CONVEXITY OF STRATA IN DIAGONAL PANTS GRAPHS OF SURFACES

J. ARAMAYONA, C. LECUIRE, H. PARLIER & K. J. SHACKLETON

ABSTRACT. We prove a number of convexity results for strata of the diagonal pants graph of a surface, in analogy with the extrinsic geometric properties of strata in the Weil-Petersson completion. As a consequence, we exhibit convex flat subgraphs of every possible rank inside the diagonal pants graph.

1. INTRODUCTION

Let S be a connected orientable surface, with empty boundary and negative Euler characteristic. The *pants graph* $\mathcal{P}(S)$ is the graph whose vertices correspond to homotopy classes of pants decompositions of S, and where two vertices are adjacent if they are related by an elementary move; see Section 2 for an expanded definition. The graph $\mathcal{P}(S)$ is connected, and becomes a geodesic metric space by declaring each edge to have length 1.

A large part of the motivation for the study of $\mathcal{P}(S)$ stems from the result of Brock [4] which asserts that $P(S)$ is quasi-isometric to $\mathcal{T}(S)$, the Teichmüller space of S equipped with the Weil-Petersson metric. As such, $P(S)$ (or any of its relatives also discussed in this article) is a combinatorial model for Teichmüller space.

By results of Wolpert [12] and Chu [7], the space $\mathcal{T}(S)$ is not complete. Masur [8] proved that its completion $\hat{\mathcal{T}}(S)$ is homeomorphic to the *augmented Teichmüller space* of S, obtained from $\mathcal{T}(S)$ by extending Fenchel-Nielsen coordinates to admit zero lengths. The completion $\hat{\mathcal{T}}(S)$ admits a natural *stratified structure*: each stratum $\mathcal{T}_C(S) \subset \hat{\mathcal{T}}(S)$ corresponds to a multicurve $C \subset S$, and parametrizes surfaces with *nodes* exactly at the elements of C. By Wolpert's result [11] on the convexity of length functions, $\mathcal{T}_C(S)$ is convex in $\hat{\mathcal{T}}(S)$ for all multicurves $C \subset S$.

The pants graph admits an analogous stratification, where the stratum $\mathcal{P}_C(S)$ corresponding to the multicurve C is the subgraph of $\mathcal{P}(S)$ spanned by those pants decompositions that contain C. Moreover, Brock's quasi-isometry between $\mathcal{P}(S)$ and $\mathcal{T}(S)$ may be chosen so that the image of $\mathcal{P}_C(S)$ is contained in $\mathcal{T}_C(S)$.

In light of this discussion, it is natural to study which strata of $\mathcal{P}(S)$ are convex. This problem was addressed in [1, 2], where certain families of strata in $\mathcal{P}(S)$ were proven to be totally geodesic; moreover, it is conjectured that this is the case for all strata of $\mathcal{P}(S)$, see Conjecture 5 of [1]. As was observed by Lecuire, the validity of this conjecture is equivalent to the existence of only finitely many geodesics between any pair of vertices of $\mathcal{P}(S)$; we will give a proof of the equivalence of these two problems in the Appendix.

The main purpose of this note is to study the extrinsic geometry of strata in certain graphs of pants decompositions closely related to $\mathcal{P}(S)$, namely the *diagonal pants graph* $\mathcal{DP}(S)$ and the *cubical pants graph*. Concisely, $\mathcal{DP}(S)$ (resp. $\mathcal{CP}(S)$) is obtained from $\mathcal{P}(S)$ by adding an edge

1

of length 1 (resp. length $\sqrt{k}$) between any two pants decompositions that differ by k *disjoint* elementary moves. Note that Brock's result [4] implies that both $\mathcal{DP}(S)$ and $\mathcal{CP}(S)$ are quasi-isometric to $\mathcal{T}(S)$, since they are quasi-isometric to $\mathcal{P}(S)$. These graphs have recently arisen in the study of metric properties of moduli spaces; indeed, Rafi-Tao [10] use $\mathcal{DP}(S)$ to estimate the Teichmüller diameter of the thick part of moduli space, while $\mathcal{CP}(S)$ has been used by Cavendish-Parlier to give bounds on the Weil-Petersson diameter of moduli space.

As above, given a multicurve $C \subset S$, denote by $\mathcal{DP}_C(S)$ (resp. $\mathcal{CP}_C(S)$) the subgraph of $\mathcal{DP}(S)$ (resp. $\mathcal{CP}(S)$) spanned by those pants decompositions which contain C. Our first result is:

Theorem 1.1. *Let S be a sphere with punctures and $C \subset S$ a multicurve. Then $\mathcal{DP}_C(S)$ is convex in $\mathcal{DP}(S)$.*

We remark that, in general, strata of $\mathcal{DP}(S)$ are not totally geodesic; see Remark 4.9 below.

The proof of Theorem 1.1 will follow directly from properties of the *forgetful maps* between graphs of pants decompositions. In fact, the same techniques will allow us to prove the following more general result:

Theorem 1.2. *Let Y be an essential subsurface of a surface S such that Y has the same genus as S. Let C be the union of a pants decomposition of $S \setminus Y$ with all the boundary components of Y. Then:*

  (1) $\mathcal{DP}_C(S)$ is convex in $\mathcal{DP}(S)$.
  (2) If Y is connected, then $\mathcal{P}_C(S)$ and $\mathcal{CP}_C(S)$ are totally geodesic inside $\mathcal{P}(S)$ and $\mathcal{CP}(S)$, respectively.

Next, we will use an enhanced version of the techniques in [1] to show an analog of Theorem 1.1 for general surfaces, provided the multicurve C has *deficiency* 1; that is, it has one curve less than a pants decomposition. Namely, we will prove:

Theorem 1.3. *Let $C \subset S$ be a multicurve of deficiency 1. Then:*

  (1) $\mathcal{DP}_C(S)$ is convex in $\mathcal{DP}(S)$.
  (2) $\mathcal{CP}_C(S)$ is totally geodesic in $\mathcal{CP}(S)$.

We observe that part (2) of Theorem 1.3 implies the main results in [1].

We now turn to discuss some applications of our main results. The first one concerns the existence of convex flat subspaces of $\mathcal{DP}(S)$ of every possible rank. Since $\mathcal{DP}(S)$ is quasi-isometric to $\mathcal{P}(S)$, the results of Behrstock-Minsky [3], Brock-Farb [5] and Masur-Minsky [9] together yield that $\mathcal{DP}(S)$ admits a quasi-isometrically embedded copy of $\mathbb{Z}^r$ if and only if $r \leq \left[\frac{3g+p-2}{2}\right]$. If one considers $\hat{T}(S)$ instead of $\mathcal{DP}(S)$, one obtains *isometrically* embedded copies of $\mathbb{Z}^r$ for the exact same values of r. In [2], convex copies of $\mathbb{Z}^2$ were exhibited in $\mathcal{P}(S)$, and it is unknown whether higher rank convex flats appear. The corresponding question for the diagonal pants graph is whether there exist convex copies of $\mathbb{Z}^r$ with a modified metric which takes into account the edges added to obtain $\mathcal{DP}(S)$; we will denote this metric space $\mathcal{D}\mathbb{Z}^r$. Using Theorems 1.1 and 1.2, we will obtain a complete answer to this question, namely:

Corollary 1.4. *There exists an isometric embedding $\mathcal{D}\mathbb{Z}^r \to \mathcal{DP}(S)$ if and only if $r \leq \left[\frac{3g+p-2}{2}\right]$.*

Our second application concerns the *finite geodesicity* of different graphs of pants decompositions. Combining Theorem 1.1 with the main results of [1, 2], we will obtain the following:

Corollary 1.5. *Let S be the six-times punctured sphere, and let P, Q be two vertices of $\mathcal{P}(S)$. Then there are only finitely many geodesics in $\mathcal{P}(S)$ between P and Q.*

The analogue of Corollary 1.5 for $\mathcal{DP}(S)$ is not true. Indeed, we will observe that if S is a sphere with at least six punctures, one can find pairs of vertices of $\mathcal{DP}(S)$ such that there are infinitely many geodesics in $\mathcal{DP}(S)$ between them. However, we will prove:

Corollary 1.6. *Given any surface S, and any k, there exist points in $\mathcal{DP}(S)$ at distance k with only a finite number of geodesics between them.*

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2. Preliminaries

Let S be a connected orientable surface with empty boundary and negative Euler characteristic. The complexity of S is the number $\kappa(S) = 3g - 3 + p$, where g and p denote, respectively, the genus and number of punctures of S.

2.1. **Curves and multicurves.** By a curve on S we will mean a homotopy class of simple closed curves on S; we will often blur the distinction between a curve and any of its representatives. We say that a curve $\alpha \subset S$ is essential if no representative of α bounds a disk with at most one puncture. The geometric intersection number between two curves α and β is defined as

$$i(\alpha, \beta) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}.$$

If $\alpha \neq \beta$ and $i(\alpha, \beta) = 0$ we say that α and β are disjoint; otherwise, we say they intersect. We say that the curves α and β fill S if $i(\alpha, \gamma) + i(\beta, \gamma) > 0$ for all curves $\gamma \subset S$.

A multicurve is a collection of pairwise distinct and pairwise disjoint essential curves. The deficiency of a multicurve C is defined as $\kappa(S) - |C|$, where $|C|$ denotes the number of elements of C.

2.2. **Graphs of pants decompositions.** A pants decomposition P of S is a multicurve that is maximal with respect to inclusion. As such, P consists of exactly $\kappa(S)$ curves, and has a representative P' such that every connected component of $S \setminus P'$ is homeomorphic to a sphere with three punctures, or pair of pants.

We say that two pants decompositions of S are related by an elementary move if they have exactly $\kappa(S) - 1$ curves in common, and the remaining two curves either fill a one-holed torus and intersect exactly once, or else they fill a four-holed sphere and intersect exactly twice. Observe that every elementary move determines a unique subsurface of S of complexity 1; we will say that two elementary moves are disjoint if the subsurfaces they determine are disjoint.

As mentioned in the introduction, the *pants graph* $\mathcal{P}(S)$ of S is the simplicial graph whose vertex set is the set of pants decompositions of S, considered up to homotopy, and where two pants decompositions are adjacent in $\mathcal{P}(S)$ if and only if they are related by an elementary move. We turn $\mathcal{P}(S)$ into a geodesic metric space by declaring the length of each edge to be 1.

The diagonal pants graph $\mathcal{DP}(S)$ is the simplicial graph obtained from $\mathcal{P}(S)$ by adding an edge of unit length between any two vertices that differ by $k \geq 2$ disjoint elementary moves.

The cubical pants graph $\mathcal{CP}(S)$ is obtained from $\mathcal{P}(S)$ by adding an edge *of length* $\sqrt{k}$ between any two edges that differ by $k \geq 2$ disjoint elementary moves.

3. THE FORGETFUL MAPS: PROOFS OF THEOREMS 1.1 AND 1.2

The idea of the proof of Theorems 1.1 and 1.2 is to use the so-called *forgetful maps* to define a distance non-increasing projection from the diagonal pants graph to each of its strata.

3.1. **Forgetful maps.** Let S_1 and S_2 be connected orientable surfaces with empty boundary. Suppose that S_1 and S_2 have equal genus, and that S_2 has at most as many punctures as S_1. Choosing an identification between the set of punctures of S_2 and a subset of the set of punctures of S_1 yields a map $\tilde{\psi} : S_1 \to \tilde{S}_1$, where $\tilde{S}_1$ is obtained by forgetting all punctures of S_1 that do not correspond to a puncture of S_2. Now, $\tilde{S}_1$ and S_2 are homeomorphic; by choosing a homeomorphism $\varphi : \tilde{S}_1 \to S_2$, we obtain a map $\psi = \varphi \circ \tilde{\psi} : S_1 \to S_2$. We refer to all such maps ψ as *forgetful maps*.

Observe that if $\alpha \subset S_1$ is a curve then $\psi(\alpha)$ is a (possibly homotopically trivial) curve on S_2. Also, $i(\psi(\alpha), \psi(\beta)) \leq i(\alpha, \beta)$ for al curves $\alpha, \beta \subset S$. As a consequence, if C is a multicurve on S_1 then $\psi(C)$ is a multicurve on S_2; in particular ψ maps pants decompositions to pants decompositions. Lastly, observe that if P and Q are pants decompositions of S_1 that differ by at most k pairwise disjoint elementary moves, then $\psi(P)$ and $\psi(Q)$ differ by at most k pairwise disjoint elementary moves. We summarize these observations as a lemma.

Lemma 3.1. *Let $\psi : S_1 \to S_2$ be a forgetful map. Then:*
  (1) If P is a pants decomposition of S_1, then $\psi(P)$ is a pants decomposition of S_2.
  (2) If P and Q are related by k disjoint elementary moves in S_1, then $\psi(S_1)$ and $\psi(S_2)$ are related by at most k disjoint elementary moves in S_2. $\qquad\square$

In light of these properties, we obtain that forgetful maps induce distance non-increasing maps

$$\mathcal{P}(S_1) \to \mathcal{P}(S_2)$$

and, similarly,

$$\mathcal{DP}(S_1) \to \mathcal{DP}(S_2) \quad \text{and} \quad \mathcal{CP}(S_1) \to \mathcal{CP}(S_2).$$

3.2. **Projecting to strata.** Let S be a sphere with punctures, C a multicurve on S, and consider $S \setminus C = X_1 \sqcup \ldots \sqcup X_n$. For each i, we proceed as follows. On each connected component of $S \setminus X_i$ we choose a puncture of S. Let $\bar{X}_i$ be the surface obtained from S by forgetting all punctures of S in $S \setminus X_i$ except for these chosen punctures. Noting that $\bar{X}_i$ is naturally homeomorphic to X_i, as above we obtain a map

$$\phi_i : \mathcal{P}(S) \to \mathcal{P}(X_i),$$

for all $i = 1, \ldots, n$. We now define a map

$$\phi_C : \mathcal{P}(S) \to \mathcal{P}_C(S)$$

by setting

$$\phi_C(P) = \phi_1(P) \cup \ldots \cup \phi_n(P) \cup C.$$

5

Abusing notation, observe that we also obtain maps

$$\phi_C : \mathcal{DP}(S) \to \mathcal{DP}_C(S) \quad \text{and} \quad \phi_C : \mathcal{CP}(S) \to \mathcal{CP}_C(S).$$

The following is an immediate consequence of the definitions and Lemma 3.1:

Lemma 3.2. *Let $C \subset S$ be a multicurve. Then*

    (1) If P is a pants decomposition of S that contains C, then $\phi_C(P) = P$.
    (2) For all $P, Q \in \mathcal{DP}(S)$, $d_{\mathcal{DP}(S)}(\phi_C(P), \phi_C(Q)) \le d_{\mathcal{DP}(S)}(P, Q)$. $\square$

3.3. **Proof of Theorem 1.1.** After the above discussion, we are ready to give a proof of our first result:

Proof of Theorem 1.1. Let P and Q be vertices of $\mathcal{DP}_C(S)$, and let ω be a path in $\mathcal{DP}(S)$ between them. By Lemma 3.2, $\phi_C(\omega)$ is a path in $\mathcal{DP}_C(S)$ between $\phi_C(P) = P$ and $\phi_C(Q) = Q$, of length at most that of ω. Hence $\mathcal{DP}_C(S)$ is convex. $\square$

Remark 3.3 (Strata that are not totally geodesic)**.** If S has at least 6 punctures, there exist multicurves C in S for which $\mathcal{DP}_C(S)$ fails to be totally geodesic. For instance, let $C = \alpha \cup \beta$, where α cuts off a four-holed sphere $Z \subset S$ and $\beta \subset Z$. Let $P, Q \in \mathcal{DP}_C(S)$ at distance at least 3, and choose a geodesic path

$$P = P_0, P_1, \ldots, P_{n-1}, P_n = Q$$

in $\mathcal{DP}_C(S)$ between them. For each $i = 1, \ldots, n-1$, let P_i' be a pants decomposition obtained from P_i by replacing the curve β with a curve $\beta' \subset Z$ such that P_i and P_i' are related by an elementary move for all i. Observing that P and P_1' (resp. P_{n-1}' and Q) are related by two disjoint elementary moves, and therefore are adjacent in $\mathcal{DP}(S)$, we obtain a geodesic path

$$P = P_0, P_1', \ldots, P_{n-1}', P_n = Q$$

between P and Q that lies entirely outside $\mathcal{DP}_C(S)$ except at the endpoints. In particular, $\mathcal{DP}_C(S)$ is not totally geodesic.

3.4. **Proof of Theorem 1.2.** Let Y be a subsurface of S such that Y has the same genus as S. Each component of ∂Y bounds in S a punctured disc. As in Section 3.2, we may define a projection

$$\phi_Y : \mathcal{P}(S) \to \mathcal{P}(Y),$$

this time by choosing one puncture of S in each punctured disc bounded by a component of ∂Y. Let C be the union of a pants decomposition of $S \setminus Y$ with all the boundary components of Y. We obtain a map:

$$\phi_C : \mathcal{P}(S) \to \mathcal{P}_C(S)$$

by setting $\phi_C(P) = \phi_Y(P) \cup C$. Again abusing notation, we also have maps

$$\phi_C : \mathcal{DP}(S) \to \mathcal{DP}_C(S) \quad \text{and} \quad \phi_C : \mathcal{CP}(S) \to \mathcal{CP}_C(S).$$

Proof of Theorem 1.2. We prove part (2) only, since the proof of part (1) is completely analogous to that of Theorem 1.1. Since Y is connected, Lemma 3.1 implies that the maps $\phi_C : \mathcal{P}(S) \to \mathcal{P}_C(S)$ and $\phi_C : \mathcal{CP}(S) \to \mathcal{CP}_C(S)$ are distance non-increasing.

    We first prove the result for the pants graph $\mathcal{P}(S)$. Suppose, for contradiction, that there exist $P, Q \in \mathcal{P}_C(S)$ and a geodesic $\omega : P = P_0, P_1, \ldots, P_n = Q$ in $\mathcal{P}(S)$ from P to Q such that $P_i \in \mathcal{P}_C(S)$

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if and only if $i = 0, n$. Let $\tilde{P}_i = \phi_C(P_i)$ for all i, noting that $\tilde{P}_0 = P$ and $\tilde{P}_n = Q$. Since P contains C but P_1 does not, then $\tilde{P}_1 = \phi_C(P_1) = P_0$. Therefore, the length of the projected path $\phi_C(\omega)$ in $\mathcal{P}(S)$ is strictly smaller than that of ω, which gives the desired contradiction.

We now treat the case of the cubical pants graph $\mathcal{CP}(S)$. Again seeking a contradiction, suppose that there exist $P, Q \in \mathcal{CP}_C(S)$ and a geodesic $\omega : P = P_0, P_1, \ldots, P_n = Q$ in $\mathcal{CP}(S)$ from P to Q such that $P_i \in \mathcal{CP}_C(S)$ if and only if $i = 0, n$. Let $\tilde{P}_i = \phi_C(P_i)$ for all i; again, $\tilde{P}_0 = P$ and $\tilde{P}_n = Q$. Since $P_0 \in \mathcal{CP}_C(S)$ but $P_1 \notin \mathcal{CP}_C(S)$, the number of disjoint elementary moves between P_0 and $\tilde{P}_1$ is strictly smaller than the number of elementary moves between P_0 and P_1, and thus the edge $P_0\tilde{P}_1$ is shorter than the edge P_0P_1. Therefore, the length of the projected path $\phi_C(\omega)$ in $\mathcal{CP}(S)$ is strictly smaller than that of ω, which finishes the proof of the theorem. □

Remark 3.4. If S has at least 3 punctures (and genus at least 1), then there are multicurves $C \subset S$ such that $\mathcal{DP}_C(S)$ is not totally geodesic. One such example is $C = \alpha \cup \beta$, where α cuts off a four-holed sphere $Z \subset S$, and $\beta \subset Z$; compare with Remark 3.3.

4. CONVEX FAREY GRAPHS: PROOF OF THEOREM 1.3

The main goal of this section is to prove that, for any surface S and any multicurve $C \subset S$ of deficiency 1, the stratum $\mathcal{DP}_C(S)$ is convex in $\mathcal{DP}(S)$. This is the equivalent theorem for $\mathcal{DP}(S)$ of the main result of [1], which states that any isomorphic copy of a Farey graph in $\mathcal{P}(S)$ is totally geodesic; compare with Remark 4.9 below.

4.1. **Geometric subsurface projections.** We recall the definition and some properties of subsurface projections, as introduced by Masur-Minsky [9], in the particular case where the subsurface F has complexity 1.

Let $C \subset S$ be a deficiency 1 multicurve; as such, $S \setminus C$ contains an incompressible subsurface F of complexity 1, thus either a one-holed torus or a four-holed sphere. Let $\alpha \subset S$ be a curve that intersects F essentially. Let a be a connected component of $\alpha \cap F$; as such, a is either a curve in F, or an arc with endpoints on ∂F. Now, there is exactly one curve γ_a in F that is disjoint from a, and we refer to the pants decomposition $\gamma_a \cup C$ as a projection of α.

We write $\pi_C(\alpha)$ to denote the set of all projections of α, each counted once; we note that $\pi_C(\alpha)$ depends only on α and C. If $D \subset S$ is a multicurve, then $\pi_C(D)$ is the union of the projections of the elements of D. In the special case where a multicurve D does not intersect F essentially, we set $\pi_C(D) = \emptyset$. Finally, observe that if D contains a curve β that is contained in F, then $\pi_C(D) = \beta \cup C$.

We will need the following notation. Given an arc $a \subset F$ with endpoints in ∂F, we call it a *wave* if its endpoints belong to the same boundary component of F; otherwise, we call it a *seam*. These two types of arcs are illustrated in Figure 1. Observe that any arc in a one-holed torus is a seam.

4.2. **Preliminary lemmas.** Unless specified, C will always be a deficiency 1 multicurve on S, and F the unique (up to isotopy) incompressible subsurface of complexity 1 in $S \setminus C$. Observe that, in this case, $\mathcal{DP}_C(S)$ is isomorphic to $\mathcal{P}(F)$. We will need some lemmas, similar to those previously used in [1]. First, since any two disjoint arcs in a one-holed torus project to curves that intersect at most once, we have:
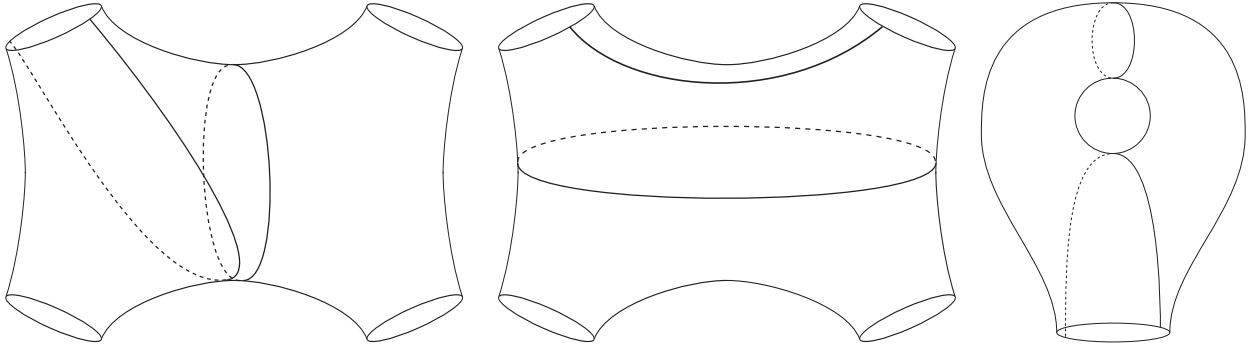
FIGURE 1. A wave and a seam on a four holed sphere, a wave on a torus (and their projections)

Lemma 4.1. *Suppose F is a one-holed torus, and let $\alpha, \beta \subset S$ be curves with $i(\alpha, \beta) = 0$. Then $\pi_C(\alpha \cup \beta)$ has diameter at most 1 in $\mathcal{P}(F)$.* $\qquad\qquad\square$

For F a four-holed sphere, we need the following instead:

Lemma 4.2. *Let F be a four-holed sphere, and let $\alpha, \beta \subset F$ be curves with $i(\alpha, \beta) \leq 8$. Then $d_{\mathcal{P}(F)}(\alpha, \beta) \leq 2$.*

Proof. It is well-known that curves in F correspond to elements of $\mathbb{Q} \cup \{\infty\}$ expressed in lowest terms. Moreover, if $\alpha = p/q$ and $\beta = r/s$, then $i(\alpha, \beta) = 2|ps - rq|$. Up to the action of $\mathrm{SL}(2, \mathbb{Z})$, we may assume that $\alpha = 0/1$ and $\beta = r/s$. Since $i(\alpha, \beta) \leq 8$, then $r \leq 4$; suppose, for concreteness, that $r = 4$. We are looking for a curve $\gamma = u/v$ with $i(\alpha, \gamma) = i(\beta, \gamma) = 2$. Since $i(\alpha, \gamma) = 2$, then $u = 1$. Since $\gcd(r, s) = 1$, then $s = 4k + 1$ or $s = 4k + 3$, for some k. In the former case, we set $v = k$ and, in the latter case, we set $v = k + 1$. $\qquad\qquad\square$

As a direct consequence of the above, we obtain:.

Lemma 4.3. *Suppose F is a four-holed sphere, and let $\alpha, \beta \subset S$ be curves with $i(\alpha, \beta) = 0$. Then, $\pi_C(\alpha \cup \beta)$ has diameter at most 2 in $\mathcal{P}(F)$.* $\qquad\qquad\square$

We will also need the following refinement of Lemma 9 of [1]:

Lemma 4.4. *Let P be a pants decomposition of S with $\partial F \not\subset P$ and let $\alpha_1 \in P$ be a curve that essentially intersects F. Then there exists $\alpha_2 \in P \setminus \alpha_1$ that essentially intersects F. Furthermore α_2 can be chosen so that α_1 and α_2 share a pair of pants.*

Proof. We prove the first assertion. Suppose, for contradiction, that α_1 is the only element of P that essentially intersects F. Since $\partial F \not\subset P$ and $P \setminus \alpha_1$ has deficiency 1, then $P \setminus \alpha_1 \cup \partial F$ is a pants decomposition of S, which is impossible since its complement contains F, which has complexity 1. Thus there is a curve $\tilde{\alpha} \neq \alpha_1$ of P essentially intersecting F.

To show that a second curve α_2 can be chosen sharing a pair of pants with α_1, consider any path $c : [0, 1] \longrightarrow F$ with $c(0) \in \alpha_1$ and $c(1) \in \tilde{\alpha}$ and choose α_2 to be the curve on which $c(\tau)$ lies, where $\tau = \min\{t | c(t) \in P \setminus \alpha_1\}$. Thus there is a path between α_1 and α_2 which intersects P only at its endpoints. Therefore, α_1 and α_2 share a pair of pants. $\qquad\qquad\square$

The next lemma will be crucial to the proof of the main result. In what follows, d_C denotes distance in $\mathcal{DP}_C(S) = \mathcal{P}_C(S)$

Lemma 4.5. *Let F be a four-holed sphere. Let P, P' be pants decompositions at distance 1 in $\mathcal{DP}(S)$ and let $\alpha \in P$, $\alpha' \in P'$ be curves related by an elementary move. Suppose both curves intersect F essentially, and let a be an arc of α on F. Then at least one of the following two statements holds:*

 i) There exists $\tilde{\alpha} \in P \cap P'$ and an arc $\tilde{a}$ of $\tilde{\alpha}$ on F such that

$$d_C(\pi_C(a), \pi_C(\tilde{a})) \leq 1.$$

 ii) The curve α' satisfies

$$d_C(\pi_C(a), \pi_C(\alpha')) \leq 2.$$

Remark 4.6. *The example illustrated in Figure 2 shows that case ii) does not cover everything. The two arcs labelled a and a', which can indeed be subarcs of curves related by an elementary move, produce projected curves that are distance 3.*

Proof of lemma 4.5. First, we choose representatives of α and α' that realize $i(\alpha, \alpha')$; abusing notation, we denote these representatives by α and α'.

Let a (resp. a') be any arc of α (resp. α') in F. First, observe that if $i(a, a') \leq 1$ then $i(\pi_C(a), \pi_C(a')) \leq 8$. The same is true if at least one of a or a' is a wave. In either case, Lemma 4.2 yields that $d_C(\pi_C(a), \pi_C(a')) \leq 2$ and thus the result follows.

Therefore, it suffices to treat the case where $i(a, a') = 2$ and both a, a' are seams. In this case, we will show that conclusion i) holds.
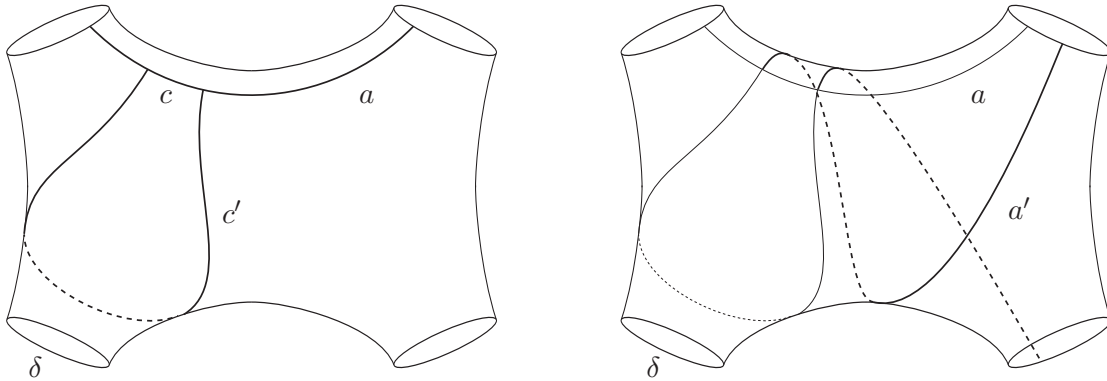
FIGURE 2. The arcs a and a', shown in the four-holed sphere F.

Consider the subarcs $c \subset a$ and $c' \subset a'$ between the two intersection points of a and a'. As α and α' intersect twice and are related by an elementary move, they fill a unique (up to isotopy) incompressible four-holed sphere $X \subset S \setminus (P \cap P')$; in particular, they have algebraic intersection number 0. As the endpoints of a' must lie on different boundary components of F, by checking the

different possibilities, the only topological configuration of the two arcs is the one shown in Figure 2.

Denote δ the curve in the homotopy class of $c \cup c'$. This curve is a boundary curve of F, as is illustrated in Figure 2, but is also one of the boundary curves of X. This is because δ is disjoint from all the boundary curves of X, and is not homotopically trivial since $i(\alpha, \alpha') = 2$. In X, the curve α' and the arc c determine two boundary curves of X, namely δ and another one we shall denote $\tilde{\delta}$ (see Figure 3).
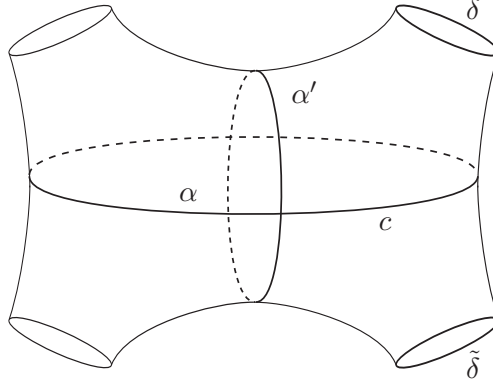
FIGURE 3. The surface X determined by α and α'

Note that the curve $\tilde{\delta}$ by definition shares a pair of pants with α in P and with α' in P'. Now, in F the arc $\tilde{c}$, illustrated in Figure 4, is an essential subarc of $\tilde{\delta}$. Now clearly

$$d_C(\pi_C(\tilde{c}), \pi_C(a)) = 1;$$

see again Figure 4. As $\tilde{\delta} \in P \cap P'$, we can conclude that i) holds and thus the lemma is proved. $\square$

4.3. **Proof of Theorem 1.3.** We will prove the following result from which Theorem 1.3 will easily follow. Again, d_C denotes distance in $\mathcal{DP}_C(S)$ (or, equivalently, in $\mathcal{CP}_C(S)$, since both are isomorphic as C is a deficiency 1 multicurve).

Theorem 4.7. *Let $C \subset S$ be a deficiency 1 multicurve, and let F be the unique (up to isotopy) incompressible surface of complexity 1 in $S \setminus C$. Let $P, Q \in \mathcal{DP}_C(S)$, and let $P = P_0, \ldots, P_n = Q$ be a path in $\mathcal{DP}(S)$ between them. If $\partial F \not\subset P_k$ for at least one k, then $n > d_C(P, Q)$.*

Proof of Theorem 4.7. Let $P = P_0, \ldots, P_n = Q$ be a path in $\mathcal{DP}(S)$, namely P_{n+1} differs from P_n by disjoint elementary moves. It suffices to consider the case that ∂F is not contained in P_k for every $k \notin \{0, n\}$. The proof relies on projecting to curves in $\pi_C(P_k)$ to produce a path $\tilde{P}_0, \ldots, \tilde{P}_n$ in $\mathcal{DP}_C(S)$, in such a way that $d(\tilde{P}_i, \tilde{P}_{i+1}) \leq 1$. The strict inequality will be due to the fact that, since P_0 contains ∂F but P_1 does not, then $\pi_C(P_1) = P_0$. We define $\tilde{P}_k$ inductively, as follows. First, since $P_0 \in \mathcal{DP}_C(S)$, then $\pi_C(P_0) = P_0$, and we set $\tilde{P}_0 := P_0$.
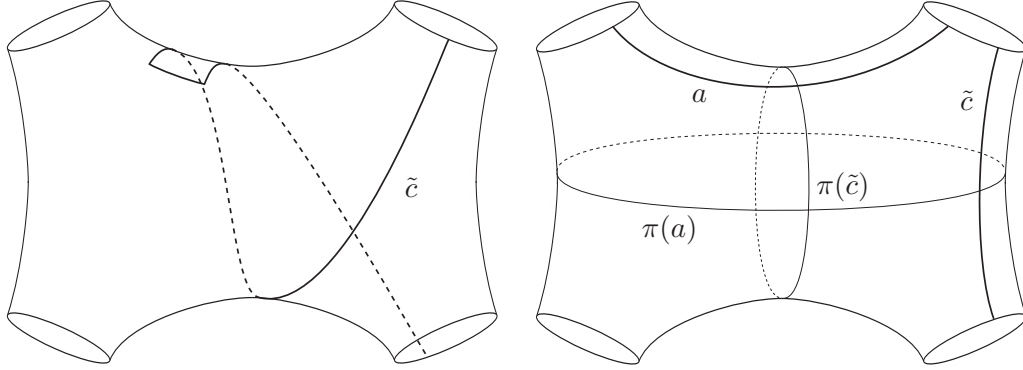
FIGURE 4. The surface X determined by α and α'

We now define $\tilde{P}_1$. Let $\alpha = P_0 \setminus C$. As P_1 does not contain C, we are in one of the following two situations:

(1) $\partial F \subset P_1$. In this case, $\alpha \in P_1$ and thus $\pi_C(P_1) = P_0$.
(2) P_1 intersects at least one component of ∂F. Since F has complexity 1 and edges in $\mathcal{DP}(S)$ correspond to disjoint elementary moves, we deduce that $\alpha \in P_1$. Therefore, $P_1 \cap F$ consists of α plus a collection of arcs, each of which is disjoint from α. Therefore, $\pi_C(P_1) = P_0$.

In either case, we define $\tilde{P}_1 := \pi_C(P_1) = P_0$. Proceeding inductively, suppose that we have constructed $\tilde{P}_k$ by projecting the arc a_k of the curve $\alpha_1^k \in P_k$. Let α_2^k the curve of P_k whose existence is guaranteed by Lemma 4.4. We begin with the easier case when F is a one holed torus.

The case where F is a one holed torus

We proceed as follows for $k \leq n-1$. At step $k+1$, at least α_1^k or α_2^k belong to P_{k+1}. We set $\tilde{P}_{k+1}$ to be the projection of any arc of such curve. By Lemma 4.1, we have $d_C(\tilde{P}_k, \tilde{P}_{k+1}) \leq 1$. Since Q contains ∂F but P_{n-1} does not, then $\pi_C(P_{n-1}) = Q$ and therefore $\tilde{P}_{n-1} = Q$. In this way we obtain the desired path, which has length at most $n-2$.

The case where F is a four holed sphere

We proceed as follows for $k \leq n-2$. There are several cases to consider:

1. If $\alpha_1^k \in P_{k+1}$, then we define $\tilde{P}_{k+1} := \tilde{P}_k$.

If $\alpha_1^k \notin P_{k+1}$, denote by α_1^{k+1} the curve in P_{k+1} that intersects α_1^k essentially.

2. Suppose that α_1^{k+1} intersects F essentially. As α_1^{k+1} and α_1^k are related by an elementary move, and since α_1^k shares a pair of pants with α_2^k, it follows that α_2^k also belongs to P_{k+1} and that it shares a pair of pants with α_1^{k+1}.

i. If $d(\pi_C(a_k), \pi_C(\alpha_1^{k+1})) > 2$, then by Lemma 4.5, there exists a curve $\tilde{\alpha}$ in $P_k \cap P_{k+1}$ (not necessarily α_2^k) with an arc $\tilde{a}$ such that $d(\pi_C(a_k), \pi_C(\tilde{a})) = 1$. In this case we set $\tilde{P}_{k+1} := C \cup \pi_C(\tilde{a})$, noting that $d(\tilde{P}_k, \tilde{P}_{k+1}) = 1$.

ii. Otherwise we have $d(\pi_C(a_k), \pi_C(\alpha_1^{k+1})) \leq 2$. Now, at least one of α_1^{k+1} or α_2^k belongs to P_{k+2}. If α_1^{k+1} belongs to P_{k+2}, we define $\tilde{P}_{k+2}$ to be the projection of any arc of α_1^{k+1}. Since $d(\pi_C(a_k), \pi_C(\alpha_1^{k+1})) \leq 2$, we have $d(\tilde{P}_k, \tilde{P}_{k+2}) \leq 2$. If α_2^k belongs to P_{k+2}, we define $\tilde{P}_{k+2}$ to be the projection of any arc of α_2^k. We have $d(\tilde{P}_k, \tilde{P}_{k+2}) \leq 2$ by Lemma 4.3. If $d(\tilde{P}_k, \tilde{P}_{k+2}) < 2$ we set $\tilde{P}_{k+1} = \tilde{P}_k$, otherwise, i.e. in the case that $d(\tilde{P}_k, \tilde{P}_{k+2}) = 2$, we choose any $\tilde{P}_{k+1}$ that contains ∂F and is adjacent to both $\tilde{P}_k$ and $\tilde{P}_{k+2}$.

3. Finally, suppose that α_1^{k+1} does not essentially intersect F. By Lemma 4.4, P_{k+1} contains a curve $\tilde{\alpha}$ which essentially intersects F and which shares a pair of pants with α_2^k. The curve $\tilde{\alpha}$ may or may not belong to P_k but, in either case, observe that it is disjoint from α_1^k. As such, by Lemma 4.3 we have:

$$d(\pi_C(a_k), \pi_C(\tilde{\alpha})) \leq 2.$$

As before, at least $\tilde{\alpha}$ or α_2^k belongs to P_{k+1} so we choose any arc of $\{\tilde{\alpha}, \alpha_2^k\} \cap P_{k+2}$ to produce $\tilde{P}_{k+2}$, which is at distance at most 2 from $\tilde{P}_k$.

This process will provide us with all $\tilde{P}_k$ up until either $k = n$ or $k = n - 1$. We conclude by noticing that if $\tilde{P}_{n-1}$ was obtained as the projection of a curve in P_{n-1} then, as in the initial step, $\tilde{P}_{n-1} = \pi_C(P_{n-1}) = P_n$ and we set $\tilde{P}_n := P_n$. $\qquad\square$

Rephrasing Theorem 4.7, we obtain the following property of geodesics in $\mathcal{DP}(S)$ and $\mathcal{CP}(S)$:

Corollary 4.8. *Let $C \subset S$ be a deficiency 1 multicurve, and let F be the unique (up to isotopy) incompressible surface of complexity 1 in $S \setminus C$. Let $P, Q \in \mathcal{DP}_C(S)$ and let ω be any geodesic from P to Q (in $\mathcal{DP}(S)$ or in $\mathcal{CP}(S)$). Then every vertex of ω contains ∂F.*

We are now in a position to prove Theorem 1.3:

Proof of Theorem 1.3. To show part (1), let $P, Q \in \mathcal{DP}_C(S)$, and let $P = P_0, \ldots, P_n = Q$ be a geodesic in $\mathcal{DP}(S)$ between P and Q. Thus we have assumed that $d_{\mathcal{DP}(S)}(P, Q) = n$. By Corollary 4.8, $\partial F \subset P_i$ for all i, and thus $P_i \cap F$ consists of a unique curve α_i. Let $\tilde{P}_i = \alpha_i \cup C$, observing that $\tilde{P}_0 = P_0$ and $\tilde{P}_n = Q$. Then, $P = \tilde{P}_0, \ldots, \tilde{P}_n = Q$ is a geodesic from P to Q (since $d_{\mathcal{DP}(S)}(P_n, P_{n+1}) \geq 1$) which is entirely contained in $\mathcal{DP}_C(S)$, and hence $\mathcal{DP}_C(S)$ is convex.

Remark 4.9. If a component F' of $S \setminus F$ is not a pair of pants then we can replace P_1 by a pants decomposition P_1' obtained by doing an elementary move on P_1 in F'. Thus $P = \tilde{P}_0, P_1', \tilde{P}_2 \ldots, \tilde{P}_n = Q$ is a geodesic but does not lie in $\mathcal{DP}_C(S)$, in particular $\mathcal{DP}_C(S)$ is not totally geodesic (compare with Remark 3.3).

We now prove part (2). Let $\omega : P = P_0, \ldots, P_n = Q$ be a geodesic in $\mathcal{CP}(S)$ between P and Q; in particular, we have assumed that $d_{\mathcal{CP}(S)} \geq n$. We want to show that $C \subset P_i$ for all i. Again by Corollary 4.8, we know $\partial F \subset P_i$, and hence $P_i \cap F$ is a curve α_i. Setting $\tilde{P}_i = \alpha_i \cup C$, we have $d_{\mathcal{CP}(S)}(\tilde{P}_i, \tilde{P}_{i+1}) \leq 1$ and $d_{\mathcal{CP}(S)}(\tilde{P}_0, \tilde{P}_n) \leq n$. Since ω is a geodesic, and since $\tilde{P}_0 = P_0$ and $\tilde{P}_n = P_n$,

we get $d_{\mathcal{CP}(S)}(P_i, P_{i+1}) = 1 = d_{\mathcal{CP}(S)}(\tilde{P}_i, \tilde{P}_{i+1})$ for any $i < n$. It follows that P_i and P_{i+1} differ by only one elementary move which turns α_i into α_{i+1}. In particular, if $C \subset P_i$ then $C \subset P_{i+1}$. Since $C \subset P_0$, we conclude that $C \subset P_i$ for any $i \leq n$. Hence $\omega \subset \mathcal{CP}_C(S)$ and thus $\mathcal{CP}_C(S)$ is totally geodesic. $\square$

5. CONSEQUENCES

Let S be a connected orientable surface of negative Euler characteristic and with empty boundary. Let $R = R(S)$ be the number $[\frac{3g+p-2}{2}]$ where g and p are, respectively, the genus and number of punctures of S. Note that R is the maximum number of pairwise distinct, pairwise disjoint complexity 1 subsurfaces of S. As mentioned in the introduction, by work of Behrstock-Minsky [3], Brock-Farb [5], and Masur-Minsky [9], R is precisely the *geometric rank* of the Weil-Petersson completion $\hat{T}(S)$ (and thus also of the pants graph $\mathcal{P}(S)$, by Brock's result [4]).

We will say that a multicurve $D \subset S$ is *rank-realizing* if $S \setminus D$ contains R pairwise distinct, pairwise disjoint incompressible subsurfaces of S, each of complexity 1. We have:

Proposition 5.1. *Let $D \subset S$ be a rank-realizing multicurve. Then $\mathcal{DP}_D(S)$ is convex in $\mathcal{DP}(S)$.*

Proof. Let $X_1, \ldots, X_R$ be the R complexity 1 incompressible subsurfaces in $S \setminus D$. Let P, Q be any two vertices of $\mathcal{DP}_C(S)$, and denote by α_i (resp. β_i) the unique curve in P (resp. Q) that is an essential curve in X_i. Consider any path ω between P and Q. As in the proof of Theorem 1.3, we may project ω to a path ω_i in $\mathcal{DP}(X_i)$ from α_i to β_i, such that $\text{length}(\omega_i) \leq \text{length}(\omega)$ for all i. In fact, by repeating vertices at the end of some of the ω_i if necessary, we may assume that all the ω_i have precisely $N + 1$ vertices, with $N \leq \text{length}(\omega)$.

Now, if $\omega_{i,j}$ denotes the j-th vertex along ω_i, then $P_j = \omega_{1,j} \cup \ldots, \omega_{R,j} \cup D$ is a vertex of $\mathcal{DP}_D(S)$. In this way, we obtain a path $P = P_0, \ldots, P_N = Q$ that is entirely contained in $\mathcal{DP}_D(S)$ and has length $N \leq \text{length}(\omega)$, as desired. $\square$

As a consequence, we obtain our first promised corollary. Here, $\mathcal{DZ}^r$ denotes the graph obtained from the cubical lattice $\mathbb{Z}^r$ by adding the diagonals to every k-dimensional cube of $\mathbb{Z}^r$, for $k \leq r$.

Corollary 1.4. *There exists an isometric embedding $\mathcal{DZ}^r \to \mathcal{DP}(S)$ if and only if $r \leq R$.*

Proof. We exhibit an isometric embedding of $\mathcal{DZ}^R$, as the construction for $r \leq R$ is totally analogous. Let D be a rank-realizing multicurve, and let $X_1, \ldots, X_R$ be the R incompressible subsurfaces of complexity 1 in $S \setminus D$. For $i = 1, \ldots, R$, choose a bi-infinite geodesic ω_i in $\mathcal{DP}(X_i)$. Let $\omega_{i,j}$ denote the j-th vertex along ω_i. Then subgraph of $\mathcal{DP}(S)$ spanned by $\{\omega_{i,j} \cup D : 1 \leq i \leq R, j \in \mathbb{Z}\}$ is isomorphic to $\mathcal{DZ}^R$, and is convex in $\mathcal{DP}(S)$ by Proposition 5.1.

Conversely, every k-dimensional cube in $\mathcal{DZ}^r$ singles out a unique (up to isotopy) collection of k pairwise distinct, pairwise disjoint incompressible subsurfaces of S, each of complexity 1. Thus the result follows. $\square$

Theorem 1.3 also implies the following:

Corollary 1.6. *Given any surface S, and any k, there exist points in $\mathcal{DP}(S)$ at distance k with only a finite number of geodesics between them.*

Proof. Let D be a rank-realizing multicurve, and let $X_1, \ldots, X_R$ be the R incompressible subsurfaces of complexity 1 in $S \setminus D$. Let $P \in \mathcal{DP}_D(S)$ and let α_i be the unique curve of P that essentially intersects X_i. For all $i = 1, \ldots, R$, choose a geodesic ray ω_i in $\mathcal{P}(X_i)$ issued from α_i, and denote by $\omega_{i,j}$ the j-th vertex along ω_i. Let $Q_k = \omega_{1,k} \cup \ldots, \omega_{R,k} \cup D$. By Proposition 5.1, $d_{\mathcal{DP}(S)}(P, Q_k) = k$ for all k.

By Theorem 4.7, any geodesic path in $\mathcal{DP}(S)$ between P and Q_k is entirely contained in $DP_D(S)$. Furthermore, it follows from the choice of Q_k that the projection to $\mathcal{DP}(X_i)$ of a geodesic path in $\mathcal{DP}(S)$ between P and Q_k is a geodesic path in $\mathcal{DP}(X_i)$ between α_i and $\omega_{i,k}$. The result now follows because $\mathcal{DP}(X_i)$ is isomorphic to a Farey graph, and there are only finitely many geodesic paths between any two points of a Farey graph. $\square$

<div align="center">APPENDIX: STRONG CONVEXITY VS. FINITE GEODESICITY</div>

In this section, we prove:

Proposition 5.2. *Let S be a connected orientable surface of negative Euler characteristic. We denote by $\mathcal{C}(S)$ either $\mathcal{P}(S)$ or $\mathcal{DP}(S)$ or $\mathcal{CP}(S)$. The following two properties are equivalent:*

(1) Given a simple closed curve $\alpha \subset S$, the subset $\mathcal{C}_\alpha(S)$ of $\mathcal{C}(S)$ spanned by the vertices corresponding to pants decompositions containing α is totally geodesic,

(2) Given two vertices $P, Q \in \mathcal{C}(S)$, there are finitely many geodesic segments joining P to Q in $\mathcal{C}(S)$.

Proof. Let us first show that $1) \Rightarrow 2)$. More precisely, we will show that $[\text{not } 2)] \Rightarrow [\text{not } 1)]$. Consider an infinite family $\{\omega_n, n \in \mathbb{N}\}$ of distinct geodesic arcs joining two points $P, Q \in \mathcal{C}(S)$ and denote the vertices of ω_n by $\omega_{n,i}$, $0 \le i \le q(n)$, where $d_{\mathcal{C}(S)}(\omega_{n,i-1}, \omega_{n,i}) = 1$ when $\mathcal{C}(S) = \mathcal{P}(S)$ or $\mathcal{DP}(S)$ and $d_{\mathcal{C}(S)}(\omega_{n,i-1}, \omega_{n,i}) = \sqrt{k}$, for some integer $k \le [\frac{3g+p-2}{2}]$, when $\mathcal{C}(S) = \mathcal{CP}(S)$. Notice that $q(n) = d_{\mathcal{C}(S)}(P, Q)$ if $\mathcal{C}(S) = \mathcal{P}(S)$ or $\mathcal{DP}(S)$ but $q(n) \le d_{\mathcal{C}(S)}(P, Q)$ may depend on n for $\mathcal{C}(S) = \mathcal{CP}(S)$. In the latter case, we first extract a subsequence such that $q(n)$ does not depend on n. Let $0 < i_0 < q(n) = q$ be the smallest index such that $\{\omega_{n,i_0}\}$ contains infinitely many distinct points. Let us extract a subsequence such that $\omega_{n,i}$ does not depend on n for $i < i_0$ and $\{\omega_{n,i_0}\} \ne \{\omega_{m,i_0}\}$ for any $n \ne m$. Extract a further subsequence such that the set of leaves of ω_{n,i_0-1} that are not leaves of ω_{n,i_0} does not depend on n (namely we fix the subsurfaces in which the elementary moves happen). Extract a subsequence one last time so that for any i, $\omega_{n,i}$ converges in the Hausdorff topology to a geodesic lamination $\omega_{\infty,i}$.

Since all the pants decompositions $\{\omega_{n,i_0}\}$ are distinct, there is a leaf α of $\{\omega_{n,i_0-1}\}$ which is not a leaf of any $\{\omega_{n,i_0}\}$ and such that the leaves α_n of the pants decompositions $\{\omega_{n,i_0}\}$ that intersect α form an infinite family. The lamination ω_{∞,i_0} contains α and some leaves spiraling towards α. Since ω_{n,i_0} and ω_{n,i_0+1} are adjacent vertices, we have $i(\omega_{n,i_0}, \omega_{n,i_0+1}) \le 3g + p - 2$ for any n. It follows that ω_{∞,i_0+1} also contains α. If, furthermore, ω_{∞,i_0+1} contains some leaves spiraling towards α, then ω_{∞,i_0+2} contains α, and the same holds for ω_{∞,i_0+3}, etc. On the other hand, $\omega_{\infty,q} = Q$ does not contain any leaves spiraling towards α. It follows that there is $i_1 > i_0$ such that ω_{∞,i_1} contains α as an isolated leaf. Then for, n large enough, ω_{n,i_1} contains α, i.e. $\omega_{n,i_1} \subset \mathcal{C}_\alpha(S)$. Since $\omega_{n,i_0-1} \subset \mathcal{C}_\alpha(S)$ and $\omega_{n,i_0} \not\subset \mathcal{C}_\alpha(S)$, the geodesic segment joining ω_{n,i_0-1} to ω_{n,i_1} and passing through $\{\omega_{n,i}, i_0 \le i \le i_1 - 1\}$ is not contained in $\mathcal{C}_\alpha(S)$. Thus we have proved that 1) is not satisfied.

15

University of Tokyo IPMU, Japan 277-8583

E-mail address: kenneth.shackleton@impu.jp

URL: http://ipmu.jp/kenneth.shackleton/