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Complex Analysis and Operator Theory

On the Strong Kreiss Resolvent Condition

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Abstract It is shown that a Banach algebra element satisfies the strong Kreiss resolvent condition if and only if some (hence any) power of it does.

Keywords Banach algebra · Resolvent · The Laplace transform · The Hankel contour

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1 Introduction

Let A be a (complex) Banach algebra with norm $\|\cdot\|$ and unit element. We denote the spectrum of $a \in A$ by $\sigma(a)$, and the resolvent function of a by $R(a,\lambda) := (\lambda - a)^{-1}, \lambda \notin \sigma(a)$. Let us recall (see, e.g., [10,12]) that an element $a \in A$ with spectrum in the unit disc is said to satisfy the strong Kreiss resolvent condition with constant $M \geq 1$ if

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$$\|R^n(a,\lambda)\| \le \frac{M}{(|\lambda|-1)^n} \quad \text{for all} \quad |\lambda|>1, \quad \text{and} \quad n=1,2,\dots. \qquad \text{[SR]}$$

We recall that the condition [SR] is equivalent to the condition

$$\|e^{za}\| \le Me^{|z|}, \quad \text{for all } z \in \mathbb{C}. \tag{1.1}$$

In this article we prove that if the condition [SR] holds for an element $a \in A$, then it also holds for the element a^m, for any integer $m \ge 2$ (with constant depending on m). It means that if the condition (1.1) holds, then there exists $M_m \ge 1$ such that

$$\|e^{za^m}\| \le M_m e^{|z|} \quad \text{for all } z \in \mathbb{C}. \tag{1.2}$$

We also prove the converse statement: if the condition [SR] holds for an element a^m for some integer $m \ge 2$, then it also holds for the element a. (See Theorems 3.1 and 3.3 below.)

It was proved in [7] that if an operator T on a Hilbert space satisfies the strong Kreiss resolvent condition, then so does the operator T^m for any $m \in \mathbb{N}$.

This natural relation between the properties of powers, in the case of the classical Kreiss resolvent condition (only $n=1$ in [SR] above), was observed in [2] and [6]. The same relationship also holds in the class of operators satisfying the uniform Kreiss resolvent condition

$$\left\|\sum_{k=0}^{n} \frac{a^k}{\lambda^{k+1}}\right\| \le \frac{M_0}{|\lambda|-1}, \quad \text{for all} \quad |\lambda|>1, \quad \text{and} \quad n=1,2,\dots,$$

which is larger than the class of [SR]-operators [6]. So the present paper completes naturally this series of results, which can obviously be formulated also for Banach algebras. Moreover, these results are particularly interesting for the [SR] class in view of the above exponential inequalities (1.1) and (1.2). Actually, in the present paper we work with these exponential formulations of the [SR] condition.

To illustrate the strength of the above three Kreiss type resolvent conditions, let us mention their consequences for the Cesàro means

$$M_n(a) := \frac{1}{n}\sum_{k=0}^{n-1} a^k, \quad n=1,2,\dots.$$

First, the classical Kreiss resolvent condition, for an element $a \in A$, does not imply boundedness of $M_n(a)$, but it does imply that

$$\|[M_n(a)]^2\| \le \text{const}, \quad n=1,2,\dots,$$

see [12]. Next, the stronger [SR] condition, for an element $a \in A$, yields that

$$\|M_n(a)\| \le \text{const}, \quad n=1,2,\dots,$$

because the [SR] condition implies the uniform Kreiss resolvent condition [6], which is, finally, equivalent to the uniform boundedness of all $M_n(\lambda a)$, $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$, by [9].

The [SR] condition, though weaker than power boundedness of the element a, still has (the same) important consequences for the underrelaxed elements (i.e., *the strict* convex combinations of 1 and a), namely, that they are power-bounded and the consecutive differences of their powers converge to zero, with order not exceeding $\mathrm{O}(n^{-1/2})$. The exponent $-1/2$ is universal for all elements a satisfying [SR]. On the other hand, it is not known whether the [SR] condition can replace the power boundedness in the well-known Esterle–Katznelson–Tzafriri theorem. See [10].

The elegant relations between the properties of powers, like in the three resolvent conditions mentioned above, do not always hold (in both directions) with respect to some other natural properties, for example, the uniform ergodicity (i.e., the norm convergence of $M_n(a)$ as $n \to \infty$). See [5,8]. So this seems to justify our present result.

To indicate the ideas of the present paper, let us consider the implication (1.1) $\Rightarrow$ (1.2) for $m = 2$. The Weierstrass formula (see., e.g., [1, p. 219 and Example 3.14.15]) says that

$$e^{za^2} = \frac{1}{2\sqrt{\pi |z|}} \int_0^\infty e^{-s^2/(4|z|)} [e^{se^{i\theta}a} + e^{-se^{i\theta}a}]\, ds, \quad z \in \mathbb{C},\ z = |z|e^{2i\theta} \neq 0, \tag{1.3}$$

for any element $a \in A$. In addition, if $a \in A$ satisfies (1.1), then we have

$$\|e^{za^2}\| \le \frac{M}{\sqrt{\pi |z|}} \int_0^\infty e^{-s^2/(4|z|)} e^s\, ds \le \frac{M}{\sqrt{\pi |z|}} \int_{-\infty}^\infty e^{-s^2/(4|z|)} e^s\, ds = 2Me^{|z|}.$$

Hence we get (1.1) $\Rightarrow$ (1.2), for $m = 2$. So, our main idea, in this direction, consists in obtaining a generalization of (1.3) for the higher powers of $a \in A$. See Theorem 3.1 below.

For the proof of the implication (1.2) $\Rightarrow$ (1.1), first of all we prove that (1.2) implies the estimate

$$\|C_m(za)\| \le \tilde{M}_m e^{|z|},\ z \in \mathbb{C}, \quad C_m(za) := \frac{1}{m} \sum_{k=0}^{m-1} e^{\nu_k za}, \tag{1.4}$$

where $\nu_k = e^{i2\pi k/m}$, $k = 0, 1, \ldots, m-1$. The function $C_m(za)$ is known as a hyperbolic function of order m, see [4, Sect. 18.2]. Then estimate (1.1) follows from (1.4) by elementary calculations (see Proposition 2.5 and Corollary 2.6). On the other hand, for the proof of the implication (1.2) $\Rightarrow$ (1.4) we use the power series expansion

$$C_m(za) = \sum_{l=0}^{\infty} \frac{(za)^{ml}}{(ml)!}, \quad z \in \mathbb{C}, \quad a \in A, \tag{1.5}$$

and an appropriate $(m-1)$-times integral representation of this function (see (2.13) below). This representation allows us to represent $C_m(za)$ as an integral of the exponential $\exp[z^m a^m / m z_1 \dots z_{m-1}]$, $z_j \in \mathbb{C}$, $j = 1, \dots, m-1$.

2 Auxiliary Results

Let $L_1(\mathbb{R}_+; e^{\sigma t})$, $\sigma \in \mathbb{R}$, be the space of all measurable functions $f : \mathbb{R}_+ \to \mathbb{C}$ such that

$$\int_0^\infty |f(t)| e^{\sigma t}\, dt < \infty.$$

They have the Laplace transform

$$(\mathcal{L}g)(\lambda) := \int_0^\infty e^{-\lambda t} g(t)\, dt, \quad \mathrm{Re}(\lambda) > \omega, \quad g \in L_1(\mathbb{R}_+; e^{-\omega t}).$$

Let $\alpha \in (0, 1)$. By [13, Ch. 9, Sect. 11], there is a function $f_{t,\alpha} \in L_1(\mathbb{R}_+)$ such that

$$e^{-t\lambda^\alpha} = \int_0^\infty e^{-s\lambda} f_{t,\alpha}(s)\, ds, \quad t > 0, \quad \mathrm{Re}\,\lambda > 0, \tag{2.1}$$

where the branch of λ^α is taken so that $\mathrm{Re}\,(\lambda^\alpha) > 0$ for $\mathrm{Re}\,(\lambda) > 0$. Namely,

$$f_{t,\alpha}(s) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{s\lambda} e^{-t\lambda^\alpha}\, d\lambda, \quad \sigma > 0, \quad s, t > 0. \tag{2.2}$$

Deforming the path yields the explicit formula

$$f_{t,\alpha}(s) = \frac{1}{\pi} \int_0^\infty e^{sr\cos\theta} e^{-tr^\alpha \cos(\alpha\theta)} \sin(st\sin\theta - tr^\alpha \sin(\alpha\theta) + \theta)\, dr,$$

where $\theta \in [\pi/2, \pi]$ is arbitrary (see [13, Ch. 9, Sect. 11]). In particular, for $\theta = \pi$, we have

$$f_{t,\alpha}(s) = \frac{1}{\pi} \int_0^\infty e^{-sr} e^{-tr^\alpha \cos(\pi\alpha)} \sin(tr^\alpha \sin(\pi\alpha))\, dr. \tag{2.3}$$

One can show [13, Ch. 9, Sect. 11] that for all $\alpha \in (0, 1)$, $t > 0$ the function $f_{t,\alpha}$ is positive and for $\alpha = 1/2$ it has the following explicit form

$$f_{t,1/2}(s) = \frac{t}{2\sqrt{\pi} s^{3/2}} e^{-t^2/(4s)}. \tag{2.4}$$

From (2.3) we see that the estimate

$$f_{t,\alpha}(s) \le \frac{1}{\pi} \int_0^\infty e^{-sr} e^{r^\alpha} \, dr, \quad s > 0, \ t \in (0, 1), \tag{2.5}$$

is true. On the other hand, using (2.2) and the simple inequality

$$2^{\beta-1}(u^\beta + v^\beta) \le (u+v)^\beta, \quad u, \ v \ge 0, \quad \beta \in (0, 1),$$

we have the estimate

$$\begin{aligned} f_{t,\alpha}(s) &\le \frac{e^{\sigma s}}{\pi} \int_0^\infty e^{-t \mathrm{Re}\,(\sigma + iu)^\alpha} \, du \le \frac{e^{\sigma s}}{\pi} \int_0^\infty e^{-t(\sigma^2+u^2)^{\alpha/2} \cos(\pi\alpha/2)} \, du \\ &\le \frac{e^{\sigma s}}{\pi} e^{-t\sigma^\alpha 2^{\alpha/2-1} \cos(\pi\alpha/2)} \int_0^\infty e^{-tu^\alpha 2^{\alpha/2-1} \cos(\pi\alpha/2)} \, du, \quad s, t > 0. \end{aligned}$$

Consequently, for any $\sigma > 0$ we obtain the inequality

$$f_{t,\alpha}(s) \le a_\alpha e^{\sigma s} e^{-b_\alpha \sigma^\alpha t}, \quad s > 0, \ t \ge 1, \tag{2.6}$$

for some constants $a_\alpha, b_\alpha > 0$. From (2.5) and (2.6) we conclude that, for any $s > 0$, $\omega > 0$, the function $t \mapsto f_{t,\alpha}(s) \in L_1(\mathbb{R}_+; e^{\omega t})$.

The following statement is quite analogous to [1, Proposition 1.6.8].

Proposition 2.1 *Let $u \in L_1(\mathbb{R}_+; e^{-\omega t})$ for all $\omega > 0$, and let for some $\alpha \in (0, 1)$,*

$$v(s) := \int_0^\infty f_{t,\alpha}(s) u(t) dt, \quad \mathrm{Re}(\lambda) > 0.$$

Then

$$(\mathcal{L}v)(\lambda) = (\mathcal{L}u)(\lambda^\alpha), \quad \mathrm{Re}(\lambda) > 0.$$

Proof Fubini's theorem, estimate (2.6) for small $\sigma > 0$, and equation (2.1) give

$$(\mathcal{L}v)(\lambda) = \int_0^\infty e^{-\lambda s} \int_0^\infty f_{t,\alpha}(s)u(t)dtds$$
$$= \int_0^\infty u(t) \int_0^\infty e^{-\lambda s} f_{t,\alpha}(s)dsdt = \int_0^\infty e^{-\lambda^\alpha t}u(t)dt = (\mathcal{L}u)(\lambda^\alpha), \quad \mathrm{Re}(\lambda) > 0.$$

□

For $\tau > 0$ and $\beta > 1$ define the function

$$u_{\tau,\beta}(t) := \chi(t-\tau)\frac{(t-\tau)^{\beta-2}}{\Gamma(\beta-1)}, \quad t > 0,$$

where $\chi = \chi_{\mathbb{R}_+}$ denotes the characteristic function of $\mathbb{R}_+$ on $\mathbb{R}$, and Γ is the gamma function. Then the Laplace transform is

$$(\mathcal{L}u_{\tau,\beta})(\lambda) = \frac{e^{-\lambda\tau}}{\Gamma(\beta-1)}\int_0^\infty e^{-\lambda t}t^{\beta-2}\,dt = \frac{e^{-\lambda\tau}}{\lambda^{\beta-1}}, \quad \mathrm{Re}(\lambda) > 0. \tag{2.7}$$

Let us define, for $\beta > 1$ and $s, \tau > 0$, the function

$$Q_\beta(s;\tau) := \int_0^\infty f_{t,1/\beta}(s)u_{\tau,\beta}(t)\,dt = \frac{1}{\Gamma(\beta-1)}\int_\tau^\infty f_{t,1/\beta}(s)(t-\tau)^{\beta-2}dt. \tag{2.8}$$

Note that the function Q_β is positive because $f_{t,1/\beta}(s)$ is. From Proposition 2.1 and (2.7), (2.8) we have the following statement.

Corollary 2.2 *The relation*

$$\int_0^\infty e^{-\lambda s}Q_\beta(s;\tau)\,ds = \frac{e^{-\tau\lambda^{1/\beta}}}{\lambda^{(\beta-1)/\beta}}, \quad \mathrm{Re}(\lambda) > 0, \quad \tau > 0,$$

is true.

Proposition 2.3 *The inequality*

$$\int_0^\infty Q_\beta(s;\tau)e^\tau\,d\tau \le \beta e^s, \quad s > 0,$$

is true.

Proof First of all, using Corollary 2.2 we obtain

$$\int_0^\infty e^\tau \int_0^\infty e^{-\lambda s} Q_\beta(s;\tau)ds\,d\tau = \frac{1}{\lambda^{(\beta-1)/\beta}} \int_0^\infty e^\tau e^{-\tau\lambda^{1/\beta}}\,d\tau$$
$$= \frac{1}{\lambda^{(\beta-1)/\beta}(\lambda^{1/\beta}-1)}, \quad \mathrm{Re}\,(\lambda) > 1.$$

Then, because $Q_\beta(s;\tau)$ is a positive function, using Fubini's theorem and Corollary 2.2 once again, we have

$$\int_0^\infty e^{-\lambda s} \int_0^\infty Q_\beta(s;\tau)e^\tau d\tau\,ds = \frac{1}{\lambda^{(\beta-1)/\beta}(\lambda^{1/\beta}-1)}, \quad \mathrm{Re}(\lambda) > 1.$$

Then, using the relation

$$\lim_{\tau\to+\infty} \int_{\sigma-i\tau}^{\sigma+i\tau} \frac{e^{s\lambda}\,d\lambda}{\lambda-2\sigma} = 0, \quad s > 0,\ \sigma > 0,$$

and the Laplace inversion formula, we have

$$\int_0^\infty Q_\beta(s;\tau)e^\tau d\tau = \lim_{\tau\to+\infty} \frac{1}{2\pi i} \int_{\sigma-i\tau}^{\sigma+i\tau} \frac{e^{s\lambda}\,d\lambda}{\lambda(1-\lambda^{-1/\beta})}$$
$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{s\lambda} \left[\frac{1}{\lambda(1-\lambda^{-1/\beta})} - \frac{1}{\lambda-2\sigma}\right] d\lambda, \quad \sigma > 1,$$

and by the Cauchy theorem we obtain

$$\int_0^\infty Q_\beta(s;\tau)e^\tau d\tau = \beta e^s + \frac{1}{2\pi i}\left\{\int_{-\infty}^0 + \int_0^{-\infty}\right\} \frac{e^{s\lambda}\,d\lambda}{\lambda(1-\lambda^{-1/\beta})}$$
$$= \beta e^s + \frac{1}{2\pi i}\int_0^\infty \frac{e^{-s\rho}}{\rho(1-\rho^{-1/\beta}e^{-i\pi/\beta})}d\rho - \frac{1}{2\pi i}\int_0^\infty \frac{e^{-s\rho}}{\rho(1-\rho^{-1/\beta}e^{i\pi/\beta})}d\rho$$
$$= \beta e^s + \frac{1}{2\pi i}\int_0^\infty \frac{e^{-s\rho}}{\rho^{1-1/\beta}}\left(\frac{1}{\rho^{1/\beta}-e^{-i\pi/\beta}} - \frac{1}{\rho^{1/\beta}-e^{i\pi/\beta}}\right) d\rho$$
$$= \beta e^s - \frac{\beta\sin(\pi/\beta)}{\pi}\int_0^\infty \frac{e^{-sr^\beta}}{(r^2-2r\cos(\pi/\beta)+1)}dr \le \beta e^s, \quad s > 0.$$

□

Using expression (2.3) we can obtain the following statement.

Lemma 2.4 *The integral representation*

$$Q_\beta(s;\tau) = \frac{1}{\pi}\int_0^\infty \frac{e^{-sr}e^{-\tau r^{1/\beta}\cos(\pi/\beta)}}{r^{(\beta-1)/\beta}} \sin(\tau r^{1/\beta}\sin(\pi/\beta) + \pi(\beta-1)/\beta)\, dr \tag{2.9}$$

holds.

Proof We have

$$\begin{aligned} Q_\beta(s;\tau) &= \frac{1}{\pi\Gamma(\beta-1)}\int_\tau^\infty \left\{\int_0^\infty e^{-sr}e^{-tr^{1/\beta}\cos(\pi/\beta)}\sin(tr^{1/\beta}\sin(\pi/\beta))\, dr\right\}(t-\tau)^{\beta-2}dt \\ &= \frac{1}{\pi\Gamma(\beta-1)}\int_0^\infty e^{-sr}e^{-\tau r^{1/\beta}\cos(\pi/\beta)}F(r;\tau)\, dr, \end{aligned} \tag{2.10}$$

where the integral

$$\begin{aligned} F(r;\tau) := &\int_0^\infty e^{-tr^{1/\beta}\cos(\pi/\beta)}\sin((t+\tau)r^{1/\beta}\sin(\pi/\beta))\, t^{\beta-2}dt \\ &= \cos(\tau r^{1/\beta}\sin(\pi/\beta))\int_0^\infty e^{-tr^{1/\beta}\cos(\pi/\beta)}\sin(tr^{1/\beta}\sin(\pi/\beta))t^{\beta-2}dt \\ &\quad + \sin(\tau r^{1/\beta}\sin(\pi/\beta))\int_0^\infty e^{-tr^{1/\beta}\cos(\pi/\beta)}\cos(tr^{1/\beta}\sin(\pi/\beta))t^{\beta-2}dt. \end{aligned}$$

Then (see [11, 2.5.30.8]),

$$F(r;\tau) = \frac{\Gamma(\beta-1)}{r^{(\beta-1)/\beta}}\sin(\tau r^{1/\beta}\sin(\pi/\beta) + \pi(\beta-1)/\beta). \tag{2.11}$$

Then, from (2.10), (2.11) we obtain expression (2.9). □

Let $\nu_k = e^{i2\pi k/m}$, $k = 0, 1, \ldots, m-1$, where $m \in \mathbb{N}$ and $m \geq 2$. Let for $a \in A$ the function $C_m(za)$ be defined by (1.4). Define for $l = 0, 1, \ldots, m-1$ the analytic functions

$$C_{m,0}(za) := C_m(za), \quad C_{m,l}(za) := \frac{1}{m}\sum_{k=0}^{m-1}\frac{e^{\nu_k za}}{\nu_k^l}, \quad z \in \mathbb{C}.$$

Proposition 2.5 (cf. [4, Sect. 18.2, (10)]) *The formulas*

$$e^{\nu_p za} = \sum_{l=0}^{m-1} \nu_l^p C_{m,l}(za), \quad p = 0, 1, \dots, m-1,$$

hold.

Proof Indeed, using the relations

$$\nu_k^l = \nu_l^k, \quad \sum_{l=0}^{m-1} \nu_l^n = 0, \quad n = \pm 1, \pm 2, \dots, \pm(m-1),$$

we have

$$\begin{aligned}\sum_{l=0}^{m-1} \nu_l^p C_{m,l}(za) &= \sum_{k=0}^{m-1} e^{\nu_k za} \left(\frac{1}{m} \sum_{l=0}^{m-1} \frac{\nu_l^p}{\nu_k^l} \right) \\ &= \sum_{k=0}^{m-1} e^{\nu_k za} \left(\frac{1}{m} \sum_{l=0}^{m-1} \nu_l^{p-k} \right) = e^{\nu_p za}.\end{aligned}$$

□

Corollary 2.6 *If for some* $m = 2, 3, \dots,$ *the element* $a \in A$ *satisfies the condition*

$$\|C_m(za)\| \le M e^{|z|} \quad z \in \mathbb{C}, \tag{2.12}$$

then

$$\|e^{za}\| \le \left(M \sum_{l=0}^{m-1} \|a^l\| \right) e^{|z|}, \quad z \in \mathbb{C}.$$

Proof Note that for $C_{m,l}(za)$ we have the expressions

$$C_{m,l}(za) = \frac{1}{(l-1)!} \int_0^z (z-s)^{l-1} a^l C_m(sa)\, ds, \quad z \in \mathbb{C}, \quad l = 1, \dots, m-1.$$

Then from Proposition 2.5, using the simple estimate

$$\frac{1}{(l-1)!} \int_0^t (t-s)^{l-1} e^s\, ds \le e^t, \quad t \ge 0, \quad l = 1, 2, \dots,$$

we have

$$
\begin{aligned}
\|e^{za}\| &\le \|C_m(za)\| + \sum_{l=1}^{m-1} \frac{\|a^l\|}{(l-1)!} \left\| \int_0^z (z-s)^{l-1} C_m(sa)\, ds \right\| \\
&\le Me^{|z|} + M \sum_{l=1}^{m-1} \frac{\|a^l\|}{(l-1)!} \int_0^{|z|} (|z|-s)^{l-1} e^s\, ds \\
&\le Me^{|z|} + M \sum_{l=1}^{m-1} \|a^l\| e^{|z|} = M \sum_{l=0}^{m-1} \|a^l\| e^{|z|}, \quad z \in \mathbb{C}.
\end{aligned}
$$

□

For $\rho > 0$ consider the Hankel contour

$$\gamma_\rho := \gamma_\rho^{(-)} \cup \gamma_\rho^{(0)} \cup \gamma_\rho^{(+)},$$

where

$$\gamma_\rho^{(-)} = (-\infty, -\rho), \quad \gamma_\rho^{(0)} = \{z = \rho e^{i\phi},\ \phi \in [-\pi, \pi]\}, \quad \gamma_\rho^{(+)} = (-\rho, -\infty).$$

Then, using the identities

$$
\begin{aligned}
1 &= (2\pi)^{(1-m)/2} \sqrt{m} \prod_{k=1}^{m-1} \Gamma\left(\frac{k}{m}\right), \\
\frac{(mn)!}{n!} &= (2\pi)^{(1-m)/2} \sqrt{m} m^{mn} \prod_{k=1}^{m-1} \Gamma\left(n + \frac{k}{m}\right), \quad n \in \mathbb{N},
\end{aligned}
$$

and the integral representation of the gamma function [3, Sect. 1.6],

$$\frac{1}{\Gamma\left(n + \frac{k}{m}\right)} = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{e^z}{z^{n+k/m}}\, dz, \quad n = 0, 1, \ldots,$$

we obtain

$$
\begin{aligned}
C_m(za) &= \frac{1}{m} \sum_{k=1}^{m} e^{\nu_k za} = \sum_{n=0}^{\infty} \frac{(za)^{mn}}{(mn)!} = \frac{1}{i^{m-1}(2\pi)^{(m-1)/2}\sqrt{m} m^{(m-1)/2}} \\
&\quad \times \int_{\gamma_\rho} \cdots \int_{\gamma_\rho} \frac{e^{(z_1+\cdots+z_{m-1})/m}}{\prod_{j=1}^{m-1} z_j^{j/m}} \exp\left[(za)^m / m z_1 \ldots z_{m-1}\right] dz_1 \cdots dz_{m-1}.
\end{aligned}
\tag{2.13}
$$

3 Main Results

Now we are able to give the promised generalization of (1.3) for an integer $m \geq 2$.

Theorem 3.1 *For $a \in A$ and an integer $m \geq 2$ the following formula*

$$e^{sa^m} = \int_0^\infty Q_m(s;\tau)C_m(\tau a)\,d\tau, \quad s > 0, \tag{3.1}$$

where $Q_\beta(s;\tau)$ is defined by (2.8), is true. Moreover, if

$$\|e^{\nu_k sa}\| \leq Me^s, \quad s \geq 0, \quad k = 1, 2, \ldots, m,$$

then the estimate

$$\|e^{sa^m}\| \leq mMe^s, \quad s \geq 0,$$

holds. In particular, if the element a satisfies $[SR]$ condition, then

$$\|e^{za^m}\| \leq mMe^{|z|}, \quad z \in \mathbb{C}.$$

Proof Indeed, using Corollary 2.2 and the equality

$$R(a^m, \mu^m) = \frac{1}{m\mu^{m-1}}\sum_{k=1}^m R(\nu_k a, \mu) = \frac{1}{m\mu^{m-1}}\sum_{k=1}^m \int_0^\infty e^{-\mu\tau}e^{\tau\nu_k a}\,d\tau, \quad \mu > \|a\|,$$

we have for the Laplace transform, for all $\lambda > \|a\|^m$:

$$\begin{aligned}\int_0^\infty e^{-\lambda s}\int_0^\infty Q_m(s;\tau)C_m(\tau a)\,d\tau\;ds &= \int_0^\infty \left(\int_0^\infty e^{-\lambda s}Q_m(s;\tau)\,ds\right)C_m(\tau a)\,d\tau \\ &= \frac{1}{\lambda^{(m-1)/m}}\int_0^\infty e^{-\tau\lambda^{1/m}}C_m(\tau a)\,d\tau = \frac{1}{m\lambda^{(m-1)/m}}\sum_{k=1}^m\int_0^\infty e^{-\tau\lambda^{1/m}}e^{\nu_k\tau a}\,d\tau \\ &= \frac{1}{m\lambda^{(m-1)/m}}\sum_{k=1}^m R(\nu_k a, \lambda^{1/m}) = R(a^m, \lambda) = \int_0^\infty e^{-\lambda s}e^{sa^m}\,ds.\end{aligned}$$

Now (3.1) follows by the uniqueness theorem for the Laplace transform [1, Sect. 1.7].

Next, using the property $Q_\beta(s;\tau) > 0$, the assumption $\|C_m(\tau a)\| \leq Me^\tau$, $\tau \geq 0$, and Proposition 2.3, we have

$$\|e^{sa^m}\| \leq \int_0^\infty Q_m(s;\tau)\|C_m(\tau a)\|\,d\tau \leq M\int_0^\infty Q_m(s;\tau)e^\tau\,d\tau \leq mMe^s, \quad s \geq 0.$$

From this, the last statement of the theorem obviously follows. □

Observe that when $m = 2$, the formula (3.1) reduces to the Weierstrass formula (1.3), by (2.8) with $\beta = 2$, and (2.4).

The equation (2.13) allows us to prove the following statement.

Theorem 3.2 *Let for an integer $m \geq 2$ the element a^m satisfy the $[SR]$ condition*

$$\|e^{za^m}\| \leq M_m e^{|z|}, \quad z \in \mathbb{C}.$$

Then there exists a constant $c_m > 0$ such that the estimate

$$\|C_m(za)\| \leq c_m M_m e^{|z|}, \quad z \in \mathbb{C},$$

holds.

Proof Let $z \in \mathbb{C}$, $z \neq 0$, be fixed, and choose $\rho = |z|$ in the Hankel contour γ_ρ. Then from (2.13) we obtain

$$\begin{aligned}
\|C_m(za)\| &\leq \frac{M_m}{(2\pi)^{(m-1)/2}\sqrt{m}m^{(m-1)/2}} \\
&\quad \times \int_{\gamma_\rho} \cdots \int_{\gamma_\rho} \frac{e^{\mathrm{Re}\,(z_1+\cdots+z_{m-1})/m}}{\prod_{j=1}^{m-1} |z_j|^{j/m}} \exp[\rho^m/m|z_1|\ldots|z_{m-1}|]\,|dz_1|\cdots|dz_{m-1}| \\
&\leq \frac{M_m\, e^{\rho/m}}{(2\pi)^{(m-1)/2}\sqrt{m}m^{(m-1)/2}} \int_{\gamma_\rho} \cdots \int_{\gamma_\rho} \frac{e^{\mathrm{Re}\,(z_1+\cdots+z_{m-1})/m}}{\prod_{j=1}^{m-1} |z_j|^{j/m}}\,|dz_1|\cdots|dz_{m-1}| \\
&= \frac{M_m\, e^{\rho/m} 2^{(m-1)/2}}{\pi^{(m-1)/2}\sqrt{m}m^{(m-1)/2}} \cdot \prod_{j=1}^{m-1} L_{m,j}(\rho),
\end{aligned}$$

with the integrals

$$L_{m,j}(\rho) := \frac{1}{2}\int_{\gamma_\rho} \frac{e^{\mathrm{Re}\,\tau/m}}{|\tau|^{j/m}}\,|d\tau| = m^{1-j/m}\int_{\rho/m}^{\infty} \frac{e^{-s}}{s^{j/m}}\,ds + \pi\rho^{1-j/m} I_0(\rho/m),$$

where $I_0(\cdot)$ is the modified Bessel function

$$I_0(\rho) = \frac{1}{\pi}\int_0^{\pi} e^{\rho\,\cos\phi}\,d\phi.$$

Note that this function is increasing for $\rho > 0$, since it has positive derivative with respect to ρ on $(0, \infty)$.

Next, we have the estimates

$$\int_{\rho/m}^{\infty} \frac{e^{-s}}{s^{j/m}}\, ds \le \int_{0}^{\infty} \frac{e^{-s}}{s^{j/m}}\, ds = \Gamma\left(1 - \frac{j}{m}\right), \quad \rho > 0,$$

and

$$\int_{\rho/m}^{\infty} \frac{e^{-s}}{s^{j/m}}\, ds \le \frac{m^{j/m}}{\rho^{j/m}} \int_{\rho/m}^{\infty} e^{-s}\, ds = \frac{m^{j/m} e^{-\rho/m}}{\rho^{j/m}}, \quad \rho \in (m, \infty).$$

Then for $\rho \in (0, m)$, we deduce

$$\begin{aligned} L_{m,j}(\rho) &\le m^{1-j/m}\Gamma(1 - j/m) + \pi\rho^{1-j/m} I_0(\rho/m) \\ &\le m^{1-j/m}\,(\Gamma(1 - j/m) + \pi I_0(1)). \end{aligned}$$

So, for $\rho \in (0, m)$ we have

$$\|C_m(za)\| \le \frac{(2/\pi)^{(m-1)/2}}{\sqrt{m}} \prod_{j=1}^{m-1} (\Gamma(1 - j/m) + \pi I_0(1))\; M_m e^{\rho}.$$

For $\rho \ge m$ we have

$$\begin{aligned} L_{m,j}(\rho) &\le \frac{1}{\rho^{j/m}}[me^{-\rho/m} + \pi\rho I_0(\rho/m)] \\ &\le \frac{\sqrt{m}\sqrt{\rho}}{\rho^{j/m}}\left[1 + \pi\frac{\sqrt{\rho}}{\sqrt{m}} I_0(\rho/m)\right] \le c\frac{\sqrt{m}\sqrt{\rho}}{\rho^{j/m}} e^{\rho/m}, \\ &\qquad c = 1 + \pi \sup_{t\ge 1} \sqrt{t}e^{-t} I_0(t), \end{aligned}$$

and then we deduce

$$\begin{aligned} \|C_m(za)\| &\le M_m c^{m-1} \frac{e^{\rho/m} 2^{(m-1)/2}}{\pi^{(m-1)/2}\sqrt{m}m^{(m-1)/2}} \cdot \prod_{j=1}^{m-1} \left(\frac{\sqrt{m}\sqrt{\rho}}{\rho^{j/m}} e^{\rho/m}\right) \\ &= c^{m-1}\frac{(2/\pi)^{(m-1)/2}}{\sqrt{m}} \cdot M_m e^{\rho}, \quad \rho \ge m. \end{aligned}$$

The theorem is proved. □

From Theorem 3.2 and Corollary 2.6 we have the following statement.

Theorem 3.3 *Let for an integer $m \ge 2$ the element a^m satisfy the $[SR]$ condition*

$$\|e^{za^m}\| \le M_m e^{|z|}, \quad z \in \mathbb{C}.$$

Then the estimate

$$\|e^{za}\| \le \left(c_m M_m \sum_{l=0}^{m-1} \|a^l\| \right) e^{|z|}, \quad z \in \mathbb{C},$$

holds, where c_m is the constant from Theorem 3.2.

Let us conclude by the following remark. Let the element a satisfy the [SR] condition. Then by Theorem 3.1 we have

$$\|e^{za^m}\| \le M_m e^{|z|}, \quad z \in \mathbb{C}, \quad M_m \le mM, \quad m \in \mathbb{N}. \tag{3.2}$$

Then, from the integral representation

$$a^m = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{za^m}}{z^2}\, dz$$

and (3.2) we obtain $\|a^m\| \le eM_m, m \in \mathbb{N}$. It means, in particular, that if $\sup_{m\in\mathbb{N}} M_m < \infty$, then the element a is power-bounded: $\sup_{m\in\mathbb{N}} \|a^m\| < \infty$. On other hand, there exist examples (see [12] for details) of Banach algebras A and elements $a \in A$ which satisfy the [SR] condition, but $\overline{\lim}_{m\to\infty} \|a^m\|/\sqrt{m} > 0$. It means that, in general, the constants M_m in (3.2) are not bounded, and may satisfy the condition $\overline{\lim}_{m\to\infty} M_m/\sqrt{m} > 0$.

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