On Excess in Finite Coxeter Groups

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Abstract

For a finite Coxeter group W and w an element of W the *excess* of w is defined to be $e(w) = \min\{\ell(x)+\ell(y)-\ell(w) \mid w = xy,\ x^2 = y^2 = 1\}$ where ℓ is the length function on W. Here we investigate the behaviour of $e(w)$, and a related concept reflection excess, when restricted to standard parabolic subgroups of W. Also the set of involutions inverting w is studied. (MSC2000: 20F55)

1 Introduction

This paper, continuing the investigations begun in [6] and [7], studies further properties of excess in Coxeter groups. First we recall the definition of excess.

Suppose W is a Coxeter group with length function ℓ, and set

$$\mathcal{W} = \{w \in W \mid w = xy \text{ where } x, y, \in W \text{ and } x^2 = y^2 = 1\}.$$

Then for $w \in \mathcal{W}$, the excess of w is

$$e(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy, x^2 = y^2 = 1\}.$$

The length function is not additive in general. The relationship between $\ell(w_1w_2)$ and $\ell(w_1) + \ell(w_2)$ for various special cases of $w_1, w_2 \in W$ appears in several well-known results. For example, it is an important fact that if W_J is a standard parabolic subgroup of W, then there is a set X_J of so-called distinguished right coset representatives for W_J in W with the property that $\ell(wx) = \ell(w) + \ell(x)$ for all $w \in W_J$, $x \in X_J$ ([9], Proposition 1.10). There is a parallel statement to this for double cosets of two standard parabolic subgroups of W ([5], Proposition 2.1.7). Also, when W is finite it possesses an element w_0, the longest element of W, for which $\ell(w_0) = \ell(w) + \ell(ww_0)$ for all $w \in W$ ([5], Lemma 1.5.3). Another special feature of finite Coxeter groups is that every element can be written (in possibly many ways) as a product xy where $x^2 = y^2 = 1$, and it seems natural to ask the extent to

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which additivity of length, as measured by excess, is achieved. The hope is that investigation into excess will yield useful additions to the techniques available for the study of Coxeter groups.

The main result in [6] asserts that every element in $\mathcal{W}$ is W-conjugate to an element whose excess is zero. In a similar vein, [7] shows that if W is a finite Coxeter group, then every W-conjugacy class possesses at least one element which simultaneously has minimal length in the conjugacy class and excess equal to zero. The present paper explores other properties of excess in finite Coxeter groups. So from now on we assume W is finite, which implies that $\mathcal{W} = W$. Moreover, every element $w \in W$ may be written as xy, where $x^2 = y^2 = 1$ and $L(w) = L(x) + L(y)$, where L is the reflection length function on W. This fact is established for finite Weyl groups in Carter [3] (and easily verified for the remaining finite Coxeter groups). This leads to the related notion of reflection excess. For $w \in W$ its reflection excess $E(w)$ is defined by

$$E(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy, x^2 = y^2 = 1, L(w) = L(x) + L(y)\}.$$

Clearly $E(w) \geq e(w)$. However $E(w)$ and $e(w)$ can be markedly different – see for example Proposition 3.3 of [7].

The first issue we address here is how excess and reflection excess behave on restriction to standard parabolic subgroups of W – as is well-known, such subgroups are Coxeter groups in their own right. If W_J is a standard parabolic subgroup of W and $w \in W_J$, we let $e_J(w)$ (respectively $E_J(w)$) be the excess of w (respectively reflection excess of w) considered as an element of W_J.
Our main results are as follows.

Theorem 1.1 *Let W_J be a standard parabolic subgroup of W and let $w \in W_J$. Then $E_J(w) = E(w)$.*

We remark that the proof of Theorem 1.1 is very short and elementary, whereas its sister statement for excess requires a lengthy case-by-case analysis. More than that there is a shock in store as we now see.

Theorem 1.2 *Let W_J be a standard parabolic subgroup of W and let $w \in W_J$. If W has no irreducible factors of type D_n, then $e_J(w) = e(w)$.*

The assumption that there be no direct factors of type D_n in Theorem 1.2 cannot be omitted. In Section 3 we give an example with W of type D_{12}, W_J of type D_{11} and an element w of W_J for which $e_J(w) = 60$ but $e(w) = 46$. However there are a number of positive results, to be found in Section 3, for W of type D_n provided we restrict W_J.

For $w \in W$, the set $\mathcal{I}_w$, which is defined as follows,

$$\mathcal{I}_w = \{x \in W \mid x^2 = 1, w^x = w^{-1}\}$$

is intimately connected with $e(w)$ and $E(w)$. This is because if x and y are elements of W, with $x^2 = y^2 = 1$, such that $xy = w$, then $w^x = (xy)^x = yx = w^{-1}$ and similarly $w^y = w^{-1}$. Therefore $x, y \in \mathcal{I}_w$. Thus $\mathcal{I}_w$ is always non-empty.

For $X \subseteq W$, we define in Section 2 a certain subset $N(X)$ of the positive roots of W. The Coxeter length (or just length) of X, denoted by $\ell(X)$, is defined to be $|N(X)|$ (see [10]). A consequence of our next theorem is that for all $w \in W$, $\ell(w) \leq \ell(\mathcal{I}_w)$.

Theorem 1.3 *For all $w \in W$, $N(w) \subseteq N(\mathcal{I}_w)$.*

This paper is arranged as follows. Our next section gathers together relevant background material while reviewing much of the standard notation used for Coxeter groups. Theorems 1.1 and 1.2 are established in Section 3; the former, being an easy consequence of Lemma 2.4(ii), is proved first. Then Lemma 3.1 gives criteria for recognizing when two involutions fail to be a spartan pair (see Definition 2.3 for the definition of a spartan pair). With this result to hand we then prove the pivotal Propositions 3.2 and 3.3. In fact Propositions 3.2 and 3.3 mark the parting of the ways for type A, B and type D. Corollary 3.5 and Proposition 3.4 combine to pin down the 2-cycles $(\overset{*}{a}\ \overset{*}{b})$ of spartan pairs. With this information we are then able to complete, in Theorems 3.6 and 3.7, the proof of Theorem 1.2. All is not lost for type D, as Theorem 3.8 demonstrates, with various conditions which guarantee that $e_J(w) = e(w)$.

Our final section investigates $N(\mathcal{I}_w)$ for $w \in W$. Proposition 4.2 and Lemma 4.3 reveal that, under certain circumstances, $N(\mathcal{I}_w) = \Phi^+$ (though this is not always the case) and the balance of this section presents a proof of Theorem 1.3.

2 Background Results and Notation

We briefly recall the standard notation used for finite Coxeter groups W and their root systems. To begin with, by definition, W has a presentation of the form

$$W = \langle R \mid (rs)^{m_{rs}} = 1, r, s \in R\rangle$$

where $m_{rs} = m_{sr} \in \mathbb{N}$, $m_{rr} = 1$ and $m_{rs} \geq 2$ for $r, s \in R, r \neq s$. We put $R = \{r_1, \ldots, r_n\}$ – the r_i are called the fundamental reflections of W. The length of an element w of W, denoted by $\ell(w)$, is defined to be

$$\ell(w) = \begin{cases} \min\{l \mid w = r_{i_1} \cdots r_{i_l}, r_{i_j} \in R\} \text{ if } w \neq 1 \\ 0 \text{ if } w = 1. \end{cases}$$

An element t of W is a *reflection* if t is conjugate to some fundamental reflection r. We let T denote the set of all reflections in W. The *reflection length* of an element w of W, denoted by $L(w)$, is defined to be

$$L(w) = \begin{cases} \min\{L \mid w = t_1 \cdots t_L, t_i \in T\} \text{ if } w \neq 1 \\ 0 \text{ if } w = 1. \end{cases}$$

Taking V to be a real euclidean vector space with basis $\Pi = \{\alpha_r \mid r \in R\}$ and norm $||\ \ ||$, we define a symmetric bilinear form $\langle\ ,\ \rangle$ on V by

$$\langle \alpha_r, \alpha_s\rangle = -||\alpha_r||\ ||\alpha_s|| \cos\left(\frac{\pi}{m_{rs}}\right), (r, s \in R).$$

Now for $r, s \in R$ we define

$$\alpha_s \cdot r = \alpha_s - 2\frac{\langle \alpha_r, \alpha_s\rangle}{\langle \alpha_r, \alpha_r\rangle}\alpha_r,$$

which extends to an action of W on V. This action is faithful and respects $\langle\ ,\ \rangle$ (see [9]). We remark that traditionally the action of a Coxeter group on its root system is on the left, but since in this paper we will largely be working with permutation groups, which usually act on the right, we have chosen to act on the right throughout. The following subset of V

$$\Phi = \{\alpha_r \cdot w \mid r \in R, w \in W\}$$

is the root system of W. Setting $\Phi^+ = \{\sum_{r\in R} \lambda_r\alpha_r \in \Phi \mid \lambda_r \geq 0 \text{ for all } r\}$ and $\Phi^- = -\Phi^+$ we have the fundamental fact that Φ is the disjoint union $\Phi^+\dot{\cup}\Phi^-$ (see [9] again), the sets Φ^+ and Φ^- being referred to, respectively, as the positive and negative roots of Φ. Let α be a positive root. Then $\alpha = \alpha_r \cdot w$ for some $w \in W$ and $r \in R$. Define $r_\alpha = w^{-1}rw$. Then $\alpha \cdot r_\alpha = -\alpha$. Such an element as r_α is called a *reflection* of W.
For X a subset of W we define

$$N(X) = \{\alpha \in \Phi^+ \mid \alpha \cdot w \in \Phi^- \text{ for some } w \in X\}.$$

If $X = \{w\}$, we write $N(w)$ instead of $N(\{w\})$. Clearly $N(X) = \cup_{w\in X} N(w)$. The Coxeter length of X, $\ell(X)$, is defined to be $\ell(X) = |N(X)|$ – for more on the Coxeter length of subsets of Coxeter groups, see [10]. The connection between $\ell(w)$ and the root system of W is contained in our next lemma.

Lemma 2.1 *Let $w \in W$ and $\alpha \in \Phi^+$.*

(i) If $\ell(r_\alpha w) > \ell(w)$ then $\alpha \cdot w \in \Phi^+$ and if $\ell(r_\alpha w) < \ell(w)$ then $\alpha \cdot w \in \Phi^-$. In particular, $\ell(r_\alpha w) < \ell(w)$ if and only if $\alpha \in N(w)$.
(ii) $\ell(w) = |N(w)|$.

Proof Parts (i) and (ii) are, respectively, Propositions 5.7 and 5.6 of [9]. □

Lemma 2.2 *Let $g, h \in W$. Then*

$$N(gh) = N(g) \setminus [-N(h) \cdot g^{-1}] \cup [N(h) \setminus N(g^{-1})] \cdot g^{-1}.$$

Hence $\ell(gh) = \ell(g) + \ell(h) - 2 \mid N(g^{-1}) \cap N(h) \mid$.

Proof See Lemma 2.2 of [6]. □

For J a subset of R define $W_J = \langle J\rangle$. Such a subgroup of W is referred to as a standard parabolic subgroup. Standard parabolic subgroups are Coxeter groups in their own right with root system

$$\Phi_J = \{\alpha_r \cdot w \mid r \in J, w \in W_J\}$$

(see Section 5.5 of [9] for more on this). A conjugate of a standard parabolic subgroup is called a parabolic subgroup of W, and a cuspidal element of W is an element not contained in any proper parabolic subgroup of W.

Definition 2.3 *Let $w \in W$. We call (x, y) a spartan pair for w if $x, y \in W$, $x^2 = y^2 = 1$, $w = xy$ and $\ell(x) + \ell(y) - \ell(w) = e(w)$.*

A consequence of Lemma 2.2 is that if $x, y \in W$ with $x^2 = y^2 = 1$ and $w = xy$, then (x, y) is a spartan pair for w if and only if $2|N(x) \cap N(y)| = e(w)$. Letting $V_\lambda(w)$ denote the λ-eigenspace of w ($\lambda \in \mathbb{R}$) we introduce the following subset $\mathcal{J}_w$ of $\mathcal{I}_w$, $w \in W$.

$$\mathcal{J}_w = \{x \in W \mid x^2 = 1, w^x = w^{-1}, V_1(w) \subseteq V_1(x)\}.$$

Lemma 2.4 *Suppose that $w \in W$. Then*

(i) $e(w)$ is the sum of the excesses and $E(w)$ is the sum of the reflection excesses of the projections of w into the irreducible direct factors of W; and

(ii) $\mathcal{J}_w$ is the set of x such that $w = xy$ where $x^2 = y^2 = 1$ and $L(w) = L(x) + L(y)$.

Proof Since $\ell(w)$, respectively $L(w)$, is the sum of the lengths, respectively reflection lengths, of the projections of w into the irreducible direct factors of W, (i) follows easily. For (ii), see Lemma 3.2(i) of [7]. $\square$

In view of Lemma 2.4(i), irreducible finite Coxeter groups appear frequently in our proofs. Such groups have been classified by Coxeter [4] (see also [9]).

Theorem 2.5 *An irreducible finite Coxeter group is either of type $A_n (n \geq 1)$, $B_n (n \geq 2)$, $D_n (n \geq 4)$, $\mathrm{Dih}(2m)$ (a dihedral group of order $2m$, $m \geq 5$), E_6, E_7, E_8, F_4, H_3 or H_4.*

We shall employ the following explicit descriptions of the Coxeter groups of types A_n, B_n and D_n and their root systems. First, $W(A_n)$ may be viewed as being $\mathrm{Sym}(n+1)$ with the set of fundamental reflections given by $\{(12), (23), \ldots, (n\ n+1)\}$, while elements of $W(B_n)$ can be thought of as signed permutations of $\mathrm{Sym}(n)$. A cycle in an element of $W(B_n)$ is of negative sign type if it has an odd number of minus signs, and positive sign type otherwise. We take $\{(\overset{+}{1}\overset{+}{2}), (\overset{+}{2}\overset{+}{3}), \ldots, (\overset{+}{n-1}\ \overset{+}{n}), (\overset{-}{n})\}$ to be the fundamental reflections in $W(B_n)$. An element w expressed as a product $g_1 g_2 \cdots g_k$ of disjoint signed cycles is *positive* if the product of all the sign types of the cycles is positive, and negative otherwise. The group $W(D_n)$ consists of all positive elements of $W(B_n)$ and we take the fundamental reflections of $W(D_n)$ to be $r_1 = (\overset{+}{1}\overset{+}{2})$, $r_2 = (\overset{+}{2}\overset{+}{3})$, …, $r_{n-1} = (\overset{+}{n-1}\ \overset{+}{n})$, $r_n = (\overset{-}{n-1}\ \overset{-}{n})$. Even if w is positive, it may contain negative cycles, which we wish on occasion to consider separately, so when considering elements of $W(D_n)$ we sometimes work in $W(B_n)$.

Let $\{e_i\}$ be an orthonormal basis with respect to the form $\langle\ ,\ \rangle$ for V. For $\sigma \in W(A_n)$ define $e_i \cdot \sigma = e_{i\sigma}$ – note that our permutations and signed permutations will always act on the right. The roots for $W(A_n)$ are $\pm(e_i - e_j)$ for $1 \leq i < j \leq n$, with the positive roots being $\{e_i - e_j \mid 1 \leq i < j \leq n\}$. The positive roots of $W(B_n)$ are of the form $e_i \pm e_j$ for $1 \leq i < j \leq n$ and e_i for $1 \leq i \leq n$. The positive roots of $W(D_n)$ are of the form $e_i \pm e_j$ for $1 \leq i < j \leq n$.

3 Excess and Standard Parabolic Subgroups

The main aim of this section is to prove Theorems 1.1 and 1.2. So let J be a subset of R.

Proof of Theorem 1.1 Let $w \in W_J$ and $x, y \in W$ with $x^2 = y^2 = 1$ and $xy = 1$. Then, by Lemma 2.4(ii), $x \in \mathcal{J}_w$. Therefore $V_1(w) \subseteq V_1(x)$. Let $U = \{v \in V \mid W_J \subseteq \mathrm{Stab}(v)\}$. Then for all $u \in U$, $u \in V_1(w) \subseteq V_1(x)$. Hence $U \subseteq V_1(x)$ and so $x \in W_J$. Thus $\mathcal{J}_w|_{W_J} = \mathcal{J}_w$. Hence $E_J(w) = E(w)$. $\square$

We direct our attention to Theorem 1.2 – first we must establish a number of preliminary results about spartan pairs.

Lemma 3.1 *Suppose x and y are involutions in W. If z is an involution centralizing both x and y, such that $\ell(zx) < \ell(x)$ and $\ell(zy) < \ell(y)$, then $|N(zx) \cap N(zy)| < |N(x) \cap N(y)|$. Hence (x, y) is not a spartan pair for $w = xy$.*

Proof By Lemma 2.2 and the observation that for an involution σ, $N(\sigma) = -\sigma N(\sigma)$, we obtain

$$\begin{aligned} N(zx) &= N(z) \setminus (-N(x) \cdot z) \ \dot\cup \ (N(x) \setminus N(z)) \cdot z \\ &= [-(N(z) \setminus N(x)) \ \dot\cup \ (N(x) \setminus N(z))] \cdot z. \end{aligned}$$

Similarly

$$N(zy) = [-(N(z) \setminus N(y)) \ \dot\cup \ (N(y) \setminus N(z))] \cdot z.$$

Notice that $\ell(zx) = \ell(z) + \ell(x) - 2|N(z) \cap N(x)|$. Hence $\frac{1}{2}|N(z)| - |N(z) \cap N(x)| = \frac{1}{2}(\ell(zx) - \ell(x))$, and the same is true for y. Therefore

$$\begin{aligned} |N(zx) \cap N(zy)| &= |N(z) \setminus (N(x) \cup N(y))| + |(N(x) \cap N(y)) \setminus N(z)| \\ &= |N(z)| - |N(z) \cap N(x)| - |N(z) \cap N(y)| + |N(x) \cap N(y)| \\ &= |N(x) \cap N(y)| - \tfrac{1}{2}(\ell(y) - \ell(zy)) - \tfrac{1}{2}(\ell(x) - \ell(zx)) \\ &< |N(x) \cap N(y)|. \end{aligned}$$

If (x, y) were a spartan pair for $w = xy$, then $e(w) = 2|N(x) \cap N(y)|$. But $zx, zy \in \mathcal{I}_w$ with $(zx)(zy) = w$ and $2|N(zx) \cap N(zy)| < e(w)$, a contradiction. Therefore (x, y) cannot be a spartan pair for w. $\square$

For the rest of this section, W_n is a Coxeter group of type A_{n-1}, B_n or D_n; the elements of W_n are therefore cycles or signed cycles of $\mathrm{Sym}(n)$. The notation $W = W(n_1, \ldots, n_k)$ means W is of type W_k with support $\{n_1, \ldots, n_k\}$. Suppose that W_J is a maximal parabolic subgroup of W_n. Then for some m with $1 \leq m \leq n$, we may assume that W_J is of the form $\mathrm{Sym}(1, 2, \ldots, m) \times W(m+1, \ldots, n)$. Note that the case $m = n$ is not included if W is of type A_{n-1}. If W is of type D_n, the length preserving graph automorphism means that it is not necessary to consider separately the case $W_J = \langle (\overset{+}{1}\overset{+}{2}), (\overset{+}{2}\overset{+}{3}), \ldots, (\overset{+}{n-2}\ \overset{+}{n-1}), (\overset{-}{n-1}\ \overset{-}{n}) \rangle$, as this will be covered by the case $m = n$. We will abuse notation and deem $D(n_1, n_2)$ and $D(n_1, n_2, n_3)$ to be of types $A_1 \times A_1$ and A_3 respectively.

We remark that involutions in W only contain cycles of the form $(\overset{+}{a}\ \overset{+}{b}), (\bar{a}\ \bar{b}), (\overset{+}{a})$ and $(\bar{a})$. That is, 1-cycles and positive 2-cycles.

For $u \in W$, the *positive support* of u, denoted $\mathrm{supp}^+(u)$, is the set of all $a \in \{1, \ldots, n\}$ for which $e_a \cdot u \neq e_a$. So $\mathrm{supp}^+(u)$, in the case of type B and D, differs from the support of u as a permutation of the set $\{\pm 1, \ldots, \pm n\}$ by only considering the elements of $\{1, \ldots, n\}$ which are moved by u.

Proposition 3.2 *Suppose W_n is of type A_{n-1} or B_n and let $w \in W_n$. If (x, y) is a spartan pair for w, then $\mathrm{supp}^+(x) \cup \mathrm{supp}^+(y) \subseteq \mathrm{supp}^+(w)$.*

Proof Suppose for a contradiction that there exists $i \in \mathrm{supp}^+(y) \setminus \mathrm{supp}^+(w)$. Then $e_i \cdot y = \pm e_j$ for some j with either $i \neq j$ or $e_i \cdot y = -e_i$. Now $e_i \cdot w = e_i$ forces $e_i \cdot x = e_i \cdot y$. Define a positive root α as follows:

$$\alpha = \begin{cases} e_i - e_j & \text{if } e_i \cdot y = e_j,\ j > i; \\ e_j - e_i & \text{if } e_i \cdot y = e_j,\ j < i; \\ e_i + e_j & \text{if } e_i \cdot y = -e_j, j \neq i; \text{ and} \\ e_i & \text{if } e_i \cdot y = -e_i. \end{cases}$$

Then $\alpha \cdot x = \alpha \cdot y = -\alpha$. This means that r_α centralizes both x and y. Moreover $\ell(r_\alpha x) < \ell(x)$ and $\ell(r_\alpha y) < \ell(y)$. By Lemma 3.1 this contradicts the fact that (x, y) is a spartan pair. Hence $\mathrm{supp}^+(y) \subseteq \mathrm{supp}^+(w)$. The same argument with x and w^{-1} implies that $\mathrm{supp}^+(x) \subseteq \mathrm{supp}^+(w^{-1}) = \mathrm{supp}^+(w)$. Therefore $\mathrm{supp}^+(x) \cup \mathrm{supp}^+(y) \subseteq \mathrm{supp}^+(w)$. $\square$

Proposition 3.3 *Suppose W_n is of type D_n and $w \in W_n$. If (x, y) is a spartan pair for w, then $|\mathrm{supp}^+(y) \setminus \mathrm{supp}^+(w)| \leq 1$ and if $i \in \mathrm{supp}^+(y) \setminus \mathrm{supp}^+(w)$ then $e_i \cdot y = e_i \cdot x = -e_i$. Furthermore $\mathrm{supp}^+(y) \setminus \mathrm{supp}^+(w) = \mathrm{supp}^+(x) \setminus \mathrm{supp}^+(w)$.*

Proof Suppose $i \in \mathrm{supp}^+(y) \setminus \mathrm{supp}^+(w)$ is such that $e_i \cdot y = \pm e_j$ for some $j \neq i$. Then $e_i \cdot x = e_i \cdot y$ and we define the positive root α as in the proof of Proposition 3.2, noting that the possibility $\alpha = e_i$ does not occur, and so α is indeed a root of D_n. Again, $\ell(r_\alpha x) < \ell(x)$ and $\ell(r_\alpha y) < \ell(y)$, contradicting the fact that (x, y) is a spartan pair.
Therefore, $\mathrm{supp}^+(y) \setminus \mathrm{supp}^+(w) \subseteq \{i \mid e_i \cdot y = -e_i\}$. Suppose $\{i, k\} \subseteq \mathrm{supp}^+(y) \setminus \mathrm{supp}^+(w)$ with $i \neq k$. Let $\beta = e_i + e_k \in \Phi^+$. Then $\beta \cdot y = \beta \cdot x = -\beta$ and hence $\ell(r_\beta x) < \ell(x)$ and $\ell(r_\beta y) < \ell(y)$, contradicting the fact that (x, y) is a spartan pair. Hence $\mathrm{supp}^+(y) \setminus \mathrm{supp}^+(w)$ contains at most one element i, and $e_i \cdot y = -e_i$. Since $e_i \cdot x = e_i \cdot y$, we have $\mathrm{supp}^+(y) \setminus \mathrm{supp}^+(w) \subseteq \mathrm{supp}^+(x) \setminus \mathrm{supp}^+(w)$. Repeating the argument with x and w^{-1} gives the reverse inclusion, forcing $\mathrm{supp}^+(y) \setminus \mathrm{supp}^+(w) = \mathrm{supp}^+(x) \setminus \mathrm{supp}^+(w)$. $\square$

Note that there are examples of spartan pairs (x, y) for w where $\mathrm{supp}^+(y)$ is not contained in $\mathrm{supp}^+(w)$. These examples are the source of (infinitely many) cases in which $e_J(w) > e(w)$. One such is the following: $w = (\overset{+}{2}\ \overset{+}{4}\ \overset{+}{6}\ \overset{+}{8}\ \overset{+}{10}\ \overset{-}{12}\ \overset{+}{11}\ \overset{+}{9}\ \overset{+}{7}\ \overset{+}{5}\ \overset{-}{3}) \in D_{12}$. As a product of fundamental reflections $w = [468.10.3456789.10.11.12.10.987654323579.11]$ where 10 is the branch node of the D_{12} diagram, and in this expression for w we have written i instead of r_i. Clearly w lies in a standard parabolic subgroup W_J of type D_{11}. It can be shown that $e_J(w) = 60$, whereas $e(w) = 46$, given by the spartan pair (x, y) where $x = (\overset{-}{1})(\overset{+}{2}\ \overset{+}{3})(\overset{+}{4}\ \overset{+}{5})(\overset{+}{6}\ \overset{+}{7})(\overset{+}{8}\ \overset{+}{9})(\overset{+}{10}\ \overset{+}{11})(\overset{-}{12})$, $y = (\overset{-}{1})(\overset{-}{2})(\overset{+}{3}\ \overset{+}{4})(\overset{+}{5}\ \overset{+}{6})(\overset{+}{7}\ \overset{+}{8})(\overset{+}{9}\ \overset{+}{10})(\overset{+}{11}\ \overset{+}{12})$.

Proposition 3.4 *Suppose (x, y) is a spartan pair for $w \in W_n$. If $(\overset{+}{a_1} \cdots \overset{+}{a_k})$ and $(\overset{\varepsilon_1}{b_1} \cdots \overset{\varepsilon_k}{b_k})$ are disjoint w-cycles for which $(\overset{+}{a_1} \cdots \overset{+}{a_k})^y = (\overset{\varepsilon_1}{b_1} \cdots \overset{\varepsilon_k}{b_k})^{-1}$, then $\max\{a_i\} > \min\{b_i\}$.*

Proof Without loss of generality, assume $a_1 y = \pm b_k$, …, $a_i y = \pm b_{k+1-i}$, …, $a_k y = \pm b_1$. Let $T = \{1, \ldots, n\} \setminus \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$. Then $y = y_1 y_2$ for some involution y_1 with

$\mathrm{supp}(y_1) \subseteq T$ and $y_2 = (\overset{\rho_1}{a_1}\ \overset{\rho_1}{b_k})\cdots(\overset{\rho_k}{a_k}\ \overset{\rho_k}{b_1})$. Define $z = (\overset{\rho_k}{a_1}\ \overset{\rho_k}{b_1})(\overset{\rho_{k-1}}{a_2}\ \overset{\rho_{k-1}}{b_2})\cdots(\overset{\rho_1}{a_k}\ \overset{\rho_1}{b_k})$. Now

$$\begin{aligned}
yz &= y_1 y_2 z = y_1 \prod_{i=1}^{k}(\overset{\rho_i}{a_i}\ \overset{\rho_i}{b_{k+1-i}}) \prod_{i=1}^{k}(\overset{\rho_{k+1-i}}{a_i}\ \overset{\rho_{k+1-i}}{b_i}) \\
&= y_1 \prod_{i=1}^{\lfloor k/2 \rfloor}(\overset{\rho_i\rho_i}{a_i}\ \overset{\rho_i\rho_i}{a_{k+1-i}}) \prod_{i=1}^{\lfloor k/2 \rfloor}(\overset{\rho_i\rho_{k+1-i}}{b_i}\ \overset{\rho_i\rho_{k+1-i}}{b_{k+1-i}}) \\
&= y_1 \prod_{i=1}^{\lfloor k/2 \rfloor}(\overset{+}{a_i}\ \overset{+}{a_{k+1-i}}) \prod_{i=1}^{\lfloor k/2 \rfloor}(\overset{\rho_i\rho_{k+1-i}}{b_i}\ \overset{\rho_i\rho_{k+1-i}}{b_{k+1-i}}).
\end{aligned}$$

Therefore yz is an involution. Next we show that xz is an involution. We know that $w^y = w^{-1}$. Hence for $1 < i \leq k$,

$$\begin{aligned}
e_{a_{i-1}} &= e_{a_i} \cdot ywy \\
&= \rho_i e_{b_{k+1-i}} \cdot wy \\
&= \varepsilon_{k+1-i}\rho_i e_{b_{k+2-i}} \cdot y \\
&= \rho_{i-1}\rho_i\varepsilon_{k+1-i} e_{a_{i-1}}.
\end{aligned}$$

Therefore $\rho_{i-1}\rho_i = \varepsilon_{k+1-i}$. Similarly $\rho_k\rho_1 = \varepsilon_k$. This allows us to calculate $e_{a_j} \cdot zwz$ and $e_{b_j} \cdot zwz$ for $1 \leq j \leq k$:

$$\begin{aligned}
e_{a_j} \cdot zwz &= \rho_{k+1-j}\varepsilon_j\rho_{k-j} e_{a_{j+1}} = e_{a_{j+1}} = e_{a_j} \cdot w \\
e_{b_j} \cdot zwz &= \rho_{k+1-j}\rho_{k-j} e_{a_{j+1}} = \varepsilon_j e_{b_{j+1}} = e_{b_j} \cdot w
\end{aligned}$$

Hence $zwz = w$ and so z centralizes $x = wy$.

Suppose for a contradiction that $\max\{a_i\} < \min\{b_i\}$. We will show that $\ell(zy) < \ell(y)$ and $\ell(zx) < \ell(x)$. Write $z_i = \prod_{t=1}^{i}(\overset{\rho_{k+1-t}}{a_t}\ \overset{\rho_{k+1-t}}{b_t})$. Now

$$\begin{aligned}
(e_{a_{i+1}} - \rho_{k-i}e_{b_{i+1}}) \cdot z_i y &= (e_{a_{i+1}} - \rho_{k-i}e_{b_{i+1}}) \cdot y = (e_{a_{i+1}} \cdot \prod_{i=1}^{k}(\overset{\rho_i}{a_i}\ \overset{\rho_i}{b_{k+1-i}}) - \rho_{k-i}e_{b_{i+1}}) \\
&= \rho_{i+1}e_{b_{k-i}} - \rho_{k-i}\rho_{k-i}e_{a_{k-i}} = -e_{a_{k-i}} \pm e_{b_{k-i}} \in \Phi^-.
\end{aligned}$$

Hence by Lemma 2.1 $\ell(zy) = \ell(z_k y) < \ell(z_{k-1}y) < \cdots < \ell(z_1 y) < \ell(y)$. Similarly

$$(e_{a_{i+1}} - \rho_{k-i}e_{b_{i+1}}) \cdot z_i x = (-e_{a_{k-i}} \pm e_{b_{k-i}}) \cdot w^{-1} = -e_{a_{k-i-1}} \pm e_{b_{k-i-1}} \in \Phi^-$$

and so $\ell(zx) < \ell(x)$. But by Lemma 3.1 this contradicts the fact that (x, y) is a spartan pair. Therefore $\max\{a_i\} > \min\{b_i\}$. $\square$

Corollary 3.5 *Suppose $w \in W_n$ is contained in a maximal parabolic subgroup W_J of W_n which is of the form* $\mathrm{Sym}(1, 2, \ldots, m) \times W(m+1, \ldots, n)$. *If (x, y) is a spartan pair for w, then for every 2-cycle $(\overset{*}{a}\ \overset{*}{b})$ of x or y, either $\{a, b\} \subseteq \{1, \ldots, m\}$ or $\{a, b\} \subseteq \{m+1, \ldots, n\}$.*

Proof Since $x = wy$, it is enough to prove the result for 2-cycles $(\overset{*}{a}\ \overset{*}{b})$ of y. Without loss of generality, assume $a < b$. Suppose that $\{a,b\} \not\subseteq \{m+1,\ldots,n\}$. Then $a \in \{1,\ldots,m\}$. If b lies in the same w-cycle as a, then b is forced to lie in $\{1,\ldots,m\}$ (since $w \in W_J$) and we are done. If b lies in a different w-cycle then since $w^y = w^{-1}$, the w-cycles containing a and b respectively are of the form $(\overset{+}{a_1}\cdots\overset{+}{a_k})$ and $(\overset{\varepsilon_1}{b_1}\cdots\overset{\varepsilon_k}{b_k})$. By Proposition 3.4, we see that $\max\{a_i\} > \min\{b_i\}$. Since $\max\{a_i\} \leq m$, we get $\{b_1,\ldots,b_k\} \subseteq \{1,\ldots,m\}$. In particular, $\{a,b\} \subseteq \{1,\ldots,m\}$. □

Theorem 3.6 *Suppose $W = W(A_n)$ and $w \in W_J \leq W$. Then $e_J(w) = e(w)$.*

Proof We may assume that W_J is maximal and hence of the form $\mathrm{Sym}(1,2,\ldots,m) \times \mathrm{Sym}(m+1,\ldots,n)$ for $1 \leq m \leq n-1$. Let (x,y) be a spartan pair for w. Note that since $x = wy$, x and y lie in the same right W_J-coset. It is therefore enough to show that $y \in W_J$. By Corollary 3.5, for every 2-cycle $(a\ b)$ of y, either $\{a,b\} \subseteq \{1,\ldots,m\}$ or $\{a,b\} \subseteq \{m+1,\ldots,n\}$. Hence y is a product of commuting reflections each of which lies in W_J, forcing $y \in W_J$. Thus every spartan pair for w is already to be found in W_J and so $e_J(w) = e(w)$. □

Theorem 3.7 *Suppose $W = W(B_n)$ and $w \in W_J \leq W$. Then $e_J(w) = e(w)$.*

Proof We may assume that W_J is maximal and hence of the form $\mathrm{Sym}(1,2,\ldots,m) \times B(m+1,\ldots,n)$ for $1 \leq m \leq n$. Let (x,y) be a spartan pair for w. Again we will show that $y \in W_J$. By Corollary 3.5, for every 2-cycle $(\overset{*}{a}\ \overset{*}{b})$ of y, either $\{a,b\} \subseteq \{1,\ldots,m\}$ or $\{a,b\} \subseteq \{m+1,\ldots,n\}$. Let $S = \{a \in \{1,\ldots,m\} \mid e_a \cdot y \in \Phi^-\}$. Assume for a contradiction that S is nonempty. Now define $z = \prod_{a\in S}(\bar{a})$. Repeated applications of Lemma 2.1 show that $\ell(zy) < \ell(y)$ and furthermore $\ell(zx) < \ell(z)$ (since $e_a \cdot x = e_a \cdot yw^{-1}$ and $e_a \cdot w^{-1} \in \Phi^+$ for all $a \in \{1,\ldots,m\}$). Note also that z centralizes y. To show that z centralizes x, suppose $a \in S$. Then $e_a \cdot y = -e_b$ for some $b \in \{1,\ldots,m\}$. So $(e_a \cdot w)\cdot y = e_a \cdot yw^{-1} = -e_b \cdot w^{-1} \in \Phi^-$. Therefore $a \in S$ implies $aw \in S$ and hence for any w-cycle $(\overset{+}{a_1}\cdots\overset{+}{a_k})$ with $\{a_1,\ldots,a_k\} \subseteq \{1,\ldots,m\}$, either $e_{a_i}\cdot z = e_{a_i}$ for all $1 \leq i \leq k$ or $e_{a_i}\cdot z = -e_{a_i}$ for all $1 \leq i \leq k$. Consequently z centralizes w. We deduce that z centralizes x as well as y, and so by Lemma 3.1, (x,y) cannot be a spartan pair, a contradiction. Thus S is empty and $e_a \cdot y \in \Phi^+$ for all $a \in \{1,\ldots,m\}$. Taken with the fact that the 2-cycles of y do not interchange elements of $\{1,\ldots,m\}$ and $\{m+1,\ldots,n\}$, this shows that $y \in W_J$. Hence $x = wy \in W_J$. Thus every spartan pair for w is contained in W_J, which implies $e_J(w) = e(w)$. □

Proof of Theorem 1.2 By Lemma 2.4(i), it is enough to prove the result for W an irreducible finite Coxeter group not of type D_n. Types A_n and B_n have been dealt with in Theorems 3.6 and 3.7. Theorem 1.2 trivially holds for dihedral groups. Types E_6, E_7, E_8, F_4, H_3 and H_4 have been checked using the computer algebra package MAGMA[2]. We discuss the details of these calculations using $W = W(E_8)$ as an example. If for each maximal parabolic subgroup W_J of W we know that for all $K \subset J$ and for all $w \in W_K$, $e_J(w) = e_K(w)$, then it suffices to verify Theorem 1.2 for all the maximal parabolic subgroups of W – that is, that

$e_J(w) = e(w)$ for all maximal parabolic subgroup W_J of W. So, for example, if W_J is of type A_n, then we may apply Theorem 3.6 for $W_K \subseteq W_J$. However we must beware of standard parabolic subgroups of W of type D_n – these must be checked directly. Among the standard parabolic subgroups of W, the most challenging calculation occurs when W_J is of type E_7. Set $E = \{y \mid y \in W_J \text{ and } y \text{ has order greater than } 2\}$. We note that $|E| = 2,892,832$. We need to check that $e(y) = e_J(y)$ for all $y \in E$ (this is because we know that $e(w) = 0 = e_J(w)$ for all $w \in W_J \backslash E$). Below we give the MAGMA code which was used for the calculations in the groups of types E_6, E_7, E_8, F_4, H_3 and H_4 (with `ans = {0}` being the output obtained in all cases).

In the routine H denotes the standard parabolic subgroup of the Coxeter group W.

```
E:={y: y in H |Order(y) gt 2};
ans:={ };for x in E do NH:=Normalizer(H,sub<W|x>);
CW:=Centralizer(W,x);
CH:=Centralizer(H,x);
SH:=Sylow(NH,2);
RH:=sub<W|SH,CH>;
TT:=Transversal(RH,CH);
for i:=1 to #TT do
if x^TT[i] eq x^-1 then inverter:=TT[i];end if;
end for;
CosH:={c*inverter : c in CH};CosW:={c*inverter : c in CW};
tempW:={y: y in CosW |Order(y) eq 2};
tempH:={y: y in CosH |Order(y) eq 2};
lenx:=CoxeterLength(W,x);
minW:=Min({CoxeterLength(W,x*y)+ CoxeterLength(W,y) - lenx : y in tempW});
minH:=Min({CoxeterLength(W,x*y)+ CoxeterLength(W,y) - lenx : y in tempH});
ans:=ans join {minH - minW};end for;
```

The final result in this section shows that excess does behave well in type D_n with respect to certain parabolic subgroups and cycle types of elements.

Theorem 3.8 *Suppose $W = W(D_n)$ and $w \in W_J \leq W$, where W_J is of the form* $\mathrm{Sym}(1,2,\ldots,m) \times D(m+1,\ldots,n)$. *Write $w = w_1 w_2$, where $w_1 \in \mathrm{Sym}(1,2,\ldots,m)$ and $w_2 \in D(m+1,\ldots,n)$. If either $m = n$, or w_2 contains a 1-cycle, or w_2 consists only of even, positive cycles, then $e_J(w) = e(w)$.*

Proof Let (x, y) be a spartan pair for w. We will show that $y \in W_J$. By Corollary 3.5, for every 2-cycle $(\overset{*}{a}\ \overset{*}{b})$ of y, either $\{a, b\} \subseteq \{1, \ldots, m\}$ or $\{a, b\} \subseteq \{m+1, \ldots, n\}$. Let $S = \{a \in \{1, \ldots, m\} \mid e_a \cdot y \in \Phi^-\}$.
We first consider the case where $|S|$ is even. Define $z = \prod_{a \in S} (\bar{a}) \in W$. Now

$$N(z) = \{e_a \pm e_b \mid 1 \leq a < b \leq n, a \in S\}.$$

Since $e_a \cdot y \in \Phi^-$ for every $a \in S$, at least one of $e_a + e_b$ and $e_a - e_b$ will be in $N(y)$ for all $b > a$. Therefore $\ell(zy) = \ell(z) + \ell(y) - 2|N(z) \cap N(y)| \leq \ell(y)$. Similarly $\ell(zx) \leq \ell(z)$. Moreover, by the same reasoning as that in the proof of Theorem 3.7, z centralizes both x and y. Therefore, by Lemma 3.1, we either have a contradiction (forcing S to be empty and

$x, y \in W_J$) or another spartan pair (zx, zy), this time contained in W_J. Hence $e_J(w) = e(w)$.

We are left with the possibility that $|S|$ is odd. If $m = n$, then since $y \in W(D_n)$ we must have $|S|$ even. So $m < n$. Suppose, for a contradiction, that w_2 consists only of even, positive cycles. By thinking of W as a subgroup of $W(B_n)$, we can write $y = y_1 y_2$, where $\mathrm{supp}^+(y_1) \subseteq \{1, \ldots, m\}$ and $\mathrm{supp}^+(y_2) \subseteq \{m+1, \ldots, n\}$. Since $|S|$ is odd, y_1 contains an odd number of sign changes, and since $y \in W(D_n)$, y_2 must also contain an odd number of sign changes. However $y_2 w_2 y_2 = w_2^{-1}$. (This is because for every 2-cycle $(\overset{*}{a}\ \overset{*}{b})$ of y, either $\{a, b\} \subseteq \{1, \ldots, m\}$ or $\{a, b\} \subseteq \{m+1, \ldots, n\}$.) But w_2 consists only of positive, even cycles. This means there are two conjugacy classes in $D(m+1, \ldots, n)$ with that signed cycle type, but only one in $B(m+1, \ldots, n)$. Therefore any element of $B(m+1, \ldots, n) \setminus D(m+1, \ldots, n)$ (such as y_2) must interchange the conjugacy classes. This contradicts the fact that w_2^{-1} is conjugate in $D(m+1, \ldots, n)$ to w_2. Hence if w_2 consists only of even, positive cycles, $|S|$ must be even.

Therefore the only possibility remaining is that w_2 contains a 1-cycle $(\overset{+}{b})$ or $(\overset{-}{b})$. Define $z = (\overset{-}{b}) \prod_{a \in S} (\overset{-}{a}) \in W$. Now

$$N(z) = \{e_a \pm e_c \mid 1 \leq a < c \leq n, a \in S\} \cup \{e_b \pm e_c \mid b < c \leq n\}.$$

Let $\lambda = |\{e_a \pm e_c \mid 1 \leq a < c \leq m, a \in S\}|$. Then $|N(z)| = \lambda + 2|S|(n-m) + 2(n-b)$. Now for $c > m$ and $a \in S$, $e_a + e_c \in N(y)$ and $e_a - e_c \in N(y)$. For $c \leq m$, at least one of $e_a + e_c$, $e_a - e_c \in N(y)$. Hence

$$\begin{aligned} |N(z) \cap N(y)| &\geq \tfrac{1}{2}\lambda + 2|S|(n-m) = \tfrac{1}{2}(\lambda + 4|S|(n-m)) \\ &> \tfrac{1}{2}\left(\lambda + 2|S|(n-m) + 2(n-b)\right) = \ell(z). \end{aligned}$$

Hence $\ell(zy) < \ell(y)$. Similarly $\ell(zx) < \ell(x)$. An almost identical argument to that in the proof of Theorem 3.7 shows that z centralizes both x and y. By Lemma 3.1 (x, y) cannot be a spartan pair, a contradiction. Hence S is empty. This implies that $x, y \in W_J$ and hence $e_J(w) = e(w)$. □

4 The Set of Roots $N(\mathcal{I}_w)$

The main result of this section is the proof of Theorem 1.3. First we look at the case when w is a cuspidal element. We require a technical lemma before we can start the proof of Proposition 4.2.

Lemma 4.1 *Suppose $w \in W$ and let $\Omega_1, \ldots, \Omega_k$ be the set of $\langle w \rangle$-orbits on Φ. For $x \in \mathcal{I}_w$, if $\Omega_i \cdot x \cap \Phi^- \neq \emptyset$, then $\Omega_i \cap \Phi^+ \subseteq N(\mathcal{I}_w)$.*

Proof Suppose that $\Omega_i \cdot x \cap \Phi^- \neq \emptyset$. Then for any $\alpha \in \Omega_i$, there is an integer j such that $\alpha \cdot w^j x \in \Phi^-$. If $\alpha \in \Phi^+$, then $\alpha \in N(w^j x)$. Since $w^j x \in \mathcal{I}_w$, we get $\alpha \in N(\mathcal{I}_w)$. This holds for each $\alpha \in \Omega_i \cap \Phi^+$, so the lemma is proved. □

Proposition 4.2 *If w is a cuspidal element of W, then $N(\mathcal{I}_w) = \Phi^+$.*

Proof Assume that w is cuspidal in W and let $\Omega_1, \ldots, \Omega_k$ be the set of $\langle w \rangle$-orbits on Φ. For $x \in \mathcal{I}_w$, suppose $\beta \in \Omega_i \cdot x$ and write $\beta = \alpha \cdot x$. Then for any integer j, $\beta \cdot w^j = \alpha \cdot x w^j = \alpha \cdot w^{-j} x \in \Omega_i \cdot x$. Therefore $\Omega_i \cdot x$ is also a $\langle w \rangle$-orbit. Suppose $\Omega = (\beta, \beta \cdot w, \ldots, \beta \cdot w^k)$ is any $\langle w \rangle$-orbit of roots. Then $v = \sum_{i=0}^{k} \beta \cdot w^i$ is a fixed point of w. It is well-known (and follows from Ch V §3.3 of [1]) that the stabilizer of any non-zero $v \in V$ is a proper parabolic subgroup of W. Since w is cuspidal, therefore, we must have $v = 0$. Hence Ω must contain both positive and negative roots. In particular, for $1 \leq i \leq k$, $\Omega_i \cdot x$ must contain at least one negative root. Therefore, by Lemma 4.1, $\Omega_i \cap \Phi^+ \subseteq N(\mathcal{I}_w)$ for $1 \leq i \leq k$. Hence $N(\mathcal{I}_w) = \Phi^+$, and the result holds. □

Lemma 4.3 *Suppose W has a non-trivial centre. Then for all $w \in W, N(\mathcal{I}_w) = \Phi^+$.*

Proof Let w_0 be the non-trivial central involution. Let $x \in \mathcal{I}_w$. Then $xw_0 \in \mathcal{I}_w$. Now

$$\begin{aligned} N(xw_0) &= N(x) \setminus [-N(w_0) \cdot x] \cup [N(w_0) \setminus N(x)] \cdot x \\ &= N(x) \setminus [\Phi^- \cdot x] \cup \left[\Phi^+ \setminus N(x)\right] \cdot x \\ &= \emptyset \cup \Phi^+ \setminus N(x) = \Phi^+ \setminus N(x). \end{aligned}$$

Hence $N(\mathcal{I}_w) \supseteq N(xw_0) \cup N(x) = \Phi^+$. Therefore $N(\mathcal{I}_w) = \Phi^+$. □

Lemma 4.4 *Suppose W is of type A_{n-1}. Then for all $w \in W$, $N(w) \subseteq N(\mathcal{I}_w)$.*

Proof Now W is isomorphic to $\mathrm{Sym}(n)$, so we may write w as a product of disjoint cycles. Consider a cycle $\lambda = (a_1 a_2 \ldots a_m)$ of w. Let $\Lambda = \{a_1, \ldots, a_m\}$. We will define σ_λ and $\sigma'_\lambda \in \mathrm{Sym}_\Lambda \cap \mathcal{I}_\lambda$. If m is odd, let $\sigma_\lambda = \sigma'_\lambda = (a_2 a_m)(a_3 a_{m-1}) \cdots (a_{(m+1)/2} a_{(m+3)/2})$. If m is even, let $\sigma_\lambda = (a_1 a_m)(a_2 a_{m-1})(a_3 a_{m-2}) \cdots (a_{m/2} a_{m/2+1})$ and $\sigma'_\lambda = (a_2 a_m)(a_3 a_{m-1}) \cdots (a_{m/2} a_{m/2+2})$. Note that every involution in $\mathrm{Sym}_\Lambda \cap \mathcal{I}_\lambda$ is of the form σ_λ or σ'_λ for some choice of a_1. Further, note that σ_λ contains the 2-cycle $(a_{\lceil m/2 \rceil} a_{\lceil m/2 \rceil + 1})$, which is of the form $(a\; aw)$.

Let $e_a - a_b \in N(w)$. Suppose a and b appear in the same cycle λ of w. Write $w = \lambda w'$ where $\mathrm{supp}(w') \cap \Lambda = \emptyset$. Let τ be an arbitrary element of $\mathcal{I}_{w'} \cap \mathrm{Sym}(\mathrm{supp}(w'))$. If a, b are separated by an even number of a_i, then by a judicious choice of a_1, we can ensure that (ab) is a 2-cycle of σ_λ. Otherwise we can ensure that (ab) is a 2-cycle of σ'_λ. Let $x = \sigma_\lambda \tau$ or $\sigma'_\lambda \tau$ accordingly. Then $x \in \mathcal{I}_w$ and $e_a - e_b \in N(x)$.
Now suppose that a and b appear in different cycles λ_1 and λ_2 of w. We may choose σ_{λ_1} and σ_{λ_2} such that σ_{λ_1} contains the 2-cycle $(a\; aw)$ and σ_{λ_2} contains the 2-cycle $(b\; bw)$. Writing $w = \lambda_1 \lambda_2 w'$ and choosing any $\tau \in \mathcal{I}_{w'} \cap \mathrm{Sym}(\mathrm{supp}(w'))$ we see that for $x = \sigma_{\lambda_1} \sigma_{\lambda_2} \tau \in \mathcal{I}_w$, $(e_a - e_b) \cdot x = (e_a - e_b) \cdot w \in \Phi^-$. Hence $e_a - e_b \in N(w)$ and therefore $N(w) \subseteq N(\mathcal{I}_w)$. □

Note that there are examples in type A_{n-1} where $N(\mathcal{I}_w) \neq \Phi^+$. For instance, in $W(A_6) \cong \mathrm{Sym}(7)$, for $w = (1234)(567)$ we have $e_4 - e_5 \notin N(\mathcal{I}_w)$.

Proposition 4.5 *Let W be of type D_n. For all $w \in W$, $N(w) \subseteq N(\mathcal{I}_w)$.*

We work in the environment of $W(B_n)$, as we will be dividing w into a product of cycles, some of which may be negative. We write $\mathcal{I}_w^B$ for $\mathcal{I}_w$ when we are considering w as an element of $W(B_n)$, and $\mathcal{I}_w^D = \mathcal{I}_w^B \cap W(D_n)$. Let $W(B_w)$ be the Coxeter group of type B over $\mathrm{supp}(w)$ and define $\overline{\mathcal{I}}_w = \mathcal{I}_w^B \cap W(B_w)$. For $w \in W(B_n)$, we write $\hat{w}$ for the corresponding element of Sym(n). For example, if $w = (\overset{+}{1}\ \overset{-}{2})$, then w is an even, negative cycle and $\hat{w} = (1\ 2)$. For every $w \in W(B_n)$, $\mathcal{I}^B_w \neq \emptyset$. If w is negative, then for any $\tau \in \mathcal{I}^B_w$, exactly one of τ and $w\tau$ is negative, so $\mathcal{I}^B_w$ contains both positive and negative elements. If w is positive, then $w \in W(D_n)$, and so $\mathcal{I}^D_w$ is non-empty, whence $\mathcal{I}^B_w$ contains at least one positive element.

Before we can prove Proposition 4.5 we need the following lemma.

Lemma 4.6 *Let g be an m-cycle of $W(B_n)$.*

(i) Suppose m is odd and write $\hat{g} = (a_1 a_2 \cdots a_m)$. Then there exists $\tau_g \in \overline{\mathcal{I}}_g$ such that τ_g is positive and $\hat{\tau}_g = (a_2 a_m)(a_3 a_{m-1}) \cdots (a_{(m+1)/2} a_{(m+3)/2})$.

(ii) Let $a \in \operatorname{supp}(g)$. Then there exist $\tau^+_{g,a} \in \overline{\mathcal{I}}_g$ and $\tau^-_{g,a} \in \overline{\mathcal{I}}_g$ such that $(\overset{+}{a})$ is a cycle of $\tau^+_{g,a}$ and $(\overset{-}{a})$ is a cycle of $\tau^-_{g,a}$. Moreover, $\tau^+_{g,a}$ is negative if and only if m is even and g is negative; $g\tau^+_{g,a}$ is negative if and only if m is odd and g is negative; $\tau^-_{g,a}$ is positive if and only if m is even and g is positive.

Proof We consider the cases m odd and m even separately.

Suppose first that m is odd. Since $\overline{\mathcal{I}}_g$ is non-empty, there exists $\tau \in \overline{\mathcal{I}}_g$, and $\hat{\tau}$ inverts $\hat{g}$ by conjugation. Hence $\hat{\tau}$ is of the form $(a_2 a_m)(a_3 a_{m-1}) \cdots (a_{(m+1)/2} a_{(m+3)/2})$ for some labelling $(a_1 \cdots a_m)$ of $\hat{g}$. The τ resulting from any choice of labelling for g are conjugate via an element of the centralizer of g, and hence for any labelling $(a_1 a_2 \cdots a_m)$ of $\hat{g}$, there exists $\tau \in \overline{\mathcal{I}}_g$ with $\hat{\tau} = (a_2 a_m)(a_3 a_{m-1}) \cdots (a_{(m+1)/2} a_{(m+3)/2})$. Let z be the central involution in $W(B_g)$, so $z = (\overset{-}{a_1})(\overset{-}{a_2}) \cdots (\overset{-}{a_m})$ and z is negative. Exactly one of τ and $z\tau$ is positive. Let τ_g be the positive one. Moreover, setting $a_1 = a$, define $\tau^+_{g,a} = \tau_g$ and $\tau^-_{g,a} = z\tau_g$. since τ_g is a positive involution containing exactly one 1-cycle, the 1-cycle must be positive. Hence $\tau^+_{g,a}$ contains $(\overset{+}{a})$ and is positive, and $\tau^-_{g,a}$ contains $(\overset{-}{a})$ and is negative. Finally $g\tau^+_{g,a}$ is negative if and only if g is negative. This establishes part (i), and (ii) for m odd.

Now suppose that m is even. Then g is conjugate either to $h = (\overset{+}{a_1}\ \overset{+}{a_2} \cdots \overset{+}{a_m})$ or to $(\overset{-}{a_1}\ \overset{+}{a_2} \cdots \overset{+}{a_m}) = (\overset{-}{a_1})h$. Let $\sigma = (\overset{+}{a_1})(\overset{+}{a_{m/2}})(\overset{+}{a_2}\overset{+}{a_m})(\overset{+}{a_3}\overset{+}{a_{m-1}}) \cdots (\overset{+}{a_{m/2}}\overset{+}{a_{m/2+2}})$. Then $\sigma \in \overline{\mathcal{I}}_h$, and hence $(\overset{-}{a_1})\sigma \in \overline{\mathcal{I}}_{(\overset{-}{a_1})h}$. Therefore $\overline{\mathcal{I}}_g$ contains an element τ with two 1-cycles, at least one of which is positive. By choice of labelling of g, we can ensure that $(\overset{+}{a})$ is a 1-cycle of τ. In addition, τ is positive if and only if g is positive. Let z be the central involution in $W(B_g)$. Note that because m is even, z is positive. Now set $\tau^+_{g,a} = \tau$ and $\tau^-_{g,a} = z\tau$. Then $\tau^+_{g,a}$ is negative if and only if g is negative, $g\tau^+_{g,a}$ is always positive, and $\tau^-_{g,a}$ is positive if and only if g is positive. This establishes part (ii) for m even, so completing the proof of the lemma. $\square$

We may now give the

Proof of Proposition 4.5 Suppose $\alpha = \pm e_a \pm e_b \in N(w)$. We must find a positive $x \in \mathcal{I}^B_w$ such that $\alpha \in N(x)$. There are two cases to consider, depending on whether a and b are in the same or distinct cycles of w.

We first consider the case that $\{a, b\} \subseteq \operatorname{supp}(g)$ for some m-cycle g of w. Suppose m is odd, then by Lemma 4.6 and judicious choice of a_1, there exists $\tau_g \in \overline{\mathcal{I}}_g$ such that (ab) is a 2-cycle of $\hat{\tau}_g$. Hence $\alpha \cdot \tau_g = \pm\alpha$. Therefore either $\alpha \in N(\tau_g)$ or $\alpha \in N(\tau_g g)$. Replacing τ_g by $\tau_g g$ if necessary, we have $\alpha \in N(\tau_g)$ and τ_g is positive whenever g is positive. Write $w = gw'$. If τ_g is positive, then choose a positive $\sigma \in \overline{\mathcal{I}}_{w'}$. If τ_g is negative, then g is negative, forcing w' to be negative (recall that w is positive), thus we may choose a negative $\sigma \in \overline{\mathcal{I}}_{w'}$. Finally, let $x = \tau_g\sigma$. Then $x \in \mathcal{I}_w^B$, x is positive and $\alpha \cdot x = -\alpha$. Hence $\alpha \in N(x)$. If m is even, just pick any positive $\sigma \in \mathcal{I}_w^B$. Let z be the central involution in $W(B_g)$. Since m is even, z is positive. Then $z\sigma \in \mathcal{I}_w$ and $\alpha \in N(\sigma) \cup N(z\sigma) \subseteq \mathcal{I}_w$.

We must now consider the case where a and b appear in different cycles of w. So assume $a \in \operatorname{supp}(g_1)$, where g_1 has length m_1, and $b \in \operatorname{supp}(g_2)$, where g_2 has length m_2. Write $w = g_1 g_2 w'$. Let $\tau = \tau^+_{g_1,a} g_1 \tau^+_{g_2,b} g_2$. Note that $\alpha \cdot \tau = \alpha \cdot w$. If τ is positive, then let σ be any positive element of $\overline{\mathcal{I}}_{w'}$. If τ is negative and w' is negative, then let σ be any negative element of $\overline{\mathcal{I}}_{w'}$. Set $x = \tau\sigma$. Then x is a positive element of $\mathcal{I}_w^B$, and $\alpha \in N(x)$. The only case not covered is where τ is negative and w' is positive. Hence $g_1 g_2$ is positive. Without loss of generality, we must have $\tau^+_{g_1,a} g_1$ negative and $\tau^+_{g_2,b} g_2$ positive, which means m_1 is odd, and g_1 is negative; hence m_2 is even and g_2 is negative. In this case we let σ be any positive element of $\overline{\mathcal{I}}_{w'}$ and set $x = \tau^-_{g_1,a}\tau^-_{g_2,b}\sigma$. We see that x is a positive element of $\mathcal{I}_w^B$. Furthermore $\alpha \cdot x = -\alpha$ and hence $\alpha \in N(x)$. For each possibility we have found a positive $x \in \mathcal{I}_w^B$ such that $\alpha \in N(x)$, so the result holds. $\square$

Proposition 4.7 *Suppose W is finite with trivial centre. Then for all $w \in W, N(w) \subseteq N(\mathcal{I}_w)$.*

Proof It is enough to consider the finite irreducible Coxeter groups W with trivial centre. These are precisely the Coxeter groups of type A_n, $n \geq 1$, D_n, $n > 4$ and n odd, E_6 and $\mathrm{Dih}(2m)$ for m odd. In $\mathrm{Dih}(2m)$ the non-trivial conjugacy classes are either classes of reflections or cuspidal classes. In either case the result is trivially true. For E_6 the result was checked with a computer using MAGMA[2]. Types A_n and D_n have been dealt with in Lemma 4.4 and Proposition 4.5 respectively. $\square$

Theorem 1.3 is an immediate consequence of Lemma 4.3 and Proposition 4.7.

Theorem 1.3 raises the question as to whether, for $w \in W$, $N(w) \subseteq N(x)$ for some $x \in \mathcal{I}_w$. However we do not have to look very far before alighting upon the following example. Choose W to be the Coxeter group $W(A_4) \cong \mathrm{Sym}(5)$ and let $w = (235)$. Now $N(w) = \{e_2 - e_5, e_3 - e_4, e_3 - e_5, e_4 - e_5\}$. Also $\mathcal{I}_w = \{(23), (35), (25), (14)(23), (14)(35), (14)(25)\}$ and we have

$x \in \mathcal{I}_w$	$N(x)$
(23)	$\{e_2 - e_3\}$
(35)	$\{e_3 - e_4, e_3 - e_5, e_4 - e_5\}$
(25)	$\{e_2 - e_3, e_2 - e_4, e_2 - e_5, e_3 - e_5, e_4 - e_5\}$
(14)(23)	$\{e_1 - e_2, e_1 - e_3, e_1 - e_4, e_2 - e_3, e_2 - e_4, e_3 - e_4\}$
(14)(35)	$\{e_1 - e_2, e_1 - e_3, e_1 - e_4, e_2 - e_4, e_3 - e_4, e_3 - e_5\}$
(14)(25)	$\{e_1 - e_3, e_1 - e_4, e_1 - e_5, e_2 - e_3, e_2 - e_4, e_2 - e_5, e_3 - e_4, e_3 - e_5\}$.

From the above we observe that for each $x \in \mathcal{I}_w$, $N(w) \not\subseteq N(x)$.

References

[1] N. Bourbaki. *Lie Groups and Lie Algebras Ch. 4–6* Springer-Verlag (2002).

[2] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., 24 (1997), 235–265.

[3] R.W. Carter. *Conjugacy Classes in the Weyl Group*, Compositio. Math. 25, Facs. 1 (1972), 1–59.

[4] H.S.M. Coxeter. *The complete enumeration of finite groups of the form* $R_i^2 = (R_iR_j)^{k_{ij}} = 1$, J. London Math. Soc. 10 (1935), 21–25.

[5] M. Geck and G. Pfeiffer. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*, LMS Monographs, New Series 21, (2000).

[6] S.B. Hart and P.J.Rowley. *Involution Products in Coxeter Groups*, J. Group Theory 14 (2011), no. 2, 251–259.

[7] S.B. Hart and P.J.Rowley. *Zero Excess and Minimal Length in Finite Coxeter Groups*, J. Group Theory 15 (2012), no. 4, 497–512.

[8] S.B. Hart and P.J.Rowley. *Corrigendum to Involution products in Coxeter groups [J. Group Theory 14 (2011), no. 2, 251–259]* J. Group Theory 17 (2014), no. 2, 379–380.

[9] J.E. Humphreys. *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, 29 (1990).

[10] S.B. Perkins and P.J.Rowley. *Coxeter Length*, J. Alg 273 (2004) 344–358.