Results. Math. 66 (2014), 481–490

1422-6383/14/030481-10
published online May 11, 2014
DOI 10.1007/s00025-014-0388-0

Results in Mathematics

J-Tangent Affine Hyperspheres

Zuzanna Szancer

Abstract. In this paper we study J-tangent affine hyperspheres. Under some additional conditions we give a local characterization of 3-dimensional J-tangent affine hyperspheres.

Mathematics Subject Classification (2010). 53A15, 53D15.

Keywords. Affine hypersurface, almost contact structure, affine hypersphere.

1. Introduction

Centro-affine real hypersurfaces with a J-tangent transversal vector field were first studied by Cruceanu in [1]. He proved that such hypersurfaces $f\colon M^{2n+1} \to \mathbb{C}^{n+1}$ can be locally expressed in the form

$$f(x_1, \ldots, x_{2n}, z) = Jg(x_1, \ldots, x_{2n}) \cos z + g(x_1, \ldots, x_{2n}) \sin z,$$

where g is some smooth function defined on an open subset of $\mathbb{R}^{2n}$. He also showed that if the induced almost contact structure is Sasakian then a hypersurface must be a hyperquadric. The latter result was generalized in [3] to arbitrary hypersurfaces with J-tangent transversal vector field.

Since the class of centro-affine hypersurfaces with a J-tangent transversal vector field is quite large, the question arises whether there are affine hyperspheres with a J-tangent Blaschke normal field. A nontrivial 3-dimensional example was provided in [4]. The main purpose of this paper is to give a local characterization of 3-dimensional J-tangent affine hyperspheres with involutive contact distribution $\mathcal{D}$.

In Sect. 2 we briefly recall basic formulas of affine diferential geometry and recall the notion of an affine hypersphere.

In Sect. 3 we recall the notion of a J-tangent transversal vector field, a definition of the induced almost contact structure as well as some results obtained in [3].

Section 4 contains the main results of this paper. We prove that there are no improper J-tangent affine hyperspheres and we give a local representation of 3-dimensional J-tangent affine hyperspheres under additional condition that the contact distribution is involutive.

2. Preliminaries

We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [2]. Let $f\colon M \to \mathbb{R}^{n+1}$ be an orientable connected differentiable n-dimensional hypersurface immersed in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then for any transversal vector field C we have

$$\mathrm{D}_X f_* Y = f_*(\nabla_X Y) + h(X,Y)C \tag{2.1}$$

and

$$\mathrm{D}_X C = -f_*(SX) + \tau(X)C, \tag{2.2}$$

where X, Y are vector fields tangent to M. It is known that ∇ is a torsion-free connection, h is a symmetric bilinear form on M, called the second fundamental form, S is a tensor of type $(1,1)$, called the shape operator, and τ is a 1-form, called the transversal connection form.

We assume that h is nondegenerate so that h defines a semi-Riemannian metric on M. If h is nondegenerate, then we say that the hypersurface or the hypersurface immersion is nondegenerate. In this paper we assume that f is always nondegenerate. We have the following

Theorem 2.1 ([2], Fundamental equations). *For an arbitrary transversal vector field C the induced connection ∇, the second fundamental form h, the shape operator S, and the 1-form τ satisfy the following equations:*

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{2.3}$$

$$(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \tag{2.4}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{2.5}$$

$$h(X,SY) - h(SX,Y) = 2d\tau(X,Y). \tag{2.6}$$

The Eqs. (2.3), (2.4), (2.5), and (2.6) are called the equations of Gauss, Codazzi for h, Codazzi for S and Ricci, respectively.

For a hypersurface immersion $f\colon M \to \mathbb{R}^{n+1}$ a transversal vector field C is said to be equiaffine (resp. locally equiaffine) if $\tau = 0$ (resp. $d\tau = 0$).

When f is nondegenerate, there exists a canonical transversal vector field C, called the affine normal (or the Blaschke normal field). The affine normal is uniquely determined up to sign by the following conditions: the metric volume form ω_h of h is ∇-parallel and coincides with the induced volume form Θ, where ω_h is defined by $|\omega_h(X_1,\ldots,X_n)| = |\det[h(X_i,X_j)]|^{1/2}$ and Θ is defined by $\Theta(X_1,\ldots,X_n) = \det[f_*X_1,\ldots,f_*X_n,C]$ for tangent vectors X_i (i=1,...,n). The affine immersion f with the Blaschke normal field C is called a Blaschke hypersurface. In this case fundamental equations can be rewritten as follows

Theorem 2.2 ([2], Fundamental equations). *For a Blaschke hypersurface f, we have the following fundamental equations:*

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{2.7}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \tag{2.8}$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X), \tag{2.9}$$

$$h(X,SY) = h(SX,Y). \tag{2.10}$$

A Blaschke hypersurface is called an affine hypersphere if $S = \lambda I$, where $\lambda = const.$

If $\lambda = 0$ f is called an improper affine hypersphere, if $\lambda \neq 0$ f is called a proper affine hypersphere.

For simplicity we shall omit f_* in front of vector fields in most cases.

3. Induced Almost Contact Structures

Let $\dim M = 2n+1$ and $f\colon M \to \mathbb{R}^{2n+2}$ be a nondegenerate affine hypersurface. We always assume that $\mathbb{R}^{2m} \simeq \mathbb{C}^m$ is endowed with the standard complex structure J. In particular, if $m = n+1$ we have

$$J(x_1,\ldots,x_{n+1},y_1,\ldots,y_{n+1}) = (-y_1,\ldots,-y_{n+1},x_1,\ldots,x_{n+1}).$$

Let C be a transversal vector field on M. We say that C is J-tangent if $JC_x \in f_*(T_xM)$ for every $x \in M$. We also define a distribution $\mathcal{D}$ on M as the biggest J invariant distribution on M, that is,

$$\mathcal{D}_x = f_*^{-1}(f_*(T_xM) \cap J(f_*(T_xM)))$$

for every $x \in M$. It is clear that $\dim \mathcal{D} = 2n$. A vector field X is called a $\mathcal{D}$-field if $X_x \in \mathcal{D}_x$ for every $x \in M$. We use the notation $X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$-fields. We say that the distribution $\mathcal{D}$ is nondegenerate if h is nondegenerate on $\mathcal{D}$.

Recall that a $(2n+1)$-dimensional manifold M is said to have an almost contact structure if there exist on M a tensor field φ of type (1,1), a vector field ξ and a 1-form η which satisfy

$$\varphi^2(X) = -X + \eta(X)\xi, \tag{3.1}$$

$$\eta(\xi) = 1 \tag{3.2}$$

for every $X \in TM$.

Let $f\colon M \to \mathbb{R}^{2n+2}$ be a nondegenerate hypersurface with a J-tangent transversal vector field C. Then we can define a vector field ξ, a 1-form η and a tensor field φ of type (1,1) as follows:

$$\xi := JC, \tag{3.3}$$

$$\eta|_{\mathcal{D}} = 0 \text{ and } \eta(\xi) = 1, \tag{3.4}$$

$$\varphi|_{\mathcal{D}} = J|_{\mathcal{D}} \text{ and } \varphi(\xi) = 0. \tag{3.5}$$

It is easy to see that (φ, ξ, η) is an almost contact structure on M. This structure is called the almost contact structure on M induced by C (or simply induced almost contact structure).

For an induced almost contact structure we have the following theorem

Theorem 3.1 ([3]). *If (φ, ξ, η) is the induced almost contact structure on M then the following equations hold:*

$$\eta(\nabla_X Y) = -h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X), \tag{3.6}$$

$$\varphi(\nabla_X Y) = \nabla_X \varphi Y + \eta(Y)SX - h(X, Y)\xi, \tag{3.7}$$

$$\begin{aligned}\eta([X, Y]) = {} & -h(X, \varphi Y) + h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X)) \\ & + \eta(Y)\tau(X) - \eta(X)\tau(Y),\end{aligned} \tag{3.8}$$

$$\varphi([X, Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X - \eta(X)SY + \eta(Y)SX, \tag{3.9}$$

$$\eta(\nabla_X \xi) = \tau(X), \tag{3.10}$$

$$\eta(SX) = h(X, \xi) \tag{3.11}$$

for every $X, Y \in \mathcal{X}(M)$.

4. *J*-Tangent Affine Hyperspheres

An affine hypersphere with a transversal J-tangent Blaschke normal field we call a J-tangent affine hypersphere.

It is obvious that the standard hypersphere $S^{2n+1}(r)$ in $\mathbb{R}^{2n+2}$

$$x_1{}^2 + x_2{}^2 + \ldots + x_{2n+2}{}^2 = r^2$$

is a J-tangent affine hypersphere, since it is an affine hypersphere and the affine normal field is orthogonal to it. The next example shows that there are also other J-tangent affine hyperspheres:

Example 4.1 ([4]). Let us consider the affine immersion f defined as follows:

$$f\colon \mathbb{R}^3 \ni (x, y, z) \mapsto \begin{bmatrix} \sin x & \sinh y \\ -\cos x & \sinh y \\ \cos x & \cosh y \\ \sin x & \cosh y \end{bmatrix} \cos z + \begin{bmatrix} \cos x & \cosh y \\ \sin x & \cosh y \\ -\sin x & \sinh y \\ \cos x & \sinh y \end{bmatrix} \sin z \in \mathbb{R}^4$$

with the transversal vector field

$$C\colon \mathbb{R}^3 \ni (x, y, z) \mapsto -f(x, y, z) \in \mathbb{R}^4.$$

f is a J-tangent affine hypersphere, since C is the affine normal field (what can be shown by straightforward computations) and $JC = f_z \in f_*TM$. Moreover, in the canonical coordinates on $\mathbb{R}^3$ we have

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, h is not positive definite.

As an immediate consequence of Theorem 3.1 we have the following:

Theorem 4.1. *There are no improper J-tangent affine hyperspheres.*

Proof. From Theorem 3.1 (formula (3.11)) we have $\eta(SX) = h(X, \xi)$ for all $X \in \mathcal{X}(M)$. Thus, if $S = 0$ then, $h(X, \xi) = 0$ for every $X \in \mathcal{X}(M)$, which contradicts nondegeneracy of h (since $\xi \neq 0$.) □

Now we can state the main result of this paper:

Theorem 4.2. *Let $f\colon M \mapsto \mathbb{R}^4$ be a J-tangent affine hypersphere with involutive distribution $\mathcal{D}$. Then f can be locally expressed in the form:*

$$f(x,y,z) = \lambda^{-\frac{5}{8}} \begin{bmatrix} \sin\sqrt{\lambda}x & \sinh\sqrt{\lambda}y \\ -\cos\sqrt{\lambda}x & \sinh\sqrt{\lambda}y \\ cos\sqrt{\lambda}x & \cosh\sqrt{\lambda}y \\ \sin\sqrt{\lambda}x & \cosh\sqrt{\lambda}y \end{bmatrix} \cos\lambda z + \lambda^{-\frac{5}{8}} \begin{bmatrix} \cos\sqrt{\lambda}x & \cosh\sqrt{\lambda}y \\ \sin\sqrt{\lambda}x & \cosh\sqrt{\lambda}y \\ -\sin\sqrt{\lambda}x & \sinh\sqrt{\lambda}y \\ \cos\sqrt{\lambda}x & \sinh\sqrt{\lambda}y \end{bmatrix} \sin\lambda z \in \mathbb{R}^4 \tag{4.1}$$

for some $\lambda > 0$.

We split the proof of Theorem 4.2 into several lemmas.

Lemma 4.1. *For every $X, Y \in \mathcal{D}$ we have*

$$h(\varphi X, \varphi Y) = -h(X, Y).$$

Proof. Since $\mathcal{D}$ is involutive and $\ker\eta = \mathcal{D}$ we have $\eta([X, Y]) = 0$ for all $X, Y \in \mathcal{D}$. Now using the formula (3.8) we obtain

$$h(X, \varphi Y) = h(Y, \varphi X) \tag{4.2}$$

for every $\mathcal{D}$-fields X and Y. Setting $Y := \varphi Y$ in (4.2) and using the fact that $\varphi^2 = -I$ on $\mathcal{D}$ we immediately get

$$-h(X, Y) = h(\varphi X, \varphi Y).$$

□

Lemma 4.2. *For every $x \in M$ there exists a neighbourhood U of x and a $\mathcal{D}$-field $X \in \mathcal{X}(U)$, $X \neq 0$ such that $h(X, X) = 1$, $h(\varphi X, \varphi X) = -1$, $h(X, \varphi X) = 0$.*

Proof. First observe that h is nondegenerate on $\mathcal{D}$, it means that for every $x \in M \quad h_x$ is nondegenerate on $\mathcal{D}_x$. We will prove it by contradiction, namely, suppose there exists $x \in M$ such that h_x is degenerate on $\mathcal{D}_x$. Now, we can find $w \in \mathcal{D}_x$, $w \neq 0$ such that $h_x(v, w) = 0$ for every $v \in \mathcal{D}_x$. From formula (3.11) we also have $h_x(\xi_x, w) = 0$. Since every vector $t \in T_xM$ can be expressed in the form

$$t = \alpha v + \beta \xi_x,$$

where $\alpha, \beta \in \mathbb{R}$, $v \in \mathcal{D}_x$, we obtain

$$h_x(w, t) = \alpha h_x(w, v) + \beta h_x(w, \xi_x) = 0$$

for all $t \in T_xM$. We have that h_x is nondegenerate on T_xM so it follows that $w = 0$, which contradicts the assumption.

Now we show that for every $x \in M$ we can find a $\mathcal{D}$-field Z such that

$$h_x(Z_x, \varphi Z_x) \neq 0.$$

Assume that there exists $x \in M$ such that for all $Z_x \in \mathcal{D}_x$ we have

$$h_x(Z_x, \varphi Z_x) = 0.$$

Then, for any $v, w \in \mathcal{D}_x$ we have:

$$\begin{aligned} 0 = h_x(v + w, \varphi v + \varphi w) &= h_x(v, \varphi v) + h_x(w, \varphi v) + h_x(v, \varphi w) \\ &+ h_x(w, \varphi w) = h_x(w, \varphi v) + h_x(v, \varphi w). \end{aligned}$$

Applying Lemma 4.1 we obtain

$$h_x(v, w) = 0$$

for all $v, w \in \mathcal{D}_x$, which contradicts nondegeneracy of h on $\mathcal{D}$. Let x be an arbitrary point of M and let $Z \in \mathcal{D}$ be such that $h_x(Z_x, \varphi Z_x) \neq 0$. Then there exists a neighbourhood U of x such that $h(Z, \varphi Z) \neq 0$ on U. Without loss of generality we can assume that $h(Z, \varphi Z) > 0$ on U (if $h(Z, \varphi Z) < 0$ we can replace Z by φZ). Now, we can define another vector field Y by the formula

$$Y := \alpha Z + \beta \varphi Z,$$

where

$$\alpha = \sqrt{\sqrt{h(Z, Z)^2 + h(Z, \varphi Z)^2} + h(Z, Z)}$$

and

$$\beta = \sqrt{\sqrt{h(Z, Z)^2 + h(Z, \varphi Z)^2} - h(Z, Z)}.$$

It is obvious that α and β are smooth and positive functions on U. Moreover

$$\begin{aligned} h(Y, \varphi Y) &= (\alpha^2 - \beta^2) h(Z, \varphi Z) - 2\alpha\beta h(Z, Z) \\ &= 2h(Z, Z) h(Z, \varphi Z) - 2h(Z, \varphi Z) h(Z, Z) = 0 \end{aligned}$$

and

$$\begin{aligned} h(Y,Y) &= \alpha^2 h(Z,Z) + \beta^2 h(\varphi Z, \varphi Z) + 2\alpha\beta h(Z,\varphi Z) \\ &= 2(h(Z,Z))^2 + 2\alpha\beta h(Z,\varphi Z) > 0 \end{aligned}$$

since α, β and $h(Z,\varphi Z)$ are positive functions on U. It is easy to verify that $X := \frac{Y}{\sqrt{h(Y,Y)}}$ has the required properties. □

Lemma 4.3. *For X from Lemma 4.2* the following equalities hold:

$$\nabla_\xi X = -\lambda\varphi X, \quad \nabla_\xi \varphi X = \lambda X, \quad \nabla_\xi \xi = 0, \quad \nabla_X \xi = -\lambda\varphi X,$$
$$\nabla_{\varphi X}\xi = \lambda X, \quad \nabla_X X = 0, \quad \nabla_X \varphi X = \xi, \quad \nabla_{\varphi X}\varphi X = 0, \quad \nabla_{\varphi X} X = \xi,$$

where λ is some positive constant.

Proof. From Theorem 3.1 we easily get

$$\nabla_\xi X,\ \nabla_\xi \xi,\ \nabla_X \xi,\ \nabla_{\varphi X}\xi,\ \nabla_X X,\ \nabla_{\varphi X}\varphi X \in \mathcal{D}.$$

Since $\dim \mathcal{D} = 2$, there are two smooth functions α, β defined on U such that

$$\nabla_\xi X = \alpha X + \beta\varphi X. \tag{4.3}$$

Now (3.7) implies

$$\nabla_\xi \varphi X = \alpha\varphi X - \beta X. \tag{4.4}$$

Moreover (3.7) and (3.11) imply

$$\nabla_\xi \xi = 0.$$

Since f is an affine hypersphere, we have $S = \lambda I$, where λ is a constant. We can assume that $\lambda > 0$ (otherwise we can change the sign of the Blaschke normal field). Let ω_h be the volume form for h. Then (since f is a Blaschke hypersurface) we have in particular

$$\begin{aligned} 0 = (\nabla_\xi \omega_h)(X,\varphi X,\xi) &= \xi(\omega_h(X,\varphi X,\xi)) - \omega_h(\nabla_\xi X,\varphi X,\xi) - \omega_h(X,\nabla_\xi\varphi X,\xi) \\ &\quad -\omega_h(X,\varphi X,\nabla_\xi \xi) = -\omega_h(\nabla_\xi X,\varphi X,\xi) - \omega_h(X,\nabla_\xi \varphi X,\xi) \end{aligned}$$

because $\nabla_\xi \xi = 0$ and $\omega_h(X,\varphi X,\xi) = \sqrt{\lambda} = \mathrm{const}$. Now, using (4.3) and (4.4), we obtain

$$\begin{aligned} (\nabla_\xi \omega_h)(X,\varphi X,\xi) &= -\omega_h(\alpha X,\varphi X,\xi) - \omega_h(\beta\varphi X,\varphi X,\xi) \\ &\quad -\omega_h(X,\alpha\varphi X,\xi) + \omega_h(X,\beta X,\xi) = -2\alpha\sqrt{\lambda}. \end{aligned}$$

Thus $\alpha = 0$.
We also have

$$\nabla_X \xi = -\varphi S X = -\lambda\varphi X$$

and

$$\nabla_{\varphi X}\xi = \lambda X.$$

From the Coddazi equation for h (2.8) we have

$$\begin{aligned}(\nabla_\xi h)(X,\varphi X) &= -h(\nabla_\xi X,\varphi X) - h(X,\nabla_\xi\varphi X)\\ &= -h(\nabla_\xi X,\varphi X) - h(\xi,\nabla_X\varphi X) = (\nabla_X h)(\xi,\varphi X),\end{aligned}$$

so $h(\xi,\nabla_X\varphi X) = -\lambda - 2\beta$. Again from (3.7) we get

$$\nabla_X\varphi X = \varphi(\nabla_X X) + \xi.$$

Thus $-\lambda - 2\beta = h(\xi,\nabla_X\varphi X) = h(\xi,\varphi(\nabla_X X)+\xi) = \lambda$, that is, $\beta = -\lambda$. Summarizing the above consideration, we obtain

$$\nabla_\xi X = -\lambda\varphi X, \quad \nabla_\xi\varphi X = \lambda X, \quad \nabla_\xi\xi = 0, \quad \nabla_X\xi = -\lambda\varphi X, \quad \nabla_{\varphi X}\xi = \lambda X.$$

Since $\nabla_X X \in \mathcal{D}$ and $\nabla_{\varphi X}\varphi X \in \mathcal{D}$, we have

$$\nabla_X X = pX + q\varphi X, \quad \nabla_{\varphi X}\varphi X = aX + b\varphi X,$$

where p, q, a, b are smooth functions on U. Hence

$$\begin{aligned}\nabla_X\varphi X &= \varphi(\nabla_X X) + \xi = p\varphi X - qX + \xi,\\ \nabla_{\varphi X}X &= \xi - \varphi(\nabla_{\varphi X}\varphi X) = \xi - a\varphi X + bX.\end{aligned}$$

Now, using the fact that

$$(\nabla_X\omega_h)(X,\varphi X,\xi) = 0,$$

we get $p = 0$. Similary from

$$(\nabla_{\varphi X}\omega_h)(X,\varphi X,\xi) = 0$$

we get $b = 0$. Again using the Coddazi equation for h (2.8) we have

$$(\nabla_X h)(X,\varphi X) = (\nabla_{\varphi X}h)(X,X).$$

Thus $q = 0$. In a similar way, using the equality

$$(\nabla_{\varphi X}h)(\varphi X,X) = (\nabla_X h)(\varphi X,\varphi X),$$

we can show that $a = 0$. Thus we have

$$\nabla_X X = 0, \quad \nabla_{\varphi X}\varphi X = 0, \quad \nabla_X\varphi X = \xi, \quad \nabla_{\varphi X}X = \xi.$$

The proof of lemma is completed. □

Now we can return to the proof of the main theorem

Proof of Theorem 4.2. From Lemma 4.3, since ∇ is a torsion free connection, we immediately get

$$[\xi,X] = [\xi,\varphi X] = [X,\varphi X] = 0.$$

Now, Frobenius' theorem implies that there exists a local coordinates system (x, y, z) on U such that $\frac{\partial}{\partial x} = X$, $\frac{\partial}{\partial y} = \varphi X$, $\frac{\partial}{\partial z} = \xi$. In these coordinates f satisfies the following differential equations:

$$f_{xx} = C = -\lambda f = -J\xi = -Jf_z, \tag{4.5}$$

$$f_{xy} = f_z, \tag{4.6}$$

$$f_{xz} = -\lambda f_y, \tag{4.7}$$

$$f_{yy} = -C = \lambda f = Jf_z, \tag{4.8}$$

$$f_{yz} = \lambda f_x, \tag{4.9}$$

$$f_{zz} = -\lambda Jf_z = -\lambda^2 f. \tag{4.10}$$

The Eq. (4.9) can be easily obtained from (4.7) and (4.10). Moreover the equation (4.6) can be determined from the remaining, as well.

From (4.10) we get

$$f(x, y, z) = c_1(x, y)\cos\lambda z + c_2(x, y)\sin\lambda z,$$

where c_1, c_2 are smooth functions with values in $\mathbb{R}^4$. Now, from (4.5), (4.7) and (4.8) we obtain

$$\begin{cases} c_{1xx} = -\lambda c_1 \\ c_{1yy} = \lambda c_1 \\ Jc_{1x} = c_{1y} \\ Jc_2 = c_1 \end{cases}$$

Solving this system of equations we get

$$c_1(x, y) = (ae^{\sqrt{\lambda}y} + be^{-\sqrt{\lambda}y})\cos\sqrt{\lambda}x + (-Jae^{\sqrt{\lambda}y} + Jbe^{-\sqrt{\lambda}y})\sin\sqrt{\lambda}x$$

and $c_2(x, y) = -Jc_1(x, y)$, where $a = (a_1, a_2, a_3, a_4)^T$, $b = (b_1, b_2, b_3, b_4)^T \in \mathbb{R}^4$. Since f must be an affine hypersphere with $S = \lambda I$, the affine normal field C must have a form $C = -\lambda f$. It is obvious that $\tau = 0$. By straightforward computations we obtain:

$$\theta\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big) = 4\lambda^3 \det[a\, b\, Ja\, Jb] \quad \text{and} \quad \omega_h\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big) = \sqrt{\lambda},$$

so f is an affine hypersphere if and only if

$$\det[a\, b\, Ja\, Jb] = \frac{1}{4}\lambda^{-\frac{5}{2}}.$$

Now, it is sufficient to find an affine J-invariant transformation A such that $\det A = 1$. Let

$$A = \lambda^{\frac{5}{8}} \begin{bmatrix} a_3 + b_3 & -a_1 + b_1 & a_1 + b_1 & a_3 - b_3 \\ a_4 + b_4 & -a_2 + b_2 & a_2 + b_2 & a_4 - b_4 \\ -a_1 - b_1 & -a_3 + b_3 & a_3 + b_3 & -a_1 + b_1 \\ -a_2 - b_2 & -a_4 + b_4 & a_4 + b_4 & -a_2 + b_2 \end{bmatrix}.$$

A is J-invariant and $\det A = 4\lambda^{\frac{5}{2}} \det[a\, b\, Ja\, Jb] = 1$. It is not difficult to verify that $A^{-1} \circ f$ has the form (4.1), what completes the proof of the theorem. $\square$

References

[1] Cruceanu, V.: Real hypersurfaces in complex centro-affine spaces. Results Math. **13**, 224–234 (1988)

[2] Nomizu, K., Sasaki, T.: Affine Differential Geometry. Cambridge University Press, Cambridge (1994)

[3] Szancer, M., Szancer, Z.: Real hypersurfaces with an induced almost contact structure. Colloquium Math. **114**, 41–51 (2009)

[4] Szancer, Z.: Real hypersurfaces with a special transversal vector field. Ann. Polonici Math. **105.3**, 239–252 (2012)

Zuzanna Szancer
Department of Applied Mathematics
University of Agriculture in Krakow
253c Balicka Street
30-198 Krakow, Poland
e-mail: Zuzanna.Szancer@ur.krakow.pl

Received: April 12, 2013.
Accepted: April 26, 2014.