Essays on Credit Risk

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Submitted to the
Faculty of Economics
Università della Svizzera italiana

for the degree of
Ph.D. in Finance

June 2013

Contents

Introduction

The research consists of two major topics. The first one is related to reduced-form credit risk models and in particular identifies an idiosyncratic and a systematic component in Credit Default Swap returns. The remuneration for the undiversifiable component is therefore estimated. The second topic consists in a deep analysis of structural credit risk models, with a specific focus on the reasons for their failure in the explanation of observed credit spreads, both in levels and in differences. This issue is tackled from two different perspectives: corporate bonds and CDS. The following three chapters will be part of the research.

In the first chapter, we study the role of systematic risk in the credit market and in CDS spreads in particular. The studied variables are the mark-to-market returns of a protection-seller of CDS contracts on a set of investment grade firms. A simple linear market model is proposed to describe the returns of the single CDS contracts using the market index (CDX index) as a systematic factor. The ability of the systematic factor to price CDS in the cross section dimension is also assessed. Consequently, the risk premium per unit of systematic risk is computed and compared to previous studies. Also the evolution through time of the performance of this simple linear model is investigated, with a specific focus on the behaviour during the financial crisis. Furthermore, a decomposition of the risk between systematic and specific is proposed and the changes in the proportion of the two, both on the entire distribution and on the tails, is investigated. Finally, a decomposition between specific and systematic risk is investigated for every different industrial sector. The previous literature discovered a common systematic factor in credit returns (see e.g. Collin-Dufresne et al. (2001) or Berndt and Obreja (2010)). In this work we propose a simple framework to model this common factor and verify if it is actually priced by the market.

In the second chapter the most important structural credit risk models are studied in the framework of a state-space framework. In particular, all the structural models can be represented in a state space form, where the unobserved variables are asset values and asset volatilities. Such latent quantities are measured, possibly with an additional noise, by two observed quantities: equity prices and volatilities implied in equity options. In general the measurement equations are nonlinear and therefore, in place of a standard Kalman

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filter, some nonlinear filtering techniques are required in order to eliminate the observational noise. In the work we verify to which extent such sophisticated filtering approach outperforms simple numerical inversion of measurement equations, performed under the hypothesis of absence of observational noise.

One of the testable implications of the presence of observational noise contaminating the measurement equations is the better performances of the filtering technique in periods of stress, when liquidity and limits of arbitrage may induce strong deviations of equity prices and volatilities from the theoretical values implied by their assets. This difference between the pre- and post-crisis period is actually observed and supports the presence of observational noise. The improvement given by the use of bivariate state space models (based on both equity prices and volatilities) with respect to a univariate one, already proposed in the literature, is also investigated.

On the common ground given by bivariate state space models, a comparison between the performance of the most commonly used structural models is performed. In particular, the ability to explain observed CDS spreads is studied. The considered models are the simple Merton's Model, the extension to stochastic interest rates, the endogenization of the default threshold and the inclusion of a jump component in the stochastic process for the latent asset values. The improvement given by stochastic interest rates and endogenous default, per se, is not substantial. Only the introduction of jumps is able to reduce the difference between predicted and observed spread. Finally, the jump risk premium implied in credit spreads will be investigated and a decomposition of the default intensity between a jump and a diffusion-to-default component will be analyzed.

The study enlarges the literature about the empirical testing of structural credit risk models (such as Jones et al. (1983), Huang and Huang (2003) and Eom et al. (2004)). Furthermore, we extend the use of the state space framework for the estimation of structural credit risk models. This idea has been proposed by Duan and Fulop (2009) in the univariate case and we extend it to the case of two latent state variables.

The last chapter extends the study of the performances of structural credit risk models using a large dataset of actual execution prices of more than 4.000 corporate bonds along ten years. The main objective is to verify the role of market-specific liquidity measures and market-wide indicators of limits of arbitrage in the explanation of the errors made by structural models. The equilibrium between the market values of assets and all types of liabilities, and therefore the accuracy of structural models, are guaranteed by non-arbitrage relationships. Therefore, in periods of adverse liquidity conditions and/or strong limits to arbitrage forces, the performances of structural models are supposed to deteriorate. We assess the performances of structural models by verifying the ability of stock and implied volatility returns to explain the changes in credit spreads. In a second step, we verify the dependence of the errors left by structural models on the liquidity measures specific to the two markets involved in the non-arbitrage relationships, i.e. the equity and corporate bond market. If the performances of structural models were affected by liquidity conditions, then one would expect higher errors in presence of adverse liquidity conditions.

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The errors of structural models are actually found to significantly depend on liquidity indicators, and in particular from the liquidity conditions in the credit rather than in the equity market. Furthermore, if also limits of arbitrage have a role in the poor performance of structural models, then the residuals left by both structural variables and liquidity indicators should significantly depend on market-wide indicators of restriction to arbitrage forces. Also this prediction is verified since the average errors of structural models are found to depend on market-wide liquidity indicators such as the TED and LIBOR-OIS spread.

The work relies on the empirical testing literature mentioned in chapter two and extends it identifying some specific drivers that describe the underperformance of structural credit risk models.

Systematic and specific risk in the CDS market

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Abstract

In this paper we study the role of systematic risk in the credit market and in CDS spreads in particular. We propose a simple linear market model to describe the returns of the single CDS contracts using the market index (CDX index) as a factor. In the cross section dimension, the CDX factor is priced and a significant risk premium of 37 basis points is estimated and seems to be compatible with previous studies. However this simple model does not seem to be enough to describe the entire cross section of realized returns since a significant intercept is present, especially during the crisis period. We also study the evolution of systematic risk before and after the financial crisis. Systematic risk seems to considerably increase during the crisis when measured by the average R^2 of time series regressions, the variance explained by the first principal components and the increase in the extreme tail dependence. Finally we propose a simple decomposition of the variance to separate the systematic and specific component. A strong increase in the systematic fraction is observed during the crisis. The sector with the highest systematic fraction of variance, and therefore highest sector fragility, is the financial one.

1　Introduction

During the recent financial crisis, we observed a dramatic increase in the credit spreads quoted on the market. This is true for both the spreads implicit in the corporate bond market and those paid to buy protection in the CDS market. This rise in the spreads can be interpreted as an increase in the risk neutral default probability of the considered companies since such a dramatic change in the expected recovery rate does not seem plausible.

One first question that arises is therefore whether such high values of the default probability can be considered systematic or specific and thus diversifiable. Another question concerns the change in the role of systematic components of credit risk during the crisis period. We investigate such questions in the CDS market. This choice is justified by the high liquidity of the credit derivatives market and by the huge volume of outstanding contracts. According to the ISDA Market Survey of June 2010, the total outstanding notional covered by CDS contracts amounts to more than 26 trillions in the first half of 2010. Although this number is huge, it is well below the historical maximum of outstanding notional, which is equal to more than 62 trillions in the second half of 2007. Another justification of the choice can also come from the wider and wider use of indexes of CDS contracts. Here we focus only on the US market and consider only the CDX as a comprehensive index.

The first contribution of the paper is therefore to propose a model that separates the systematic and specific components of risk in the CDS market. The second objective is to analyze in depth the evolution of systematic credit risk before and during the financial crisis. This is done both on the entire sample of firms and considering the same data grouped by sector. The change in risk is investigated also with respect to the "extreme systematic" risk, measured in terms of extreme tail dependence.

The remainder of the paper is organized as follows. In section 2 we briefly review the literature on the topic. In section 3 we describe the data we are using with some of its most important statistical properties. In section 4 we propose a statistical characterization for the single CDS returns based on a linear dependence on the returns of the CDX market index. In section 5 we present the results of the estimated model in the time series and cross-section dimension, in section 6 we show some tests for the linear model and finally in section 7 we decompose credit risk into its two components. Section 8 concludes.

2 Review of the literature

In this paper we rely on several different strands of literature. Since we are applying a linear model for the characterization of returns, we rely first of all on the classical asset pricing literature related to the Capital Asset Pricing Model such as Sharpe [1964], Lintner [1965] and Mossin [1966]. Nevertheless the classical CAPM literature is based on the equilibrium condition in capital markets. Given the fact that here we are considering only a relatively small segment of the entire financial market (the credit market), it seems inappropriate to invoke such a strong condition. Therefore we are simply assuming that the returns in the credit market are linearly dependent on the returns of a (sufficiently diversified) credit market index and that pricing in this market can be done relying only on the (weaker) non-arbitrage condition. This statistical characterization is analogous to the Arbitrage Pricing Theory usually applied to the equity market, whose literature is originated by the works by Ross [1976] and Roll and Ross [1980].

The second big strand of literature is obviously referred to credit risk. As is well known, we can identify two big families of credit risk models: structural and reduced-form. In this paper we are directly modeling credit spreads, neglecting in this way all the other corporate and market information available to agents and thus assuming that all the relevant information is efficiently incorporated into observed credit spreads. Thus the literature we are mainly referring to is the one of reduced-form models. The first two articles in this group are Altman [1968] and Beaver [1968]. In both cases, the default probability is directly modeled through the use of balance-sheet variables. The first examples of reduced-form models based on market data, instead, are in Jarrow and Turnbull [1992] and Jarrow and Turnbull [1995]. A more recent reference for reduced-form models is Duffie and Singleton [1999]. In all the last three articles the relevant variable is the intensity (Hazard Rate) of a stochastic process that models the arrival of default events.

A number of articles directly address the modeling of credit spreads through the use of a set of explanatory variables. Among these, it is worth mentioning Collin-Dufresne, Goldstein, and Martin [2001], Ericsson, Jacobs, and Oviedo [2009] and Berndt and Obreja [2010]. The first article shows that the variables used in structural models (e.g. implied volatility or leverage) have a relatively small explanatory power on corporate bond premia and that a strong systematic factor is still present in the residual of these regressions. The second article reaches a very different conclusion, showing that volatility, leverage and risk-free rate explain quite well CDS spreads and leave no systematic factor in the residuals. However, this second study is performed on a much shorter time series and CDS spreads are used in place of corporate bond spreads. Finally, the article

by Berndt and Obreja [2010] tries to explain the residual common factor found by Collin-Dufresne et al. [2001] building a "catastrophic factor" as the difference between the spreads of tranches with different seniority in CDO products. Such factor actually seems to be strongly significant and to eliminate the common behaviour in model residuals.

There is also an increasing literature that investigates the presence of a systematic risk and fragility factors in credit markets. First of all, Jarrow, Lando, and Yu [2005] theoretically discuss the equivalence between the risk-neutral and physical measure showing that the two measure are asymptotically equivalent for "well diversified portfolios". An exact equivalence is based on stronger utility arguments. Das, Duffie, Kapadia, and Saita [2007] verify that the correlation between defaults is higher than can be explained using observable factors. They argue that an economy-wide frailty factor is present that makes default events more correlated. A similar finding is presented by Duffie, Eckner, Horel, and Saita [2009]. They show that the high value of extreme correlation in defaults can be explained introducing a common unobserved systemic factor. In this case they also estimate this latent factor as a component in the default intensity process. Finally Bhansali, Gingrich, and Lonstaff [2008] estimate a model in which the default intensity is a linear combination of three latent Poisson processes. The first corresponds to specific default risk, the second to the risk of a sector-wide default and the third to systemic risk. We also have to consider the literature on systematic risk which is more closely related to linear models such as Sharpe [1963].

A strand which is related to these articles is the one that investigates the presence and the size of a risk premium for undiversifiable credit risk. In Fama and French [1989] a systematic undiversifiable risk in corporate bonds is identified, which can be partially explained using macro variables and indicators of the business cycle in particular. Also Duffee [1999] verifies the presence of a positive price for credit risk. In this case it is estimated from the intensity of the arrival of defaults, which is modeled as a square root diffusion process. The same result is also found by Elton, Gruber, Agrawal, and Mann [2001] and Liu, Longstaff, and Ravit [2002]. In this last case the estimation of the premium is based on the Interest Rate Swap market, in which a default risk of the counterparty is implicit.

Other articles define the risk premium as the ratio between risk-neutral and physical default probabilities. This measure, which is indeed proportional to the state price per unit probability, can be interpreted as the premium required by a risk averse agent buying credit risk. This measure is used among others by Berndt, Douglas, Duffie, Ferguson, and Schranz [2004] and Driessen [2005]. In the first case an estimation of the physical default probability is obtained using the Expected Default Frequencies from the

KMV model, while in the second case the actual default frequencies from Moody's and S&P are used. In all the mentioned articles a ratio of roughly 2 is obtained. Finally it is worth citing the articles by Hull, Predescu, and White [2005] and Lonstaff, Giesecke, Schaefer, and Strebulaev [2010]. In both cases the premium is estimated from corporate bond prices and is significantly different from zero. The only article in contrast with this literature is Dichev [1998], in which no positive relationship between default risk (measured by Altman [1968] Z-Score and Ohlson [1980] model) and subsequent returns is found. Therefore this last work is the only which finds no remuneration for bearing default risk.

3 Data

3.1 The CDX index

As a proxy for the systematic credit risk, we use the Markit CDX index Investment Grade with maturity 5 years. This is calculated as the average of the CDS spreads of the most liquid 125 reference entities. In order to eliminate the spreads of firms that have been downgraded, for which a credit event took place and the ones that are not sufficiently liquid any more, the CDX index rolls its constituents every six months. The continuous series of CDX spreads provided by Bloomberg do not account for the variation in the spread simply due to the roll over and only continue the series with the spread of the new on-the-run series. The series used here, on the other hand, concatenates the percentage changes of the 15 on-the-run series, removing in this way the spurious variation only due to the roll over. The source is in this case Bloomberg.

The time period considered here coincides with the existence of the CDX index. It begins in October 2003 and ends in September 2010. All the following analysis is based on a measure of the performance realized by a potential investor that sells protection in the CDS or CDX market. Such performance is computed accordingly to the ISDA Standard Model explained in detail in Appendix and is expressed in percentage of the standard notional amount of the contract. In general, very similar results are obtained when the simple first difference of the series is used in place of returns.

The considered period is a very particular one and includes the heavy credit crisis of 2008-2009. The evolution of the CDX spread is represented in figure 1 together with the evolution of the S&P500 Index.

It's easy to verify the sharp increase in CDX spread and the corresponding fall of the stock market starting in August 2007. This close relationship is confirmed by a negative correlation between the two series of returns equal

to -45.16% and -60.16% for the daily and two-weeks returns respectively. We also split the considered time period into two subperiods of equal length, the first goes from October 29, 2003 to April 13, 2007 and the second from April 16, 2007 to September 30, 2010. As far as daily changes are concerned, the correlation increases in absolute value from -0.3494 in the first period to -0.4615 in the second, as one might expect in periods of crisis.

In table 3.1, we present some descriptive statistics of the daily time series of the CDX spread in levels.

	N	Mean	St. Deviation	Min.	Median	Max.
CDX Index	1807	102.98	74.21	28.48	60.95	352.60

Table 1: Descriptive statistics

The minimum has been reached in February 2007 and the maximum in December 2008.

We tested the null hypothesis of unit root of the CDX spread process using an augmented Dickey-Fuller test. The null hypothesis cannot be rejected at all the standard confidence levels (the ADF t-statistic is -2.34). The CDX spread process therefore seems to be non-stationary. The same result is confirmed by the Phillips-Perron and the KPSS test.

If we consider the first difference of the process, the null of unit root is rejected at any confidence level (the ADF t-statistic is -27.47). We can conclude that the series is integrated of order 1.

We also investigated the Granger causality relationships between equity market (here represented by the S&P 500 Index) and the credit market (CDX Index) using again daily data. The strength of results partially depends on the number of lags we use in the vector autoregression model. However we observe in general a strong causality from the equity market to the CDX market. In particular granger-causality probabilities are virtually zero in this direction, so that we can always reject the null that Equity market does not cause (in a Granger sense) Credit market. In the opposite direction the causality relation depends on the number of lags used but in general it seems much weaker. For example, the Granger causality probabilities with 5 lags (one week) are the following.

Variable	CDX	S&P 500
CDX	0.0000	0.0000
S&P 500	0.0412	0.0003

Table 2: Granger Causality probabilities between CDX and S&P500

We can see that at 5% confidence level we can always reject the null of

absence of causality in both direction but at 1% level causality is rejected only from the CDX to the S&P 500 but not in the opposite direction.

We also investigated the cointegration between the time series of the CDX spread and the S&P500. Both the CDX index and the S&P 500 are integrated of order 1. Since we are considering only two variables, we can test cointegration by simply implementing the two step Engle-Granger procedure. After estimating a regression on the two variables in levels, we use the augmented Dickey-Fuller test on the residuals to verify their stationarity. The unit root hypothesis in the residuals can never be rejected. Therefore no strong cointegration relation between the S&P 500 and the CDX spread is present. The test based on Johansen procedure gives the same result accepting the null of no cointegration relations. This seems quite reasonable since we expect the equity index to be much more persistent than the CDX. The latter, although statistically I(1), from an economic point of view can show a behaviour closer to stationarity. This fact prevents the existence of a long-term equilibrium relationship between the levels of the two series.

We also briefly investigate the process of price discovery between stock and credit market. In the literature, such task is achieved using the measure introduced in Gonzalo and Granger [1995], and more recently applied by Blanco, Brennan, and Marsh [2005] in the credit spreads context, or by means of the intervals proposed by Hasbrouck [1995]. Unfortunately in our case both these procedures are not applicable since the time series of CDX and S&P500 index are not cointegrated.

Therefore, we simply verify the dependence of contemporaneous returns of the CDX index on lagged returns of the S&P500 index and vice versa. We consider only the most significant lags, that is the first two. The two equations we are estimating are the following

$$r_{CDX,t} = b_0 + b_1 r_{SP,t-1} + b_2 r_{SP,t-2} + \varepsilon_{1,t} \tag{1}$$

$$r_{SP,t} = c_0 + c_1 r_{CDX,t-1} + c_2 r_{CDX,t-2} + \varepsilon_{2,t} \tag{2}$$

where $r_{CDX,t-k}$ and $r_{SP,t-k}$ are the CDX and S&P500 index return respectively, at the kth lag. The results are the following

	Coefficient	t-stat
b_0	-2.779e-05	-0.5760
b_1	0.0396	11.0623
b_2	0.0199	5.5442

Table 3: Estimated coefficients of equation (1)

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	Coefficient	t-stat
c_0	3.1801e-05	0.0996
c_1	-0.2920	-1.9263
c_2	-0.2594	-1.7110

Table 4: Estimated coefficients of equation (2)

We can see that the S&P500 index actually anticipates the CDX index both at lag 1 and at lag 2. The significance of the first lag is also noticeable. On the other hand there is no economic significance in the negative coefficients of the second estimation, in which the ability of CDX index to anticipate S&P500 is assessed. We can conclude that the credit market can be predicted using information from the equity market while the opposite is not true, at least in the considered sample. This is also fully consistent with the results about Granger causality exposed above. A reason for this could be the presence of more "informed traders" in the equity than in the credit derivatives market, which induces a delay in the complete incorporation of the information in credit market prices. Also this result can be interpreted as a lack of efficiency in the credit market. It is quite surprising to observe such a dependence on a daily horizon, given that one would expect the information to flow from a market to another on an intraday horizon. A similar result, but referred to corporate bonds, is obtained by Kwan [1996], who shows that equity market actually anticipates the corporate bond market, supporting the hypothesis that more informed traders are present in the equity market.

A simpler alternative explanation of this finding can come from the presence of stale quotes in the CDS contracts underlying the CDX index which automatically induces correlation between the CDX index and lagged equity returns.

According to standard structural models (such as Merton [1974]) another determinant beyond equity value affects the default event: volatility. In figure 2 we represent the evolution of the VIX and CDX Index. The common trend is evident and it is also confirmed by the correlation between the two series, which is equal to 39.41% for daily changes and 56.48% for two-weeks changes. Also in this case, the correlation in the second subperiod (40.39%) is higher than in the first (33.37%).

We also compared the behaviour of the CDX index and traditional bond indexes. In particular we built an artificial index as the difference between daily returns in the Citigroup GBI Corporate BBB 1-5 Years and the Citigroup GBI Treasury 1-5 Years. The daily returns of the resulting index have a correlation equal to 62.51% with the CDX Index. This indicates that the spreads on the bond and credit derivative market are highly but not perfectly correlated. Of course, in this case, the imperfect correlation can also come

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from different maturity (5 years Vs. 1-5 years) and rating.

Finally, we consider the returns of the CDX index at 1-day frequencies. For the computation of such returns refer to section A. The first interesting result, as already documented in Bystrom [2008] for the Itraxx indexes, is the presence of a strong autocorrelation (13.91%) in the first differences of the daily CDX spreads. The same result (13.7%) is obtained when we use the daily mark-to-market returns of the market value of a CDX contract expressed as a percentage of the notional. Some autocorrelation (3.21% and 3.06% for the first differences and the mark-to-market percentage variations respectively) is present also at the second lag.

We verify this by estimating the following simple AR(2) model

$$\Delta CDX_t = a_0 + a_1 \Delta CDX_{t-1} + a_2 \Delta CDX_{t-2} + \varepsilon_t \qquad (3)$$

and we get the coefficients in table 5.

	Coefficient	t-stat
a_0	0.0512	0.4563
a_1	0.1373	5.8285
a_2	0.0130	0.5502

Table 5: Estimated coefficients of equation (3)

The results are very similar when we use the percentage mark-to-market returns in place of the simple first difference. The regression coefficient is still strongly significant.

We also evaluated the evolution in time of such autocorrelation. In particular, we evaluate it in the two subperiods defined above. The first order autocorrelation is significantly higher in the first period (21.69%) than in the second (13.56%). Similar results are derived from the estimation of equation (3). In particular

	10/2003-7/2007		7/2007-9/2010	
	Coefficient	t-stat	Coefficient	t-stat
a_0	-0.0170	-0.5615	0.1223	0.5491
a_1	0.2169	6.5041	0.1356	4.0631
a_2	0.0276	0.8271	0.0123	0.3692

Table 6: Estimated coefficients of equation (3)

In both periods the lagged difference is statistically significant but the coefficient and its significance is reduced in the second period.

Also in this case, this apparent inefficiency can be justified by the presence of stale quotes in the CDX index and in particular in the early stages of its life, since this phenomenon is vanishing as the volume of transactions and the number of agents in the market is increasing.

Of course this analysis is based on quoted mid prices and does not consider the presence of bid-ask spreads and the costs required to implement an arbitrage strategy between the index and the single CDS.

The most important descriptive statistics of the time series of daily CDX returns are summarized in the following table.

	N	Ann. St. Dev.	Skew.	Kurt.	Min.	Max.
CDX Index	1806	3.33%	-0.7024	22.51	-2.15%	1.55%

Table 7: Descriptive statistics for the CDX Index

It's important to notice the strong negative value of the skewness, especially if compared with the skewness of the S&P500 index in the same period, which is -0.2622, and the extremely high level of kurtosis. The distribution of CDX returns is very far from normality and the Jarque-Bera null hypothesis of normality is strongly rejected (with a p-value of less than 0.001).

If we extend the horizon and consider the 15-days differences and returns the presence of autocorrelation completely disappears. Only a negative autocorrelation of about 2% is present for the first and second lag. The estimation of equation ((3)) for 15-days differences and mark-to-market variations gives in this case totally insignificant coefficients (not reported).

The entire autocorrelation function of daily returns is represented in figure 3 and shows a strongly significant coefficient at the first lag.

If we move to the autocorrelation function of squared returns, we can deduce the clear presence of strong persistence in variance (see figure 4). So, in analogy to the equity market, we can hypothesize the presence of a GARCH structure also in the returns from credit default swap market. There are no dramatic differences between the particular specifications of the GARCH model. We explored the use of different specifications such as the EGARCH and the Glosten-Jagannathan-Runkle (GJR), and different distributions for the innovations such as Normal and Student's t, but the improvement in the fit of the data is not decisive. Also concerning the choice of ARCH and GARCH parameters the simplest choice of $p = q = 1$ seems to be enough. Therefore, in figure 5 we present the autocorrelation function of the filtered residuals of a parsimonious GARCH(1,1) model. Much of the structure of the returns is removed and residuals are quite close to serial uncorrelation.

Unfortunately, this procedure doesn't have a particular impact on the normal-

ity of returns. In fact the values of skewness and kurtosis are not significantly modified with respect to the non-filtered series and the Jarque-Bera hypothesis of normality is still strongly rejected (with a p-value still less than 0.001).

3.2 A Markov Switching model

We have already mentioned the apparent presence of two clearly separated time periods in the considered interval. In particular, the first "State of the World" ends at the beginning of the sub-prime crisis in the summer of 2007 while the second goes from the beginning of the crisis to the end of the period. We consider such a separation several times in the paper. We now investigate the statistical reliability of our choice. In order to do so, we fit a two-state Markov switching model on the time series of returns for the CDX index.

We can clearly fit a two states model and observe a neat break between state 1 and 2 in the summer of 2007. The first of the two states is characterized by credit returns with a slightly higher mean and a much smaller variance. The graph of the time series of the (smoothed) probabilities of being in state 1 or 2 is presented in figure 6. We observe the apparent change of state during the summer of 2007 and a smaller switch to the "bad" state around May 2005. We can identify this episode with the automotive sector crisis, and in particular with the downgrade of GM bonds from investment grade to junk, which had an impact on the entire credit market.

This finding confirms the presence of a Markov switching model in the credit market already investigated by Alexander and Kaeck [2008] for the iTraxx index in the European market.

3.3 Single CDS contracts

The CDS spreads used here are relative to 5-years contracts on 238 reference entities. These companies are selected from the population of the S&P500 based on the liquidity of the quoted CDS spreads. All the spreads used are *quoted* and not *transaction* spreads and the source is Thompson Datastream.

In figure 7 we represent the rating distribution of the sample as provided by S&P and observed at the end of the period.

The median rating is BBB+. One has to consider that these ratings are observed after the financial crisis of 2008 and 2009. Consequently, they are arguably lower than the ratings of the same companies at the beginning of the period.

The distribution by GICS sector of the companies is the following

GICS Sector	N
Materials	21
Industrials	34
Consumer Discretionary	42
Telecommunication	10
Energy	23
Consumer Staples	25
Financials	31
Utilities	19
Information Technology	12
Health Care	21

Table 8: Sector distribution of the analyzed companies

The average spread of the 238 CDS on the considered period is 108.46 bps, which is very similar to the average of the CDX index spread (102.98). Therefore the set of companies used here seems to have a level of risk indeed similar to the CDX Investment Grade index. As said before, the liquidity of single CDS contracts is smaller than that of the CDX index. We therefore avoid the use of daily data and concentrate only on 2-weeks differences and returns, removing in this way all the high-frequency noise coming from stale quotes.

The average pairwise correlation between the 238 CDS is equal to 0.43, which is quite a large number, especially when compared to the average correlation between the equity returns of the same 238 companies, which is 0.34. So the analysis of the systematic components for the panel of CDS returns seems particularly important.

We also implemented a simple principal component analysis on the panel of two-weeks returns for the 238 companies. The first component explains the 35.76% of the total variance and the second the 13.38%. It is also interesting to notice that the first principal component has a correlation equal to 78.14% with the CDX Index. This indicates that the first principal component captures quite accurately systematic credit risk and, at the same time, that the choice of the CDX index as a proxy for market credit risk seems suitable.

4 A linear Model

We propose a statistical characterization of the returns of every single Credit Default Swap based on a simple linear model. In particular, the return of a single CDS is assumed to be

$$r_{i,t} = E\left(r_{i,t}\right) + \beta_i\left(r_{CDX,t} - E\left(r_{CDX,t}\right)\right) + \varepsilon_{i,t} \tag{4}$$

where

- $r_{i,t}$ is the change in mark-to-market value of the position of a protection seller on the ith CDS, expressed as a percentage of the notional

- $r_{CDX,t}$ is the change in mark-to-market value of the position of a protection seller on the CDX North America Investment Grade index, expressed as a percentage of the notional

- β_i is the sensitivity of the ith CDS to the CDX index

- $E(\cdot)$ indicates the expectation under the risk neutral measure

- $\varepsilon_{i,t}$ is an idiosyncratic component, specific to asset i. It is assumed to be independent of the variable $r_{CDX,t}$ and such that $E(\varepsilon_{i,t} \cdot \varepsilon_{j,t}) = 0$ for $i \neq j$.

As said above, the spread of a new CDS contract is chosen in such a way that the expected return under the risk neutral measure is equal to zero for both parties. Therefore all the risk neutral expectations in (4) are zero for every CDS and for the CDX index.

Thus equation (4) reduces to

$$r_{i,t} = \beta_i r_{CDX,t} + \varepsilon_{i,t} \tag{5}$$

If a sufficiently large number of CDS is present on the market, we can argue that a portfolio can be constructed in such a way that all the idiosyncratic components are diversified away. The perfect diversification can be obtained only asymptotically, in presence of an infinite number of contracts.

If this condition is verified, we can invoke the Arbitrage Pricing Theorem (Harrison and Pliska [1981]) or simply take the physical expectation of equation (5) to find, under absence of asymptotic arbitrage opportunities, the following pricing relation.

$$E^P\left(r_{i,t}\right) = E^P\left(r_{CDX,t}\right)\beta_i + \eta_i \tag{6}$$

where $E^P\left(\cdot\right)$ indicates the expectation computed under the *physical* measure. This is the theoretical pricing equation for any CDS on the market. Notice that in this relationship no intercept is present. The reason is again that, under the risk neutral measure, the expected return of a position in a CDS is zero and not the risk-free interest rate. This is in strict analogy to what happens in futures markets.

4.1 Testable Equations

In order to estimate the model, we need the empirical counterparties of equations (5) and (6).

The first can be estimated in the time series dimension using the following regression

$$r_{i,t} = \alpha_i + \beta_i r_{CDX,t} + \varepsilon_{i,t} \tag{7}$$

The model predicts that all the intercepts α_i are zero for every CDS i.

The second equation has to be tested in the cross sectional dimension using the following regression

$$\overline{r_{i,t}} = \gamma + \lambda \beta_i + \eta_i \tag{8}$$

where

- $\overline{r_{i,t}}$ is the time average of the mark to market performance in the considered period

- β_i is the coefficient estimated in equation (7) for the ith CDS.

In this case the model predicts the intercept γ to be zero and the slope λ to be significantly positive.

4.2 Measurement Error

In the considered framework, two econometric issues are present. The standard errors and the t-statistics of the estimated coefficients are incorrect, given the possible presence of heteroskedasticity and cross correlation in the residuals of regression (8). Furthermore, as observed by Black, Jensen, and Scholes [1972], the coefficients β used in equation (8) are not directly observable and may change over time. This induces a measurement error which causes bias and inconsistency in the coefficients γ and λ estimated above.

In the same article, the authors also suggest a simple procedure to overcome the problem of measurement error: the grouping of single return series into portfolios. This procedure should also reduce the effects of time-varying betas. The grouping should ideally be based on a variable that is correlated with the coefficients β but measured independently from them. Due to the relatively short time series it seems impossible to use values of the betas estimated on previous non-overlapping periods, as often proposed in the literature.

	Coefficient	t-stat
a_0	-0.2298	-2.3877
a_1	0.7202	10.9570
a_2	0.8607	3.7898

Table 9: Estimated parameters of equation (9)

Standard structural models of credit risk, such as Merton [1974], postulate the presence of a one-to-one relationship between debt and equity. We therefore propose to construct the groups using the equity betas as instrumental variables. We compute, for every CDS reference entity, the regression coefficients of equity returns on the returns of the market index S&P500 in the same period. These quantities are certainly measured independently from "credit" betas and so the only thing we have to verify in order to justify their use as instruments is the presence of correlation with equity betas. It turns out that the cross section correlation between credit and equity β is equal to 57.8% for the single CDS and 89.29% for grouped data.

We go slightly more in depth considering that, according to Merton model, the relationship between debt and equity volatility also depends on the leverage of the firm. Thus, we want to confirm the validity of equity beta as an instrument for credit beta estimating the following regression.

$$\beta = a_0 + a_1 \beta_{Equity} + a_2 Leverage + \varepsilon \tag{9}$$

The estimated coefficients are in table (9).

As expected, both equity beta and leverage seem to be significant. This is true also jointly, since the p-value of the F statistic is negligible. The cross-section R^2 is in this last regression equal to 37.25% and the null of zero cross-correlation of residuals is strongly accepted.

Therefore the 238 companies are sorted on the basis of their equity beta and then grouped into 30 equally weighted portfolios.

We tackle the other econometric issue, the incorrect t-statistics, by correcting the standard error of the estimated coefficients (and consequently the t-statistic) with the Eicher-White method.

5 Estimation and Results

5.1 Time series estimation

5.1.1 Single CDS

We first estimated equation (7) separately for every CDS without grouping into portfolios.

The intercept α is not univocally significant along the sample, while the slope β is always significant. In order to give an aggregate representation of the significance of the coefficients α and β, we have to compute an aggregate point estimate and an aggregate t-statistic. The point estimate, presented in the following table, is obtained by simply averaging the coefficients of the 238 single regressions. The t-statistic is obtained dividing the point estimate by the cross-section standard deviation of the vector of estimated coefficients. The same method is used in Collin-Dufresne et al. [2001].

The estimated coefficients are

	Coefficient	t-stat
α	1.37e-04	4.3714
β	0.8009	17.2933

Both coefficients are strongly significant. The significance of the intercept can be interpreted against the validity of this simple linear model. It seems reasonable that such a simple specification is not able to capture all the structure present in the panel of CDS data and that the neglected information (missing variables or non-linearities) is included into the constant. On the other hand, the strong significance of the intercept, supports the dependence of each CDS, in the time series dimension, on a 'credit market factor', here proxied by the CDX index.

The $R^2 s$ of the 238 regressions in (7) are distributed between 3.03% and 57.87% with an average of 32.56% (see figure 8). The F statistic is significant at a 5% level for all the 238 regressions and only in 3 cases it is not significant at the 1% level.

We also estimated regression (7) as a pooled regression. The point estimates are not different from the ones presented above while the t-statistics are changing. In particular, the significance of the intercept is reduced, although it remains strong, and the significance of the slope is considerably amplified. We have in particular

	Coefficient	t-stat
α	1.37e-04	2.1759
β	0.8009	80.023

As already said, the model is very simple and it appears natural that some autocorrelation is still present in the residuals. In particular, according to the Durbin-Watson test, the autocorrelation is statistically zero only for 71.0% of the examined residual series at a 95% confidence level. The F-statistics computed on the single regressions confirm the significance of the regression since none of the p-values of the test is larger than 5%. Residuals still present strong characteristics of non-normality (the average kurtosis in the residuals is 16.68 and the average skewness -0.52).

In analogy with the panel of CDS returns, we also implemented a principal component analysis on the panel of the residuals from the simple linear regressions described above. The first component explains the 21.86% of the total variance and the second the 6.20%. This is significantly less than the results obtained on the panel of returns (35.76% and 13.38%, respectively). This means that the model is able to eliminate a considerable fraction of the systematic risk. Of course, all the principal components are now orthogonal to the returns on the CDX index.

The relatively short time series seem to make the estimation of time-varying coefficients very unreliable. We therefore repeat the procedure described above on the two periods considered above and justified by the estimation of the Markov Switching model in section 3. The results of the time-series estimations on the two subperiods are presented in the following table

	10/2003-7/2007		7/2007-9/2010	
	Coefficient	t-stat	Coefficient	t-stat
α	3.52e-05	1.6076	2.19e-04	3.5388
β	0.9628	10.4774	0.7965	17.0526

If we use the pooled regression technique, we obtain larger t-statistics for the slope coefficients and slightly smaller for the intercepts but, as above, the results of the significance tests are not modified. The average R^2 is equal to 22.68% in the first period and 34.95% in the second.

If we consider the principal components of the residuals on the two subperiods separately we still observe a reduction in the variance explained by the first principal component with respect to the case of the returns. In the first subperiod the first principal component goes from 36.67% for the returns to 10.70% for the residuals. In the second subperiod the first principal component goes from 36.96% for the returns to 24.32% for the residuals. In both periods the systematic structure left in the residuals is less than in the

original returns, but the effectiveness is much higher in the first period. This simple linear model is probably not able to effectively take into consideration all the sources of systematic risk in the crisis period.

5.1.2 Grouped data

All the results presented in the previous section are based on single CDS time series. The presence of measurement error already described in section 4 suggests the use of portfolios of CDS instead of single contracts. This will be particularly relevant for the cross-section analysis presented in next section. Nevertheless, we also present some time-series evidences based on 30 portfolios created according to the Beta measured in the equity market. This guarantees the independence of measurement from the credit beta preserving the correlation between the two, as verified above.

The results at time series level and obtained with the same procedure described in the previous section are reported in the following table.

	Coefficient	t-stat
α	1.31e-04	3.2253
β	0.8088	7.6510

The point estimates are very close to the case of ungrouped data, while we observe a decrease in the t-statistic of the slope, although it remains strongly significant. The average R^2 of the 30 regressions on grouped data is 56.09%. The F statistics is always significant at the 5% and 1% level. Also in this case, according to Durbin-Watson test, the autocorrelation in residuals is statistically zero for only 23 portfolios out of 30 (76.67%).

If we repeat the analysis on the two sub-periods we get

	10/2003-7/2007		7/2007-9/2010	
	Coefficient	t-stat	Coefficient	t-stat
α	2.31e-05	1.0066	2.23e-04	2.7515
β	0.9544	9.6376	0.8051	7.5193

Also in this case, we observe estimates very similar to those based on ungrouped data, apart from the t-statistic of slope in the second subperiod, which is lower. The average R^2 increases from 52.83% in the first period to 57.22% in the second.

Both on the entire sample and on the two subsamples we ran pooled regressions. The results confirm the tendency observed in ungrouped data: the significance of the intercept is decreased while the significance of the slope strongly increases. The point estimates are still not distinguishable.

5.2 Cross section estimation

The next step in the estimation of the model is the assessment of the relationship between the exposure to systematic credit risk (β) and the remuneration for bearing such a risk. As a consequence of the already mentioned error in measurement of the βs, we estimate the cross section equation (8) only for the 30 portfolios of CDS created according to the equity beta.

One first issue one has to care about is if the coefficients β present a sufficient degree of dispersion across assets to make the cross section estimation reliable. Our sample seems to present a satisfying dispersion of betas, as testified by the histogram in figure 9.

In figure 10 we also represent the scatter plot of the average quoted spread during the considered period versus the β estimated from equation (7). We present the dispersion for both the single CDS and the 30 portfolios. The relationship is strongly positive. This means that the protection against default for companies with a bigger (linear) exposition to systematic credit risk is more expensive. Nevertheless, we cannot directly interpret the CDS spread as a risk premium and therefore we cannot conclude that the a premium for bearing systematic risk exists and is positive.

Indeed, in a world in which agents are risk neutral the CDS spread is just the compensation for the expected loss, where the expectation is computed using the risk-neutral default probabilities. In the real world, agents may use physical default probabilities (in general lower than risk neutral ones) and include a risk premium in the spread they require for bearing default risk or equivalently for unexpected loss, which is neglected by risk neutral agents. In general we don't know the degree of risk aversion of the agents in the market and therefore from the presented dependence we can only conclude that companies with higher systematic credit risk have a higher risk-neutral expected loss or are characterized by a higher compensation for unexpected loss.

We then move to consider realized returns rather than spreads, and in particular we examine the cross section relationship expressed in equation (8). Graphically, we observe in figure 11 the presence of a positive relationship between systematic risk (β) and returns realized in the period. Nevertheless datapoints seem quite dispersed around the fitted line and the relationship appears quite weak.

The estimated coefficients are the following

	Coefficient	t-stat
γ	-0.1847e-03	-4.3801
λ	0.1430e-03	2.4643

Also the F statistic is very high and its p-value is less than 1%. The cross-section R^2 on the 30 groups is equal to 36.32%. The coefficients in the regression are referred to two-weeks returns. If we annualize them, we get an intercept equal to -48 bps and a slope equal to 37 bps. We can interpret the slope as a risk premium paid to the bearer of systematic credit risk per unit of beta. This premium is only the remuneration for the unexpected loss that a risk averse agent requires.

We can compare our estimate with the results in some of the articles mentioned in section 2. Liu et al. [2002] estimated a premium equal to 45 basis points per year. However, their estimate is based on the spread between Interest Rate Swaps and Treasury rates at a 10 years maturity. So they consider a longer horizon and only firms in the financial sector. Hull et al. [2005] estimate a premium of 66 basis point for investment grade bonds but all the maturities they are considering are greater than 6 years. Finally a work by Lonstaff et al. [2010] proposes a risk premium equal to about 80 basis points, which is more than twice our estimation. However, this last study is based on secular bond data (almost 150 years). Consequently, we argue that their sample is quite different from ours. For instance, the vast majority of listed companies at the beginning of their sample, and before the crisis of the 1870s, was in the railroad sector while today such sector is almost completely absent from the benchmark indexes.

The estimation of the premia on the CDS rather than on the corporate bonds market arises questions about the presence of a remuneration for the default risk of the counterparty and not only of the reference entity. In this case we can rely on the work by Arora, Gandhi, and Longstaff [2011] in which the presence of a counterparty premium is proved to be statistically significant but almost negligible from the economic point of view. According to the article, both the statistical significance and the economic magnitude increase after the Lehman collapse but they still remain quite small. The reason for this may be the presence of margin requirements and collateralization conventions usually adopted in the market, which should strongly reduce counterparty risk. The presence of a risk premium automatically shows that the theoretical "sufficient diversification" condition given in Jarrow et al. [2005] is actually not verified in the market.

The presence of a credit risk premium can also be justified by the results in section 3, where a considerable correlation between equity and credit market is shown. This fact reduces the diversification opportunities for the investors, who consequently require a premium for investing in the credit market.

In conclusion, the presence of a positive risk premium seems to be justified and its numerical estimation seems quite reasonable compared to other studies, especially when we consider that our estimation period is characterized by the presence of the financial crisis, which lowers the realized returns and therefore the realized risk premia.

5.2.1 A comparison with the Equity Premium

In order to have an idea of the reliability of the estimated premium, we apply the method just described to the computation of the equity risk premium for the same sample of companies in the same period.

In order to reduce the effects produced by the above mentioned measurement error in betas, we use again the grouping approach based on 30 portfolios. We still need an instrumental variable that is correlated to the equity beta but measured independently on it. In analogy with the previous paragraphs, we use the credit beta as instrumental variable and form the portfolios based on it.

The results of the time series estimation are presented in the following table. As before, the point estimate is the average of the single estimations and the t-statistic is based on the cross-sectional standard deviation of the estimated betas.

	Coefficient	t-stat
α	0.0026	11.7843
β	1.1874	8.7234

The average R^2 of the 30 portfolios is in this case 71.79%. The F statistics is always strongly significant.

When we consider the cross section relationship of equation (8), we estimate the following coefficients

	Coefficient	t-stat
γ	0.0024	5.5992
λ	0.0011	2.8957

Both the coefficients are significant. The cross-section R^2 on the 30 groups is equal to 24.46%. If we annualize the estimated coefficients, we get an estimated equity risk premium equal to 2.45% per year.

We can derive some conclusions from the comparison between the credit and equity linear model. First, the fraction of cross-sectional variance explained by the model is greater in the credit model case, where the R^2 is more than 36%. This may indicate that the use of the systematic returns as unique risk factor is more appropriate in the credit market than in the equity market or,

equivalently, that the impact of the idiosyncratic component is less important in the credit market, more dependent on systematic factors.

Second, in both cases a significant cross-section intercept is present. This means that the simple linear model is not able to capture the total cross-sectional variability in a complete way. It seems reasonable to conclude that the linear specification is not appropriate or that some other explanatory variable is missing.

Third, we observe that the relatively small credit risk premium comes with a relatively small risk premium on the equity market, at least if compared to other standard studies such as Mehra and Prescott [1985]. This may be a consequence of the sample of firms or of the time period in which the estimations are performed.

6   Some tests of the model

There are many ways to test a model like the proposed one. Here we present some simple tests in the cross-sectional dimension. The main purpose is to verify if systematic credit risk is actually priced and if it is able to explain the cross section of CDS returns.

In particular, if we consider equation (6), the model predicts an intercept γ statistically equal to zero and a generally positive coefficient λ. The estimations for the intercept γ and the coefficient λ, respectively, are presented in the following two tables. Results are referred to grouped data and annualized.

γ	Coefficient	t-stat
10/2003-7/2007	-0.0006	-0.3306
7/2007-9/2010	-0.0097	-7.9230
10/2003-9/2010	-0.0048	-4.3801

λ	Coefficient	t-stat
10/2003-7/2007	0.0039	1.7706
7/2007-9/2010	0.0036	2.7594
10/2003-9/2010	0.0037	2.4643

The CDX factor is priced on the entire sample at standard confidence levels. On the pre-crisis period the relationship appears weaker and the factor is priced only at the 90% level, while in the second sub-period the t-statistic is 2.76.

On the other hand, the estimated intercept seems to be insignificant in the pre-crisis period but becomes significantly negative during the crisis and on the entire sample. This means that during periods of stress the model seems

to be too simple and some of the unexplained information is captured by the constant. It is possible that the impact of systematic risk becomes less elementary in those periods, involving non-linear relationships or stronger extreme connections.

6.1 Impact of other factors

We want to verify if the estimated premium actually comes from the remuneration of systematic non-diversified credit risk or from other sources of risk. One possibility we can probably exclude is liquidity risk. CDS are synthetic instruments, available in theoretically unlimited quantities, which eliminates the exposure to liquidity conditions. Nevertheless we tried to verify the role of some liquidity factor. One issue here is that many commonly used liquidity factors are both liquidity and credit factors. For instance spreads such as the TED or the Libor-OIS spread cannot be considered as pure liquidity indicators. In fact, they incorporate a reward for the credit risk of the financial intermediaries that are operating on the Libor market, which are not risk-free and whose default probability significantly increased during the crisis. In this way the proxy for liquidity risk would include also some components of credit risk, and would reduce the orthogonality between the two risks. On the other hand, an indicator like Pastor-Stambaugh seems difficult to compute on frequencies higher than one month, as in our case. This is because these indicators are in essence regression coefficients, whose estimation on 10 working days only appears unreliable.

We used therefore two bid-ask spreads: the bid-ask of the 30-year on-the-run T-Bond and the bid-ask spread of the 3 months Eurodollar future rate. These quantities can be considered as indicators of the liquidity in the fixed-income market at short and long maturities and do not include any considerable counterparty risk. It turns out that they are both insignificant and not priced and their effect on the estimated premium is in general negligible.

We also add other factors to the statistic characterization of credit returns proposed in equation (4). We added an indicator of economic cycle and other indicators of interest rate conditions. In particular, we take the slope of the term structure of interest rates (10-2 years spread) as indicator of business cycle and the observed zero interest rates at different maturities.

Thus the additional regressors included are

- the changes in the bid-ask spread of the US Eurodollar deposits with maturity 3 months

- the changes in the bid-ask spread of the on-the-run US 30-year T-bond

- the changes in interest rates at all the maturities from 3 months to 10 years

- the change of the difference between 10 and 2 years interest rates (Slope)

- the returns of a broad BBB bond index

We therefore consider the regression

$$r_{i,t} = \alpha_i + \beta_{1,i} r_{CDX,t} + \beta_{2,i} X_t + \varepsilon_{i,t} \tag{10}$$

where X_t is the generic additional factor used.

As before, we first estimate the time-series regressions for the 30 groups and then the cross-section one including the CDX index and each of the mentioned regressors separately. The increase in the R^2 for grouped data is only marginal (1-2% on average). The regressors that give the best improvement are long-term interest rate, liquidity indicators and the BBB bond index, although the increase is always less than 5%.

We then look at the significance of the cross-section coefficient estimating the following equation.

$$\overline{r_{i,t}} = \gamma + \lambda_1 \beta_{1,i} + \lambda_2 \beta_{2,i} + \eta_i \tag{11}$$

We observe that the new factors are in general not significantly priced. In particular, liquidity factors are statistically insignificant in the cross-section dimension. A strong exception is the BBB bond index, which seems to be statistically priced. This is probably due to the high correlation between the BBB and the CDX index already mentioned above. Another priced factor seems to be the change in the slope of the term structure, but all the considered variables do not significantly affect the cross-section slope coefficients estimated for the CDX index which remains close to 40 bps. Also in this case, the only exception is the BBB bond index, for which a collinearity issue may be present.

The detail of the cross-section regressions are presented in the following table

Additional Factor	λ_2	t-stat	λ_1	t-stat
Eurodollar Bid-Ask	0.0093	0.8967	0.0045	3.8618
T-Bond Bid-Ask	-6.005e-5	-0.0969	0.0037	3.8857
Slope	0.0027	3.7350	0.0041	5.2164
6 months zero rate	0.0018	1.2605	0.0045	3.7563
12 months zero rate	0.0019	1.2228	0.0044	3.9023
10 years zero rate	-0.0035	-1.7953	0.0040	4.4372
BBB Index returns	0.0171	5.7769	-0.0080	-3.2192

All the t-statistics are corrected for heteroskedasticity in the residuals according to the Eicher-White method.

7 Systematic risk

One of the goals of this work is to disentangle the systematic component of credit risk from the specific one. There are many possibilities to measure systematic credit risk. The first and most obvious one is just given by the average R^2 of equation (4) across single assets or portfolios. Of course, when the systematic component of risk is high the ability of the market index to explain (on average) the variability of the single CDS will be greater. This means that the behaviour of single credit spreads are strongly driven by common factors and therefore that systematic risk is high.

In the period considered here, we observe an increase of the average R^2 of regression (4) for single CDS and grouped data. We split the time period into two as in the previous section and verify that the average R^2 increases from 52.83% in the first period to 57.22% in the second.

Another indication of the increase in systematic risk comes from the principal component analysis of the panel of mark to market variations and residuals from equation (7). As far as returns are concerned, the fraction explained by the first principal component increases from 68.7% in the first period to 78% in the second. If we consider residuals, the increase is more dramatic. The weight of the first component is only 30.12% in the first period and it climbs to 54.73% in the second period. Similar results are obtained if we consider the sum of the first 2 or 3 principal components.

From the first result we can conclude, as before, that the systematic credit risk has increased in the second period. On the other hand, from the second result we conclude that the component of the variance *unexplained* by the model increases. This means that the simple linear model considered here is able to capture the systematic component of risk in the first non-crisis period but it fails in the second. During the crisis, a considerable portion of the variance remains unexplained by this simple linear model. The presence

of structure in the residuals is also confirmed by the big difference between the weight of the first principal component with respect to the second and the following ones, which does not happen in the first subperiod.

7.1 Variance decomposition

If we observe equation (5) and given the independence properties of ε, we have that

$$V(r_{i,t}) = \beta_i^2 V(r_{CDX,t}) + V(\varepsilon_{i,t}) \tag{12}$$

where $V(\cdot)$ indicates the variance operator. The quantity $\beta_i^2 V(r_{CDX,t})$ can be interpreted as the systematic component of the total variance of asset i, while $V(\varepsilon_{i,t})$ is the specific component. The only thing to compute is the evolution of the variance of the CDX index and of each residual series through time.

Given the GARCH structure observed in credit returns described above, we computed the conditional variances of $r_{CDX,t}$ and $\hat{\varepsilon}_t$ for each of the 10 GICS sectors according to a GARCH(1,1) model. The results of the decomposition are presented in figure 12.

Figure 13 represents the ratio between systematic and total variance in each sector.

We observe a general increase in the systematic component for every sector, which is particularly evident during the crisis period. The financial sector shows the highest fraction of systematic risk compared to all other sectors.

This last indicator can also be interpreted as a measure of the fragility of a sector. This interpretation is related to the literature concerning the systematic risk and the frailty factor mentioned in section 2.

The indicator proposed here is much more easily obtained than those discussed above and simply gives a measure of the variability in credit returns of one sector that can be explained by one single observed systematic factor. Of course, if a sector has a very high fraction of its variance explained by one factor, then it seems plausible to consider it as more vulnerable. From our analysis emerges that the financial sector shows the highest degree of fragility.

One interesting possibility for future research is to verify the correlation between the latent frailty factor identified by the previously cited articles and the measures here computed.

7.2 Tail dependence

The existence of a systematic component of credit risk originates from the presence of a fraction of risk in the returns of CDS that is not diversifiable. In the previous paragraphs, the presence of non-diversifiable risk has been essentially investigated through the standard Pearson linear correlation, either directly or through the computation of the linear regression coefficients β in equation (7).

It seems worthy to verify the existence of commonality in credit returns beyond the linear Pearson coefficient. In particular, during the last crisis period the existence of extreme dependence in financial markets has become evident. Therefore we studied the dependence between credit returns through the use of copulae.

These functions allow us to model separately the marginal and joint distributions of returns. In particular, a key result in the theory, Sklar's theorem, relates marginal and joint distributions of a random vector in the following way

$$H(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)) \tag{13}$$

where

- $H(x_1, x_2, ..., x_n)$ is the joint distribution function computed at points $x_1, x_2, ..., x_n$

- $C(u_1, u_2, ..., u_n)$ is the copula function, which is a function $C : [0,1]^n \to [0,1]$

- $F_i(x_i)$ is the ith marginal distribution function computed at the point x_i.

In our case we use as marginal distribution the empirical CDF combined with a Generalized Pareto distribution for the extreme quantiles. We also assume that the variances of each margin have a GARCH(1, 1) structure, which agrees with the empirical findings of section 3. This quite complicated structure of margins has been successfully applied to the equity market (see e.g. McNeil and Frey [2000] or Nystrom and Skoglund [2002]) and, due to the similarities detected before, we extended it to the credit market.

As far as the joint distribution function is concerned, we have to use a copula function that allows for the presence of extreme tail dependence. This measure indicates the comovement between two random variables when extreme events occur, that is when the observed values come from the tails of the distributions. In particular, it describes the limiting fraction of data from one variable exceeding a certain quantile given that the other variable

has exceeded the same quantile. So this quantity can be interpreted as a measure of the "extreme systematic risk" present in the market.

For some copula functions, such as the Gaussian copula, the tail dependence is zero. This means that such copulae show asymptotic independence and no correlation between extreme events. Consequently, we decided to study extreme correlation using two other copulae: Clayton and Gumbel. They both belong to the family of Archimedean copulae and are characterized by the presence of extreme tail dependence. In particular, the Clayton copula shows extreme lower tail dependence (in the left tail) while Gumbel Copula shows extreme upper tail dependence (in the right tail). Our work in this case is quite similar to what Longin and Solnik [2001] applied to the equity market.

We compute the coefficients of upper and lower tail dependence in the pre-crisis and post-crisis period for all the possible pairs of sector return. In this way we obtain a matrix of tail correlation coefficients for all the 10 GICS sectors considered. We then compute the simple average of all the obtained correlation coefficients. We observe a considerable increase in both the lower and upper tail correlation. In particular the left tail coefficient increases from 0.43 to 0.55 (+26.24%) and the right tail coefficient increases from 0.15 to 0.6 (+288.1%). For a comparison, the standard linear Pearson correlation coefficient increases from 0.696 to 0.776 (+11.59%) only. This means that the increase in the non-extreme component of systematic risk is much lower than the increase in the extreme systematic risk for the credit market. The results are summarized in the following table.

	10/2003-7/2007	7/2007-9/2010	% Change
Linear Pearson Correlation	0.6955	0.7762	+11.59%
Lower tail dep. coefficient	0.4318	0.5451	+26.24%
Upper tail dep. coefficient	0.1535	0.5958	+288.1%

Table 10: Changes in correlations

7.3 Sector analysis

We now analyze in detail the systematic risk of every sector. We start considering the evolution of the beta coefficient in equation (7) computed for every sector (Table 11). The average returns of the companies in every sector used for the regression are equally weighted.

Sector	10/2003-7/2007	7/2007-9/2010
Materials	1.2111	0.8367
Financials	0.6140	1.3458
Utilities	1.1242	0.5765
Health Care	0.7022	0.4041
Consumer Discretionary	1.3350	1.1267
Energy	1.0795	0.7161
Information Tech.	1.2357	0.7403
Telecomm.	1.7112	0.7951
Industrials	0.6337	0.5749
Consumer Staples	0.5676	0.4270

Table 11: Beta of the sectors

It's easy to notice a sharp increase in the systematic exposure to systematic risk of the financial sector in the second period, compensated by a decrease in all the other sectors.

Given the particular evolution of the systematic exposure of the financial sector, we repeated the time series and cross section analysis presented above for the single CDS contracts present in each of the 10 sectors. Cross section coefficients γ and λ are reported in tables 12 and 13.

Sector	10/2003-7/2007		7/2007-9/2010		10/2003-9/2010	
	$\gamma \cdot 10^4$	t-stat	$\gamma \cdot 10^4$	t-stat	$\gamma \cdot 10^4$	t-stat
Materials	-2.47	-3.94	-1.81	-1.33	-2.1	-1.59
Financials	0.55	0.39	-3.83	-3.81	-3.9	-3.83
Utilities	-1.14	-1.66	-0.37	-0.35	-0.52	-0.50
Health Care	-0.68	-2.72	-0.32	-1.49	-0.34	-1.52
Consumer Discr.	-3.77	-4.48	-4.33	-2.94	-4.49	-3.10
Energy	-2.21	-4.03	-1.3	-1.41	-1.43	-1.58
Information Tech.	-3.21	-3.59	-4.86	-5.41	-4.83	-5.45
Telecomm.	-1.92	-1.99	-1.01	-0.68	-1.19	-0.80
Industrials	-0.95	-3.05	-0.3	-0.71	-0.33	-0.77
Consumer Staples	-0.67	-0.68	1.22	1.32	1.19	1.28

Table 12: γ of the sectors in the three periods

In this case some sectors include only a small number of companies. This makes the grouping approach not feasible. We therefore implement an analysis based on the single CDS and then correct the standard errors (and consequently the t-stats) according to Shanken [1992].

| | 10/2003-7/2007 | | 7/2007-9/2010 | | 10/2003-9/2010 | |
Sector	λ	t-stat	λ	t-stat	λ	t-stat
Materials	34	4.21	29	0.80	38	1.08
Financials	-22	-0.47	75	5.11	78	4.09
Utilities	26	2.50	16	0.40	22	0.59
Health Care	-7	-1.22	-35	-3.70	-33	-3.53
Consumer Discr.	58	5.87	81	2.89	85	3.09
Energy	32	3.97	15	0.57	20	0.77
Information Tech.	81	5.35	202	7.36	197	7.44
Telecomm.	21	2.34	16	0.43	21	0.58
Industrials	7	0.91	-22	-1.46	-21	-1.36
Consumer Staples	-35	-1.09	-161	-3.64	-158	-3.57

Table 13: λ of the sectors in the three periods (in bps)

In general we observe a significantly negative value for γ. This agrees with results in section 5 and implies the relatively limited ability of the model to describe the entire cross-section of returns. This is true for the majority of the sectors.

If we consider the slope coefficient λ we observe significant and positive values for the financial, IT and consumer discretionary sector. On the other hand, in the health care and Consumer Staples sectors the risk premium realized in the considered period is negative.

8   Conclusions

In this paper we investigated the systematic risk in credit market for a sample of investment grade firms. The studied variables are the mark-to-market returns of CDS contracts and their market index, the CDX index, obtained according to the ISDA Standard Model. Qualitatively similar results are obtained using the simple first difference in the quoted CDS (or CDX) spread.

Some preliminary investigation of the statistical properties of the returns time series shows a strongly significant autocorrelation, which tends to decrease in the second part of the sample. Some predictability is present also when lagged equity returns are used. Credit returns show a clear GARCH structure, a strong non-normality and seem to follow a two-state Markow Switching model.

A linear market model is proposed to describe the returns of the single CDS position using the market index as a factor. A strong dependence in the time series dimension is found. In the cross section dimension, the CDX factor is

priced and a significant risk premium of 37 basis is estimated and seems to be compatible with previous studies.

However, some evidence is present against the simple linear model here proposed. In particular, during the crisis and on the entire sample the presence of a null intercept is rejected.

The evolution of systematic risk before and after the financial crisis is also investigated. Systematic risk seems to considerably increase during the crisis when measured by the average R^2 of time series regression, by the variance explained by the first principal components and by the increase in the extreme tail dependence, which can be interpreted as an indicator of extreme systematic risk.

Finally a simple decomposition of the variance is proposed to separate the systematic and specific component. A strong increase in the systematic fraction is observed during the crisis and the sector with the highest systematic fraction of variance, and therefore highest sector fragility, is the financial one.

9 Figures
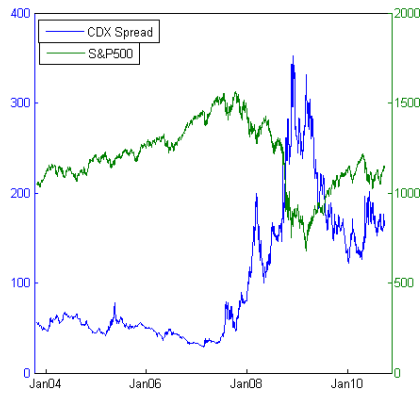
Figure 1: Evolution of the CDX North America Investment Grade spread
and the S&P500 Index
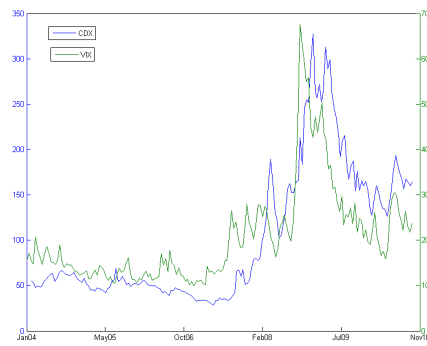
Figure 2: Evolution of the CDX North America Investment Grade spread
and the VIX Index
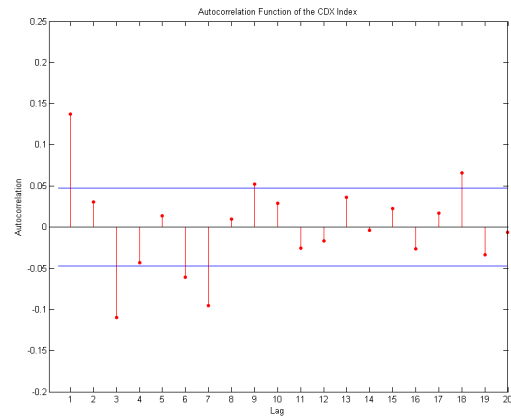
Figure 3: Autocorrelation Function of the CDX Index returns
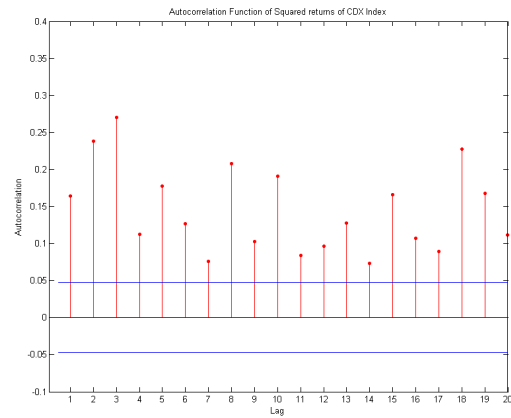
Figure 4: Autocorrelation Function of the squared CDX Index returns
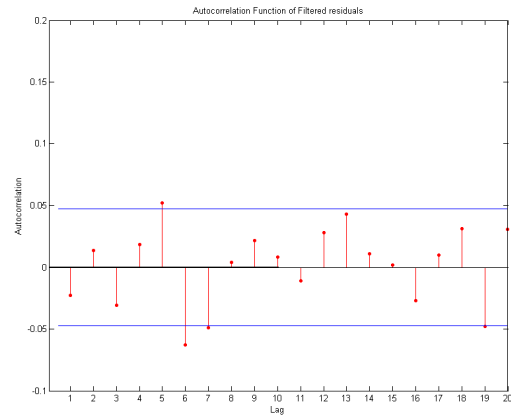
Figure 5: Autocorrelation Functionn of GARCH filtered residuals
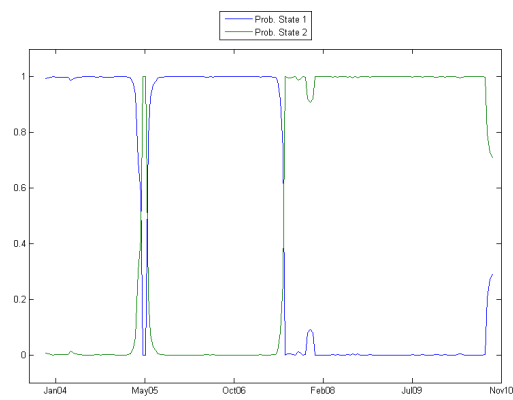
Figure 6: Smoothed probability of State 1 and 2 in a Markov switching model

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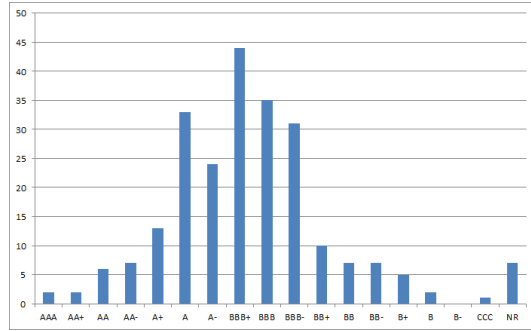
Figure 7: Rating distribution of the sample
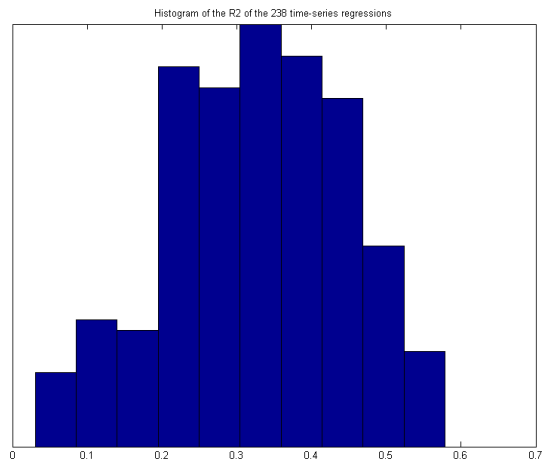
Figure 8: Histogram of the R^2 of the 238 time series regressions
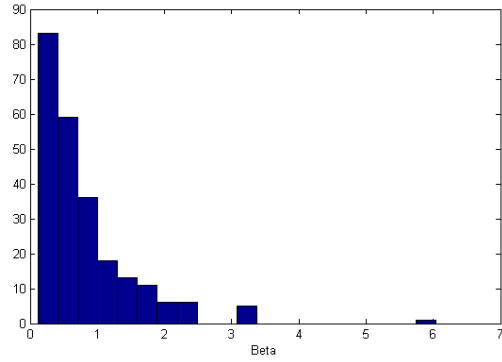
Figure 9: Histogram of the distribution of Betas.
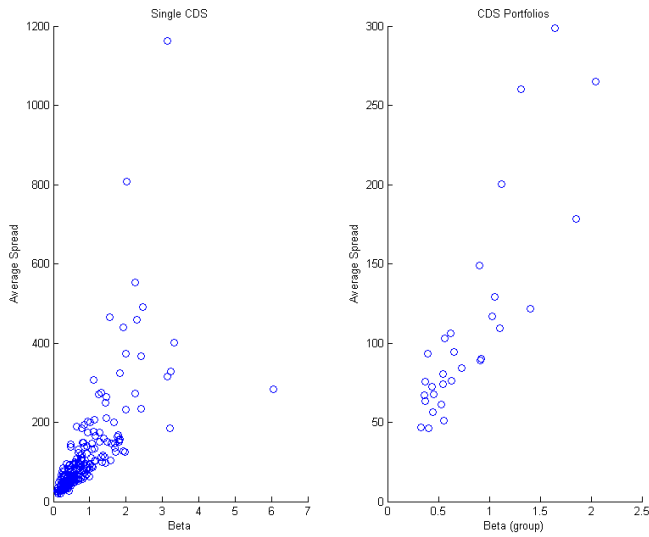
Figure 10: Scatter plot of average spreads versus Beta.
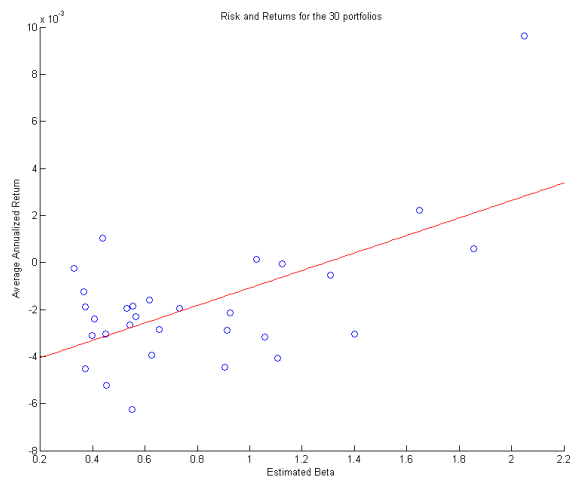
Figure 11: Scatter plot of average return versus Beta.
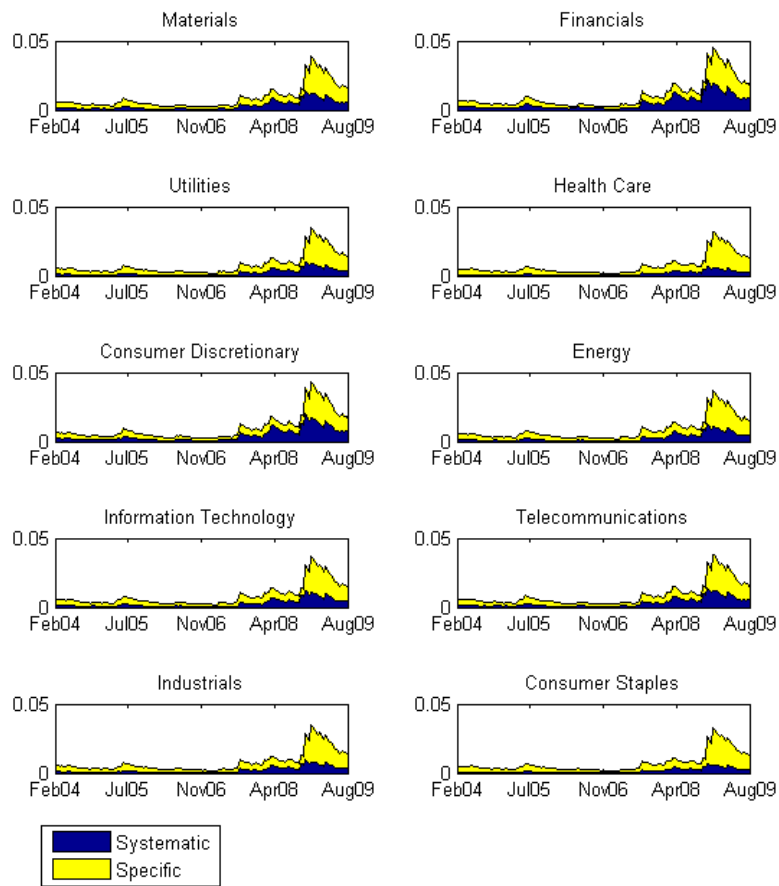
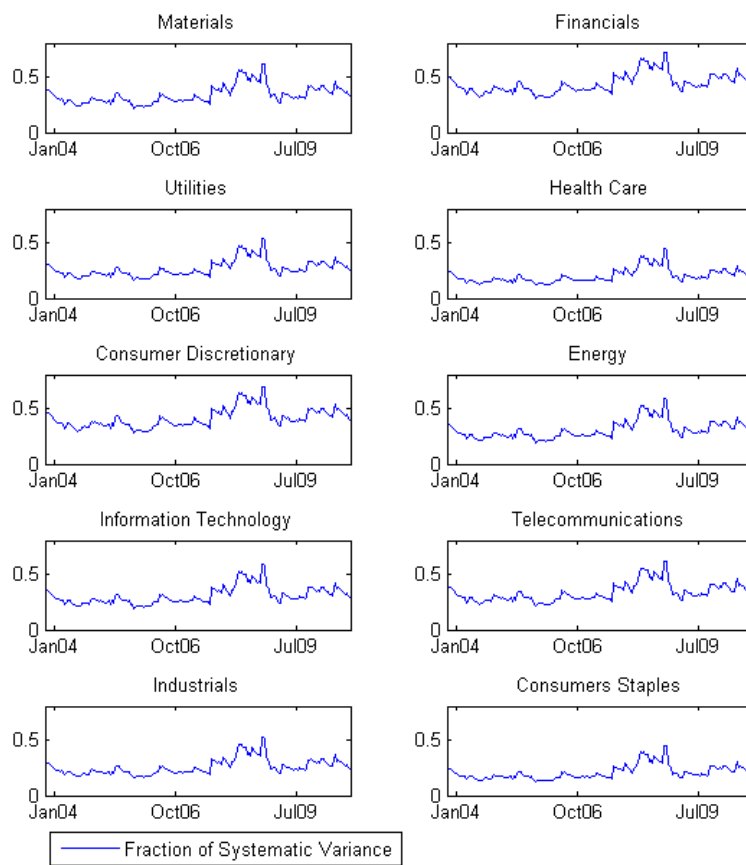Figure 12: Systematic and specific components of the variance for the 10 GICS sectors

Figure 13: Fraction of systematic variance for every sector

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Appendix

A   The ISDA Standard model

A credit default swap is an agreement between a seller and a buyer of protection against credit risk. The buyer of protection pays a constant fee (here indicated by s_0) ususally on a quarterly basis. The stream of payments continues until the maturity of the contract or until a credit event occurs, whichever is first. Credit events are accurately described in a list written by the International Swaps and Derivatives Association (ISDA). It does not include only default events but also, for instance, Repudiation (for sovereign entities), Restructuring, Failure to pay and others. On the other hand, the protection seller makes a payment only in presence of a credit event. The net payment made in case of credit event is equal to the difference between the par value and the market price of the reference bond. Also the accrual of the quarterly spread is usually considered. Both physical settlement (with the delivery of the bond) and cash settlement can take place, though physical settlement is usually more common.

The market of CDS is Over-the-Counter. So there is no compelling rule for the computation of the daily mark-to-market of the contract between the two parties. Nevertheless, the general convention used in the market is the use of the so-called "Standard Model" proposed by the ISDA. See e.g. O'Kane and Turnbull [2003] or Berndt and Obreja [2010] for a detailed description of the model, which originates from the theoretical work by Hull and White [2000].

The Market Value (MV) of a CDS contract is given by three components: the expected present value of the stream of payments made by the buyer of protection (define it Premium Leg), the expected present value of the accrual payment at default (define it Accrual Leg) and the expected present value of the payment made by the seller of protection (define it Protection Leg), that is

$$MV = PremiumLeg + AccrualLeg - ProtectionLeg \qquad (14)$$

Therefore *PremiumLeg* is the expected present value of receiving a stream of payments equal to s_0 for the residual life of the contract. Using the same notation of O'Kane and Turnbull [2003], this is equal to

$$PremiumLeg = s_0 \cdot RPV01 \qquad (15)$$

where $RPV01$ is the risky present value of 1 dollar received at all the future payment dates. It is defined "risky" since it is not the simple risk free present value of an annuity with known number of payments but instead the end of the stream of payments is random, depending on the occurrence of a default event. The quantity is equal to

$$RPV01 = \sum_{n=1}^{N} \Delta(t_{n-1}, t_n) Z(t_V, t_n) \left[Q(t_V, t_n) + \frac{1}{2} (Q(t_V, t_{n-1}) - Q(t_V, t_n)) \right] \quad (16)$$

where

- t_V is the time at which the valuation is done

- $\Delta(t_{n-1}, t_n)$ is the date count fraction between t_{n-1} and t_n

- $Q(t_V, t_n)$ is the risk neutral surviving probability between t_V and t_n

- $Z(t_V, t_n)$ is the risk free discount factor between t_V and t_n

We see that this quantity can also be interpreted as a stochastic residual life of the contract. In the case of a default-free company $RPV01$ equals the residual life of the contract and is a deterministic quantity, while in general it is a random quantity that depends on the default probability of the reference entity.

The other assumption of the model is that the credit event arrival process follows a Poisson process of (risk neutral) intensity $\lambda(t)$. It follows that the risk-neutral conditional survival probabilities between t_V and t_n is

$$Q(t_V, t_n) = \exp \left[-\int_{t_V}^{t_n} \lambda(s) ds \right] \quad (17)$$

AccrualLeg is the expected present value of the accrual payment made by the protection buyer. This is the fraction of the spread s_0 that corresponds to the interval of time between the last premium payment and the default time. It is given by

$$AccrualLeg = \int_{t_V}^{t_N} \Delta(t_{n_s}, t_s) Z(t_V, s) Q(t_V, s) \lambda(s) ds \quad (18)$$

where t_{n_s} is the payment date immediately before the generic instant t_s.

On the other hand, *ProtectionLeg* is the expected present value of the terminal payment only in case of default. It is given by

$$ProtectionLeg = (1 - R) \int_{t_V}^{t_N} Z(t_V, s) Q(t_V, s) \lambda(s) \, ds \qquad (19)$$

where R is the recovery rate, which is conventionally assumed to be equal to 40%.

Thus the only inputs for the computation of the mark to market value are a complete term structure of interest rates, the spread of every contract, the risk neutral surviving probabilities Q or the risk neutral default intensity λ. As in O'Kane and Turnbull [2003], we extract risk neutral default intensities and surviving probabilities from the current value of the spread of a newly issued CDS, assuming that credit events follow a Poisson counting process. The usual assumption for the practical implementation of the ISDA model is that the function $\lambda(s)$ is piecewise constant, which makes straightforward the computation of the integral in (17).

The model allows us to compute the value of the premium and protection leg, and therefore the market value of CDS and CDX contracts, at any date. Returns are computed assuming that every two weeks the position in the CDS is closed realizing the market value calculated in (14). At the same date, a new CDS contract is created at the current market spread, and it will be closed two weeks later. In this way we can correctly use the quoted CDS spreads, which are referred to newly issued contracts, and at the same time keep the maturity always close to 5 years. The time series of market values obtained in this way is then divided by the notional of the contract. We therefore obtain, since the market value at inception of a CDS contract is zero, the series of two-weeks variations in market values expressed in percentage of the notional. This time series is computed for every CDS and for the CDX index and is the basis of all subsequent analysis.

Due to the presence of stale quotes for some CDS contracts, the frequency used for the calculation of the returns corresponds to 2 weeks. The final sample therefore consists of 180 observations.

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Structural Credit Risk Models in presence of Observational Noise [*]

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Abstract

In this paper we analyze some of the most important structural credit risk models in a state-space framework. We consider asset values and asset volatilities as state variables while equity values and equity volatilities are the observable variables. We verify that the filtering technique outperforms the standard variance restriction and that the improvement is particularly significant in crisis periods. Furthermore, the observation of implied volatilities improves the performance of the simple univariate state space framework already proposed in the literature. Within the bivariate setting, we compare the credit spreads implied by the models with the observed CDS spreads. The simple Merton's model is the one that performs worst. The extension to stochastic interest rates and the endogenization of the default only marginally improves the performance of the fit. Only the inclusion of a jump component and an endogenous default considerably improves the performance. Even in this case, however, a part of the market spreads remains unexplained. We therefore compute the value of the jump-to-default intensity able to explain observed CDS spreads and discuss the evolution over time of average implied intensity, jump-to-default risk premium and diffusion-to-default probability. In periods of low volatility, the majority of the default probability is constituted by jump-to-default probability, while in periods of high volatility also the possibility of a diffusion-to-default event becomes not negligible.

1   Introduction

The introduction of structural credit risk models dates back to the first work by Black and Scholes [1973] based on the contingent claim analysis of corporate liabilities. A refinement of this idea is in Merton [1974] and has become the first and simplest structural model of credit risk. The idea is to consider all the liabilities of a company as derivative contracts written on the unobservable value of its assets. Equity, in particular, has the same payoff of a call option while debt has the same payoff of a risk-free investment plus a short position in a put option on the assets.

Since then, many extensions have been introduced into the framework and several empirical tests of the models have been proposed. The results of such tests gave rise to the so-called credit spread puzzle, that is the inability of these simple structural models, when calibrated to historical equity prices and volatilities, to explain the observed credit spreads. This has been verified for corporate bond spreads for instance by Jones, Mason, and Rosenfeld [1983] and by Eom, Helwege, and Huang [2004], among others. In some cases also huge errors in estimated spreads are observed. In the majority of empirical tests, structural credit risk models have been documented to underestimate the observed spreads, especially for the shortest debt maturities, and only in a few studies (e.g. Eom et al. [2004] for some models) an overestimation is verified. These results are in striking contrast with the elegance of the models and their totally plausible assumptions. On the other hand, the usual alternative to structural models, reduced-form models, works better in explaining credit spreads but often lacks of a clear economic rationale.

In this work, we want to investigate more in depth this issue taking also into consideration the existence of observational noise in stock prices and volatilities. This is done through the use of a bivariate state-space framework which extends the idea proposed by Duan and Fulop [2009]. The objectives of the paper are therefore to introduce a bivariate state-space framework in order to filter latent asset and volatility processes. This setup allows finding these latent variables using standard nonlinear filtering techniques. On this common ground we then asses the performance of some of the most popular structural models. We compare the ability of the considered models to explain observed credit spreads. We concentrate on CDS and not on corporate bond spreads. The use of CDS instead of corporate bond spreads is becoming more and more frequent for the test of structural models and, as argued for example by Huang and Zhou [2008], has several advantages. For instance, it allows to ignore the presence of fixed or floating coupons in debt issues and does not need to distinguish the bond issues based on seniority. Furthermore, if we use bond spreads we have to consider the possible presence of embedded optionalities, guarantees and liquidity, which is in general very

low for corporate bond issues.

So we can argue that the CDS spread gives a closer approximation to the "true" default spread, neglecting additional components and premia present in the corporate bond spread. Furthermore, as claimed for instance in Blanco, Brennan, and Marsh [2005], a theoretical non-arbitrage relationship between corporate bond and CDS spreads should hold in non-crisis periods, while such a relationship is not verified in crisis times due to stronger limits of arbitrage (see e.g. Fontana [2010]). The use of CDS spreads reduces these issues and gives a more consistent measure of default risk through the different market and arbitrage conditions.

Our results seem to confirm the "structural" underestimation of observed credit spreads even though the size of the error seems smaller than in previous studies. The model that performs best is Leland [2006], which contemplates the possibility of a sudden jump to default, and produces a mean absolute error around 25 bps. This quantity roughly corresponds to half of the total variation. Nevertheless, a considerable component of the observed CDS spreads is not explained by any of the considered models. We therefore concentrate on the best performing model and compute the "implied" jump-to-default intensity, that is the particular jump intensity able to explain the observed market spreads.

We find that implied intensities, and consequently risk premia, are on average higher than the ones proposed in previous studies, at least after the crisis, and are strongly time varying. Another related finding is the relative behaviour of jump-to-default and diffusion-to-default probabilities during the considered period. In particular, the percentage of the default probability explained by the pure diffusion component seems to be negligible in "quiet" periods but becomes relevant in high volatility periods, when it accounts for more than 40% of the total default probability. Also the jump-to-default probability increases during the crisis period but to a much lower extent.

The reminder of the paper is organized as follows. In section 2 we briefly review the literature about structural models and their testing. In section 3 we describe the impact of observational noise on structural credit risk models. In section 4, we describe the most important techniques used to estimate structural models, while in section 5 we compare the performance of the considered models under the state-space framework. In section 6 we derive the implied jump-to-default intensities and the evolution in time of diffusion- and jump-to-default probabilities. Section 7 concludes.

2   Review of the literature

As observed by Leland [2009], the literature about structural credit models is so huge that an exhaustive overview seems impossible. In this section we concentrate on the evolution of the empirical literature on the performances of structural credit risk models while we refer to appendix A for a detailed review of each considered structural model.

Two of the first empirical articles are Jones et al. [1983] and Jones, Mason, and Rosenfeld [1984]. Here the ability of the contingent claim analysis to price corporate debt is tested for investment and non-investment grade firms. The result is that, for high-rating companies, the models provide a strong underestimation and no improvement is achieved with respect to the use of the risk-free term structure or, equivalently, that the predicted credit spread is zero. A better fit of the observed credit spreads is achieved for junk bonds. Furthermore, an improvement in the fit is obtained when a stochastic component in interest rates and the effect of corporate taxation are considered.

Also the second article, Jones et al. [1984], documents a relevant underestimation of the spreads. The result is confirmed also by Ogden [1987], who finds an average underestimation of the Merton [1974] model of 104 bps on average, and by Lyden and Saraniti [2000], who extend the result also to the Longstaff and Schwartz [1995] model.

The work by Lyden and Saraniti [2000] supports the negative performance of the models with average absolute errors between 80 and 90 bps for Merton [1974] model. They also find that the results are not improved when the possibility of default before maturity, industry-specific recovery rate or stochastic interest rates are considered. The simple Merton's models seems even able to outperform the Longstaff and Schwartz [1995] model with stochastic interest rates.

Elton, Gruber, Agrawal, and Mann [2001] find that the pure default component, measured by structural models, only partially explains the credit spread on corporate bonds. They also document that another fraction is explained by the different taxation on corporate and government bonds. They finally argue that the residual part of the spreads is a premium for systematic risk and they document that it strongly depends on the classical Fama and French [1992] factors.

A similar study is the one by Delianedis and Geske [2001], who verify that the fraction of corporate bond spreads explained by structural models is in general small and particularly insignificant for high-rating firms. The explained fraction goes from 5% for AAA firms to 20% for BBB firms. They also find that neither differential taxation nor stochastic recovery rates

improve the quality of the estimation. Only liquidity indicators, market risk factors (such as market volatility) and the presence of jumps can give a considerably better fit of the spreads. They also perform an analysis similar to ours and compute the jump size implicit in observed spreads. Their jumps do not necessarily drive the firm to default and have an a priori intensity λ. They find, for $\lambda = 1$ (i.e. an expectation of one jump per year), an implied jump size $k = 20\%$.

Huang and Huang [2003] present an investigation about the ability of structural models to explain corporate bond spreads. They confirm the difficulties of the models to justify observed spreads. In particular the fraction of spreads that can be explained by the models is around 20/30% for long and even less for short maturities. They identify the unexplained part of the spreads with a "liquidity" factor and verify that this is much stronger for high-rated bonds. They implement a relatively large set of models and verify that their performances are quite similar across the models.

Eom et al. [2004] verify the ability of five structural models to predict corporate bond spreads. They find that the models by Merton [1974] and Geske [1977] considerably underestimate observed spreads. On the other hand, Longstaff and Schwartz [1995], Leland and Toft [1996] and Collin-Dufresne and Goldstein [2001] models overestimate spreads. In general, they verify a poor ability of the models to explain spreads and in some cases they observe also huge errors in the prediction.

A more recent empirical investigation is due to Shaefer and Strebulaev [2008]. The authors confirm the poor performances of structural models in predicting spreads but provide evidence of their ability to give accurate hedge ratios for corporate bonds. They justify this with the presence of credit and non-credit factors in corporate bond spreads. The first category is actually captured by structural models and this ability is enough to compute correct equity hedge ratios, since non-credit components are not affected by the returns of the underlying stock.

Finally, we have to mention the introduction of nonlinear filtering techniques in the estimation of structural models. A fundamental work in this direction is Duan and Fulop [2009], where a particle filter is implemented to recover the unobserved state process that represents the asset value of the firm. The state space is univariate and the volatility is assumed to be a model parameter constant in time. A similar model is presented in Huang and Yu [2010]. The innovation in this last work is the frequency of the time series (daily) and the bayesian algorithm used for the estimation.

Although a large part of the literature agrees on the poor performances of structural models in the explanation of credit spreads, the magnitude and even the sign of the error are still not clear. Our contribution to the existing

literature consists in the proposal of a new bivariate state-space framework, which gives a new common framework to assess the predictions of the models, and in an estimation of the jump risk premium implied in CDS spreads. The filtering technique allows us to remove a large component of observational noise and therefore, in analogy with Huang and Yu [2010], we are able to increase the frequency of the estimation. Consequently, our estimation is at weekly frequency while much of the previous literature is based on monthly frequency.

3 Structural Credit risk models and observational noise

The most important distinction in credit risk models is between structural and reduced-form models.

Reduced-form credit risk models assume that a jump process governs the arrival of default events and only the intensity of such process is the object of the model. The calibration of the model is performed only on past observed credit spreads and no data from equity market or from the capital structure of the firm is used.

In structural credit risk models, on the other hand, the different components of the corporate liabilities of a company, such as equity and debt with different seniority, are priced as contingent claims on some relevant observed or unobserved corporate quantities. A link between some structural quantities of the company and the market value of its liabilities is therefore identified.

A first and most important group of structural models for corporate entities identifies such corporate quantities with unobserved "stock" variables (such as the market value of the assets of the company). Another group of models is based on not necessarily unobserved "flow" quantities (such as the value of EBIT in Goldstein, Ju, and Leland [2001]). In this work we will concentrate on the first category of structural models and try to statistically describe the latent processes.

In appendix A we briefly summarize the structural models considered here. A common characteristic of the models we are considering in this paper is the use of the asset value (V) and asset volatility (σ^V) as latent quantities required to price corporate liabilities. Each of the models provides a specific closed-form mapping from the latent variables to the observed equity quantities. So in general

$$S_t = h_1\left(V_t, \sigma_t^V, \psi_t\right)$$
$$\sigma_t^S = h_2\left(V_t, \sigma_t^V, \psi_t\right) \tag{1}$$

where the (possibly nonlinear) functions h_1 and h_2 map the unobserved asset value and asset volatility to the observed equity price (S) and equity volatility (σ^S). The vector ψ_t contains exogenous inputs such as interest rate or asset payout.

In presence of a fully efficient and perfectly liquid market it is immediate to obtain the values of the unobserved variables V and σ^V from the observed variables S and σ^S. In this case we just need to invert the equity pricing functions h_1 given in equations (16), (22), (23), (30), and (36) of the appendix and the generic mapping h_2 from asset to equity volatility given in equation (19). This procedure, usually called variance restriction, involves in general numerical inversion but is straightforward, given the one-to-one closed-form mapping between unobserved assets and observed equity.

The first justification for the presence of a noise that perturbs equation (1) is the misalignment between the different liabilities and their theoretical value based on the assets. Even if we assume that on average the estimates of the market participants are correct, and consequently that on average the market keeps equity prices in a non-arbitrage relationship with the assets of the company, it is known that strong limits to arbitrage are present and that they can prevent the equilibrium to be reached also for very long periods of time. This is particularly true in our case, since one of the two positions in the potential arbitrage strategy, the assets of the company, can be extremely illiquid and in some cases even not tradable (e.g. intangible assets).

A vast and increasing literature addresses the question of limited arbitrage opportunities (see e.g. Gromb and Vayanos [2010] for a review). Here we can summarize them into two big categories. The first one is the presence of non-fundamental shocks to the demand of corporate liabilities (equity or corporate bonds) that can be explained by non-rational behaviours of the agents or by institutional frictions. The second reason are arbitrage costs, that can arise from the remuneration for arbitrage strategies that are risky, from the costs of short selling, from the presence of leverage constraints or finally from equity capital constraints.

Ericsson and Reneby [2002] show that even if assets are not traded, it is always possible to replicate them using a risk-free bond and a risky security, such as the equity or debt of the company. This is only partially true in this context, since the debt and the equity of the company are available only in limited quantities and their liquidity is often extremely scarce limiting arbitrage strategies. But even if we assume to be able to replicate the assets

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with equity, for instance, we cannot enforce an arbitrage strategy between equity, debt and assets unless the debt of the firm is also efficiently traded. In other words, an arbitrage strategy can be implemented if at least two claims among equity, debt and assets are traded. It seems difficult to assume that all the liabilities of the company, that is equity and the entire debt (the sum of all the loans and bond issues), are efficiently traded.

This limitation in the arbitrage possibilities due to scarce liquidity of some claims can produce strong and persistent misalignments between the prices of equity and the true market value of the company. A consequence of this fact is that it is likely to observe equity prices contaminated by an error with respect to their equilibrium 'structural' values based on the asset process.

Furthermore, the observation of the balance-sheet value of the liabilities is performed only at discrete times and low frequencies (quarterly for the companies considered here and even less in general). This means that not only the observed equity price may be contaminated by noise but also the face value of debt that we use in the pricing formula. This fact can induce an even bigger observational noise on equity.

On top of this, we also have to consider the possible presence of microstructure frictions such as bid-ask bounce or discrete clustering. The impact of these issues clearly depends on the frequency at which the observations are taken and we can argue that, at the weekly frequency we are considering here, this effect is small or even negligible. Nevertheless, a theoretically correct implementation cannot neglect the impact of microstructure noise.

All these arguments justify the assumption of observational noise and therefore the problem of the estimation of V and σ^V becomes a (nonlinear) filtering problem.

4   Estimation of Structural Models

A first naive implementation of structural models is simply based on the solution of a system of two equations in two unknowns, that is equations (1). The solution of the nonlinear system requires numerical iteration and gives the two unknowns V and σ^V. It is possible to solve the system for every point in the time series and therefore obtain two vectors of estimated asset values and estimated volatilities.

4.1   Maximum likelihood

The next step towards a more rigorous statistical implementation, especially from the inferential point of view, which is completely neglected in the naive

approach, is the estimation technique due to Duan [1994] and based on likelihood maximization. In this case the presence of observational noise is still completely neglected and the only observed variable is S_t, the equity price. No observation of volatility is required.

The asset value process is assumed to follow a geometric brownian motion. Therefore the transition density under the physical measure is

$$\log\left(\frac{V_{t+1}}{V_t}\right) \sim N\left(\mu - \delta, \sigma^V\right) \tag{2}$$

In this way we can derive the log-likelihood function for a vector of asset values V_t of length n as

$$L_V\left(V_t, t = 1, ...n; \mu, \sigma^V\right) = -\frac{n-1}{2}\log\left(2\pi\right) - \frac{n-1}{2}\log\left(\sigma^{V2}\right)$$
$$-\frac{1}{2\sigma^{V2}}\sum_{t=2}^{n}\left(\log\left(\frac{V_t}{V_{t-1}}\right) - \mu + \delta\right)^2. \tag{3}$$

Duan [1994] shows that given a deterministic one-to-one transformation $y = T(x; \theta)$, it is possible to express the likelihood of $y_t, t = 1, ...n$ in terms of the likelihood of $x_t, t = 1, ...n$. In particular

$$L_Y\left(y_t, t = 1, ...n; \theta\right) = L_X\left(\hat{x}_t, t = 1, ...n; \theta\right) - \sum_{i=1}^{n}\log\left|\frac{dT_i\left(\hat{x}_i; \theta\right)}{dx_i}\right| \tag{4}$$

where $\hat{x}_i = T_i^{-1}\left(y_i; \theta\right)$. When we apply this general result to our specific case, we have $T\left(V; \theta\right) = h_1\left(V; \theta\right)$ and therefore the log-likelihood of equity given the parameters of the asset process is

$$L_S\left(S_t, t = 1, ...n; \mu, \sigma^V\right) = -\frac{n-1}{2}\log\left(2\pi\right) - \frac{n-1}{2}\log\left(\sigma^{V2}\right) - \sum_{t=2}^{n}\log\left(N\left(d_t\right)\right)$$
$$-\frac{1}{2\sigma^{V2}}\sum_{t=2}^{n}\left(\log\left(\frac{\hat{V}_t}{\hat{V}_{t-1}}\right) - \mu + \delta\right)^2 \tag{5}$$

where $\hat{V}_t$ is the inverse of the Black-Scholes formula and $d_t = \frac{\log\left(V_t/F\right) + \sigma^{V2}T/2}{\sigma^V\sqrt{T}}$.

In this way we can numerically maximize the likelihood function on μ and σ^V. With the estimated parameters we can invert the Black-Scholes pricing formula and obtain a vector of estimated asset values. If we are working

under the risk-neutral measure we can set $\mu = r - \delta$ and maximize over σ^V only.

The most important drawbacks of this model are that it completely neglects the presence of observational noise and that it does not consider the possible variability of the volatility parameter in time. This is a consequence of the strict Black-Scholes assumption of constant deterministic volatility. The time series of equity prices is therefore enough to determine the parameter σ^V and is the only required input.

4.2   A state-space framework

If we want to take into consideration the presence of an observational noise and also to exploit the observation of equity volatility (e.g. options implied volatility), we need to rearrange our model in a state-space form.

It is possible to describe a generic state-space model by a state equation and a measurement equation. The first one determines the dynamics of the usually unobserved state variables while the second relates the unobserved states to some observable variables. So in general the data generating process for the states X is

$$X_t = f(X_{t-1}, \theta) + \varepsilon_t \tag{6}$$

where θ is a vector of parameters and the noise vector ε is a multivariate normal with zero mean and covariance matrix W.

On the other hand, the measurement equation is

$$Y_t = h(X_t, \theta) + \eta_t \tag{7}$$

where the noise vector η is still a multivariate normal with zero mean and covariance matrix N.

All the considered structural models are characterized by the presence of a set of latent and a set of observed random processes. The first implementation of structural credit risk models in a state-space framework is due to Duan and Fulop [2009]. They assume that the unobserved signal X is the (log-)asset value $\log(V)$ and that the observed variable Y is the (log-)price of equity. The "true" value of equity is contaminated by a univariate trading noise η.

The filtering procedure of the latent process requires the estimation of the model parameters and in particular the asset volatility σ^V and the noise covariance N that in this case is a scalar. Duan and Fulop [2009] assume that

σ^V is constant and deterministic during the entire estimation period. The resulting model is fully equivalent to Duan [1994] with the only assumption that a noise η is present and contaminates equity prices.

In this work we model the asset value process $\{V_t\}_{t=1}^{\infty}$ and the asset volatility process $\{\sigma_t^V\}_{t=1}^{\infty}$ as two unobserved processes in discrete time.

In order to guarantee the non-negativity of the asset and volatility process, as in Duan and Fulop [2009], in the data generating system, we do not model directly V and σ^V but rather their natural logarithms.

Therefore, the joint discrete-time dynamics of the two variables are given by the following unobserved state equations

$$\left(\begin{array}{c} \log(V_t) \\ \log(\sigma_t^V) \end{array} \right) = A \cdot \left(\begin{array}{c} \log(V_{t-1}) \\ \log(\sigma_{t-1}^V) \end{array} \right) + \left(\begin{array}{c} r \\ 0 \end{array} \right) \Delta t + \left(\begin{array}{c} \varepsilon_t^1 \\ \varepsilon_t^2 \end{array} \right) \qquad (8)$$

where A is a 2×2 real transition matrix and ε is bivariate normal noise vector with zero mean and covariance matrix W. In this way we are restricting the generic functions f in equation (6) to only linear ones.

The observed variables are the price process of equity S_t and its volatility process σ_t^S, which are both function of V_t and σ_t^V possibly contaminated by noise. In particular, the observation equations are given by

$$\left(\begin{array}{c} S_t \\ \sigma_t^S \end{array} \right) = \left(\begin{array}{c} h_1\left(V_t, \sigma_t^V, \psi_t\right) \\ h_2\left(V_t, \sigma_t^V, \psi_t\right) \end{array} \right) + \left(\begin{array}{c} \eta_t^1 \\ \eta_t^2 \end{array} \right) \qquad (9)$$

We assume that η_t is a bivariate gaussian noise vector with zero mean and covariance matrix N

The functions h_1 and h_2, are different for every considered structural model and are described in appendix A for every specific case. These two functions fully identify the most commonly used structural credit risk models.

4.3   A simulation exercise

To verify the ability of the filtering procedure to capture the unobserved asset process, we cannot rely on real-world data since we do not have any information about the true latent asset process of traded stocks. We therefore implement the following simulation exercise.

We jointly simulate the asset and volatility processes according to equation (8) for 250 days, which corresponds to one year of daily observations. We then use equation (9) and a simulated noise η to get the simulated contaminated processes for S and σ^S. After that, we apply the filtering technique to recover

the process for the assets. We compute the mean squared error between the true underlying asset process and the filtered path. We do the same using the naive inversion and the UKF technique. Therefore we get two values of MSE for each simulated path.

Given the slowness of the procedure and the strong stability of the results, we repeat the procedure only 500 times and for 100 increasing values of the noise volatility, i.e. the standard deviation of process η. We finally take the average over the 500 simulations. The resulting average MSE is plotted as function of the noise volatility in Figure 1.

We clearly see that, for small values of the noise, the errors of the two procedures are comparable while for increasing noise the filtering method outperforms the naive inversion technique. The performance of the filtering procedure is therefore always at least as good as the inversion implementation. We argue that this advantage comes from two sources. The first one is the obvious ability of the particle filter to eliminate the noise perturbation. The second one is a simple numeric issue, that is the difficulty of the numeric algorithm to invert the nonlinear system (9), due to the analytic complexity of the functions involved or to functions flat around zero.

4.4 Implementation

The implementation of the Duan [1994] model is straightforward, since it involves only a simple maximization of the likelihood function in equation (5), which gives a unique estimated value for the asset variance in the considered time period.

The implementation of all the other models is based on nonlinear filtering and requires either the UKF or the Square-Root UKF. The performance of the two algorithms are fully comparable. Due to the slightly better ability of the SR-UKF to explain observed CDS spreads and to its relatively better efficiency, we present only the results obtained with this algorithm. For a more detailed description of the algorithms, refer to appendix B.

The first step is the implementation of the simple model by Duan and Fulop [2009] in a state space framework. In this case the functions f and h are both scalar-valued. In particular, the function h is given by equation (16) of the appendix. We therefore have only one latent state, the asset process and one observable, the price of equity. The state process is governed, under the risk-neutral measure, by only one unknown parameter, σ^V, to be estimated. Since the parameter is not time-varying we exclude the implementation of the asset volatility as a parameter of the model. The natural estimation technique is therefore the standard maximum likelihood procedure. Given the computational time required by the UKF algorithm to run, we do not

implement a numerical maximization of the parameter $\boldsymbol{\sigma}^V$, since at every iteration a new estimation of the UKF is required. We therefore build a grid for $\boldsymbol{\sigma}^V$, run a SRUKF filter to recover the latent process V_t and compute the likelihood at every point of the grid according to equation (5). We then take the value of $\boldsymbol{\sigma}^V$ for which the likelihood is maximum. This involves a number of computations of the filter equal only to the number of points in the grid, which we set to 300.

This implementation requires only the time series of (log) equity prices, the prior moments of the state x_0 (which in this case is simply the log-price) and the initial values of the parameters in matrix A in equation (8), which in this case is a scalar. In the more general bivariate case the state x_0 is a vector with the log-price and the log-volatility of the asset process, V_t and $\boldsymbol{\sigma}_t^V$ and A is a 2-by-2 transition matrix. We get an approximated estimate of the matrix A and the moments of x_0, required by the algorithm, from the simple inversion of function $h(\cdot)$ under the assumption of no noise. In this way we get an approximation of the time series V and $\boldsymbol{\sigma}^V$ from which we can find a proxy of all the required priors. The reliability of the estimation is not essential, since such parameters are used only as starting values for the algorithm and are changed at every iteration until convergence.

We also performed the estimation of the models using log-returns in place of log-prices in equation (8) for both asset values and volatilities. We observe that in this case the performance of the estimation is poorer. This may be due to the fact that all the structure present in the level data is lost when we take the first difference. In other words, in the time-series in levels the matrix A in equation (8) is significantly different from zero and the elements on its main diagonal are close to one. On the other hand, when we work with first differences all the elements of the matrix are close to zero and give no insight into the structure of the data generating process. Consequently, in the first difference case, the time series we are filtering is very close to pure noise, giving less reliable results and more limited information.

In some of the considered models some estimation of exogenous parameters is also needed. One can decide whether to include the parameters into the set of state variables or alternatively to estimate the parameters separately and to use them as exogenous variables. In all the models, we follow this second strategy in order to leave unchanged the dimension of the state-variable vector and avoid possible identification problems.

The first parameter needed is the dividend yield, which is required in Leland and Toft [1996] and in Leland [2006] model. In this case we assume a constant dividend yield equal to 2%, compatible with the historical mean on the Dow Jones Ind. Average index on the same period (2.44%).

In these last two models an assumption on the average coupon paid on the

outstanding debt is also required. In the Leland and Toft [1996] model we assume that no coupon is paid while in the Leland [2006] we follow the original paper and assume that the coupon C is numerically chosen in such a way that the market value of the debt is always equal to its par value.

In the stochastic interest rate model, the correlation parameter ρ between the equity and interest rate risk sources is required. This parameter is estimated empirically for every considered company and for simplicity it is assumed to be constant in time. The estimation procedure for ρ involves, in a first step, the solution of the standard Merton's model that provides us with an approximation of the asset value process V_t. After that, we compute the sample Pearson correlation between the approximated asset returns and the first difference in the short-term interest rate. In general, we observe values of ρ between 0 and 30%. So it seems that the correlations between asset returns and changes in the interest rate are positive and in general significant. The sample average of ρ is 0.18 and its cross-sectional standard deviation is 9.52%. This result supports the empirical implication of Merton's model, for which an increase in interest rates has a positive effect on the market value of assets since it increases its risk-neutral drift rate.

In this type of models we also need an estimation of the parameters in equation (20) of the appendix. In this case the estimation of the interest rate curve parameters is performed separately from the estimation of the asset and volatility processes. This procedure simplifies the filtering operation, which remains bivariate, and is fully consistent with previous literature (e.g. Huang and Zhou [2008]). We start with the discrete-time counterpart of the considered stochastic differential equation.

$$r_t = \alpha_t + \gamma_t r_{t-1} + \varepsilon_t \tag{10}$$

Coefficients α and γ are indexed with t since we allow for time-varying parameters. The estimation is performed using a Kalman filter and assuming that the processes for α_t and γ_t follow an $AR(1)$ process. Therefore

$$
\begin{aligned}
r_t &= \alpha_t + \gamma_t r_{t-1} + \varepsilon_t \\
\alpha_t &= a_1 \alpha_{t-1} + \eta_{1,t} \\
\gamma_t &= a_2 \gamma_{t-1} + \eta_{2,t}
\end{aligned}
\tag{11}
$$

Next we compute the standard deviation of the estimated residual $se_t = \text{std}(\hat{\varepsilon}_i)$ for $i = 1, ..., t$.

The parameters of equation (20) can finally be obtained as

$$k_t \;=\; -\frac{\log(\gamma_t)}{\Delta t} \tag{12}$$

$$\theta_t \;=\; \frac{\alpha_t}{1 - \gamma_t} \tag{13}$$

$$\nu_t \;=\; se_t \sqrt{\frac{-2\log(a)}{\Delta t \, (1 - \gamma^2)}}$$

where Δt is the time interval between two observations [1].

As far as models with jumps are concerned, a final required input is the risk-neutral intensity of the jump to default process, which is always assumed to be a Poisson process. We start from the historical default probability obtained from Moody's data for each rating class. After that, we transform this historical probability into a risk-neutral one using the risk premium μ obtained by Driessen [2005]. We then subtract from the entire default probability the probability of default explained by the pure risk-neutral diffusion process. This amounts to the probability of a first passage time of a geometric Brownian motion to a threshold. The result is a probability of jump-to-default, which can be easily transformed into the corresponding intensity.

As a proxy for the corporate tax rate, we assume a value of 35%, while the recovery rate, which is needed for the computation of the jump size, is obtained from Moody's historical data based on the rating class.

The models by Leland and Toft [1996] and Leland [2006] also assume that the debt is continuously repaid at a rate m. In our implementation we assume a value of $m = 0.2$, which gives an average maturity of the outstanding debt exactly equal to 5 years. This is the horizon we are considering for every model and also the horizon of the CDS spreads we are trying to fit. Default costs are set to $\alpha = 0.1$, which is 10% of the asset value but our estimation is quite robust to different choices of α.

A final element that has to be taken into consideration is a required burn-in period in the UKF procedure. The estimation of the posterior distribution based on the Unscented Kalman Filter technique requires the convergence of a Markov chain to a stationary distribution. The first estimations in the time series don't have necessarily converged yet to the stationary distribution and therefore need to be excluded from the final estimation. Consequently, we

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[1] As argued by Lo [1988] and Broze, Scaillet, and Zakoian [1995] the estimation of the parameters of a continuous time process by means of its discrete-time counterpart induces bias and inconsistency in the estimated parameter. Nonetheless the bias is known to be relatively small (see e.g. Broze et al. [1995]) and the unbiased estimators are not straightforward to implement in the case of time-varying parameters.

have to consider a number of observations to be excluded at the beginning of each estimated time series. Usually the convergence is extremely rapid and in less than 10 observations it is always reached.

5 Empirical Results

5.1 Description of the data

The sample period starts with the beginning of the liquid CDS market and corresponds to January 2003-December 2011. We include all the firms that have been part of the Dow Jones Industrial Average Index during this period. Given the difficulties to identify a correct value for the defaultable liabilities, we exclude from our analysis financial companies as in the majority of the literature, with the only exclusion of Lyden and Saraniti [2000].

In table 1 we present the considered companies and some descriptive statistics of the corresponding CDS spreads. A first characteristic of the data is the extreme dispersion in the cross-sectional and, most of all, in the time-series dimension. This is confirmed by the high difference between maxima and minima in the series. A relevant positive skewness is also present, since we observe values of the mean in general higher (and sometimes much higher) than the median.

We take equity prices, quarterly balance-sheet data, adjustment factors and the number of outstanding shares from Datastream. The time series of the debt-per-share (DPS) is then obtained as

$$DPS_t = \frac{ST_t + \frac{1}{2}LT_t}{Nosh_t} \cdot AF_t \tag{14}$$

where ST_t are short-term liabilities and LT_t long-term liabilities at time t. $Nosh$ is the number of outstanding shares and AF is the adjustment factor, which accounts for stock splits and other corporate actions.

We obtain estimates of volatility from call options with a delta of 0.4 and a maturity of two years, which is the longest still liquid maturity available. The source is in this case OptionMetrics.

The CDS spreads we are considering are referred to contracts with 5-year maturity, which is known to be the most liquid in the market, and are taken from Bloomberg. For each reference entity we take the longest possible time series of CDS spreads even though in some case the first observation is after the 1st of January 2003.

As a proxy for the risk-free rate we use the US Treasury constant maturity 5-years rate taken from the Federal Reserve Economic Database. The maturity is again chosen to match the one of the CDS spreads we are considering. The frequency of our analysis is weekly so that we are left with 470 observation dates.

Company	Weekly Obs.	Mean	St. Dev.	Min.	Median	Max.
3M	418	28.2	24.6	4.7	20.9	132.5
AT&T	418	58	60.7	8.2	40.1	361.2
Alcoa	470	151.5	185.7	15.9	48.6	1043.3
Boeing	470	62.7	55.3	6.5	54.7	330
Caterpillar	470	59.3	60.8	11.2	32.7	403.8
Chevron	418	30.2	24.3	3.5	22.1	132.8
Cisco Systems	417	41.7	33	2	36.8	170.2
Coca Cola	417	28.8	19.5	6.5	25.8	99.9
El Du Pont	470	41	32.8	8.7	24.4	215.1
Exxon Mobil	418	21.7	19.5	2.5	19.3	113.6
General Electric	447	128.5	157.6	10.9	43.8	976.1
Hewlet-Packard	470	40.4	26.6	7.8	31.6	155.9
Home Depot	418	63.5	65.5	7.3	57	353.8
Intel	185	51.7	17.4	26.3	52.3	87.3
IBM	470	33.6	19.9	6.5	30.8	133.3
J&J	418	23	18.6	2.9	14.3	76.3
Kraft Foods	470	52.8	28	13.1	51.4	178.8
McDonalds	470	31.6	13.7	10	30	78.3
Merk & CO.	418	30	17	5.8	30	82.5
Microsoft	276	24.3	12.4	7.5	26.1	55.3
Pfizer	418	33.4	28.6	3.5	22.1	132
P&G	418	35.6	29.2	5.5	33.1	149.4
United Tech.	418	37.6	28.4	8.5	31.1	148.3
Verizon Comm.	467	52.5	22.4	16.5	53.2	103.5
Wal Mart Stores	470	31.9	24	5.9	22.2	133.4
Walt Disney	470	43.9	22.2	9.8	41.4	126.3
Honeywell	418	35.8	22.4	10.5	32.3	150
Altria	418	93.4	44.7	19.2	95	218.3

Table 1: Descriptive statistics of the considered CDS

5.2 Improved estimation

With the simulation exercise proposed above, we have already verified that the filter is always at least as good as the inversion technique and, for

increasing noise volatility, it starts outperforming the variance restriction method. Another way to verify the improvement of the filtering technique over the simple variance restriction procedure in absence of observations of the latent asset process is to look at the ability to explain observed CDS spreads. We compute the squared error between the observed CDS spreads and those estimated from Merton's model and then average each of them in the cross-section dimension. In this way we obtain a time series of squared errors for the variance restriction and the filtering technique. We then test if the errors of the filtering technique are statistically higher than those of the variance restriction. For this purpose, we implement the test proposed by Diebold and Mariano [1995]. The null hypothesis we are testing is

$$H_0 : E\left(\varepsilon_{1,t}^2\right) = E\left(\varepsilon_{2,t}^2\right)$$

where $\varepsilon_{1,t}$ is the estimation error of the variance restriction and $\varepsilon_{2,t}$ is the estimation error of the filtering technique. The test statistic is

$$S = \frac{\bar{d}}{\sqrt{LRV/T}} \overset{\text{asy}}{\sim} N(0,1)$$

where

- $\bar{d} = \frac{1}{T}\sum_{T}^{j=1} d_i$
- $d_i = \varepsilon_{1,i}^2 - \varepsilon_{2,i}^2$
- $LRV = \gamma_0 + 2\sum_{j=1}^{T}\gamma_i$
- T is the number of observations
- γ_i is the autocovariance of order i of the vector d.

The value of the test on the entire sample is $S = 1.253$, which is not statistically significant at any standard confidence level (the p-value is 0.105). Nevertheless, a testable prediction of the filtering technique is the improvement in performances in periods of high "noise" in financial markets, such as periods of limited liquidity or arbitrage. We therefore split the considered period in a "crisis" period and a "non-crisis". This is objectively done by fitting a two-states Markov switching model on the time series of the TED spread, a widely used indicator of illiquidity and frictions in the financial markets. The model identifies the beginning of the crisis period with August 14th 2008 and the end with May 18th 2009. After that the levels of the TED spread considerably decreased.

We find that, if we consider the estimation errors on the crisis period only, the filtering technique significantly outperforms the naive numerical inversion.

When we compute the Diebold-Mariano test in the crisis period only we find $S = 2.267$ with a p-value of 0.0117. We can therefore reject at a 95% confidence level that the filter has the same performance of the naive inversion in crisis periods.

5.3   A comparison of the models

Once verified the ability of the filtering technique to capture the latent asset process, we assess the performances of the five considered models in the explanation of observed market CDS spreads. We have to notice that all the data we are using to fit spreads come from equity and options market and no input is taken from the bond or CDS market. This means that the test is a true out-of-sample one, while a part of the existing literature adopts different econometric techniques to fit observed spreads (see e.g. Huang and Zhou [2008] or Longstaff, Mithal, and Neis [2005]) using corporate variables.

Our results confirm the presence of the credit spread puzzle, since the percentage of explained variation is always less than 50%. A large amount of variation remains therefore unexplained. Nevertheless the error seems smaller than in previous studies (e.g. Jones et al. [1984] or Ogden [1987]), especially when we consider that all the companies in the sample are highly rated. We argue that the reason for this is twofold. First, we are working on CDS and not on corporate bond spreads, and in this way we can eliminate the presence of premia (such as liquidity) and concentrating on pure default premium. Second, the use of the filtering technique filters out the observational noise in the time series for equity and equity volatility.

The use of stochastic interest rates improves the performance only to a small extent. The endogenization of the default, per se, gives no improvement on the Merton's model. The inclusion of discontinuities in the asset process, on the other hand, improves the fit. The Leland [2006] model is the one that performs best in our analysis and gives a RMSE and a MAE of 46 and 25 bps, respectively. The model is able to explain between 40 and 50% of the total variation in the spreads.

We also have to take into account that the required risk premium, described is section 4, is assumed to be constant in time. Further research could be oriented to the design of a more efficiently estimated time-varying jump risk premium, in order to improve the performance of the fit.

Model	Root Mean Sq. Error	Mean Abs. Error
Merton	67 (17.9%)	45 (8.9%)
Stochastic IR	60 (26.2%)	42 (14.4%)
Merton + Jump	54 (34.4%)	32 (34.9%)
Leland Toft [1996]	72 (12.6%)	46 (6.3%)
Leland [2006]	46 (43.6%)	25 (48.2%)

6   Implied default intensity and jump risk premium

As suggested in the previous paragraph, the Leland [2006] model is the one that performs best when compared to the others. Nevertheless, we can say that, even for the best model, the fit is far from perfect, with a mean squared error of 36 basis points. This is particularly true for the investment grade firms considered here, characterized by small CDS spreads. Therefore a purely structural model based only on data from equity and options market does not seem to be completely satisfactory. In particular, a common characteristic that we find in the considered models is the underestimation of the observed CDS spreads. This is true for diffusion but also for jump-diffusion models, when we use a default intensity compatible with observed default rates and reasonable risk-premia.

We therefore want to verify which value of the default intensity is needed to justify the observed CDS spread. This is achieved for the single CDS and for every observation by numerically solving for the "implied" jump-to-default intensity. In this case, and by construction, the fit to the spreads is almost perfect, with a mean squared error of around one basis point.

The evolution in time of the average implied default intensity is represented in Figure 2. We observe considerably higher values of the implied intensity after the crisis and two minimum levels in summer 2003 and spring 2007. The maximum implied intensity is observed during the peak of the financial crisis, at the beginning of 2009.

The values of the implied jump-to-default intensity are much higher than the historical risk-neutralized jump-to-default intensities obtained in Section 4. We can also compare the implied intensities with the ones suggested in Leland [2006]. In the first part of the sample the implied intensity is between the one of an A and a Baa rated company, while the average rating of our sample is between Aa3 and A1. Therefore the difference is not extreme. On the other hand, after the crisis we have to require the intensities of a B-rated company to explain the observed spreads. This supports the presence of a structural change after the crisis.

The ratio of risk-neutral to physical intensities is proportional to the state

price per unit probability and therefore it is a measure of the risk premium. The evolution over time of this measure of risk premium is presented in figure 3. We observe generally low risk premia in the first part of the sample (a ratio of one implies no premium for bearing risk) while a peak is reached during the financial crisis, indicating the deterioration of the risk appetite in this period. The risk premium estimated by Driessen [2005] is equal to 2.15, which seems comparable with our estimation only for the first period of the analysis. In the second part the estimated premium seems much higher and considerably time-varying. The jump risk premium required to explain the observed CDS spreads varies a lot in time and therefore assuming it to be constant can be a serious limitation.

6.1 Diffusion-to-default and jump-to-default probability

This section compares the diffusion-to-default probability with the implied jump-to-default probability. We obtain the diffusion-to-default from the Leland [2006] model computing the probability that the asset process conditional on no jump reaches the endogenously determined default threshold V_B (see appendix A). This approach gives a more sophisticated estimate than the simple Merton's model as a consequence of the endogenization of the default event.

The jump-to-default probability can be computed from the implied intensity obtained in the previous paragraph easily providing a probability on the considered time horizon.

The jump-to-default probability in figure 5 shows a slightly increasing trend during the considered period and reaches the maximum during the financial crisis. Diffusion-to-default probability, on the other hand, remains always close to zero and only during the crisis grows sharply, as we can see from figure 4. This is arguably due to the extreme increment in the observed implied volatilities that causes a corresponding spike in the default probabilities.

If we look at the area plot of the two default probabilities we see that the role played by diffusion to default is in general very limited and only in periods of very high volatility and distress it becomes relevant (see figure 6).

This impression is confirmed when we plot the fraction of default probability constituted by diffusion-to-default (see figure 7). The ratio is close to zero in "standard times" while it increases significantly during the crisis and in correspondence of the Lehman default the explosion of implied volatilities makes it close to 0.5, which means that in this period half of the default probability came from diffusion-to-default.

7 Conclusions

In this paper we proposed a bivariate state space framework for the estimation of structural credit risk models. We considered five of the most popular models able to express the equity of a firm as a closed-form function of the value and the volatility of the assets. The implied credit spread is therefore computed from the market value of debt. We compare the explained credit spread with the market CDS spreads for each of the models.

We assume the existence of two state variables, the asset value and assets volatility, and two observable variables, the equity value and equity volatility. The recovery of the latent states amounts to a nonlinear filtering problem, which is solved using an Unscented Kalman Filter algorithm.

We find that the bivariate framework improves the performance of the simple univariate state space. Within our bivariate framework, the simple Merton's models is the one that performs worst. The extension to stochastic interest rates improves the performance of the fit only by a small amount. Also the endogenization of the default threshold proposed by Leland and Toft [1996] gives no clear improvement with respect to the standard Merton's model. Only the joint inclusion of a jump component and an endogenous default considerably improves the fit. This model has been proposed by Leland [2006] and is the one that performs best.

Even for the best performing model, however, we verify the existence of a considerable portion of the market spread that remains unexplained. We therefore compute a measure of "implied" jump-to-default intensity, that is the value of the default intensity parameter that is able to explain market spreads.

We observe an increase in the value of average implied intensity during the considered period with a peak reached around the Lehman crisis. We also compare the implied intensity with the historical one, obtaining in this way an estimate of the jump-to-default risk premium. This quantity is comparable with previous studies in the first part of the sample, while it becomes substantially higher from the beginning of the financial crisis. Furthermore, the risk aversion seems to vary a lot during time, making unplausible the assumption of constant risk premia.

As far as the evolution of diffusion-to-default and jump-to-default probabilities are concerned, we observe that in "quiet" periods, characterized by low volatility, the vast majority of the default probability is constitued by jump-to-default while, in periods of high volatility, also the possibility of a diffusion-to-default event becomes not negligible.

A promising application of the filtering technique is the estimation of the

spreads for companies with very illiquid equity or opaque flows of information and consequently with higher observational noise. We can argue that in this case the improvement over the standard maximum likelihood or variance restriction techniques would be would be even higher.
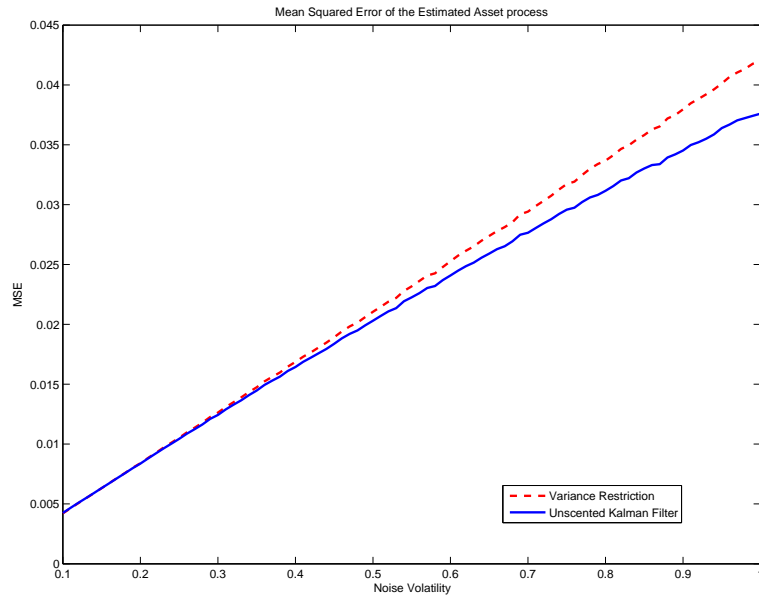
8   Figures

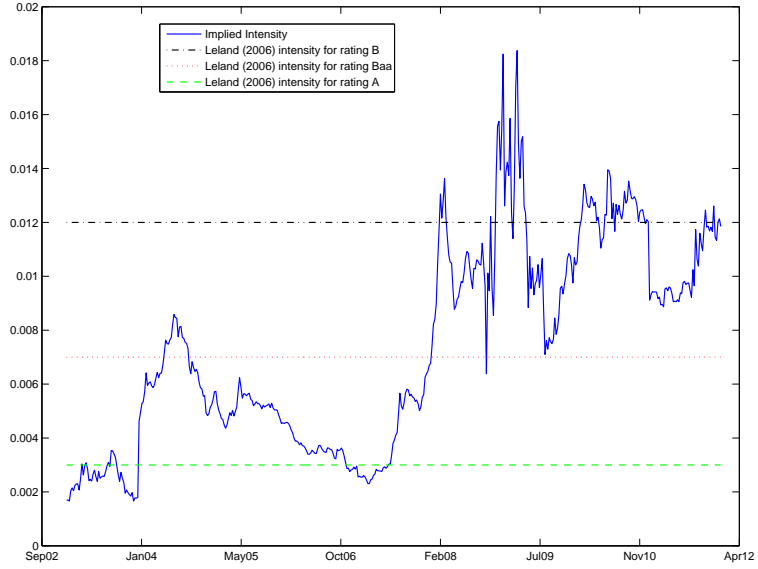Figure 1: MSE of the estimated assets as a function of the observational noise

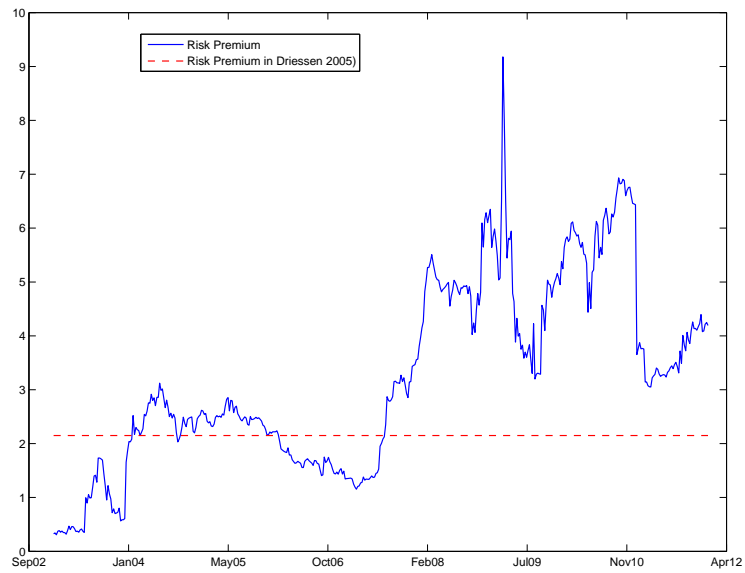Figure 2: Average jump-to-default intensity implied in the CDS spreads
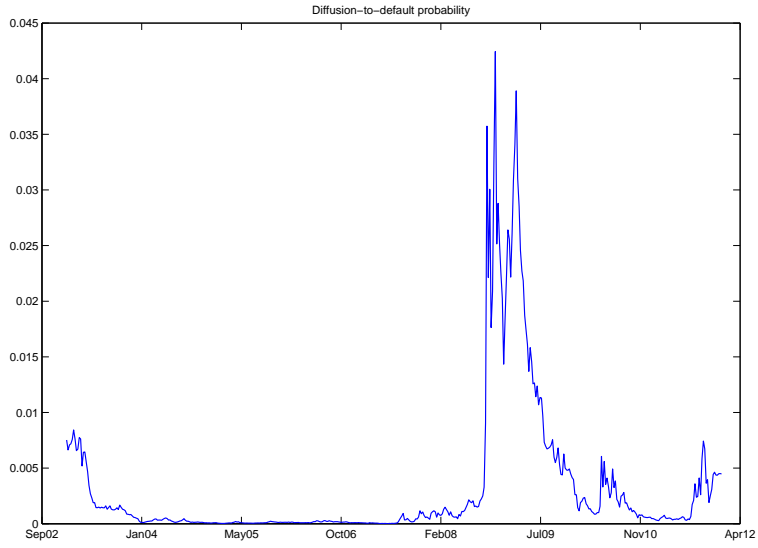

Figure 3: Average jump-to-default Risk premium

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Figure 4: Evolution of the diffusion-to-default probability
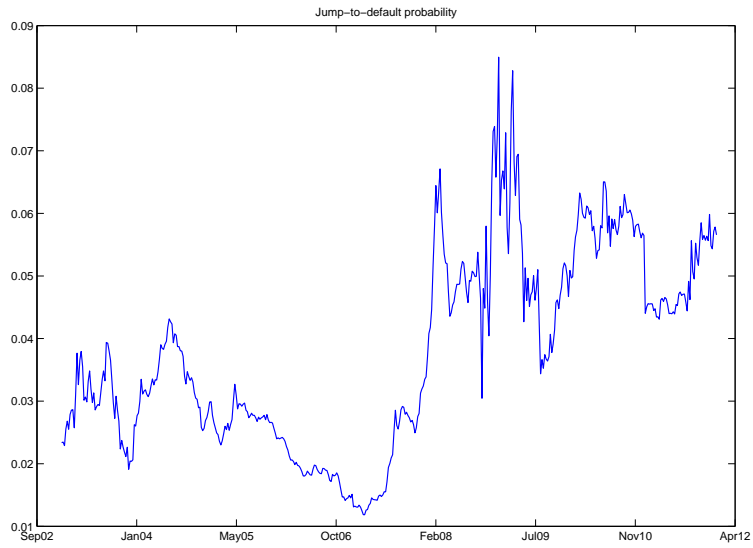
Figure 5: Evolution of the jump-to-default probability
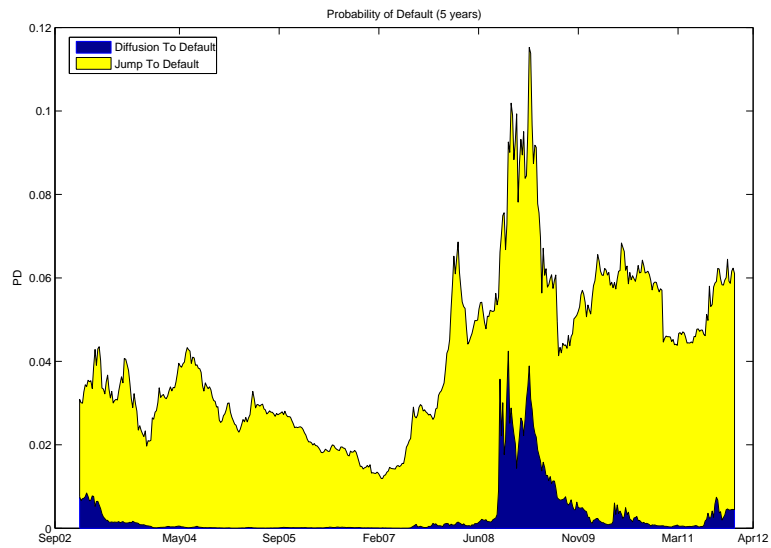
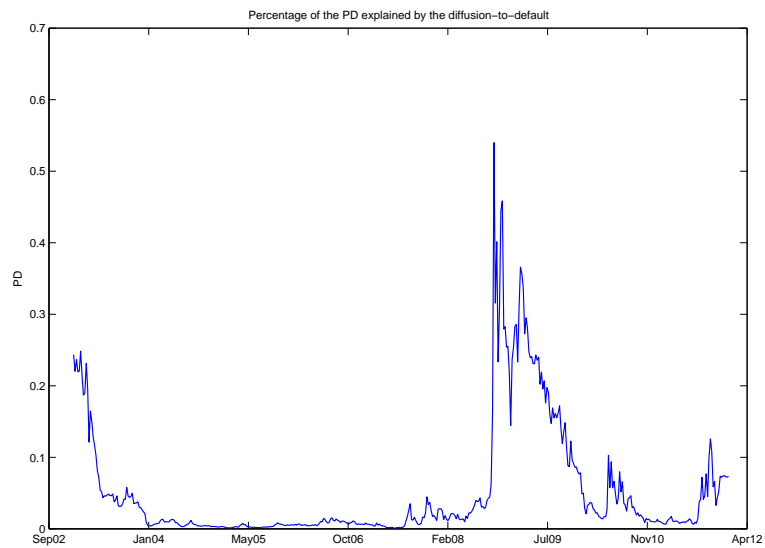Figure 6: Evolution of the diffusion-to-default and jump-to-default probability

Figure 7: Evolution of the ratio between diffusion-to-default and jump-to-default probabilities

Appendix

A Considered structural models

A.1 Merton's Model

The relevant literature in the structural credit risk models field originates from the articles by Black and Scholes [1973] and Merton [1974], in which the liabilities of a company are seen as simple European options written on the asset value process of the firm.

The first and simplest structural credit risk model is due to Merton [1974]. The capital structure of the company is extremely simplified. The debt consists of a single zero-coupon bond with maturity T and face value F. The default event can happen only at the maturity T and not before. Due to the limited liability rule, the debt at maturity will be worth $D(T) = \min(F, V_T) = F - \max(F - V_T, 0)$, where F is the face value and V the market value of the assets. This payoff is equal to a fixed amount of cash F plus a short position in a put option written on V with strike price F. So, in order to avoid arbitrage, the value of risky debt today has to be equal to $D = F \cdot e^{-rT} - p$, where p is the current price of the European put option.

On the other hand, the value of equity in T will be $S_T = \max(V_T - F, 0)$. This is exactly the payoff of a European call option written on V with strike price F. Therefore, by non-arbitrage, the price of equity today has to be equal to the price of the call written on V. This result is based on the set of assumptions that is standard in the Black-Scholes world, that is: perfect markets, continuous trading, constant volatility, deterministic and constant interest rates, infinite liquidity, Ito dynamics for the asset value process. More precisely, the assumption of the model is that V follows a geometric Brownian motion

$$\frac{dV}{V} = (r - \delta)\, dt + \sigma^V dZ \tag{15}$$

and consequently it is possible to price the put option present in the value of debt and the call option present in the value of equity by the standard Black-Scholes formula. Thus

$$S = BS\left(V, \sigma^V, r, \delta, T, F\right) \tag{16}$$

where

- S is the equity price

- r is the deterministic risk free interest rate

- δ is the average payout rate of the total assets of the company

- σ^V is the constant deterministic instantaneous volatility of asset returns

- the function $BS(\cdot)$ indicates the Black-Scholes price for a call option

All the structural models we are considering here give a (closed-form) mapping from the value and volatility of the assets to the value and volatility of the equity. In this simple model the mapping from asset value to equity value is just the standard Black-Scholes call pricing formula. In order to obtain the mapping from σ^V to σ^S we need to compute the instantaneous dynamics of equity. We apply Ito's lemma to equation (16) and get

$$dS = \frac{\partial S}{\partial V} V (r - \delta) dt + \frac{\partial S}{\partial V} V \sigma^V dZ + \frac{\partial S}{\partial t} dt + \frac{1}{2} \frac{\partial^2 S}{\partial V^2} V^2 \left(\sigma^V \right)^2 dt \qquad (17)$$

$$= \left[\frac{\partial S}{\partial V} V (r - \delta) + \frac{1}{2} \frac{\partial^2 S}{\partial V^2} V^2 \left(\sigma^V \right)^2 \right] dt + \frac{\partial S}{\partial V} V \sigma^V dZ \qquad (18)$$

We can therefore obtain the instantaneous volatility of equity from the diffusion coefficient of assets

$$S \sigma^S = \frac{\partial S}{\partial V} V \sigma^V \qquad (19)$$

In the case of the simple Merton's model the derivative on the right hand side of the last equation is just the Black-Scholes delta, that is $N(d_1)$. In more general cases a numerical derivation may be required.

An improvement of this simple framework that we are not considering here is the possibility of a default at every instant of time before the maturity of the debt. This refinement is due to Black and Cox [1976] and is also present in other important articles such as Longstaff and Schwartz [1995]. In these works equity is considered as a barrier option and the default event is triggered at the first hitting time of an exogenously determined barrier. Another factor neglected in Merton's model and developed for instance in Brennan and Schwartz [1978] is the role of corporate taxation and tax shields.

A.2   Merton's Model in presence of stochastic interest rates

Another line of evolution is the assumption of stochastic interest rates in the model. This improvement is achieved by many authors, the most important

of which are Ronn and Verma [1986], Kim, Ramaswamy, and Sundaresan [1993], Briys and de Varenne [1997] and also Longstaff and Schwartz [1995]. In our work we assume that interest rates are stochastically defined using the short-term interest rate model proposed by Vasicek [1977]. In this case the capital structure and the default schedule remain as simple as in the original Merton's model. Therefore, under the risk-neutral measure, the instantaneous interest rates dynamics are given by

$$dr = k(\theta - r)dt + \nu dZ^r \tag{20}$$

or, under a different parametrization,

$$dr = (\zeta - \beta r)dt + \nu dZ^r \tag{21}$$

where Z^r is a standard brownian motion characterized by a correlation ρ with the Brownian motion Z in equation (15) and k, θ, ν, ζ and β are real (and in general positive) constants.

The drawbacks of this simple model are that the possible shapes of the term structure are quite limited and that the probability of negative interest rates, though very small for reasonable parametrizations, is not zero. Nonetheless, the model is still widely used in practice.

According to Rabinovitch [1989], the price of a call option written on V under a stochastic interest rate determined by the Vasicek model is

$$S = VN(d_1) - FPN(d_2) \tag{22}$$

where

- $d_1 = \frac{(\sigma_{1,1} + \sigma_{1,2} - a)}{\sqrt{b}}$
- $d_2 = d_1 - \sqrt{b}$
- $P = e^{\left(\frac{1}{2}\sigma_{2,2} - B\right)}$
- $B = -\frac{1}{k}\left[\left(r_0 - \frac{k\theta - \nu q}{k}\right)\left(e^{-kT} - 1\right) - (k\theta - \nu q)T\right]$
- $a = \sigma_{1,1}/2 - B + \log(F/V)$
- $b = \sigma_{1,1} + 2\sigma_{1,2} + \sigma_{2,2}$
- $\sigma_{1,1} = \sigma^{V^2}T$
- $\sigma_{1,2} = \frac{\sigma\nu\rho}{k}\left(\frac{e^{-kT}-1}{k} + T\right)$

- $\sigma_{2,2} = \frac{v^2}{k^2} \left(T - \frac{3 + e^{-kT} \left(e^{-kT} - 4 \right)}{2k} \right)$

- q is the premium for bearing interest rate risk

Under the same assumptions of Merton [1974] model, we can consider the equity of a company as a call on the assets. Therefore equation (22) gives us the required mapping from assets V to equity S.

A.3 Merton's Model in presence of simple jumps in the asset process

Another possible extension of the simple model for the asset process proposed by Merton is the inclusion of jumps. This extension has been suggested in order to overcome the observed underestimation of credit spreads by standard Merton's model, and in particular the tendency of estimated spreads to converge to zero as the maturity of the debt tends to zero documented e.g. in Lando [2004]. In this way the process $\{V_t\}_{t=1}^{\infty}$ includes two components, a pure diffusion which is still assumed to follow a geometric brownian motion, and a pure jump component. The first implementation of a jump-diffusion model of this form is due to Merton [1976]. Unfortunately a realistic description of a jump diffusion model, characterized for example by the presence of random jumps of infinitely many sizes, does not allow for a closed-form pricing formula like equation (16).

In this work we discuss only models that give a closed-form mapping between V and S. We therefore limit ourselves to a simple characterization of jumps. Only jumps with a constant deterministic size k are allowed. The size k is computed in such a way that, in the event of a jump, the company immediately defaults and the asset value goes to a level compatible with the recovery rate on debt historically observed for the rating group of the company (from Moody's data).

Furthermore, we still assume that δ is the average payout rate of the assets and λ is the (risk-neutral) default intensity parameter. Default event is again possible only at the maturity of the debt.

In this case a very simple closed-form pricing formula for european options is available

$$S = BS\left(V, \sigma^V, r, \delta, T, F\right) \cdot \left(e^{-\lambda T}\right) + e^{-rT} \max\left(V\left(1 - k\right) - F, 0\right) \left(1 - e^{-\lambda T}\right) \quad (23)$$

which gives us the required mapping from assets to equity.

A.4 Leland and Toft [1996] Model

In all the models mentioned until now, the default threshold is exogenously determined. A fundamental improvement in the literature is the endogenization of the default point. In this perspective, the firm optimally and continuously decides whether to default on its obligations or not and consequently dynamically optimizes its capital structure. The first attempt in this direction is due to Leland [1994] who considers only infinite-life debt. A popular extension of this is the model by Leland and Toft [1996] who consider finite-maturity debt and are able to obtain a closed-form solution to the price of risky debt and, consequently, of equity. In this framework also tax advantages and costs of default are considered and have an influence on the optimal capital structure choice of the firm.

The stochastic process V^u for the unlevered value of the assets of the firm is as usually given by

$$\frac{dV^u}{V^u} = \left(\mu\left(V^u, t\right) - \delta\right) dt + \sigma dZ \tag{24}$$

where δ is the average payout rate of the assets. The company has an outstanding debt with finite maturity T and continuously substitutes a fraction of such debt with newly issued bonds. Therefore the rate of substitution is $m = F/T$. The debt also pays a continuous coupon equal to C.

A default happens whenever the process V^u reaches the barrier V_B. Opposite to the Merton's model, the threshold V_B is not exogenous but it is determined by the smooth-pasting condition

$$\left.\frac{\partial S}{\partial V^u}\right|_{V^u = V_B} = 0 \tag{25}$$

This condition drives the choice of the company, which is free to choose at any instant in time whether to default or not on its obligations.

Furthermore, a default event is now possible at any time during the life of the company and not only at debt maturity. Leland and Toft [1996] compute the optimal endogenous threshold V_B as

$$V_B = \frac{\frac{C}{r}\left(\frac{A}{rT} - B\right) - \frac{AF}{rT} - \frac{\tau Cx}{r}}{1 + \alpha x - (1 - \alpha) B} \tag{26}$$

where

- $A = 2ae^{-rT}N\left(a\sigma\sqrt{T}\right) - 2zN\left(z\sigma\sqrt{T}\right) - \dfrac{2}{\sigma\sqrt{T}}n\left(z\sigma\sqrt{T}\right) +$

  $\dfrac{2e^{-rT}}{\sigma\sqrt{T}}n\left(a\sigma\sqrt{T}\right) + (z-a)$

- $B = -\left(2z + \dfrac{2}{z\sigma^2 T}\right)N\left(z\sigma\sqrt{T}\right) - \dfrac{2}{\sigma\sqrt{T}}n\left(z\sigma\sqrt{T}\right) + (z-a) + \dfrac{1}{z\sigma^2 T}$

- $a = \dfrac{r-\delta-\sigma^2/2}{\sigma^2}$

- $z = \dfrac{\left(a^2\sigma^4 + 2r\sigma^2\right)^{1/2}}{\sigma^2}$

- $x = a + z$

- $N(\cdot)$ is the standard normal cdf

- $n(\cdot)$ is the standard normal pdf

- τ is the corporate tax rate

The market value of debt can be shown to be equal to

$$D = \frac{C}{r} + \left(F - \frac{C}{r}\right)\left(\frac{1-e^{-rT}}{rT} - I(T)\right) + \left((1-\alpha)V_B - \frac{C}{r}\right)J(T) \qquad (27)$$

where

- $I(T) = \frac{1}{rT}\left(G(T) - e^{-rT}F(T)\right)$

- $J(T) = \frac{1}{z\sigma\sqrt{T}}\left[-\left(\frac{V^u}{V_B}\right)^{-a+z}N(q_1(t))q_1(t) + \left(\frac{V^u}{V_B}\right)^{-a-z}N(q_2(t))q_2(t)\right]$

- $F(t) = N(h_1(t)) + \left(\frac{V^u}{V_B}\right)^{-2a}N(h_2(t))$

- $G(t) = \left(\frac{V^u}{V_B}\right)^{-a+z}N(q_1(t)) + \left(\frac{V^u}{V_B}\right)^{-a-z}N(q_2(t))$

- $h_1(t) = \frac{-b - a\sigma^2 t}{\sigma\sqrt{t}}$

- $h_2(t) = \frac{-b + a\sigma^2 t}{\sigma\sqrt{t}}$

- $q_1(t) = \frac{-b - z\sigma^2 t}{\sigma\sqrt{t}}$

- $q_2(t) = \frac{-b + z\sigma^2 t}{\sigma\sqrt{t}}$

- $b = \log\left(\frac{V^u}{V_B}\right)$

In the expression for the market value of debt, we can identify three components. The first element is the value of debt that comes from the present value

of all future coupons, the second part comes from the future reimbursement of principal and the last part from the expected recovery rate in the event of default.

We now want to compute the levered value of assets, that is

$$V = V^u + TS - DC \tag{28}$$

where TS indicates the tax savings from debt and DC the costs of default. From Leland [1994] the explicit expression of this last equation becomes

$$V = V^u + \frac{\tau C}{r} \left(1 - \left(\frac{V}{V_B} \right)^{-x} \right) - \alpha V_B \left(\frac{V}{V_B} \right)^{-x} \tag{29}$$

Finally, the value of equity is given by

$$S = V - D \tag{30}$$

which maps the unobserved value of assets V into the observed equity price S.

A.5 Leland [2006] Model

The most recent literature tries to combine the endogeneity of default events with the richest possible jump structure. The most important results in this direction have been obtained for instance by Hilberink and Rogers [2002], Leland [2006] and Chen and Kou [2009].

The model we implement here is Leland [2006], which is very similar to Leland and Toft [1996] with the only inclusion of a jump component in the unlevered firm value $\{V_t^u\}_{t=1}^{\infty}$. Only jumps with deterministic size $k \in [0,1]$ are allowed. k is such that in the event of a jump the company defaults. The diffusion component is a geometric brownian motion with a drift compensated for the presence of jumps in such a way that the expected instantaneous return is still $E\left(dV^u/V^u \right) = (r - \delta)\, dt$.

In particular, V^u solves the following SDE

$$\frac{dV^u}{V^u} = \begin{cases} (r - \delta + \lambda k)\, dt + \sigma dZ & \text{with probability } (1 - \lambda dt) \\ -k & \text{with probability } (\lambda dt) \end{cases}$$

where λ is the jump-to-default intensity parameter.

In this simplified jump-diffusion framework jumps are rare events that immediately drive the firm to bankruptcy. Of course, defaults can still occur also as a consequence of the diffusion process and in absence of jumps. This choice is also justified by the empirical evidence, according to which an abrupt jump to default is quite a rare event (See e.g. Collin-Dufresne, Goldstein, and Helwege [2003]).

It is also assumed that the maturity of the debt is infinity but it is continuously repaid at a proportional rate m. This means that the average maturity of the outstanding debt is $T = \int_0^\infty t\,(me^{-mt})\,dt = 1/m$. The model can also include default costs equal to α. Repaid debt is continuously replaced by new issues with the same characteristics as the retired ones.

The company optimally decides the value V_B of the asset process that triggers the default event. In particular V_B is the value of the assets that satisfies the smooth-pasting condition

$$\frac{\partial S}{\partial V^u}\bigg|_{V^u=V_B} = 0 \tag{31}$$

which can be proved to be

$$V_B = \frac{\frac{(C+mF)y(z_1)}{z_1} + \frac{\lambda(1-k)Vy(z_2)}{z_2} - \frac{\tau Cy(z_3)}{z_3}}{1 + (1-\alpha)\,y\,(z_1) + \alpha y\,(z_3)} \tag{32}$$

where

- $y(z) = \frac{\left(g-(1/2)\sigma^2\right)+\left(\left(g-(1/2)\sigma^2\right)^2+2z\sigma^2\right)^{0.5}}{\sigma^2}$

- $z_1 = r+m+\lambda$

- $z_2 = z_1 - g$

- $g = r-\delta+\lambda k$

- $z_3 = r+\lambda$

Therefore, as in Leland and Toft [1996], a default can occur whenever V_B is reached and therefore at any time before the maturity of the debt.

It is possible to show that the value of the debt is equal to the sum of three components: the debt value conditional on no default, the debt value conditional on diffusion to default and the debt value conditional on jump to default. Therefore

$$D = \int_0^\infty e^{-rt} \left[e^{-mt} \left(C + mP \right) \right] \left(1 - F\left(t \right) \right) e^{-\lambda t} dt$$

$$+ \int_0^\infty e^{-rt} \left[e^{-mt} \left(1 - \alpha \right) V_B \right] f\left(t \right) e^{-\lambda t} dt$$

$$+ \int_0^\infty e^{-rt} \left[e^{-mt} \left(1 - k \right) e^{gt} V \right] \left(1 - F\left(t \right) \right) e^{-\lambda t} dt \qquad (33)$$

where $F\left(t \right)$ and $f\left(t \right)$ are, respectively, the cumulative distribution function and the probability density function of the first passage time of the process V^u to V_B.

Solving the three integrals (see Leland [1994]) we get the market value of debt

$$D = \frac{C + mF}{z_1} \left(1 - \left(\frac{V^u}{V_B} \right)^{-y(z_1)} \right) + \left(1 - \alpha \right) V_B \left(\frac{V^u}{V_B} \right)^{-y(z_1)}$$

$$+ \frac{\lambda \left(1 - k \right) V^u}{z_2} \left(1 - \left(\frac{V^u}{V_B} \right)^{-y(z_2)} \right) \qquad (34)$$

The next step is to consider the tax savings from debt (TS) and the default costs (DC). Taking these elements into consideration, we can find, in analogy with the Leland and Toft [1996] model, the *levered* value of the asset process, which is

$$V = V^u + TS - DC \qquad (35)$$

where

- $TS = \frac{\tau C}{r + \lambda} \left(1 - \left(\frac{V^u}{V_B} \right)^{-y(z_3)} \right)$

- $DC = \alpha V_B \left(\frac{V^u}{V_B} \right)^{-y(z_3)}$

- $z_3 = r + \lambda$

Finally, the value of equity is given by

$$S = V - D \qquad (36)$$

This is the mapping from assets to equity we are looking for.

In this work we implement a filtering technique and therefore we explicitly estimate the latent asset value process. In the original work by Leland [2006] the actual estimate of the latent process V is not performed and a simple estimate of the price of debt, and consequently the assessment of the ability of the model to explain the observed term structure of corporate spreads, is performed. The original work by Leland [2006] also introduces an exogenously determined liquidity premium of 60 bps. In our case we ignore this, since we are working with CDS, a synthetic instrument, and not with corporate bonds, which are subject to stronger liquidity issues.

B Unscented Kalman Filter

If the functions f and h of the state space model in equations (6) and (7) are linear then the universally accepted estimation technique for the problem is the Kalman filter. In our case, however, the function f is actually assumed to be linear but unfortunately h is not. Therefore a more sophisticated estimation technique is required.

We can identify at least two big families of algorithms that solve our problem. The first one tries to linearize the function h around the current value of the state vector X_t. This is usually done through a Taylor approximation arrested at the first order. This family of methods is usually called Extended Kalman Filter.

A second family of techniques does not try to approximate the function h but rather the transition density function of the state vector. A consequence of this is that the entire approximation procedure is derivative-free. The approximation is carried out by simulating a set of points χ_i (called σ-points). To each of these points a weight w_i is assigned according to the likelihood of the simulated values. The set of all the couples $\{\chi_i, w_i\}$ approximates the pdf of the states. This technique is usually called Unscented Kalman Filter (UKF) and can be summarized in the following steps. For a more detailed presentation see e.g. Simandl [2006] or Bar-Shalom, Li, and Kirubarajan [2001].

Step 1

The first step in the UKF algorithm is the definition of an initial state vector x_0 with known mean $\mu_0 = E[x_0]$ and known covariance $P_0 = E\left[(x_0 - \mu_0)(x_0 - \mu_0)'\right]$.

Step 2

We then define the first set of σ-points χ_{k-1} and their corresponding weights as

$$
\begin{aligned}
\chi_0 &= \mu_0; \; w_0 = \frac{\xi}{n+\xi} \\
\chi_i &= \mu_0 + \left(\sqrt{(n+\xi)P_0}\right)_i; \; w_i = \frac{1}{2(n+\xi)} \\
\chi_j &= \mu_0 - \left(\sqrt{(n+\xi)P_0}\right)_{j-n}; \; w_j = \frac{1}{2(n+\xi)}
\end{aligned}
\tag{37}
$$

for $i = 1, ..., n$ and $j = n + 1, ..., 2n$ and where the i or j index represents the ith or j^{th} column of the matrix $\sqrt{(n + \xi) P_0}$. ξ is a scaling parameter which determines the dispersion of the σ-points around their mean and is function of three other constants. We select these constants, and consequently ξ, according to the most frequent choice in the literature (see e.g. Wan and der Merwe [2000] for a detailed discussion of the choice).

Step 3

After that we apply the measurement function $h(\cdot)$ and get a set of simulated observable variables

$$Z_k = h_k(\chi_k) \tag{38}$$

We use these points and the corresponding weights to estimate the predictive statistics of the measurement variables

- $\hat{z}_k$, the average of the measurement

- $P_{z,k}$, the covariance matrix of the points Z_k

- $P_{xz,k}$, the covariance between the points χ_k and Z_k

Step 4

Analogously to the standard Kalman gain step, we update the state estimate with the last measurement z_k

$$
\begin{aligned}
\hat{x} &= \hat{x} + K_k(z_k - \hat{z}_k) \\
P_k &= P_k - K_k P_{z,k} K'_k
\end{aligned}
\tag{39}
$$
$$\tag{40}$$

with $K_k = P_{xz,k}(P_{z,k})^{-1}$

Step 5

We then define a new vector of sigma points as

$$\begin{aligned}
\chi_{0,k} &= \hat{x}_k; \ w_0 = \frac{\xi}{n+\xi} \\
\chi_{i,k} &= \hat{x}_k + \left(\sqrt{(n+\xi)P_k}\right)_i \ ; \ w_i = \frac{1}{2(n+\xi)} \\
\chi_{j,k} &= \hat{x}_k - \left(\sqrt{(n+\xi)P_k}\right)_{j-n} \ ; \ w_j = \frac{1}{2(n+\xi)}
\end{aligned}$$

$$\text{(41)}$$

with this new set of σ-points we set $k = k+1$ and go back to step 3. We repeat until $k = n$.

Of course, if the parameters of equation (8) are unknown and need to be estimated, we have to consider them as unknown states and therefore they have to be added to the vector of states X. This is the case of our problem, since we do not know the value of the parameters in the transition matrix A of equation (8).

It is possible to implement also another and more efficient procedure, the so-called Square-Root UKF. The algorithm is very similar to the presented one, the only difference is that we do not propagate the covariance matrix P_x but its Cholesky decomposition S_k with $P_k = S_k S_k'$.

It is interesting to notice that this method does not involve any type of numerical inversion of the functions f or h. This completely eliminates numerical issues due to the numerical search of the zeros of the functions.

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Liquidity and limits of arbitrage in structural credit risk models. [*]

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Abstract

In this paper we assess the performances of structural credit risk models in explaining corporate bond spread variations. We confirm that the explanatory power is in general low but it increases when the observational frequency is reduced, supporting the claim that a transitory noise is present and affects the performances of structural models. We confirm this by verifying that the errors left by structural models significantly depend on standard liquidity indicators, from equity and bond market. We finally claim that the variability left in the residuals left by both structural models and liquidity indicators can be considered as a measure of limits of arbitrage and verify that it actually depends on global indicators of market frictions such as the TED spread or the LiborOIS.

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1 Introduction

Structural credit risk models are well known and widely investigated in finance since the seminal works by Black and Scholes [1973] and Merton [1974]. The basic idea is the use of contingent claim pricing theory for the valuation of corporate liabilities. According to this approach, equity and debt are seen as derivative contracts written on the unobserved assets of the company and can be priced accordingly. Due to the limited liability rule, for instance, equity can be interpreted as a long call option written on the assets while debt is equivalent to a certain amount of risk-free bonds plus a short position in a put option on the assets.

Different structural credit risk models have postulated different stochastic processes for the evolution of the unobserved asset values and different characteristics of equity and debt in terms of maturity, generated cash flows and seniority. The original assumption of geometric Brownian motion for the evolution of the market value of assets has been substituted by other hypotheses, including processes with non-continuous trajectories and the presence of jumps. Also the presence of a fixed debt maturity and the possibility to default only at that time, originally present in this type of models, has been replaced by more realistic assumptions, such as the possibility to default before maturity. An improvement has been obtained also for the contract characteristics of debt. The original hypothesis of bullet zero-coupon reimbursement, for instance, has been replaced by the more flexible assumption of continuous rolling of the debt during the entire life of the company.

Finally, the exogeneity of the default threshold, given only by the face value of the outstanding debt, has been replaced by an endogenously determined threshold and a framework in which the company optimally and continuously decides whether to default or not. Consequently many different pricing formulae have been proposed for corporate equity and debt, one for every model specification.

Several tests have been carried out in order to verify the performances of structural models in the prediction of corporate spreads. One of the first is Jones, Mason, and Rosenfeld [1984] and another fundamental one Eom, Helwege, and Huang [2004]. A common result of such tests is the substantial inability of structural models to explain observed credit spreads and in particular the general underestimation of observed credit spreads when calibrated on historical equity prices and volatilities. For some models, also an overestimation of the spreads is observed and in general the performances of structural credit risk models are found to be poor. In the literature, such failure is commonly referred to as "credit spread puzzle", since it appears in striking contrast with the conceptual soundness, elegance and plausibility of

the considered models.

A first motivation of this work is a new extensive and model-free test of structural models based on one of the largest available datasets of real transaction corporate bond prices. We claim the test to be model-free since we do not impose a particular pricing form for equity and debt but we simply test a linear dependency, compatible with a generic first order approximation. In other words, we are not testing if, for instance, the Merton's model works but simply if the relevant state variables proposed in the model (namely equity price and equity volatility) have an impact on debt pricing. The considered period appears particularly interesting since it includes a relatively stable period of financial markets before the crisis and the recent stressed period.

A second motivation is the investigation of the reasons for the failure of structural models. We argue that the reasons are primarily large and persistent imperfections in equity and debt markets. We verify that the errors made by structural models can partly be justified by the most commonly used liquidity indicators, which are able to explain a non-negligible part of the residuals left by structural models. We can argue that the role of illiquidity mainly comes from the credit market, which is characterized by worst liquidity conditions with respect to the equity market (see section 5.1.2 for an heuristic confirmation of this) but a rigorous proof of this conjecture appears more difficult. This finding substantially confirms the results in Barone-Adesi and Borghi [2012], where the removal of observational noise for the estimation of the unobserved assets were achieved by nonlinear filtering and the improvement on standard inversion techniques, even if not extreme, was documented.

We finally verify that the variability left in the residuals after removing also liquidity friction proxies strongly depends on market-wide indicators of trading frictions or global liquidity conditions. This finding supports the claim according to which liquidity frictions and limits of arbitrage contribute to the poor performances of structural credit risk models.

The current work focuses on the assessment of the performances of structural credit risk models in explaining the *variability* in credit spreads. The credit spread puzzle originates looking at the performances of such models in explaining the *levels* of spreads. On the other hand, good performances of structural models have been documented (see e.g. Shaefer and Strebulaev [2008]) in the prediction of hedge ratios of corporate bonds in terms of equity. We can argue that the error in levels can be a consequence of the risk aversion of agents on the market. In particular, the main factors that can explain the evidence are the following. The first is the presence of illiquidity risk premia required for buying corporate bonds (a typically illiquid asset class, see section 5.1.2). This will increase the expected returns and consequently spreads on such securities by a component that cannot be justified by the information

present in the corporate structure of the company. The second type of risk premium is required because the non-arbitrage relationships between the prices of corporate liabilities are enforced by strategies that in general are not risk-free (see below) and therefore require a premium. A third reason is that the mentioned convergence strategies are based on contracts (e.g. Repos) that may involve counterparty risk and therefore require an additional premium. Another argument (proposed by Elton, Gruber, Agrawal, and Mann [2001]) is that the observed difference in spreads can be generated by different taxation between corporate and government bonds. A last possible reason is that a remuneration for systematic undiversifiable risk is present in corporate bond yields. A less clear theoretical explanation is available for the poor performance of the model in explaining the variability of credit spreads and here we concentrate in this topic.

The paper is organized as follows. Section 2 briefly reviews the existing literature concerning structural models, their testing and the impacts of liquidity and limits of arbitrage issues on market pricing. In section 3 we describe the theoretical framework behind structural models and how liquidity deterioration and limits of arbitrage can affect their performances. In section 4 we describe the dataset used for our analysis while in section 5 we present how the analysis is performed and the results obtained. Section 6 concludes.

2 Review of the literature

This paper mainly draws on three streams of literature. The first one is the classical literature about structural credit risk models and the related empirical tests, the second is related to liquidity indicators and the last one is about limits of arbitrage.

Beyond the already mentioned works by Black and Scholes [1973] and Merton [1974], the evolution of structural credit risk models can be represented by five main steps of development. The first is the work by Black and Cox [1976], where the hypothesis to default only at the maturity of the debt is removed and default at any time is allowed through the use of barrier instead of plain vanilla options to model corporate liabilities. A second step is Brennan and Schwartz [1978], where the role of corporate taxation in the valuation of equity and debt is taken into account. A third step in this evolution is represented by the introduction of stochastic interest rates, which is presented for the first time by Ronn and Verma [1986]. The final two major improvements in this field are due to Leland [1994], where the exogenous default barrier is replaced by a threshold optimally and endogenously determined by the company, and Leland [2006], where the optimally selected default point is combined with the presence of jumps in

the stochastic evolution of asset values.

As far as empirical tests are concerned, we can find the first articles at the beginning of the '80s with the works by Jones, Mason, and Rosenfeld [1983], where the the performances of the models are for the first time questioned. A common finding is that structural models are underestimating observed bond spreads and that the error is particularly large for high-rated companies. The works by Ogden [1987] and Lyden and Saraniti [2000] further investigate the question and identify a large underestimation of spreads, between 90 and 105 bps. Elton et al. [2001] confirm these results and identify in the different taxation on corporate and government bonds another determinant of the underestimation of spreads and propose that the remaining unexplained part of the spreads is a remuneration for systematic risk, dependent on the Fama and French [1992] factors.

Also more recent works, such as Huang and Huang [2003], confirm the difficulties of the models in the explanation of corporate spreads. On average, they are able to explain less than the 30% of spread levels. The unexplained component is identified with a "liquidity" factor. Eom et al. [2004] confirm the underestimation of spreads for the majority of structural models but for some of them (e.g. Longstaff and Schwartz [1995] and Leland and Toft [1996]) an overestimation is actually identified. Large errors are in any case documented.

A final paper worth mentioning is Collin-Dufresne and Goldstein [2001]. This work is quite in line with this paper and tries to explain the changes in bond spreads by means of structural variables. The main innovations we are proposing with respect to this piece of work are the following: the use of actual transaction data instead of market makers' quotes, the use of company-specific structural variables instead of market indexes for equity and volatility, the use of company specific liquidity indicators for equity and bonds and finally the relationship with limits of arbitrage. The use of company-specific structural variables allows us to obtain a better fit to spread changes but the poor performances of structural models is generally confirmed.

The literature related to liquidity indicators is in general designed for the equity market, which is mainly based on established exchange trades and therefore provides rich information about transaction prices and volumes. The corporate bond market is mainly OTC and therefore less data is usually available. However, provided that specific information about single transactions is available, one can easily extend the equity literature to credit market. Many liquidity indicators have been proposed in the literature (see Goyenko, Holden, and Trzcinka [2009] for a nice review). Apart from intuitive indicators such as the number of trades, here we are concentrating on the one proposed by Amihud [2002].

The theory of limits of arbitrage is investigated in many articles, which date back to the '90s. One of the most important works in this field is Shleifer and Vishny [1997]. In this article, the classical assumptions about riskless and costless arbitrage strategies are removed and it is shown that in the real world an arbitrage strategy entails upfront costs from the arbitrageur and is not in general free of risk. The presence of a limited number of professional arbitrageurs working under an agency relationship with investors is also shown to generate, in crisis periods, strong deleveraging pressures on funds, which are forced to close their arbitrage strategies before the prices return to equilibrium.

The financial constraints affecting arbitrageurs are also discussed in Gromb and Vayanos [2002]. Here a model is proposed for such agents and it is shown that, in some circumstances, arbitrageurs can fail to act optimally and consequently the market can be allocatively inefficient and prices out of equilibrium can persist.

The impact that agency relationships between investors and arbitrageurs have on limits of arbitrage is also investigated in Acharya and Viswanathan [2011]. The model proposed in the article explains changes in market liquidity due to agency relationships and formalizes the deleveraging process of funds and financial institutions observed in stressed periods.

The amplification of price changes due to the unwinding of positions operated by intermediaries is also modeled in and Brunnermeier and Pedersen [2009]. In this framework, the behaviours of liquidity conditions and their impacts on limits of arbitrage are also captured by the model. For a more detailed review of the literature and an alternative model, see Gromb and Vayanos [2010].

We contribute to the existing literature by performing a wide test on the performance of structural credit risk models in the pricing of corporate bonds based on more than 4.000 securities for which we are able to obtain implied volatilities, bond and equity prices.

In a second step we investigate the errors of structural models and clearly identify a role of liquidity indicators in their explanation. A role of limits of arbitrage in the explanation of errors is also identified.

3   Theoretical framework

Structural credit risk models assume that the default of a company happens when the unobserved market value of its assets V falls below a certain threshold.

In the first and simplest model, Merton [1974], the capital structure of the company is extremely simple and consists only in equity and a zero-coupon bond with maturity T and face value F, whose current market value is $D = Fe^{-yT}$, where y is the appropriate yield-to-maturity. The assets of the company follow a geometric Brownian motion and the default can happen only at maturity T. The default event takes places when the market value of assets at maturity (V_T) is smaller than the face value of debt (i.e. $V_T < F$). Consequently, the value of equity at maturity is $\max(V_T - F; 0)$, which is exactly the payoff of a call option written on the assets of the company with strike equal to the face value of the debt. The payoff of debt at maturity is therefore $\min(F; V_T)$, which can be expressed as the payoff of a short put option on company's assets plus some risk-free debt. Consequently, and by non arbitrage, also the values today of debt and equity can be expressed using call and put option prices.

In this simple case, the default threshold is just given by the face value of liabilities while in other more sophisticated models the threshold can be a more complicated function of company parameters and can be even endogenously determined (see e.g. Leland [2006]).

In general, all structural credit risk models are able to express the market value and volatility of equity as increasing functions of the unobserved market value and volatility of assets, possibly contaminated by observational (e.g. trading) noise. Defining S_t and σ_t^S the equity price and equity volatility at time t, respectively, we can write

$$
\begin{aligned}
S_t &= h_1\left(V_t, \sigma_t^V, \psi_t\right) + \varepsilon_{1,t} \\
\sigma_t^S &= h_2\left(V_t, \sigma_t^V, \psi_t\right) + \varepsilon_{2,t}
\end{aligned}
\tag{1}
$$

where σ_t^V is the asset volatility and ψ_t is a vector of exogenous parameters such as the risk-free interest rate, the maturity of the debt, etc.

In the simplest Merton's case, h_1 is just the Black-Scholes price of a call and h_2 can be recovered by a straightforward application of Ito's lemma and is $\sigma^S = \frac{\partial S}{\partial V} \frac{V}{S} \sigma^V$. $\varepsilon_{i,t}$, $i = 1, 2$ are the white noise residuals. An obvious consequence of system (1) is that also the market value D of debt can be expressed as a function of asset value and volatility, i.e.

$$
D_t = h_3\left(V_t, \sigma_t^V, \psi_t\right) + \eta_t
\tag{2}
$$

where η_t is again white noise and, in Merton's case, h_3 is equal to a constant minus the Black-Scholes price of a put. The function h_3 in equation (2) is

still increasing in V_t but decreasing in σ^V, since the debt is equal to a *short* position in a put on the assets.

Under invertibility of functions h_1 and h_2, or equivalently if the nonlinear system (1) admits a unique solution in V and σ^V, then it is possible to express the market value of debt as

$$D_t = g\left(S_t, \sigma_t^S, \psi_t\right) + \eta_t \tag{3}$$

where the function g is increasing in S_t and decreasing in σ_t^S.

Different structural credit risk models postulate different functions h and g but must share the same signs in the dependency of debt on equity and equity volatility.

From equation (3) we deduce that, in a frictionless market, the relationship between corporate bond and the corresponding equity returns should hold by non-arbitrage. When the equity of a company is relatively overpriced (underpriced) with respect to the corresponding corporate bonds, an arbitrageur can take advantage of the situation by short selling (buying) the equity and buying (selling) the bond.

Of course, some limitations to this theoretical framework are induced by the complex capital structure of real-world firms, where there are many different types of shares (ordinary, non-voting, redeemable, management, etc) and a much more complicated debt structure (loans, bonds, hybrid instruments with many different characteristics in terms of maturity, coupon, seniority and privileges) is present.

Furthermore, practical limitations to the achievement of the theoretical equilibrium between the prices of corporate liabilities may be caused by the factors mentioned in the introduction, that is

- Liquidity risk premia

- Premia to remunerate risky arbitrage

- Counterparty risk

- Different taxation for government and corporates

- Remuneration of systematic risk in corporate bonds

Such forces, affecting the levels of spreads together with liquidity constraints or the presence of limits to arbitrage forces may transform a theoretically risk- and cost-free arbitrage into a risky and expensive (at least upfront) strategy.

3.1 Liquidity

The possibility to exploit an arbitrage opportunity clearly depends on the liquidity conditions of the considered markets (equity and corporate bonds in this case). An arbitrage opportunity may be theoretically present but the volume traded on at least one of the two markets may be so small that a practical implementation becomes impossible. Similarly, the bid-ask spread for the securities may be so large that the theoretical arbitrage opportunity vanishes in the practical implementation. A strictly related aspect is the price impact of a unitary trade in a market. If the market is very thin a theoretical arbitrage opportunity may disappear when an agent tries to exploit it and negotiates a relatively large volume. All such liquidity conditions are market specific and can be measured by the indicators described below.

The choice to use the TRACE database, which reports the transaction volumes trade by trade, and the CRSP database, which includes the daily traded volume of each security, allows us to build powerful indicators of the liquidity in the market and therefore to verify if these measures are able to explain the deviations from the equilibrium postulated by structural models. We are implementing a set of liquidity indicators to explain the errors in structural credit risk models.

A first simple and intuitive indicator of liquidity is the number of trades of every bond in a given time period.

Another useful (il)liquidity indicator is the one proposed by Amihud [2002] and given by

$$Illiq_{i,t} = \frac{1}{n} \sum_{j=1}^{n} \frac{|r_{i,j}|}{Vol_{i,j}} \tag{4}$$

where

- n is the number of trades observed in the period on which the measure is computed

- $r_{i,j}$ is the return of security i generated by trade j

- $Vol_{i,j}$ is the volume of security i in trade j.

The index can be interpreted as the impact of a unitary transaction on the absolute return of the security. In the case of very liquid securities, also large traded volumes have small impacts on the absolute return and the illiquidity index will therefore be close to zero. For illiquid securities, on the other hand, even small traded volumes have a large impact on the realized returns, which translates into a large *Illiq* index.

We are able to compute the index for both equity and bonds. In the first case, we are able to compute the original index proposed by Amihud [2002], i.e. based on transaction-by-transaction data. In the case of equities, on the other hand, transaction-level data are not available in CRSP, and we are therefore forced to rely on daily returns and volumes, which may reduce the effectiveness in the assessment of liquidity conditions.

We also propose a "systematic Amihud" index, computed, for every considered period, as the cross-section average of the single bond-specific Amihud measures. This last computation is performed by trimming the 2.5% extreme observations on each side of the distribution. This is justified by the presence of a small number of outliers characterized by extremely small traded volumes, which may bring the index to infinity and therefore affect the reliability of the estimation. The evolution of the index is in figure 1. The illiquidity spike that started in 2007 is evident and during the worst phases of the crisis a 1 mln trade could have had a price impact almost equal to 9%. This seems particularly relevant when we consider that this is the *average* impact of a trade. During the peak of the crisis, 4.81% of 1 million trades could have generated returns above 20% and 3.02% even larger than 40%. Starting from the second half of 2009 the liquidity conditions, measured in terms of price impacts, are in general very good, and even better than during the pre-crisis period.

We are excluding the indicator proposed by Lesmond, Ogden, and Trzcinka [1999] because their measure comes from the estimation of a full market model for the behaviours of investors, able to explain different liquidity conditions in the market and this may introduce too much structure in the estimation. We also tried to implement the implicit bid-ask spread proposed by Roll [1984], who relates the measure to the observed first-order autocovariance of returns computed trade-by-trade. The reason for the exclusion is that the observed autocovariance is actually zero or negative in a non-negligible number of observations, which makes impossible the use of the indicator. A possible reason of this behaviour is that the first hypothesis of the model, i.e. the informational efficiency of the market, does not hold. Another widely used liquidity indicator is the one proposed in Pastor and Stambaugh [2003], which consists in the coefficient of a regression of returns separately performed every month and tries to measure the reverse of the previous day order flow shock. We do not implement such indicator since it is available only on relatively low frequencies (typically monthly) in order to estimate the regression in a reliable way, while in our work we take into consideration also higher frequencies.

An important work with a specific focus on corporate bond market liquidity is Dick-Nielsen, Feldutter, and Lando [2012]. Here a new liquidity indicator is proposed and is essentially a linear combination of the most important

liquidity measures used in this study. Since the new factor is statistically built in a principal component framework, its specification can vary over time and is in general different if estimated in two different dates. In order to avoid such time inconsistency and to reduce the loss of information coming from the concentration of several indicators into one, we just consider as inputs the original and most important liquidity measures.

The described liquidity indicators are strictly linked to the market in which they are measured (equity and bond in this case) and are in general strongly influenced by the traded volume of a security. Other limits of arbitrage can originate outside equity and bond markets and therefore cannot be captured by the proposed indicators. Such events are described in the next section.

3.2   Limits of Arbitrage

It is known at least since the work by Shleifer and Vishny [1997] that the existence of a fully risk-free and costless arbitrage is extremely rare since many frictions are present and strongly limit arbitrage opportunities.

The entire theory behind structural credit risk models is actually based on the presence of arbitrage forces that keep the market value of assets and all types of liabilities in equilibrium. The typical convergence trading strategies that should guarantee the equilibrium between equity and corporate bond prices and returns are long-short strategies. Such strategies, in their practical implementation, require an initial amount of cash needed to enter into repo and reverse-repo contracts.

In particular, the logical schemes of the arbitrage strategies are represented in figures 2 and 3. Let's suppose that the bond of a company is underpriced with respect to the corresponding equity. In this case the arbitrageur would buy the underpriced bond and sell the equity (for simplicity, we don't represent a third position, required in a risk-free bond). Clearly, the arbitrageur in general does not hold the equity and therefore needs to enter into a reverse repo contract (right-hand side of the picture). If we assume that the cash flows from buying the bond and selling the equity are equal, it is clear that the repo contract has to be financed with new cash. This is usually achieved via a repo contract in which the bonds just bought are pledged as collateral.

Both the repo and reverse repo contracts entered include some haircuts. In the repo, the amount received versus the pledged bond is less than its market value in order to protect the counterparty from a sudden decrease of the bond price realized before a margin call for new collateral can be fulfilled. In the reverse repo contract, the market value of equity is less than the cash provided as collateral in order to protect the counterparty from a sudden

increase in equity price, realized before a margin call for new cash can be completed.

Two types of repos are usually available: the overnight and the term repo. The first is the most frequently used and is particularly common in the north American market. It needs to be rolled over every day until the strategy results in the expected profits. Commonly, no margin call is required due to the short life of the contract. Since the contract needs to be rolled over every day, also the cash required to finance the haircuts need to be found every day. The second type of contract is characterized by a longer horizon (usually two weeks). In this case no daily roll-over is required but if the underlying asset price moves in the adverse direction a possible margin call for new collateral might take place. In both cases, cash has to be provided upfront and possibly during the entire life of the strategy. Such money should come from the equity of the arbitrageur or needs to be financed on the debt market. In particularly stressed periods, the possibility to find such resources might become extremely expensive or even impossible. This means that the possibility to implement a convergence strategy crucially depends on the liquidity conditions of the interbank market. If such market dries up, strong limitations to convergence strategies may appear.

A second possible limitation to the mentioned strategies can derive from the slow convergence between the prices of the considered securities. We can assume that in the long run the price of bonds and equities can converge to equilibrium but there is no guarantee about the time in which such convergence will happen and in the short run the divergence may even increase indefinitely. In practice, arbitrageurs are usually large fund managers that are subject to funding constraints and that are in an agency relationship with investors. If they implement a convergence strategy but in the short run the underlying securities diverge even more from their equilibrium a loss is realized by the fund. The investors may rationally believe that the loss is a consequence of the poor skills of fund managers and may therefore withdrawn their funds. As a consequence of this decision, fund managers are forced to close their convergence strategies, which makes the divergence between prices even larger. This phenomenon may also be amplified by an increased risk aversion of investors, which may want to close their investments in risky assets and concentrate more funds in cash-like investments.

A reduced liquidity in the interbank market, an increased deleveraging pressure on arbitrage funds from investors together with an increased risk aversion may therefore seriously reduce the effectiveness of capital structure arbitrage and therefore reduce the performances of structural credit risk models, which are based on the assumption of equilibrium between the market values of assets and all types of liabilities.

It is also useful to notice that in periods of extreme financial stress, such as

the last financial crisis, all the three mentioned characteristics are present and therefore we can theoretically expect a worst performance of structural credit risk models.

4   Description of the Data

The first component of our data set is the TRACE (Trade Reporting and Compliance Engine) database. This database has been introduced in July 2002 by FINRA (Financial Industry Regulatory Authority) and includes over 65 million OTC transaction data for corporate bonds exchanged on the North American market in the period 2002-2011. The total number of traded bond issues is 71.000 but we are able to safely assign issue-specific characteristics (maturity, coupon, etc.) to only 24.475. The corresponding number of issuer entities is 5.198.

We take equity data from the University of Chicago Center for Research in Securities Prices (CRSP). We rely on the daily security master file, which includes more than 10.000 companies for the period from September 2000 to December 2011. We are using three fields from this database: the stock price (PRC), the conversion factor for equity price (CFACPR) accounting for stock splits and other corporate actions and finally the volume traded per day.

Implied volatility data is taken from Ivy Optionmetrics Database, which includes over 55.000 options over the period 2001-2011. We use the volatilities implied in the longest available maturities (730 day) taking the average over all strikes for both calls and puts.

The intersection between the 3 databases is performed using the CUSIP code, whose first six digits univocally define the issuer of any security. In the final dataset we managed to collect 596.757 observations with weekly frequency. Given the fact that TRACE observations are trade-by-trade and that more than one trade per day can take place, the data we are using in our analysis is an average of all the observed trades weighted on the volume of each trade.

A final input are the term structures of risk-free interest rates. We use for this purpose the zero curve computed by Datastream for all maturities between one month and fifty years. The considered bonds are denominated in four currencies: USD, EUR, GBP and CAD and therefore we use the corresponding four risk-free curves. Nevertheless, given the fact that the TRACE database is based on US transactions, we notice that the vast majority of bonds are denominated in USD, and only few issues are in the other 3 currencies.

In table 1 we present the distribution of the observed bond prices according to their rating.

Rating	%
AAA	0.78%
AA+	0.32%
AA	2.37%
AA-	4.14%
A+	7.49%
A	13.44%
A-	9.84%
BBB+	10.95%
BBB	14.69%
BBB-	10.19%
BB+	3.79%
BB	5.19%
BB-	5.41%
B+	3.25%
B	2.01%
B-	1.34%
CCC+	0.54%
CCC	0.2%
CCC-	0.01%
CC	0.04%
Defaulted	0.01%
NR	3.98%

Table 1: Distribution per rating of the weekly observations

The time frequency we are using is both weekly and monthly. We use both the last observation of every week and a weekly average. In the first case, we identify the last trading day of a specific bond and look for the equity price and implied volatility at that specific day. In a second dataset we just use weekly averages of bond prices, equity prices and implied volatilities. The use of point observations has the advantage of using all the available information at a certain time without the effects of old data. The use of weekly averages, on the other hands, has the advantage to potentially reduce some noise present in our database of bond transactions. We present both results but their differences are not particularly relevant. For the monthly frequency, we take the last observed price of every month and the corresponding equity prices and implied volatilities at that specific date.

5 Empirical Analysis

Consistently with the literature (see e.g. Collin-Dufresne, Goldstein, and Martin [2001]), we test spread differences and not levels. This is also justified by the characteristics of credit spreads, very close to non stationary unit roots, which may induce spurious results in the regression analysis we are performing

The first step of our analysis is the assessment of the performances of structural credit risk models in the explanation of price movements in corporate bonds, which is equivalent to a test of the performances of equation (2). As explained above, we do not test the exact nonlinear equation in (3) but we linearize it for a generic structural credit risk model and therefore test

$$r_{D,t} = \alpha + \beta_1 r_{S,t} + \beta_2 r_{\sigma^s,t} + \varepsilon_t \tag{5}$$

where $r_{D,t}$, $r_{S,t}$ and $r_{\sigma^s,t}$ are the returns of the risky bond spread, the corresponding equity and implied volatility, respectively.

Many different choices can be made for the measure of credit returns. The first possibility is to take the excess returns at every observation point, i.e.

$$r_{D,t} = \left(\frac{P_t + I_t}{P_{t-1} + I_{t-1}} - 1 \right) - \left(\frac{P_t^* + I_t^*}{P_{t-1}^* + I_{t-1}^*} - 1 \right) \tag{6}$$

where P_t is the clean price at time t, I_t is the accrued interest at time t and P_t^* and I_t^* are the corresponding quantities for a risk-free bond with the same characteristics of the risky one.

Another possibility is to directly rely on the changes of the spread, that is

$$r_{D,t} = (y_t - y_t^*) - (y_{t-1} - y_{t-1}^*) = s_t - s_{t-1} \tag{7}$$

where y_t is the risky yield-to-maturity, y_t^* is the risk-free yield-to-maturity and s_t is the spread at time t.

Here we rely on the second choice because we are able to obtain the yield-to-maturity directly from TRACE, while the first choice would require the computation of the accrued interest I_t, which needs the exact amount and ex-date of each coupon. The availability of such data, and ex-coupon dates in particular, does not appear complete in TRACE.

In our analysis we exclude the impact of the changes in the risk-free term structure, i.e. we subtract y^* from equation (7), which is the largest and most relevant component of risky bond prices. The quantity y^* can be

considered as the yield of an hypothetical risk-free bond with exactly the same characteristics, in terms of maturity and coupon, of the considered risky security. In this way we are able to specifically model the movements in excess of the risk-free term structure, which are exactly the ones that can be explained by structural credit risk models.

The TRACE database provides only the total yield of each bond at each observation. We use the government term structure of zero interest rates to compute, given coupon and maturity, the hypothetical yield y^* described above.

Furthermore, we have to notice that equation (5) is linear in equity and volatility while the theoretical equation in (2) is expressed by the function g, which is in general nonlinear (e.g. the Black-Scholes option price in Merton's case). The performed analysis is therefore approximated but gives us two advantages on the original test based on equation g. The first one is the reduction in the computational burden of a nonlinear estimation in equation (5) based on non-analytical functions and the second is the possibility to universally apply equation (5) to all structural models, independently from the specific functional form g that defines each model.

Exactly the same procedure is repeated at weekly and monthly frequency. If the reason for the observed poor performances of structural models comes from the presence of a non persistent noise, one would expect that the goodness of fit of equation (5) at lower frequencies improves, since this different sampling is able to remove the high frequency component of the noise.

Once equation (5) is estimated, all the variability due to structural changes in the market value of assets and asset volatility is removed. In other words, the variability left in the residuals ε of equation (5) cannot be explained by structural credit risk models and must be originated by some kind of limit to the arbitrage forces driving the prices to equilibrium.

In this paper we specifically address the role of imperfectly liquid markets. We therefore try to explain the residuals ε in equation (5), for each corporate bond, by a set of specific and systematic liquidity indicators. The considered indicators are in particular

- The Amihud index computed according to section 3.1 for each considered corporate bond

- The Amihud index computed according to section 3.1 for the equity corresponding to each considered bond

- A "systematic Amihud" index described above

- The number of trades

The Amihud indexes for bond and equity appear weakly correlated. Their linear regression coefficient is only equal to 3.49%. This means that companies with liquid equity do not necessarily have more liquid bond issues and vice versa. This seems plausible since the liquidity index is computed issue-by-issue and a company with liquid equity may have many small illiquid bond issues.

The illiquidity in the corporate bond market appears more pronounced than in the equity market. The average Amihud index for equities is equal to only 5 bps while for bonds it reaches 25 bps. This means that, for the considered sample and period, a trade equal to 1 mln of face value moves the price by 5 bps for equity and 25 for bonds.

The cross-sectional distribution of the Amihud index for bonds is strongly kurthotic and skewed to the right, indicating that, for some bond, extreme levels of illiquidity are reached.

In a second step, the equation we are estimating for the assessment of structural model residuals is

$$|\varepsilon_t| = \gamma_0 + X_t \gamma + \eta_t \tag{8}$$

where

- γ_0 is a scalar parameter
- γ is a vector of parameters
- X_t is the vector of relevant liquidity indicators described above in this paragraph.
- η_t is a residual

The residuals ε are taken in absolute value since we are interested in the *absolute* error of structural models. When markets are very liquid, we expect residuals to be close to zero but without a specific sign and, on the other hand, when liquidity conditions are poor we just expect large residuals in absolute terms but we do not have any a priori expectation about the sign of the error.

In this way we are able to remove from the residuals ε of structural credit risk models all the variability explained by adverse liquidity conditions present in the market. The variability left in the last residuals η can be identified with the limits of arbitrage different from liquidity frictions such as the ones described in section 3.2.

As a final step, we look at the absolute residuals η of equation (8) at weekly frequency, we compute their cross-sectional average and represent its graph in

figure 4. This indicator is by construction a very general measure of limits of arbitrage present in the market and different from pure liquidity conditions. Indeed, in a perfectly frictionless market, the non-arbitrage relationships between equity and credit should drive residuals ε and, a fortiori, η close to zero.

It is clear that the strongest limits of arbitrage are present around the Lehman collapse and in particular between September 2008 and February 2009. It seems reasonable to assume that exactly in this period strong limits of arbitrage could be generated for instance by relevant outflows from mutual and hedge funds, which can be considered the most relevant arbitrageurs in the market. The deleveraging process generated by these redemptions forced the arbitrageurs to close their convergence strategies between debt and equity prices and therefore even enlarged the basis between the two. Furthermore, as explained above, convergence strategies require some cash upfront to finance repo/reverse repo transactions. The extreme conditions of the interbank market in that period seem fully compatible with the observed spike. Also in this case we notice that the general arbitrage conditions in 2010 and 2011 are in general good and comparable to the ones in 2003 and 2004, implying that in this period capital structure arbitrage is more efficiently working.

From a more quantitative point of view, we try to explain this index with two indicators of limits of arbitrage and general liquidity conditions: the TED and the LiborOIS spreads.

The first is given by the difference between the 3-months libor rate and the 3-months treasury rate. If liquidity in the interbank market decreases, then the LIBOR rate will increase as the demand of funds overweights the offer. The impact of such event on the treasury market will be smaller and therefore the treasury rate will increase less than the libor, widening in this way the spread. The TED spread is not actually a pure liquidity indicator since it includes also a relevant counterparty premium. The transactions on the libor market are between two intermediaries and therefore the exchanged interest rate needs to include a remuneration for the risk that the counterparty defaults before the end of the contract.

The second indicator is very similar to the first one and is given by the difference between the Libor interbank rate and the Overnight Index Swap, which can be considered as another proxy for the risk-free interest rate. The equation we are testing is therefore

$$\text{ArbIdx} = \delta_0 + \delta_1 TED_t + \delta_2 LiborOis_t + \mu_t \tag{9}$$

where ArbIdx is the residual index described above, δ_0 and δ_1 are two

constants.

5.1 Results

5.1.1 Test of structural credit risk models

The assessment of structural credit risk models is based on the estimation of equation (5). Both the point estimates and the t-statistics for the coefficients of such equation are computed following the procedure proposed in Collin-Dufresne et al. [2001]. According to this method, one separate regression is performed for every security. In our case, we take into consideration for the test only securities with more than 20 weekly or 10 monthly observations. A vector of estimates is therefore obtained for every coefficient. Such vector includes a very small number of extreme outliers. We deal with this by removing 0.5% of extreme observations both on the left and on the right of the distribution and then work with the "clean" vector. The presented point estimate is the average of all the estimated coefficients. The t-statistic is obtained using as standard error the sample cross-section standard deviation of the estimated coefficients rescaled by the square root of the number of securities.

The estimated coefficients are reported in table 2 for both weekly averages and weekly point observations.

Coef.	Weekly Average		Weekly Point Obs.	
	Estimate	t-stat	Estimate	t-stat
α	0.0001	13.26	0.0001	8.79
β_1	-0.0089	-10.90	-0.0090	-13.61
β_2	0.0151	23.22	0.0159	26.53

Table 2: Estimated coefficients in equation (5) for weekly data

The sign of estimated coefficients corresponds to the theoretically predicted one and the significance is considerable. The coefficient β_1 is negative since an increase in equity prices positively affects the bond price and consequently lowers the observed spread. On the other hand, an increase in implied volatilities reduces the value of the debt, which is comparable to a short put option written on the assets. Consequently the observed credit spreads are increased.

When we look at the number of bonds characterized by exposure with the theoretically predicted sign, we observe that the 71.4% of bond spreads has a negative exposure to the corresponding equity and 82.7% has a positive

exposure to the volatility implied in its options. Similar results (74.6% and 83.6%, respectively) are obtained with point observations instead of weekly averages.

It is also worth noting that the impact of implied volatilities on the market value of debt is more relevant than the impact of equity prices. This appears justified by the standard option theory. If we consider that the debt is equivalent to a short put option, the result is equivalent to having vegas larger than the corresponding deltas in absolute value. For put options, this is actually the case in most conditions and in particular for high values of volatility and underlying price with respect to the strike (i.e. put out of the money). In this case, the market value of equity is positive and therefore the market value of assets larger than the face value of debt or, equivalently, the put constituting the debt is out-of-the-money. This is exactly the case in which (in absolute value) vega is larger than delta.

Alternatively, we also considered the entire sample, without excluding any extreme datapoint. In this case we looked at the median, a more robust statistic, instead of the mean. The nonparametric Wilcoxon signed rank test has been implemented in place of the standard t-test. This procedure can be considered as a straightforward extension of the test in Collin-Dufresne et al. [2001] for the median estimator. The results fully agree with the ones presented above for the trimmed mean and the three coefficients are statistically different from zero at all the standard confidence levels.

The R^2 of the regressions is substantially small and it is equal to only 14.75% for the weekly average and 16.28% for the weekly point observation. Furthermore, the intercept is always very strongly different from zero, which is not theoretically justified, and can be explained by some misspecification of the model. We can conclude that structural variables, even if significant, are able to explain only a small part of the total spread variability and confirm the presence of a "credit spread puzzle" also for spread variability and not only for their level.

We repeated exactly the same procedure on a monthly time horizon. In this case we consider only the point observation since the use of averages on such a long period may be questionable and induce some loss of information. The results are presented in table 3.

| Coef. | Monthly Point Observation | |
	Estimate	t-stat
α	0.0002	10.5667
β_1	-0.0071	-8.0522
β_2	0.0204	31.4016

Table 3: Estimated coefficients in equation (5) for monthly data

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The percentage of coefficients characterized by the predicted sign is comparable to the weekly case. In particular, we observe that, for 71.4% of the bonds, spreads changes have a negative exposure to equity returns and 85.3% has a positive exposure to the volatility implied in the corresponding options. The constant is still strongly significant.

It is interesting to note that in this case the R^2 of the regression is almost double with respect to the weekly case and equals 29.36%. This evidence supports the claim according to which the errors produced by structural models are generated by a non persistent observational noise. The use of a lower observational frequency filters out some of the high-frequency noise and improves the performance of structural models. Such results are compatible to the ones presented in Collin-Dufresne et al. [2001].

As already said, equation (5) removes all the structural effects from the changes in corporate spreads. Therefore no structural effect is present anymore in the residuals ε.

5.1.2 Impact of liquidity indicators

In a second step, we concentrate on the absolute residuals ε from equation (5), which can be interpreted as errors of structural credit risk models. We estimate regression (8) based on the previously mentioned indicators. The coefficients and t-statistics are estimated using the same method described above and are presented in table 4 for weekly and 5 for monthly data.

	Weekly Average		Weekly Point Obs.	
Variable	Estimate	t-stat	Estimate	t-stat
Constant	0.0009	6.6364	0.0008	6.4929
Amihud Bond	0.0303	2.4050	0.0423	3.3879
Amihud Equity	0.2599	0.5005	0.2537	0.6402
N. Trades	0.0000	0.7795	0.0000	0.9801
Systematic Amihud	0.0745	6.3053	0.0491	5.2296

Table 4: Estimated coefficients in equation (5) for weekly data

It is clear that liquidity indicators, both corporate-specific and systematic, have a role in the explanation of the structural models residuals. Furthermore, the estimated coefficients have the theoretically predicted signs. In particular, Amihud indexes have a positive sign, indicating that when security-specific illiquidity is high, then the errors of structural models are larger. The fact

that liquidity conditions in the bond market are statistically significant while in the equity are not, may support the idea that the scarce liquidity in the debt market has a major role in preventing convergence trading and other corporate structure arbitrages with respect to the one in equity markets. Unexpectedly, the number of trades in the considered period has no impact on the structural errors since its coefficients are neither statistically nor economically significant (the sign of the estimate is the opposite of the theoretically predicted). It is finally interesting to note that the "systematic" Amihud factor is highly significant, indicating that security-specific liquidity indicators are not sufficient to explain structural errors and also market-wide variables are required. The R^2 of the regression are 20.26% and 23.46% in the two weekly cases and 36.99% in the monthly, which indicates that market liquidity conditions actually have a non-negligible role in explaining the errors of structural models.

Coef.	Estimate	t-stat
Constant	0.0012	4.6993
Amihud Bond	0.0358	1.0368
Amihud Equity	0.0184	0.4807
N. Trades	0.0000	0.8081
Systematic Amihud	0.0506	4.7215

Table 5: Estimated coefficients in equation (8) for monthly data

5.1.3 Limits of Arbitrage

The estimated parameters of equation (9) are

Coef.	Estimate	t-stat
Constant	0.0021	32.7465
TED Spread	-0.0002	-1.0673
Libor-OIS	1.89e-05	6.4366

Table 6: Estimated coefficients in equation (9)

The R^2 is 29.71%. The coefficient of the TED spread is not economically significant, since the sign is the opposite of the expected one. This may be justified by the fact that the Libor-OIS index is very close to the TED and captures the effect of the latter. Given the fact that in the first part of the considered period the number of bonds is very small, the proposed index

appears considerably volatile. A less noisy behaviour is present starting from 2003. If we run regression (9) only from 2003, we get the estimates in table 7.

Coef.	Estimate	t-stat
Constant	0.0019	43.42
TED Spread	0.0003	2.3069
Libor-OIS	1.36e-05	7.4718

Table 7: Estimated coefficients in equation (9)

The R^2 increases in this case to 59.36% and both indicators are statistically and economically significant. This clearly confirms that limits of arbitrage and general liquidity conditions have a role in the unsatisfactory performances of structural Credit risk models.

6   Conclusion

In this paper an empirical test of structural credit risk models is performed. The quality of the dataset appears high, since it combines actual transaction prices of corporate bonds taken from TRACE, implied volatilities taken from Optionmetrics and stock prices taken from CRSP.

The test confirms the poor performances of structural credit risk models in explaining corporate bond returns (measured in this case by changes in credit spreads).

We claim that at least part of this unsatisfactory performance can be explained by the presence of transitory noise in observed volatilities, bond and equity prices.

A first heuristic support to this proposition comes from the impact of observational frequency on the performance of structural models. The quality of the fit decreases with the observational frequency, which means that when we filter out the high-frequency noise by reducing the sampling frequency, structural models work better in explaining changes in spreads.

A more accurate test is based on the analysis of the (absolute) residuals orthogonal to equity and volatility returns. These quantities depend significantly on some standard liquidity indicators taken from both the equity and bond market, such as the index proposed by Amihud [2002], and the variability explained by this measures is not negligible.

When we look at the behaviour of the average absolute residuals left after removing also the liquidity component, we notice that they closely resemble the qualitative pattern of limits of arbitrage present in the period 2001-2011.

Furthermore, such variable significantly depend on standard indicators of liquidity frictions and limits of arbitrage, such as the TED and LiborOIS spreads. Even if a relevant part of the variation in credit spreads remains unexplained, we can conclude that liquidity indicators and limits of arbitrage have a non-negligible role in this shortcoming.

7   Figures
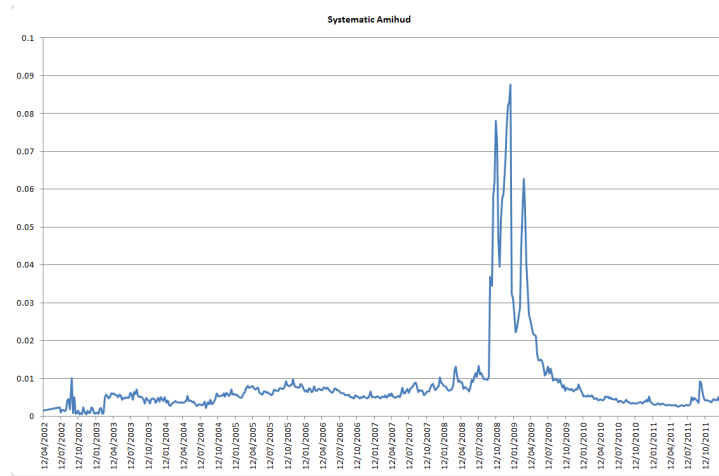
Figure 1: Evolution of the Systematic Amihud factor
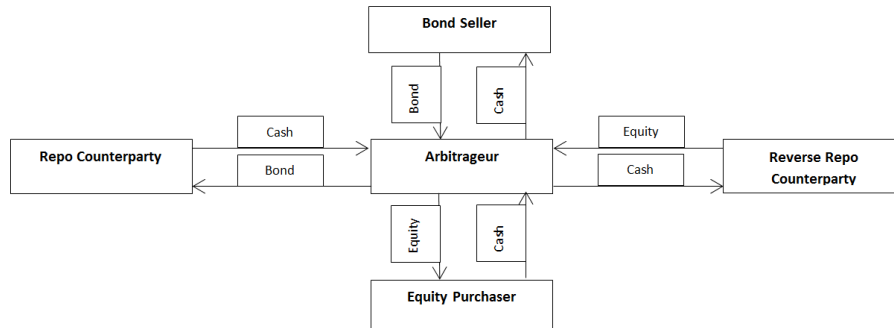
Figure 2: Convergence strategy implemented when bonds are underpriced with respect to equity
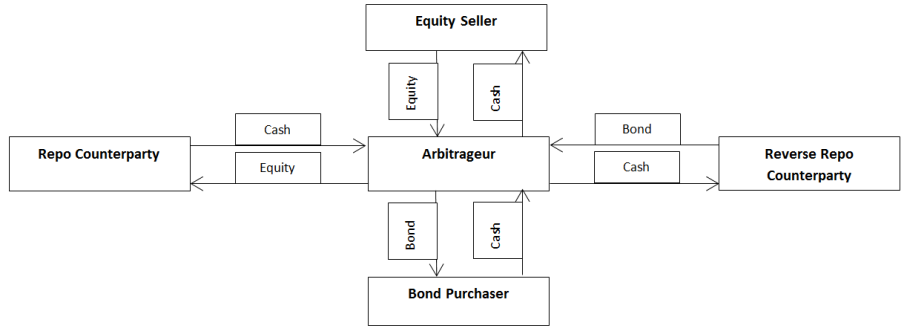
Figure 3: Convergence strategy implemented when the equity is underpriced with respect to bonds
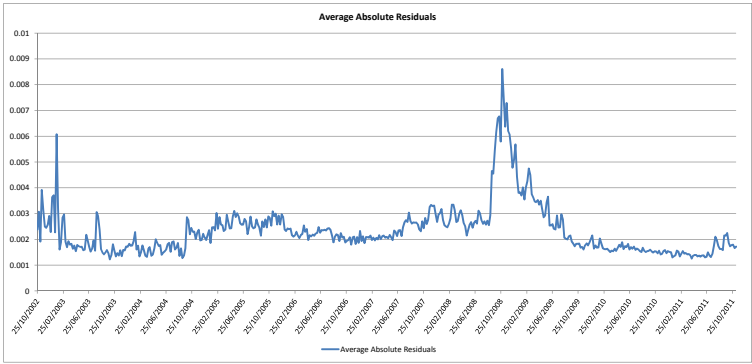
Figure 4: Evolution of the limits of arbitrage measure

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Conclusion

Two major topics are investigated in this work: the presence of a systematic and an idiosyncratic component in CDS spreads and the investigation of the so called "credit spread puzzle".

We verify that a systematic factor (here proxied by the CDX Investment Grade index) is actually priced in the cross-section of CDS returns. We also notice that the systematic component of risk increases after the financial crisis. We finally verify that the fraction of systematic risk is not the same in different industrial sectors. In particular, more "cyclical" and "systemic" factors, such as Financials or Consumer discretionary, show a much larger impact of systematic factor.

Possible directions for further research may be oriented to the inclusion of additional factors in the proposed APT model for credit spreads. A promising extension is also the modelization of credit spreads in a non-linear way, possibly including more sophisticated dependence structures than the simple one proposed here.

Regarding the second topic, we extend the literature proposing a bivariate state space model and verify that it actually improves the performances of standard inversion techniques in explaining the observed credit spreads. The improvement is particularly significant during the crisis period, characterized by a larger noise contaminating the observed equity price and equity volatility. This supports the ability of the state space model to remove the noise component and to produce better estimates of the asset value of the company and, consequently, more accurate predictions of spreads.

Further research in this direction may involve the extension of the proposed state space model to include more than two state variables, such as interest rates or dividend yields and verify their ability to explain observed credit spreads. Furthermore, the investigation is here limited to structural models able to express the measurement function in closed-form. An interesting topic could be the use of numeric techniques able to express the observed variables (such as equity prices) as a more general function of state variables. This would allow also the use of more sophisticated stochastic processes, such as long memory processes, to describe the assets of a company.

In the last chapter we identify some explicit drivers for the noise postulated

in chapter 2. In particular, we verify that the errors produced by structural credit risk models significantly depend on liquidity indicators and that their explained variability is not negligible. We finally verify that the errors left by both structural variables and liquidity indicators, are strongly correlated with market-wide measures of limits of arbitrage and/or deleveraging pressures.

Acknowledgments

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