal{RCC}8c^\circ, \mathsf{RCP}^+(\mathbb{R}^2))$ is NP-complete, and that the problem $Sat(\mathcal{RCC}8c^\circ, \mathsf{RC}^+(\mathbb{R}^2))$ is NP-hard. We showed in Sec. 5 that these problems are distinct. At the time of writing, the decidability of $Sat(\mathcal{RCC}8c^\circ, \mathsf{RC}^+(\mathbb{R}^2))$ is open.

We begin with the lower bounds, which are established by reduction of the NP-complete string graph problem (Proposition 6).

Lemma 23. *Let $\mathcal{L}$ be either of $\mathcal{RCC}8c$ or $\mathcal{RCC}8c^\circ$. Then the problems $Sat(\mathcal{L}, \mathsf{RCP}^+(\mathbb{R}^2))$ and $Sat(\mathcal{L}, \mathsf{RC}^+(\mathbb{R}^2))$ are* NP*-hard.*

Proof. Given a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$, let Φ_G be the following $\mathcal{RCC}8c$-network:

$$\begin{aligned} &c(v_i), && \text{for } v_i \in V,\\ &\mathsf{PO}(v_i, v_j), && \text{for } (v_i, v_j) \in E,\\ &\mathsf{DC}(v_i, v_j), && \text{for } (v_i, v_j) \notin E; \end{aligned}$$

and let Φ_G° be the $\mathcal{RCC}8c^\circ$-network obtained by replacing c by c° in Φ_G. It suffices to show that the following are equivalent: (a) G is a string graph; (b) Φ_G° is satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$; (c) Φ_G is satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$; (d) Φ_G is satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$; (e) Φ_G° is satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$.

(a) ⇒ (b): Let $\alpha_1, \ldots, \alpha_n$ be a string representation of G. Note that (the loci of) these arcs are closed, bounded sets. By Lemma 8, we can find $v_1, \ldots, v_n$ in $\mathsf{RCP}^+(\mathbb{R}^2)$ such that, for all i, j ($1 \leq i, j \leq n$): (i) $\alpha_i \subseteq v_i^\circ$; (ii) if $i \neq j$, then $v_i \nsubseteq v_j$; and (iii) if α_i and α_j do not intersect, then $\mathsf{DC}(v_i, v_j)$. Thus, $v_1, \ldots, v_n$ satisfies Φ_G°.

(b) ⇒ (c) and (c) ⇒ (d): Trivial.

(d) ⇒ (e): Suppose $v_1, \ldots, v_n \in \mathsf{RC}^+(\mathbb{R}^2)$ satisfies Φ_G. By Lemma 8, we can find interior-connected regions $r_1, \ldots, r_n$ in $\mathsf{RC}^+(\mathbb{R}^2)$ such that, for all i, j ($1 \leq i, j \leq n$): (i) $v_i \subseteq r_i^\circ$; (ii) if $i \neq j$, then $r_i \nsubseteq r_j$; and (iii) if $\mathsf{DC}(v_i, v_j)$, then $\mathsf{DC}(r_i, r_j)$. Thus, $r_1, \ldots, r_n$ satisfies Φ_G°. Notice that we do not require $r_1, \ldots, r_n$ to be polygons.

(e) ⇒ (a): Suppose $v_1, \ldots, v_n \in \mathsf{RC}^+(\mathbb{R}^2)$ satisfies Φ_G°. For any $e = (v_i, v_j) \in E$, select $x_e \in v_i^\circ \cap v_j^\circ$. For each i, $1 \leq i \leq n$, consider the collection of those x_e in v_i°, and let $\alpha_i \subseteq v_i^\circ$ be a Jordan arc connecting all these points. Then $\alpha_1, \ldots, \alpha_n$ is a string representation of G. □

We next establish the matching upper complexity bounds by giving criteria for satisfiability of a given constraint network Φ in terms of a type-certificate for Φ and certain planarity conditions, which ensure that the type-certificate for Φ is realizable on the plane (and not just in some topological space as in Theorem 11). Let $\Phi(r_1, \ldots, r_n)$ be an $\mathcal{RCC}8c$-constraint network and let $\bar{\boldsymbol{\tau}} = \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_m$ be a tuple of Φ-types. We proceed to construct a graph $G_{\Phi,\bar{\tau}}$. For all k ($1 \leq k \leq m$), let $V_k = \{v_{i,k} \mid 1 \leq i \leq n\}$ and let $G_k = (V_k, D_k)$ be a fresh copy of the cyclic graph of order n. Take a fresh vertex v^* and set $T = \{(v_{1,k}, v^*) \mid 1 \leq k \leq m\}$. For any i ($1 \leq i \leq n$), define the set of edges

$$C_i = \begin{cases} \{(v_{i,k}, v_{i,k'}) \mid 1 \leq k \neq k' \leq m,\ r_i \in \boldsymbol{\tau}_k \cap \boldsymbol{\tau}_{k'}\}, & \text{if } c(r_i) \in \Phi, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{25}$$

Let $G_{\Phi,\bar{\tau}} = (V, E)$, where $V = \bigcup_{k=1}^m V_k \cup \{v^*\}$ and $E = \bigcup_{k=1}^m D_k \cup T \cup \bigcup_{i=1}^n C_i$. Finally, let $S_{\Phi,\bar{\tau}}$ be the set of all pairs of distinct edges of E given by

$$S_{\Phi,\bar{\tau}} \quad = \quad \bigcup_{1 \leq i \leq n} (C_i \times T) \quad \cup \quad \bigcup_{\substack{1 \leq i \neq j \leq n \\ \mathsf{DC}(r_i, r_j) \notin \Phi}} (C_i \times C_j). \tag{26}$$

Lemma 24. *Let $\Phi(\bar{r})$ be an $\mathcal{RCC}8c$-constraint network. Then the following are equivalent*:

(a) Φ *is satisfiable over* $\mathsf{RC}^+(\mathbb{R}^2)$;

(b) *there exist a type-certificate $\bar{\boldsymbol{\tau}}$ for Φ of $3|\Phi|$ types and $S \subseteq S_{\Phi,\bar{\tau}}$ such that the design $(G_{\Phi,\bar{\tau}}, S)$ has a drawing*;

(c) Φ *is satisfiable over* $\mathsf{RCP}^+(\mathbb{R}^2)$ *by a tuple of bounded polygons, each of which has at most $3|\Phi|$ components.*

Proof. In demonstrating these equivalences, we rely in several places on Lemma 8. Accordingly, if $X_1, \ldots X_n$ is some collection of sets under consideration, we denote their 'thickenings,' obtained by Lemma 8, by $X_1^+, \ldots X_n^+$. We silently assume that any set (mentioned in the context of the discussion) not intersecting X_i also does not intersect X_i^+—and in particular, that any point (mentioned in the context of the discussion) not contained in X_i is also not contained in X_i^+. Likewise, we silently assume that X_i^+ is interior-connected if X_i is connected.

(a) ⇒ (b): Suppose $\Phi(r_1, \ldots, r_n)$ is satisfied by some $\mathfrak{a}\colon \mathcal{V} \to \mathsf{RC}^+(\mathbb{R}^2)$. By adding dummy arguments if necessary, we may assume $n \geq 3$. Let $x_1, \ldots, x_m$ be a point-certificate for Φ under $\mathfrak{a}$ and let $\bar{\boldsymbol{\tau}} = \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_m$, where $\boldsymbol{\tau}_k = \boldsymbol{\tau}(x_k, \mathfrak{a})$, for $1 \leq k \leq m$. Let $X_i = \mathfrak{a}(r_i)$, for $1 \leq i \leq n$, and let $X_1^+, \ldots, X_n^+ \in \mathsf{RC}^+(\mathbb{R}^2)$ be the thickenings of $X_1, \ldots, X_n$ guaranteed by Lemma 8. Let $d_1, \ldots, d_m$ be closed discs, centred on the respective points $x_1, \ldots, x_m$, such that $x_k \in (X_i^+)^\circ$ implies $d_k \subseteq (X_i^+)^\circ$, and $x_k \notin X_i^+$ implies $d_k \cap X_i^+ = \emptyset$. It is then obvious that none of the sets $(X_i^+)^\circ$ is disconnected by (simultaneous) removal of all the $d_1, \ldots, d_m$.

Fixing k ($1 \leq k \leq m$), take a set of points $\{y_{1,k}, \ldots, y_{n,k}\}$ lying on δd_k. Thus, for all i ($1 \leq i \leq n$), the following are equivalent:

$$r_i \in \boldsymbol{\tau}_k \qquad \Longleftrightarrow \qquad x_k \in X_i \qquad \Longleftrightarrow \qquad x_k \in (X_i^+)^\circ \qquad \Longleftrightarrow \qquad y_{i,k} \in (X_i^+)^\circ. \tag{27}$$

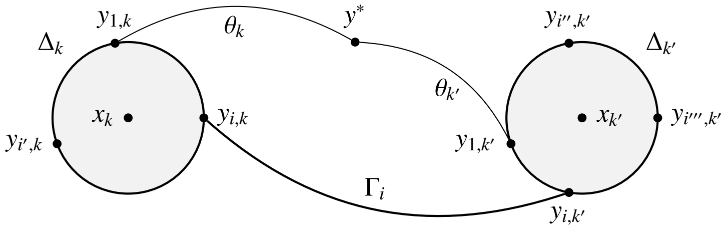


Figure 20: Three sorts of edges in the graph $G_{\Phi,\bar{\tau}}$.

Since $n \geq 3$, we can decompose each Jordan curve δd_k into a collection $\Delta_k = \{\gamma_{1,k}, \ldots, \gamma_{n,k}\}$ of Jordan arcs connecting the various points $y_{i,k}$ in a cycle. In addition, let y^* be a point lying outside all the d_k, and let θ_k be a Jordan arc connecting $y_{1,k}$ to y^*. Evidently, we may choose the θ_k in such a way that, together with the arcs $\gamma_{i,k}$, they form a plane graph; see Fig. 20. Set $\Theta = \{\theta_1, \ldots, \theta_m\}$. Fixing i ($1 \leq i \leq n$), we define a further set of edges Γ_i as follows. If $c(r_i) \notin \Phi$, then $\Gamma_i = \emptyset$. Otherwise, $(X_i^+)^\circ$ is connected. Hence, for any pair of distinct points $y_{i,k}$ and $y_{i,k'} \in (X_i^+)^\circ$, we may draw a Jordan arc $\gamma \subseteq (X_i^+)^\circ$, such that γ avoids all the disks d_h, except at the endpoints $y_{i,k}$ and $y_{i,k'}$. (This is possible because $(X_i^+)^\circ$ is not disconnected by simultaneous removal of the $d_1, \ldots, d_m$).

Recalling the definition of $G_{\Phi,\bar{\tau}} = (V, E)$, we see that the function $f\colon v_{i,k} \mapsto y_{i,k}$ and $f\colon v^* \mapsto y^*$ induces a natural map from the edges in each D_k to the arcs in Δ_k, and from the edges in T to the arcs in Θ. Furthermore, by (27), $r_i \in \boldsymbol{\tau}_k$ if and only if $y_{i,k} \in (X_i^+)^\circ$, and so f induces a natural map from the edges in each C_i to the arcs in Γ_i. That is: f is a realization of $G_{\Phi,\bar{\tau}}$. Furthermore, suppose α and β are arcs in this realization which cross at some point—i.e., which have points in common other than a common endpoint. By the construction of these arcs, and exchanging α, β if necessary, we have $\alpha \in \Gamma_i$ for some i, and either $\beta \in \Theta$, or $\beta \in \Gamma_j$ for some j such that $\mathsf{DC}(r_i, r_j) \notin \Phi$. (Indeed, $\mathsf{DC}(r_i, r_j) \in \Phi$ implies X_i^+ and X_j^+ are disjoint, whence no arc in Γ_i can intersect an arc in Γ_j.) Thus, the pair (α,β) is in the set $S_{\Phi,\bar{\tau}}$ of allowed edge crossings, and we have a drawing of $(G_{\Phi,\bar{\tau}}, S)$, for some $S \subseteq S_{\Phi,\bar{\tau}}$, as required.

(b) $\Rightarrow$ (c): Let $\bar{\boldsymbol{\tau}} = \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_m$ be a type-certificate for Φ and $S \subseteq S_{\Phi,\bar{\tau}}$. If there is a drawing of $(G_{\Phi,\bar{\tau}}, S)$ in $\mathbb{R}^2$, then there is one in the closed plane, $\mathbb{S}^2$. Let f be such a drawing. Write $f(v_{i,k}) = y_{i,k}$, $f(C_i) = \Gamma_i$ and $f(D_k) = \Delta_k$ for all i and k ($1 \leq i \leq n$ and $1 \leq k \leq m$); and write $f(v^*) = y^*$ and $f(T) = \Theta$. Thus, each Δ_k defines a Jordan curve in $\mathbb{S}^2$, with the set of allowed edge-crossings $S_{\Phi,\bar{\tau}}$ ensuring that none of these Δ_k separates any other from the point y^*. Denote by d_k the closed disc-homeomorph enclosed by Δ_k and not containing y^*. By taking the point at infinity so that it is close to y^*, we may regard f as a drawing of $(G_{\Phi,\bar{\tau}}, S)$ in $\mathbb{R}^2$: the various d_k will still be closed disc-homeomorphs, and y^* will lie outside all of them. Indeed, without loss of generality, we may assume that this drawing is rectified. (Hence, any two arcs which intersect at an internal point cross each other in the obvious sense.) We proceed to construct an assignment $\mathfrak{a}\colon \mathcal{V} \to \mathsf{RCP}^+(\mathbb{R}^2)$ of polygons satisfying Φ.

a) $r_i^\circ \in \boldsymbol{\tau}_k$ b) $r_i \in \boldsymbol{\tau}_k$ but $r_i^\circ \notin \boldsymbol{\tau}_k$

Figure 21: Wedges involving the point x_k (and disks d_k).

Without loss of generality, we may assume that Φ contains no $\mathsf{EQ}(r_i, r_j)$. By **(type)**, **(tpp-u)**, **(ntpp-u)**, we can order the variables $\bar{r} = r_1, \ldots, r_n$ in such a way that

$$j < i \quad \text{whenever} \quad \mathsf{TPP}(r_j, r_i) \in \Phi \text{ or } \mathsf{NTPP}(r_j, r_i) \in \Phi.$$

For each i ($1 \leq i \leq n$) and each k ($1 \leq k \leq m$), we select a region $W_{i,k} \in \mathsf{RCP}(\mathbb{R}^2)$ as follows:

– If $r_i^\circ \in \boldsymbol{\tau}_k$, let $W_{i,k} \subseteq d_k$ be a lozenge such that $x_k \in (W_{i,k})^\circ$ and $y_{i,k} \in \delta W_{i,k}$; see Fig. 21a.

– If $r_i \in \boldsymbol{\tau}_k$ but $r_i^\circ \notin \boldsymbol{\tau}_k$, let $W_{i,k} \subseteq d_k$ be a lozenge such that $x_k, y_{i,k} \in \delta W_{i,k}$; we may choose these $W_{i,k}$ in such a way that no two of them have intersecting interiors; see Fig. 21b.

– Otherwise, i.e., if $r_i \notin \boldsymbol{\tau}_k$, let $W_{i,k} = \emptyset$.

We refer to the $W_{i,k}$ as *wedges*. The construction of $\mathfrak{a}$ will ensure that $W_{i,k} \subseteq \mathfrak{a}(r_i)$, for any i and k; the function of these wedges is to guarantee the existential conditions of the $\mathcal{RCC}8$-constraints in Φ.

Fix any i $(1 \leq i \leq n)$. If $c(r_i) \notin \Phi$, let $Y_i = \emptyset$. Otherwise, i.e., if $c(r_i) \in \Phi$, the arcs of Γ_i connect together all the wedges $W_{i,k}$ (for varying k). Form a region $Y_i \in \mathsf{RCP}^+(\mathbb{R}^2)$ by covering all these arcs with a finite series of lozenge-shaped regions, as shown in Fig. 22. Note that, to ensure $Y_i \in \mathsf{RCP}^+(\mathbb{R}^2)$, we rely on the fact that, by construction, arcs in Γ are piecewise linear and have no line segments in common.

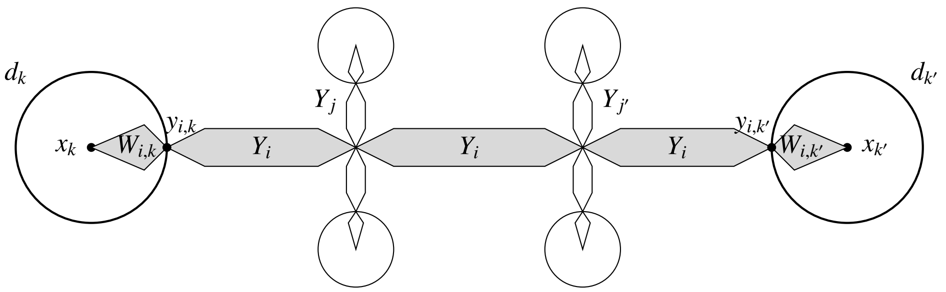


Figure 22: Connecting together wedges $W_{i,k}$ and $W_{i,k'}$.

It is immediate by construction that Y_i does not intersect the interior of any of $d_1, \ldots, d_m$. Moreover, we have

$$i \neq j \implies Y_i^\circ \cap Y_j^\circ = \emptyset, \tag{28}$$

$$\mathsf{DC}(r_i, r_j) \in \Phi \implies Y_i \cap Y_j = \emptyset. \tag{29}$$

To see (29), it suffices to show that α and β do not intersect where $\alpha \in \Gamma_i$ and $\beta \in \Gamma_j$. By definition, these arcs share no endpoints; on the other hand, if $\mathsf{DC}(r_i, r_j) \in \Phi$ then $S_{\Phi,\bar{\tau}} \cap (\Gamma_i \times \Gamma_j) = \emptyset$, so that α and β cannot cross.

Now define

$$Z_i = Y_i + \sum_{1 \leq k \leq m} W_{i,k}.$$

Thus, if $c(r_i) \in \Phi$, then Z_i is connected. (Note that Z_i will not in general be *interior*-connected, as we see from Fig. 22.) By construction:

$$r_i^\circ \in \boldsymbol{\tau}_k \iff x_k \in Z_i^\circ, \tag{30}$$

$$r_i \in \boldsymbol{\tau}_k \iff x_k \in Z_i. \tag{31}$$

By **(ec-u)**, if $\mathsf{EC}(r_i, r_j) \in \Phi$, then, for all k, the interiors of the wedges $W_{i,k}$ and $W_{j,k}$ cannot intersect. And, by **(dc-u)**, if $\mathsf{DC}(r_i, r_j) \in \Phi$, then, for all k, one of the two wedges $W_{i,k}$ and $W_{j,k}$ must be empty. By the definition of the Z_i and (28) and (29),

$$\mathsf{EC}(r_i, r_j) \in \Phi \implies Z_i^\circ \cap Z_j^\circ = \emptyset, \tag{32}$$

$$\mathsf{DC}(r_i, r_j) \in \Phi \implies Z_i \cap Z_j = \emptyset. \tag{33}$$

We now define polygons X_i $(1 \leq i \leq n)$ by recursion on i. We simultaneously ensure that, for all k $(1 \leq k \leq m)$:

$$r_i^\circ \in \boldsymbol{\tau}_k \iff x_k \in X_i^\circ, \tag{34}$$

$$r_i \in \boldsymbol{\tau}_k \iff x_k \in X_i. \tag{35}$$

And we likewise ensure that, for all j $(1 \le j \le n)$:

$$\mathsf{EC}(r_j, r_i) \in \Phi \quad \Longrightarrow \quad X_j^\circ \cap X_i^\circ = \emptyset, \text{ if } j < i, \quad \text{and} \quad Z_j^\circ \cap X_i^\circ = \emptyset, \text{ if } j > i, \tag{36}$$

$$\mathsf{DC}(r_j, r_i) \in \Phi \quad \Longrightarrow \quad X_j \cap X_i = \emptyset, \text{ if } j < i, \quad \text{and} \quad Z_j \cap X_i = \emptyset, \text{ if } j > i. \tag{37}$$

Base case: Let $X_1 = Z_1$. Conditions (34)–(37) are simply (30)–(33).

Inductive step: Suppose now that, for $1 < i \le n$, the polygons $X_1, \dots, X_{i-1}$ have been defined satisfying (34)–(37). We first take X_{i-1}^+ to be the thickening of X_{i-1} guaranteed by Lemma 8, it being supposed that $X_1^+, \dots, X_{i-2}^+$ have already been defined in previous iterations. Observe that, since all the sets in the context of discussion are bounded, we may assume that X_{i-1}^+ is a bounded element of $\mathsf{RCP}^+(\mathbb{R}^2)$. Next, we define

$$X_i \;=\; Z_i \;+\; \sum\nolimits_{\mathsf{TPP}(r_j,r_i)\in\Phi} X_j \;+\; \sum\nolimits_{\mathsf{NTPP}(r_j,r_i)\in\Phi} X_j^+. \tag{38}$$

This definition is legitimate, because $\mathsf{TPP}(r_j, r_i) \in \Phi$ or $\mathsf{NTPP}(r_j, r_i) \in \Phi$ implies $j < i$.

We must establish that (34)–(37) hold. For (34), notice that, by construction, if $r_i^\circ \in \boldsymbol{\tau}_k$ then $x_k \in Z_i^\circ \subseteq X_i^\circ$ and if $r_i \in \boldsymbol{\tau}_k$ then $x_k \in Z_i \subseteq X_i$. Conversely, suppose $x_k \in X_i^\circ$ but $r_i^\circ \notin \boldsymbol{\tau}_k$. That could happen only if either $x_k \in X_j^\circ$, for some $j < i$ with $\mathsf{TPP}(r_j, r_i) \in \Phi$, or if $x_k \in (X_j^+)^\circ$, for some $j < i$ with $\mathsf{NTPP}(r_j, r_i) \in \Phi$. In the former case, applying (34) to $j < i$, by the induction hypothesis, we have $r_j^\circ \in \boldsymbol{\tau}_k$, whence $r_i^\circ \in \boldsymbol{\tau}_k$, by **(tpp-u)**, whence $x_k \in X_i^\circ$ by (35)—contradicting $x_k \notin X_i^\circ$. In the latter case, by (34), $r_j \in \boldsymbol{\tau}_k$, whence $r_i \in \boldsymbol{\tau}_k$, by **(ntpp-u)**, and the conclusion again follows. Condition (35) follows by an almost identical argument. Conditions (36)–(37) are immediate by construction. This completes the induction.

Consider the mapping $\mathfrak{a}\colon r_i \mapsto X_i$. We show that $\mathfrak{a}$ satisfies the $\mathcal{RCC}8$-constraints in Φ. If $\mathsf{EC}(r_j, r_i) \in \Phi$ then, by **(ec-e)**, there is k with $r_j, r_i \in \boldsymbol{\tau}_k$, whence, by (34) and (35), $x_k \in X_j \cap X_i$. The existential conditions generated by constraints of the forms $\mathsf{PO}(r_j, r_i)$, $\mathsf{TPP}(r_j, r_i)$ and $\mathsf{NTPP}(r_j, r_i)$ are handled similarly using **(po-e)** and **(diff-e)**. We turn now to the universal conditions. By (38), $\mathsf{TPP}(r_j, r_i) \in \Phi$ implies $X_j \subseteq X_i$, and $\mathsf{NTPP}(r_j, r_i) \in \Phi$ implies $X_j \subseteq X_i^\circ$. If $\mathsf{EC}(r_j, r_i) \in \Phi$, then $X_j^\circ \cap X_i^\circ = \emptyset$ by (36); and if $\mathsf{DC}(r_j, r_i) \in \Phi$, then $X_j \cap X_i = \emptyset$ by (37).

Finally, we claim that, if $c(r_i) \in \Phi$, then X_i and X_i^+ are connected. By construction, Z_i is connected. But if X_j or X_j^+ is one of the summands in (38), then any component of this set will, by **(reg-e)**, contain at least one lozenge $W_{j,k}$. And since Φ contains either $\mathsf{TPP}(r_j, r_i)$ or $\mathsf{NTPP}(r_j, r_i)$, we know that $W_{j,k}$ shares x_k with the lozenge $W_{i,k}$, which is part of the connected set Z_i. Hence X_i is connected. Thus, the tuple $X_1, \dots, X_n$ of elements of $\mathsf{RCP}^+(\mathbb{R}^2)$ satisfies Φ, as required. Finally, observe that, by construction, the polygons $X_1, \dots, X_n$ are all bounded and are built up from wedges $W_{i,k}$ in $\le 3|\Phi|$ discs and connecting sausage-like regions. So, if a region is connected then it has just a single component; otherwise, all wedges within one disc form a component (as they share its central point x_k), resulting in $\le 3|\Phi|$ components.

(c) $\Rightarrow$ (a): Trivial. $\square$

It is then immediate from Lemmas 23 and 24 that we have the following result.

Theorem 25. *The problems Sat*($\mathcal{RCC}8c$, $\mathsf{RCP}^+(\mathbb{R}^2)$) *and Sat*($\mathcal{RCC}8c$, $\mathsf{RC}^+(\mathbb{R}^2)$) *coincide and are* NP*-complete.*

The case of $\mathcal{RCC}8c^\circ$ is similar to that of $\mathcal{RCC}8c$, and we briefly indicate the main differences. If Φ is an $\mathcal{RCC}8c^\circ$-network and $\bar{\boldsymbol{\tau}} = \boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_m$ a type-certificate for Φ, we first define, in place of the edge-sets C_i given in (25), the edge-sets

$$C_i^\circ = \begin{cases} \{(v_{i,k}, v_{i,k'}) \mid 1 \le k \ne k' \le m,\ r_i \in \boldsymbol{\tau}_k \cap \boldsymbol{\tau}_{k'}\}, & \text{if } c^\circ(r_i) \in \Phi, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{25°}$$

(Intuitively, the condition $r_i \in \boldsymbol{\tau}_k \cap \boldsymbol{\tau}_{k'}$ states that region r_i—but not necessarily r_i°—contains both x_k and $x_{k'}$; hence if r_i is interior-connected, then there are points in r_i° close to x_k and $x_{k'}$, and connected by an arc lying in r_i°.) This yields, in place of the graph $G_{\Phi,\bar{\boldsymbol{\tau}}}$, the graph $G_{\Phi,\bar{\boldsymbol{\tau}}}^\circ = (V, E)$ where $V = \bigcup_{k=1}^m V_k \cup \{v^*\}$ and $E = \bigcup_{k=1}^m D_k \cup T \cup \bigcup_{i=1}^n C_i^\circ$. We next define, in place of the set of allowed edge-crossings $S_{\Phi,\bar{\boldsymbol{\tau}}}$ given in (26), the the set of allowed edge-crossings

$$S_{\Phi,\bar{\boldsymbol{\tau}}}^\circ \;=\; \bigcup_{1\le i\le n} (C_i^\circ \times T) \;\cup\; \bigcup_{\substack{1\le i\ne j\le n \\ \mathsf{DC}(r_i,r_j)\notin\Phi \text{ and } \mathsf{EC}(r_i,r_j)\notin\Phi}} (C_i^\circ \times C_j^\circ). \tag{26°}$$

(Intuitively, since the arcs in C_i° are all meant to lie in the interior of r_i, arcs in C_j° can intersect those in C_i° only if r_i and r_j have interior points in common.) The proof then proceeds as for Lemma 24, but with $G^\circ_{\Phi,\bar{\tau}}$ and $S^\circ_{\Phi,\bar{\tau}}$ replacing $G_{\Phi,\bar{\tau}}$ and $S_{\Phi,\bar{\tau}}$, respectively.

Lemma 26. *Let $\Phi(\bar{r})$ be an $\mathcal{RCC}8c^\circ$-constraint network. Then the following are equivalent*:

(a) Φ *is satisfiable over* $\mathsf{RCP}^+(\mathbb{R}^2)$;

(b) *there exist a type-certificate $\bar{\boldsymbol{\tau}}$ for Φ of $3|\Phi|$ types and $S \subseteq S^\circ_{\Phi,\bar{\tau}}$ such that the design $(G^\circ_{\Phi,\bar{\tau}}, S)$ has a drawing*;

(c) Φ *is satisfiable over* $\mathsf{RCP}^+(\mathbb{R}^2)$ *by a tuple of bounded regions, each of which has at most $3|\Phi|$ components and at most $3|\Phi|^2$ i-components.*

Proof. (a) $\Rightarrow$ (b): The argument proceeds similarly to that for (a) $\Rightarrow$ (b) in the proof of Lemma 24. We suppose that $\Phi(r_1, \dots, r_n)$ is satisfied by some $\mathfrak{a}\colon \mathcal{V} \to \mathsf{RCP}^+(\mathbb{R}^2)$, where $n \geq 3$. Let $x_1, \dots, x_m$ be a point-certificate for Φ under $\mathfrak{a}$ and let $\bar{\boldsymbol{\tau}} = \boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_m$, where $\boldsymbol{\tau}_k = \boldsymbol{\tau}(x_k, \mathfrak{a})$, for $1 \leq k \leq m$. Set $X_i = \mathfrak{a}(r_i)$, for $1 \leq i \leq n$. Notice that these regions are by assumption in $\mathsf{RCP}^+(\mathbb{R}^2)$. This allows us to choose points $y_{i,k}$ so that $r_i \in \tau_k \Leftrightarrow y_{i,k} \in X_i^\circ$ (cf. (27); we have no use for the regions $X_1^+, \dots, X_n^+$, and we do not create them). The arc sets Δ_k and Θ are defined as before. However, the arc sets Γ_i are chosen slightly differently: if Φ contains $c^\circ(r_i)$, we choose γ connecting any pair of distinct points $y_{i,k}$ and $y_{i,k'} \in X_i^\circ$ so that $\gamma \subseteq X_i^\circ$. (This is possible because connected, open sets are arc-connected.) This choice of the Γ_i ensures that, even with the restricted collection of allowed crossings given by (26°), we still have a drawing of $(G^\circ_{\Phi,\bar{\tau}}, S)$, for some $S \subseteq S^\circ_{\Phi,\bar{\tau}}$, as required.

(b) $\Rightarrow$ (c): The argument proceeds similarly to that for (b) $\Rightarrow$ (c) in the proof of Lemma 24. The only significant change is that we no longer employ the 'sausage' construction of Fig. 22: we simply take each polygon Y_i to be the result of 'thickening' the closed, bounded set

$$\bigcup_{1\leq k\leq m} W_{i,k} \quad \cup \quad \bigcup_{\gamma\in C_i^\circ} \gamma$$

to a polygon, as guaranteed by Lemma 8. Thus, for any arc $\gamma \in C_i^\circ$, we have $\gamma \subseteq Y_i^\circ$. The construction of $X_1, \dots, X_n$ and the argument that this tuple satisfies Φ then proceeds essentially as for Lemma 24 and, again, the polygons are bounded and are built up from wedges $W_{i,k}$ in $3|\Phi|$ discs and connecting regions (which are interior-connected and overlap the respective wedges). So, if a region is interior-connected then it has just a single i-component; otherwise, each wedge within one disc can form an i-component, resulting in at most $3|\Phi|^2$ i-components.

(c) $\Rightarrow$ (a): Trivial. □

It is then immediate from Lemmas 23 and 26 that we have the following result.

Theorem 27. *Sat*$(\mathcal{RCC}8c^\circ, \mathsf{RCP}^+(\mathbb{R}^2))$ *is* NP-*complete.*

It is illuminating to ask why the proof of Lemma 26 fails when $\mathsf{RCP}^+(\mathbb{R}^2)$ is replaced by $\mathsf{RC}^+(\mathbb{R}^2)$. The answer lies in the innocent-looking wedges $W_{i,k}$, which may be required to connect points $y_{i,k} \in X_i^\circ$ to points $x_k \in \delta X_i$. The problem is that, for $\mathsf{RC}^+(\mathbb{R}^2)$, we cannot guarantee that these wedges exist (cf. Fig. 15), and hence, we cannot use arcs included in X_i° to ensure the satisfaction of the constraint $c^\circ(r_i)$. When dealing with $\mathcal{RCC}8c$, we could afford to be more relaxed, drawing the wedges $W_{i,k}$ and arcs Γ_i within the expanded region X_i^+, and using sausage-like regions Y_i to connect up the pieces. But sausage-like regions do not suffice for $\mathcal{RCC}8c^\circ$, since they do not secure interior-connectedness. We remark that a purported proof of Theorem 25 was presented in [21]; however, this proof contains an error.[5] Another incorrect proof of the same theorem was given in [36].[6]

[5] The first sentence on p. 105 makes a false claim.

[6] Theorem 9, p. 539; the preceding Lemma 2 is false, as shown by the first inequality *Sat*$(\mathcal{RCC}8, \mathfrak{D}) \subsetneq$ *Sat*$(\mathcal{RCC}8, \mathfrak{I})$ of Theorem 13 of the present article.

8. The fragmentation of satisfying configurations

Let Φ be a constraint network in any of the languages $\mathcal{RCC}8$, $\mathcal{RCC}8c$ or $\mathcal{RCC}8c^\circ$, and suppose Φ is satisfiable over $\mathsf{RC}^+(\mathbb{R}^n)$ or $\mathsf{RCP}^+(\mathbb{R}^n)$. What can we say about the amount of information required to store some satisfying assignment, as a function of the number of variables involved? In particular, what can we say about the extent to which Φ forces satisfying assignments to 'chop up' the space in which they are embedded? As we shall see, the answer is only loosely related to the complexity-theoretic bounds derived above.

We first consider the fragmentation of satisfying assignments for the language $\mathcal{RCC}8$ over $\mathsf{RC}^+(\mathbb{R}^n)$ and $\mathsf{RCP}^+(\mathbb{R}^n)$, for all $n \geq 1$. Bearing in mind that any satisfiable $\mathcal{RCC}8$-constraint network Φ has a point-certificate of at most $3|\Phi|$ points, Lemma 19 guarantees that if Φ is satisfiable over $\mathsf{RCP}^+(\mathbb{R})$, it can be satisfied by an assignment dividing $\mathbb{R}$ into at most $6|\Phi|^2$ intervals over each of which it is uniform.

Corollary 28. *If Φ is an $\mathcal{RCC}8$-network satisfiable over $\mathsf{RCP}^+(\mathbb{R}^n)$ ($n \geq 1$), then Φ has a satisfying assignment of bounded polyhedral regions all of whose vertices have integer coordinates in the range $[0, O(|\Phi|^2)]$.*

We next consider the fragmentation of satisfying assignments for the languages $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ over $\mathsf{RC}^+(\mathbb{R}^n)$ and $\mathsf{RCP}^+(\mathbb{R}^n)$ for all $n \geq 3$.

Corollary 29. *If Φ is an $\mathcal{RCC}8c$- or $\mathcal{RCC}8c^\circ$-network satisfiable over $\mathsf{RC}^+(\mathbb{R}^n)$ ($n \geq 3$), then Φ has a satisfying assignment of bounded polyhedral regions all of whose vertices have integer coordinates in the range $[0, O(|\Phi|^2)]$.*

Proof. Let Φ^- be the result of removing all c- or c°-constraints from Φ. We take a satisfying assignment in $\mathsf{RCP}(\mathbb{R})$ whose vertices have integer coordinates in the range $[0, O(|\Phi|^2)]$. Now cylindrify to an assignment in $\mathsf{RCP}(\mathbb{R}^3)$, and connect up regions as necessary using points whose coordinates in the second and third dimensions are bounded by $O(|\Phi|)$. For $n > 3$, cylindrify again. □

The fragmentation of satisfying assignments for $\mathcal{RCC}8c$ over $\mathsf{RC}^+(\mathbb{R})$ and $\mathsf{RCP}^+(\mathbb{R})$ has likewise already been dealt with (recall that connectedness coincides with interior-connectedness over $\mathbb{R}$, and so $\mathcal{RCC}8c^\circ$ is the same as $\mathcal{RCC}8c$). Theorem 15 tells us that $\mathcal{RCC}8c$-networks satisfiable over $\mathsf{RC}^+(\mathbb{R})$ can force regions to have infinitely many components. By contrast, Lemma 19 shows that $\mathcal{RCC}8c$-networks satisfiable over $\mathsf{RCP}^+(\mathbb{R})$ have satisfying assignments with integer coordinates in the range $[0, O(|\Phi|^2)]$.

This leaves the more interesting case of the fragmentation of satisfying assignments for the languages $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ over $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$. We consider first the language $\mathcal{RCC}8c$. Let Φ be an $\mathcal{RCC}8c$-network satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$. By Lemma 24, Φ is satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$ (by bounded polygons). But how complicated does this arrangement of polygons have to be? In terms of numbers of components, not very. Recall the terminology introduced in Sec. 3.1: if $r \in \mathsf{RC}^+(\mathbb{R}^2)$ and X is a component of r°, then X^- is an *i-component* of r (Fig. 6). As we noted, both $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$ are closed under taking i-components, and every $r \in \mathsf{RCP}^+(\mathbb{R}^2)$ is the sum of its i-components. By Lemma 24,

(r) if Φ is an $\mathcal{RCC}8c$-constraint network satisfied by a tuple of regions in $\mathsf{RC}^+(\mathbb{R}^2)$, then it is satisfied by a tuple of bounded regions in $\mathsf{RCP}^+(\mathbb{R}^2)$ each of which has at most $3|\Phi|$ components.

Similarly, by Lemma 26,

(r$^\circ$) if Φ is an $\mathcal{RCC}8c^\circ$-constraint network satisfied by a tuple of regions in $\mathsf{RCP}^+(\mathbb{R}^2)$, then it is satisfied by a tuple of bounded regions in $\mathsf{RCP}^+(\mathbb{R}^2)$ each of which has at most $3|\Phi|$ components and at most $3|\Phi|^2$ i-components.

As we might say, over $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$, $\mathcal{RCC}8c$ can force no more than linearly many components; and over $\mathsf{RCP}^+(\mathbb{R}^2)$, $\mathcal{RCC}8c^\circ$ can force no more than quadratically many i-components.

We now proceed to show that **(r)** becomes false if 'component' is replaced by 'i-component': $\mathcal{RCC}8c$-constraint networks can in general force satisfying tuples to have exponentially many i-components (Theorems 30 and 31). Likewise, **(r$^\circ$)** becomes false if '$\mathsf{RCP}^+(\mathbb{R}^2)$' is replaced by '$\mathsf{RC}^+(\mathbb{R}^2)$'—indeed radically so: there exist $\mathcal{RCC}8c^\circ$-constraint networks, satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, but only by tuples of regions some of which have infinitely many components (Theorem 32).

The following result uses the construction originally devised by Kratochvíl and Matoušek [31] to establish Proposition 2.

Theorem 30. *There exists a sequence Φ_n of $\mathcal{RCC}8c$-constraint networks, satisfiable over* $\mathsf{RCP}^+(\mathbb{R}^2)$*, such that $|\Phi_n|$ is $O(n)$, but every satisfying assignment contains some element with $2^{\Omega(n)}$ i-components.*

Proof. Let $G_3 = (V_3, E_3)$ be the planar graph given by the solid lines in Fig. 23. We generalize this graph by inserting new vertices (of degree 2) as follows. If G_i is defined for $i \geq 3$, we take $G_{i+1} = (V_{i+1}, E_{i+1})$ to be the result of inserting a new vertex u_{i+1} into the edge (u_i, y), a new vertex w_{i+1} into the edge (w_i, w), a new vertex v_{i+1} into the edge (u_i, u_{i-1}) and adding a new edge (u_{i+1}, w_{i+1}). Having defined $G_n = (V_n, E_n)$ for some fixed $n \geq 3$, let E'_n be a set of additional edges involving the vertices V_n such that $G'_n = (V_n, E_n \cup E'_n)$ is a 3-connected planar graph (such an E'_n can easily be constructed). In addition, let E''_n be the set of edges $\{(u_1, v_1), \ldots, (u_n, v_n)\}$, realized by arcs arranged as shown (for $n = 3$) by the grey lines in Fig. 23. We call the edges in $E_n \cup E''_n$ *visible*, and we call the edges in E'_n *invisible*. (Note that the latter cannot be seen in Fig. 23.) Let $G^*_n = (V_n, E_n \cup E'_n \cup E''_n)$. Thus, G^*_n is a non-planar graph; however, it has a specific realization in which the visible edges $E_n \cup E''_n$ are arranged as in Fig. 23. Let S^*_n be the set of pairs of distinct edges of G^*_n which cross in this realization. Thus, $D^*_n = (G^*_n, S^*_n)$ by definition has a drawing, and, by inspection of Fig. 23, permits *visible* edges e and e' to cross only if either: (*i*) $e = (u_i, v_i)$ for some i $(1 \leq i \leq n)$ and $e' = (x, u_0)$; or (*ii*) $e = (u_i, v_i)$ for some i $(2 \leq i \leq n)$ and e' lies on the line-segment $\overline{u_{i-2}, z}$.

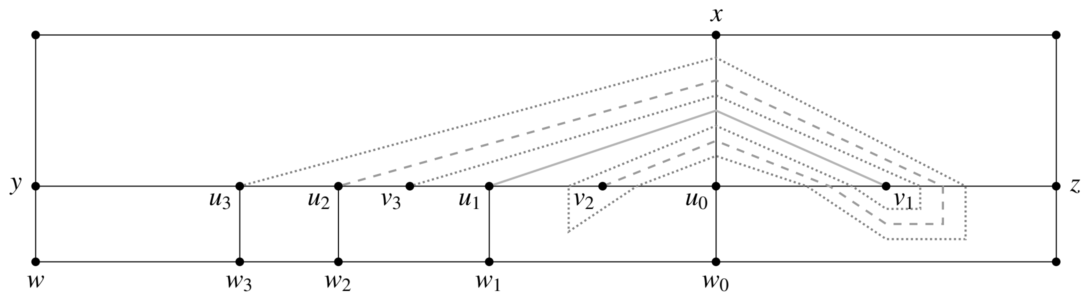


Figure 23: The construction of Kratochvíl and Matoušek [31].

By the first statement of Lemma 10, $\Theta_{D^*_n}$ is satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$. Conversely, suppose $(\bar{r}, \bar{s})$ is a satisfying assignment for $\Theta_{D^*_n}$. By the second statement of Lemma 10, this tuple weakly embeds a rectified drawing of $\Theta_{D^*_n}$, which contains a drawing of the 3-connected planar graph G'_n. By Whitney's theorem, G'_n has only one drawing in $\mathbb{S}^2$ (up to homeomorphism of $\mathbb{S}^2$ onto itself). Hence—by applying a homeomorphism if necessary—we may assume that the induced drawing of the subgraph G_n is exactly as depicted (for $n = 3$) by the solid lines in Fig. 23. Since the allowed crossings of visible edges are given by S^*_n, it is then obvious from inspection of Fig. 23 that, in the given drawing of $\Theta_{D^*_n}$, the arc realizing the (visible) edge (u_n, v_n) must cross (x, u_0) at least 2^{n-1} times. And since, in $\Theta_{D^*_n}$, regions corresponding to pairs of crossing edges are required to satisfy EC, we see that the member of the tuple $\bar{s}$ corresponding to the edge (u_n, v_n) must have at least $(2^{n-1} + 1)$ i-components—each separated from the others by these crossing points. □

The next observation shows that this is as bad as things get.

Theorem 31. *If Φ is an $\mathcal{RCC}8c$- or $\mathcal{RCC}8c^\circ$-network satisfiable over* $\mathsf{RCP}^+(\mathbb{R}^2)$*, then Φ has a satisfying assignment that can be drawn using points with integer coordinates in a square grid of size $2^{O(|\Phi|)}$.*

Proof. Immediate from the proofs of Lemmas 24 and 26 together with Propositions 1 and 5. □

Thus, over $\mathsf{RCP}^+(\mathbb{R}^2)$ (and hence also over $\mathsf{RC}^+(\mathbb{R}^2)$), $\mathcal{RCC}8c$ can force arrangements with polynomially many components, but exponentially many i-components.

Turning now to $\mathcal{RCC}8c^\circ$, a different pattern emerges. We consider first satisfiability over $\mathsf{RCP}^+(\mathbb{R}^2)$. As we have already observed, in terms of counting i-components, there is even less scope for forcing complexity than with $\mathcal{RCC}8c$: by Lemma 26, if an $\mathcal{RCC}8c^\circ$-network Φ is satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$, then Φ has a satisfying assignment in which no region has more than $3|\Phi|^2$ i-components. Of course, just because each region in a satisfying assignment has only polynomially many i-components, that says nothing about how these i-components intersect. And indeed, a simple modification of the argument of Theorem 30 shows that there exists a sequence $\{\Phi_n\}_{n\geq 1}$ of $\mathcal{RCC}8c^\circ$-networks, satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$, such that $|\Phi_n|$ is $O(n)$, but every satisfying assignment involves a drawing with $2^{\Omega(n)}$ arc-crossings. (Again, we know from Theorem 31 that this is as bad as things get.) Thus, over $\mathsf{RCP}^+(\mathbb{R}^2)$, $\mathcal{RCC}8c^\circ$ can force arrangements with polynomially many i-components, but exponentially complicated drawings.

Finally, we consider the complexity of satisfying assignments for $\mathcal{RCC}8c^\circ$-networks over $\mathsf{RC}^+(\mathbb{R}^2)$. We show that, in 2-dimensional space, $\mathcal{RCC}8c^\circ$-networks can force regions in satisfying assignments to be infinitely fragmented.

Theorem 32. *There exists an $\mathcal{RCC}8c^\circ$-formula that is satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, but only by tuples some of whose elements have infinitely many components.*

Proof. We briefly outline the underlying intuition. Using Lemma 9, we write $\mathcal{RCC}8c^\circ$-constraints whose only satisfying assignments strongly embed the plane graph in Fig. 24. By adding the constraints (39)–(45) we ensure the existence of a region t, including the point v_1, and containing points close to w_2, and a region u, including the point w_1, and containing points close to v_2. The regions t and u are both confined to the interior of the large triangle, and have no internal points in common. These constraints thus force t and u to consist of infinitely many interleaving 'fingers', arranged as indicated by the bold lines in Fig. 25. By adding the constraints (46)–(47), we ensure the existence of a region $u_0 \subseteq u$ such that infinitely many of the fingers of u contain points of u_0.

Let H be the plane graph depicted in Fig. 24. We label the points of H as $v_1, \ldots, v_7, w_1, \ldots, w_7$ and its arcs as $\alpha_0, \ldots, \alpha_7, \beta_1, \ldots, \beta_7$, the double sets of letters broadly reflecting the symmetry of H about the central arc α_0. Let H^+ be a 3-connected, plane graph extending H.

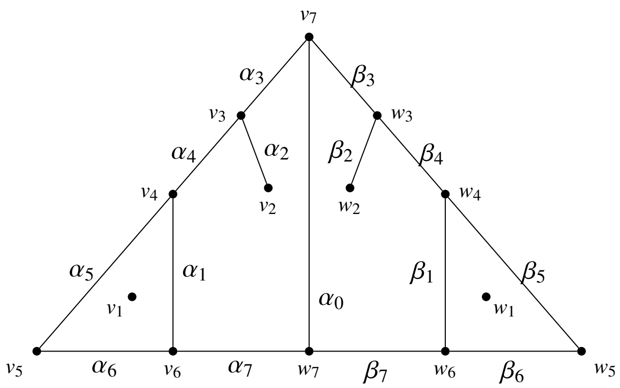


Figure 24: The plane graph H used in the proof of Theorem 32.

We may regard H and H^+ as *graphs* (as opposed to *plane graphs*) in the usual way, so that the constraint networks Ω°_H and $\Omega^\circ_{H^+}$ are as defined in Sec. 3. Let us write $\Omega^\circ_H = \Omega^\circ_H(\bar{p}\bar{q}; \bar{r}\bar{s})$, where the variables $\bar{p} = p_1, \ldots, p_7$ and $\bar{q} = q_1, \ldots, q_7$ correspond to the points (vertices) $v_1, \ldots, v_7$ and $w_1, \ldots, w_7$, respectively, and the variables $\bar{r} = r_0, r_1, \ldots, r_7$ and $\bar{s} = s_1, \ldots, s_7$ correspond to the arcs (edges) $\alpha_0, \alpha_1, \ldots, \alpha_7$ and $\beta_1, \ldots, \beta_7$, respectively. Letting the variables $\bar{p}'$ correspond to those points of H^+ not present in H, and the variables $\bar{r}'$ to those arcs of H^+ not present in H, we may write $\Omega^\circ_{H^+} = \Omega^\circ_{H^+}(\bar{p}\bar{q}\bar{p}'; \bar{r}\bar{s}\bar{r}')$. Obviously, $\Omega^\circ_{H^+}$ includes Ω°_H as a subset. We shall sometimes refer to the regions denoted by $\bar{p}\bar{q}\bar{p}'$ as *vertex-regions*, and those denoted by $\bar{r}\bar{s}\bar{r}'$ as *edge-regions*.

Now let t, u and u_0 be fresh variables, and let $\Psi(\bar{p}\bar{q}\bar{p}', \bar{r}\bar{s}\bar{r}', t, u, u_0)$ be the union of $\Omega^\circ_{H^+}$ with the constraints (39)–(47) given below. To help motivate the construction, we divide these constraints into groups. We have included some deliberate redundancies for the sake of clarity.

We shall require t and u to be interior-connected regions, in external contact with both p_2 and q_2, and with each other:

$$c^\circ(t), \qquad c^\circ(u), \tag{39}$$

$$\mathsf{EC}(p_2, t), \qquad \mathsf{EC}(q_2, t), \qquad \mathsf{EC}(p_2, u), \qquad \mathsf{EC}(q_2, u), \tag{40}$$

$$\mathsf{EC}(t, u). \tag{41}$$

In addition, t° includes p_1 and u° includes q_1:

$$\mathsf{NTPP}(p_1, t), \qquad \mathsf{NTPP}(q_1, u). \tag{42}$$

We further insist that neither t nor u intersect any of the vertex-regions $p_3, \ldots, p_7$ or $q_3, \ldots, q_7$. In addition, t may not intersect any of the edge-regions $\bar{r}, \bar{s}$, except (possibly) r_0 or r_1; and u may not intersect any of $\bar{r}, \bar{s}$, except (possibly) r_0 or s_1.

$$\mathsf{DC}(p_h, t), \quad \mathsf{DC}(q_h, t), \quad \mathsf{DC}(p_h, u), \quad \mathsf{DC}(q_h, u), \quad (3 \le h \le 7), \tag{43}$$

$$\mathsf{DC}(r_i, t), \quad \mathsf{DC}(s_i, t), \quad \mathsf{DC}(s_i, u), \quad \mathsf{DC}(r_i, u), \quad (2 \le i \le 7), \tag{44}$$

$$\mathsf{DC}(s_1, t), \qquad \mathsf{DC}(r_1, u). \tag{45}$$

Finally, we take u_0 to be a region included in u, in external contact with p_2 but disjoint from both r_0 and r_1.

$$\mathsf{TPP}(u_0, u), \qquad \mathsf{EC}(u_0, p_2), \tag{46}$$

$$\mathsf{DC}(u_0, r_0), \qquad \mathsf{DC}(u_0, r_1). \tag{47}$$

Notice that we do not require u_0 to be interior-connected.

We first show that Ψ is satisfiable. Returning to the view of H^+ as a plane graph, let $\bar{p}$, $\bar{q}$, $\bar{p}'$, $\bar{r}$, $\bar{s}$ and $\bar{r}'$ be the tuples of regions obtained by canonically thickening the vertices and edges of H^+, as in the proof of Lemma 9 (a), so that $\Omega^\circ_{H^+}$ is satisfied. Thus, each point of H lies in the interior of the corresponding vertex-region: $v_i \in p_i^\circ$ and $w_i \in q_i^\circ$ $(1 \le i \le 7)$. Furthermore, each arc of H lies in the union of the interiors of the corresponding edge- and vertex-regions: for example, $\alpha_0 \subseteq p_7^\circ \cup r_0^\circ \cup q_7^\circ$, and so on. Fig. 25 shows some representative examples.

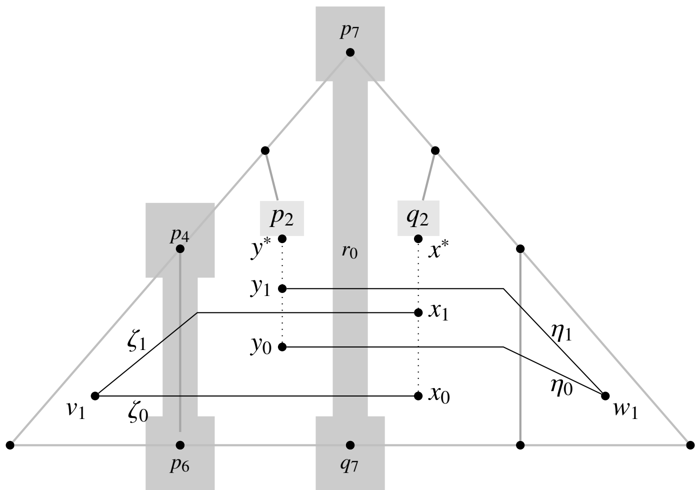


Figure 25: Constructing a satisfying assignment of Ψ in the proof of Theorem 32.

Let $\{x_i\}_{i \ge 0}$ be an infinite sequence of points, arranged as shown in Fig. 25, and having an accumulation point x^* on the boundary of q_2. For each $i \ge 0$, let ζ_i be an arc from v_1 to x_i, drawn as shown (up to $i = 1$) in Fig. 25. Let the points y_i and arcs η_i be constructed similarly, with the y_i having an accumulation point y^* on the boundary of p_2, and η_i connecting w_1 to y_i. Now let t be the result of thickening all the ζ_i and taking the infinite sum in $\mathsf{RC}(\mathbb{R}^2)$; and let u be the result of thickening all the η_i and taking the infinite sum in $\mathsf{RC}(\mathbb{R}^2)$. Clearly, this may be done in such a way that constraints (39)–(45) are satisfied. Finally, draw a small disc around each y_i, lying in the interior of u. Clearly, this may be done so that each disc maintains some fixed minimum distance from both r_0 and r_1. Let u_0 be the infinite sum, in $\mathsf{RC}(\mathbb{R}^2)$, of these discs. Then the constraints (46)–(47) are satisfied: in particular, the accumulation point y^* lies on the frontier of u_0. Thus, Ψ is satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, as required. Observe that, in this satisfying assignment, u_0 has infinitely many components, namely, the discs around the y_i.

Now suppose $\Psi(\bar{p}\bar{q}\bar{p}', \bar{r}\bar{s}\bar{r}', t, u, u_0)$ holds; we show that u_0 has infinitely many components. Since $\Psi \supseteq \Omega^\circ_{H^+} \supseteq \Omega^\circ_H$, the tuple $(\bar{p}\bar{q}\bar{p}'; \bar{r}\bar{s}\bar{r})$, by Lemma 9 (b), strongly embeds some drawing of the planar graph H. In fact, since the graph H^+ is 3-connected, it has at most one drawing in $\mathbb{S}^2$ up to isomorphism, so that, applying a homeomorphism if necessary, the induced embedding of the subgraph H is exactly as shown in Fig. 24. Thus, each point of H lies in the interior of the corresponding vertex-region: $v_i \in p_i^\circ$ and $w_i \in q_i^\circ$ $(1 \leq i \leq 7)$. Furthermore, each arc of H lies in the union of the interiors of the corresponding edge- and vertex-regions: for example, $\alpha_0 \subseteq p_7^\circ \cup r_0^\circ \cup q_7^\circ$, and so on. In what follows we assume some orientation of arcs (which should be clear from the context) and often identify arcs with their loci, to reduce clutter.

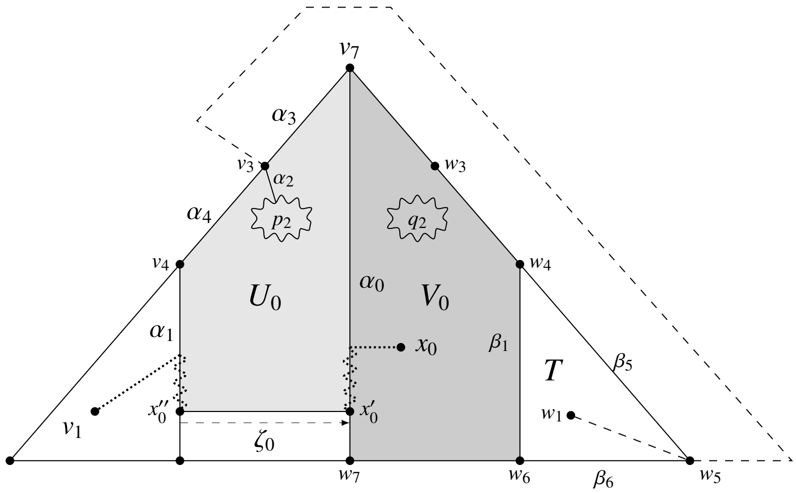


Figure 26: The construction of x_0, x_0', x_0'' and ζ_0 in the proof of Theorem 32.

Denote by V_0 the closed trapezoid indicated in Fig. 26 by the dark grey shading. From the constraints in $\Omega^\circ_{H^+}$, we have $q_2 \subseteq V_0^\circ$ and so, by (40), $V_0^\circ \cap t \neq \emptyset$, whence there exists a point $x_0 \in V_0^\circ \cap t^\circ$. On the other hand, by (42), $v_1 \in t^\circ$. Since t° is connected, by (39), there exists an arc $\zeta_0' \subseteq t^\circ$ from v_1 to x_0. By (43)–(45), ζ_0' lies entirely within the large triangle. Let x_0' be the first point of ζ_0' lying on α_0 and let x_0'' be the last point of $\zeta_0'[v_1, x_0']$ lying on α_1. Consider $\zeta_0 = \zeta_0'[x_0'', x_0']$. Thus, $\zeta_0\alpha_0[x_0', v_7]\alpha_3\alpha_4\alpha_1[v_4, x_0'']$ is a Jordan curve, which we denote by Γ_0.

We claim that Γ_0 separates the region p_2 from the point w_1. To see this, note that $\zeta_0 \subseteq t^\circ$ can intersect, by (43)–(45), neither $\alpha_2 \subseteq p_2^\circ \cup r_2^\circ \cup p_3^\circ$ nor the triangle T bounded by β_1, β_5 and β_6. Now connect p_2 to w_1 by first following α_2 upwards, and then proceeding along the dashed path in Fig. 26—i.e. leave the large triangle at v_3; re-enter it at w_5; and proceed directly to w_1. Since $\zeta_0 \cap (\alpha_2 \cup T) = \emptyset$, this path intersects Γ_0 at a single point, namely, v_3, establishing that Γ_0 separates p_2 and w_1. Denote by U_0 the closed region with frontier Γ_0 that includes p_2 (light grey shading in Fig. 26).

Now fix any $i \geq 1$, and assume that the points $x_{i-1}'' \in \alpha_1$, $x_{i-1}' \in \alpha_0$ and arc $\zeta_{i-1} \subseteq t^\circ$ have been constructed such that ζ_{i-1} connects x_{i-1}'' to x_{i-1}', and the Jordan curve $\Gamma_{i-1} = \zeta_{i-1}\alpha_0[x_{i-1}', v_7]\alpha_3\alpha_4\alpha_1[v_4, x_{i-1}'']$ is the frontier of a closed region U_{i-1} including the region p_2 in its interior, but not containing the point w_1 (dark grey area in Fig. 27). By the second constraint in (46), $U_{i-1}^\circ \cap u_0 \neq \emptyset$, whence there exists a point $y_i \in U_{i-1}^\circ \cap u_0^\circ$. By the first constraint in (46), $y_i \in u^\circ$. On the other hand, by (42), $w_1 \in u^\circ$. Since u° is connected, by (39), there exists an arc $\eta_i' \subseteq u^\circ$ from w_1 to y_i (Fig. 27). By (43)–(45), η_i' lies entirely within the large triangle.

We know that η_i' intersects Γ_{i-1}. Indeed, by (43)–(45) and (41), $\eta_i' \subseteq u^\circ$ must be disjoint from α_1, α_3, α_4 and ζ_{i-1}, and therefore intersects $\alpha_0[x_{i-1}', v_7]$. Let y_i' be the point of $\eta_i' \cap \alpha_0[x_{i-1}', v_7]$ lying nearest to v_7 (Fig. 27). Let y_i'' be the last point of $\eta_i'[w_1, y_i']$ lying on β_1, and let $\eta_i = \eta_i'[y_i'', y_i']$. Thus, $\eta_i\alpha_0[y_i', v_7]\beta_3\beta_4\beta_1[w_4, y_i'']$ is a Jordan curve, which we denote by Δ_i. We remark that y_i' need not be the first, or the last point of η_i' lying on α_0 as we travel from w_1 to y_i; it is simply the nearest to v_7.

We claim that Δ_i separates the region q_2 from the point v_1. To see this, note that $\eta_i \subseteq u^\circ$ can intersect, by (43)–(45), neither $\beta_2 \subseteq q_2^\circ \cup s_2^\circ \cup q_3^\circ$, nor the triangle S bounded by α_1, α_5 and α_6. Hence, using a mirror image of the construction

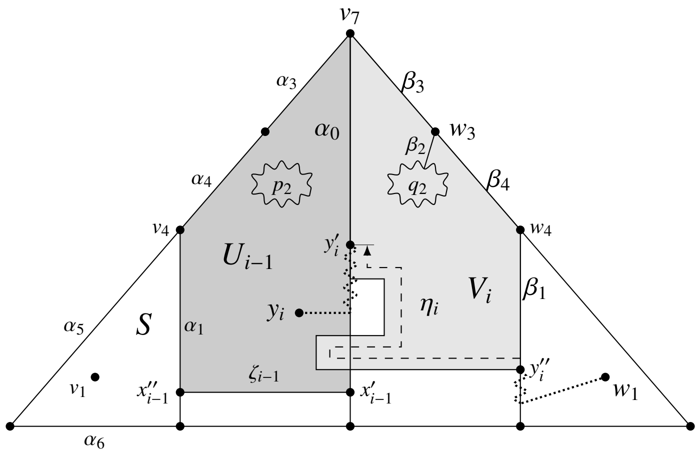


Figure 27: The construction of y_i and η_i in the proof of Theorem 32.

for Γ_0, we may connect q_2 to v_1 by a path guaranteed to intersect Δ_i at a single point, namely, w_3. Denote by V_i the regular closed region with boundary Δ_i which includes q_2 (light grey area in Fig. 27).

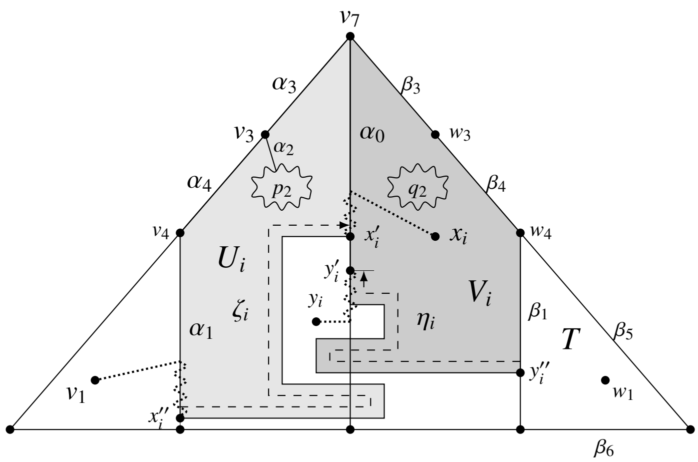


Figure 28: The construction of x_{i+1} and ζ_{i+1} in the proof of Theorem 32.

Having constructed V_i on the basis of U_{i-1}, we proceed to construct U_i on the basis of V_i. We know that $q_2 \subseteq V_i^\circ$, and that the frontier of V_i is $\Delta_i = \eta_i\alpha_0[y'_i, v_7]\beta_3\beta_4\beta_1[w_4, y''_i]$. By (40), $V_i^\circ \cap t \neq \emptyset$, whence there exists a point $x_i \in V_i^\circ \cap t^\circ$. On the other hand, by (42), $v_1 \in t^\circ$. Since t° is connected, by (39), there exists an arc $\zeta'_i \subseteq t^\circ$ from v_1 to x_i. By (43)–(45), ζ'_i lies entirely within the large triangle (Fig. 28).

We know that ζ'_i intersects Δ_i. Indeed, $\zeta'_i \subseteq t^\circ$ must be disjoint, by (43)–(45) and (41), from $\beta_1, \beta_3, \beta_4$ and η_i, and therefore intersects $\alpha_0[y'_i, v_7]$. Let x'_i be the first point of ζ'_i lying on $\alpha_0[y'_i, v_7]$ and let x''_i be the last point of $\zeta'_i[v_1, x'_i]$ lying on α_1. Let $\zeta_i = \zeta'_i[x''_i, x'_i]$. Thus, $\zeta_i\alpha_0[x'_i, v_7]\alpha_3\alpha_4\alpha_1[v_4, x''_i]$ is a Jordan curve, which we denote by Γ_i.

We claim that Γ_i separates the region p_2 from the point w_1. To see this, note that $\zeta_i \subseteq t^\circ$ can intersect, by (43)–(45), neither $\alpha_2 \subseteq p_2^\circ \cup r_2^\circ \cup p_3^\circ$ nor the triangle T bounded by β_1, β_5 and β_6. Now connect p_2 to w_1 by first following α_2 upwards, and then proceeding along the dashed path as with the construction of Γ_0 (in Fig. 26). Since $\zeta_i \cap (\alpha_2 \cup T) = \emptyset$, this path intersects Γ_i at a single point, namely, v_3, establishing that Γ_i separates p_2 and w_1. Denote by U_i the closed region with frontier Γ_i which includes p_2 (light grey shading in Fig. 28).

We have constructed points $x''_i \in \alpha_1$, $x'_i \in \alpha_0$ and an arc $\zeta_i \subseteq t^\circ$, such that ζ_i connects x''_i to x'_i, and the Jordan curve $\Gamma_i = \zeta_i\alpha_0[x'_i, v_7]\alpha_3\alpha_4\alpha_1[v_4, x''_i]$ forms the frontier of a region U_i including p_2, but not containing the point w_1.

<table>
<tr><th rowspan="2">language</th><th colspan="3">$\mathbb{R}$</th><th colspan="3">$\mathbb{R}^2$</th><th colspan="2">$\mathbb{R}^3$</th></tr>
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<tr><td>$\mathcal{RCC}8c^\circ$</td><td rowspan="2">NP [Thm. 22]</td><td rowspan="2">[Thm. 15]</td><td rowspan="2">NP [Thm. 20]</td><td>NP-hard [L. 23]</td><td>[Thm. 16]</td><td>NP [Thm. 27]</td><td colspan="2" rowspan="2"></td></tr>
<tr><td>$\mathcal{RCC}8c$</td><td colspan="3">NP [Thm. 25]</td></tr>
<tr><td>$\mathcal{RCC}8$</td><td colspan="8">NLogSpace [Thm. 12]</td></tr>
</table>

Table 1: Summary of the obtained separation and complexity results.

The construction thus repeats indefinitely, generating series of points x_i'', x_i', y_i'', y_i' and arcs ζ_i, η_i, with the properties established above, for all $i \geq 1$. We remark in passing that U_i is not necessarily a subset of U_{i-1} (as shown in Figs. 27 and 28).

By definition, $y_i \in U_{i-1}^\circ$ for all $i \geq 1$. We claim that $y_j \notin U_{i-1}$, for all $j < i-1$. To see this, note first that, by construction, the sequence of points $x_0', y_1', x_1', y_2', \ldots$ on the central arc α_0 creeps steadily upwards: hence, y_j' lies (non-strictly) below y_{i-1}', which in turn lies strictly below x_{i-1}'. Also by definition, y_j' is that point of $\eta_j' \cap \alpha_0$ lying nearest to v_7: hence η_j' cannot intersect $\alpha_0[x_{i-1}', v_7]$. On the other hand, $\eta_j' \subseteq u^\circ$ cannot intersect any of ζ_i, α_3, α_4 or α_1 either, by (43)–(45). That is, η_j' does not intersect Γ_{i-1}, and joins $w_1 \notin U_{i-1}$ to y_j. It follows that $y_j \notin U_{i-1}$, as claimed.

Consider the infinite sequence of points $y_1, y_2, \ldots$ in u_0, and the infinite sequence of closed sets $U_0, U_1, \ldots$, with respective boundaries $\Gamma_0, \Gamma_1, \ldots$. Since $\Gamma_i = \zeta_i \alpha_0[x_i', v_7]\alpha_3\alpha_4\alpha_1[v_4, x_i''] \subseteq t^\circ \cup r_0^\circ \cup p_7^\circ \cup r_3^\circ \cup p_3^\circ \cup r_4^\circ \cup p_4^\circ \cup r_1^\circ$, it does not intersect u_0, by (41) and (45)–(47). Further, we have established that, for $1 \leq j < i$, $y_j \in U_{j-1} \setminus U_{i-1}$ and $y_j \in u_0$. Thus, y_i and y_j ($i \neq j$) lie in different components of u_0. Hence, u_0 has infinitely many components. □

It is not known whether $Sat(\mathcal{RCC}8c^\circ, \mathsf{RC}^+(\mathbb{R}^2))$ is decidable. Theorem 32 suggests that resolving this issue may not be completely straightforward.

9. Conclusion

In this article, we investigated the widely studied qualitative spatial representation language known as $\mathcal{RCC}8$. This language allows us to write qualitative descriptions of spatial configurations in the form of networks of atomic constraints, with variables ranging over spatial regions, usually modelled as regular closed sets of some topological space. The satisfiability problem for this language is known to be largely independent of the topological space in question, and to have low computational complexity. Algorithms for solving this problem based on constraint satisfaction techniques have been known for some time, and their behaviour investigated in detail. Our point of departure was the observation that this formalism does not allow us to state that regions are connected (i.e. consist of a single piece), and we asked what would happen if such a facility were added. We considered two extensions of $\mathcal{RCC}8$: $\mathcal{RCC}8c$, in which we can state that a region is connected, and $\mathcal{RCC}8c^\circ$, in which we can instead state that a region has a connected interior. And we investigated the satisfiability problems for these languages over regular closed sets of low-dimensional Euclidean spaces. The work reported here is similar in spirit to Davis, Gotts and Cohn's analogous extension of $\mathcal{RCC}8$ by means of a convexity predicate [26], and partly relies on the remarkable results of Schaefer, Sedgwick and Štefankovič [24, 25] on the satisfiability problem for $\mathcal{RCC}8$ interpreted over the collection of disc-homeomorphs in the plane.

It is easy to see that, for both $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$, the satisfiability problem over the regular closed sets of $\mathbb{R}^n$ depends on n, for $n \leq 3$, but is identical for all $n \geq 3$. Less obviously, we showed that, in the case of $\mathcal{RCC}8c^\circ$, there exist finite sets of constraints that are satisfiable over $\mathsf{RC}^+(\mathbb{R}^2)$, but only by 'wild' regions having no possible physical meaning. This prompted us to consider interpretations over the more restrictive domain of non-empty, regular closed, polyhedral sets, $\mathsf{RCP}^+(\mathbb{R}^n)$. We found that the satisfiability problems for $\mathcal{RCC}8c$ (equivalently, $\mathcal{RCC}8c^\circ$) over $\mathsf{RC}^+(\mathbb{R})$ and $\mathsf{RCP}^+(\mathbb{R})$ are distinct and both NP-complete; the satisfiability problems for $\mathcal{RCC}8c$ over $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$ are identical and NP-complete; and the satisfiability problems for $\mathcal{RCC}8c^\circ$ over $\mathsf{RC}^+(\mathbb{R}^2)$ and $\mathsf{RCP}^+(\mathbb{R}^2)$ are distinct, with the latter being NP-complete. Decidability of the satisfiability problem for $\mathcal{RCC}8c^\circ$ over $\mathsf{RC}^+(\mathbb{R}^2)$ remains open. For $n \geq 3$, $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ are not interestingly different from $\mathcal{RCC}8$. The obtained separation and complexity results are collected in Table 1.

We finished by answering the following question: given that a set of $\mathcal{RCC}8c$- or $\mathcal{RCC}8c^\circ$-constraints is satisfiable over $\mathsf{RC}^+(\mathbb{R}^n)$ or $\mathsf{RCP}^+(\mathbb{R}^n)$, how complex is the simplest satisfying assignment? For both languages, we exhibited a sequence of constraints Φ_n, satisfiable over $\mathsf{RCP}^+(\mathbb{R}^2)$, such that the size of Φ_n grows polynomially in n, while the smallest configuration of polygons satisfying Φ_n cuts the plane into a number of pieces that grows exponentially. Over $\mathsf{RC}^+(\mathbb{R}^2)$, the situation is still more dramatic: we showed that, over $\mathsf{RC}^+(\mathbb{R}^2)$, $\mathcal{RCC}8c$ again requires exponentially large satisfying diagrams, while $\mathcal{RCC}8c^\circ$ can force regions in satisfying configurations to have infinitely many components.

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