Zamolodchikov–Faddeev algebra and quantum quenches in integrable field theories

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Abstract. We analyze quantum quenches in integrable models and in particular we determine the initial state in the basis of eigenstates of the post-quench Hamiltonian. This leads us to consider the set of transformations of creation and annihilation operators that respect the Zamolodchikov–Faddeev algebra satisfied by integrable models. We establish that the Bogoliubov transformations hold only in the case of quantum quenches in free theories. For the most general case of interacting theories, we identify two classes of transformations. The first class induces a change in the S-matrix of the theory but not in its ground state, whereas the second class results in a 'dressing' of the operators. We consider as examples of our approach the transformations associated with a change of the interaction in the sinh–Gordon model and the Lieb–Liniger model.

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1. Introduction

A quantum quench is an instantaneous change in the parameters that determine the dynamics of an isolated quantum system, e.g. the masses or coupling constants of its Hamiltonian. This topic has recently attracted a lot of attention as shown by the increasing number of papers addressing this issue (for a recent review, see [1] and references therein). From an experimental point of view this is a feasible way to bring the system out of equilibrium and study its evolution under the quantum mechanical natural laws, in isolation from the environment. In particular, the scientific interest in quantum quenches started growing after the experimental realization of global sudden changes of the interaction in cold atom systems, a novel technology where quantum statistical physics can be experimentally demonstrated and probed [2]–[5]. From a theoretical point of view the problem consists in preparing the system in a particular trial state, which is typically the ground state of some Hamiltonian, and studying its evolution under a different Hamiltonian [6]–[17]. Apart from providing one of the simplest and most well-posed ways to study out-of-equilibrium quantum physics, quantum quenches also give rise to a fundamental long-standing open question of central importance in statistical physics, the question of thermalization: how do extended quantum physical systems tend to thermal equilibrium starting from an arbitrary initial state?

Of particular interest is the case of $(1+1)$ dimensions where a discrimination between integrable and non-integrable systems is possible. Integrable models are models that exhibit factorization of the scattering matrix and can be solved exactly (see, for instance [18], and references therein). Their classical counterparts possess as many integrals of motion as they have degrees of freedom and this fact prevents thermalization of an arbitrary initial state, as not all of the microstates of equal energy respect the conservation of all other integrals of motion. This property is also expected to hold at the quantum level. In a seminal experiment [5] it was observed that a trapped $(1+1)$D Bose gas, initially prepared in a non-equilibrium state, does not thermalize but tends instead to a nonthermal momentum distribution. The absence of thermalization suggests as a possible reason the integrability of the system which approximates a homogeneous (1+1)D Bose gas with point-like collisional interactions, a typical integrable model, even though the confining potential used in the experiment breaks the homogeneity and therefore integrability of the system. This experiment triggered an intensive discussion about the role of non-integrability in the thermalization process. It was soon conjectured [19] that in an integrable case the system does exhibit stationary behavior for long times, described however not by the usual Gibbs ensemble but by a generalized Gibbs ensemble (GGE) where new Lagrange multipliers are introduced into the density matrix, one for each integral of motion, for accounting for their conservation (in the same way that the inverse temperature β is the Lagrange multiplier corresponding to the constraint of energy conservation):

$$\rho = Z^{-1} \exp\left(- \sum_m \lambda_m \mathcal{I}_m \right) . \tag{1}$$

This conjecture has been shown to be correct in many different special cases, by both analytical and numerical methods [17], [20]–[26]. On the other hand, it has not yet become clear whether non-integrability alone is sufficient to ensure thermalization or not [27, 28]; neither has exact thermalization been firmly demonstrated as an outcome of unitary evolution. For instance, recent analysis suggests that the behavior may be more complicated and may depend on the initial state, finite size effects and locality [29]–[31]. For recent experimental developments backed by numerical simulations we refer the reader to [32]–[34].

As regards the analytic approach to the problem, in the paper [24] it was shown that any quantum quench in an integrable quantum field theory where the initial state has the form

$$|\psi_0\rangle \sim \exp\left(\int \mathrm{d}\theta \, K(\theta) Z^\dagger(\theta) Z^\dagger(-\theta) \right) |0\rangle, \tag{2}$$

leads to stationary behavior as described by the GGE ansatz. In the expression above, $|0\rangle$ is the vacuum state of the theory, while $Z^\dagger(\theta)$ is the creation operator of the quasiparticle excitation, which satisfies the relativistic dispersion relations $E = m \cosh\theta, P = m \sinh\theta$, where θ labels the rapidity of the particle of mass m. As is clear from this expression, the function $K(\theta)$ is the amplitude relative to the creation of a pair of particles with equal and opposite rapidities.

The above form of a quench state is also called a 'squeezed coherent' state. The reason for choosing such an initial state comes from its relation with boundary integrable states (i.e. boundary states that respect the integrability of the bulk theory) and from the technical advantages that it exhibits. It is however true that this requirement is satisfied in general for quantum quenches in a free theory, bosonic or fermionic, as well as for the important cases of Dirichlet and Neumann states in integrable field theory. These states are supposed to capture the universal behavior of all quantum quenches in integrable models if renormalization group theory expectations are also applicable out of equilibrium. They were successfully used earlier [11, 12], applying a Wick rotation from real to imaginary time which allows a mapping of the original quantum quench problem to an equilibrium boundary problem defined on a Euclidean slab with boundary conditions both equal to the initial state right after the quench. Although this approach does not help in determining the expression for the initial state as a function of the quench parameters, it has led to correct predictions in certain important asymptotic limits.

It still remains to find out, from first principles, whether this assumption for the form of the initial state holds in general for any quantum quench in an integrable system or, if not, under what conditions this happens. Our method for attacking this problem begins by obtaining an understanding of the fundamental reason for this condition holding generally for free systems and then investigating whether this reason can be generalized to the integrable case. It turns out that in free systems the reason lies in the fact that the relation between the creation–annihilation operators before and after the quench is of linear Bogoliubov type, which itself is a consequence of their canonical commutation or anti-commutation relations. In integrable theories these commutation relations are replaced by the so-called Zamolodchikov–Faddeev (ZF) algebra which, assuming for simplicity that there is only one quasiparticle in the theory, can be written as

$$\begin{aligned} Z(\theta_1)Z(\theta_2) &= S(\theta_1-\theta_2)Z(\theta_2)Z(\theta_1) \\ Z(\theta_1)Z^\dagger(\theta_2) &= S(\theta_2-\theta_1)Z^\dagger(\theta_2)Z(\theta_1)+\delta(\theta_1-\theta_2). \end{aligned} \tag{3}$$

Intuitively this means that the exchange of two quasiparticles is done by the scattering matrix $S(\theta)$. Then a natural question arises: what are the possible transformations of creation–annihilation operators that respect the above algebra? This is a question of more general interest in both the abstract mathematical description of integrable field theories and their potential physical applications in concrete models. In this paper we show that, unlike in free theories, the ZF commutation relations do not admit Bogoliubov transformations and we construct several other classes of non-trivial infinitesimal transformations.

We start our presentation by first discussing the structure of initial states in global quantum quenches. Then we outline a general strategy for determining the initial state from the relation between the ZF creation–annihilation operators before and after the quench and deriving the conditions that must be satisfied by infinitesimal transformations of these operators in order to respect the ZF algebra. After showing that in interacting theories the linear Bogoliubov transformations do not leave the ZF algebra invariant, we find acceptable transformations of two types, the first of which induces a shift in the S-matrix but does not affect the ground state of the theory, while the second does not change the S-matrix but does change the ground state and the corresponding transformed ground state is, under certain conditions, of the squeezed coherent state form. We note that these are special classes of transformations and outline how more general ones can be constructed. Next we apply these ideas in two typical integrable models, the sinh–Gordon model and the Lieb–Liniger model, deriving some examples of infinitesimal transformations of their ZF operators that demonstrate the presence of the first type constructed before. Finally we summarize our findings, giving directions for their application to concrete quantum quench problems. There are also two appendices: in appendix A we discuss the squeezed states in free quantum field theories while in appendix B we discuss the derivation of the two classes of generators of the ZF algebra transformations.

2. On the initial states in global quantum quenches

A quench process consists of preparing the system in a state $|\psi_0\rangle$ that is not an eigenstate of its Hamiltonian H and letting this state unitarily evolve according to H. At time t, the expectation values of local observables $\Lambda(r)$ are given by

$$\langle\Lambda(t,r)\rangle = \langle\psi_0|\mathrm{e}^{\mathrm{i}Ht}\Lambda(r)\mathrm{e}^{-\mathrm{i}Ht}|\psi_0\rangle, \tag{4}$$

with similar expressions for higher point correlation functions. As is evident from the expression above, important information on the subsequence dynamics of the system is encoded in the initial state $|\psi_0\rangle$. Relevant features of this state can be derived on the basis of general considerations for extended quantum systems having particle excitations and for global quenches. First of all, by relativistic (or even Galilean) invariance, we can always assume that the quench state $|\psi_0\rangle$ carries no momentum. Let $Z^\dagger(p)$ be the creation operator of a particle excitation[5] of the system of momentum p and let us assume that a basis of the Hilbert space is given by the multi-particle excitations, eigenvectors of the Hamiltonian H. Then the general form of the initial state $|\psi_0\rangle$ for a global quench is given by an *infinite* superposition of multi-particle states of zero momentum:

$$|\psi_0\rangle = \sum_{n=0}^{\infty}\int \mathrm{d}p_1\cdots\mathrm{d}p_n\,\tilde{\mathcal{K}}_n(p_1,\ldots,p_n)\delta\left(\sum_{i=1}^{n}p_i\right)Z^\dagger(p_1)\cdots Z^\dagger(p_n)|0\rangle, \tag{5}$$

where $|0\rangle$ is the vacuum state of the system. This requirement is due to a thermodynamics argument related to the formulation of quench dynamics in d dimensions to the thermodynamics of a $(d+1)$ dimensional field theory in a slab geometry, where the initial state $|\psi_0\rangle$ plays the role of boundary conditions on both borders of the slab [12]. In this interpretation of the quench process, the quantity

$$\mathcal{Z}_0(\tau) = \langle\psi_0|\mathrm{e}^{-\tau H}|\psi_0\rangle \equiv \mathrm{e}^{-\mathcal{F}_0(\tau)} \tag{6}$$

plays the role of the partition function of the system with boundary conditions fixed by $|\psi_0\rangle$. For global quenches, the corresponding free energy $\mathcal{F}_0(\tau)$ must be an extensive quantity, $\mathcal{F}_0(\tau) \simeq V f_0(\tau)$, where V is the volume of the system. On the other hand, this quantity can be computed by employing the expression (5) for the initial state $|\psi_0\rangle$ and a proper normalization[6] of $\delta(0)$: then the only way to have an extensive behavior in the volume V of the system for $\mathcal{F}_0(\tau)$ is by $|\psi_0\rangle$ containing an infinite number of multi-particle states.

[5] To simplify the following formulas we assume that the system has only one neutral elementary excitation, with normalization proportional to the δ function, $\langle Z(p_1)Z^\dagger(p_2)\rangle \propto \delta(p_1-p_2)$.

[6] See [35] for an explicit regularization of this term in the one-dimensional case.

Notice that a way to automatically take into account the condition of zero momentum of the initial state $|\psi_0\rangle$ is to assume that its infinite superposition is made up of pairs of particles of equal and opposite momentum, i.e. Cooper pairs:

$$|\psi_0\rangle = \sum_{n=0}^{\infty} \int \mathrm{d}p_1 \cdots \mathrm{d}p_n \, \mathcal{K}_{2n}(p_1, \ldots, p_n) Z^\dagger(-p_1) Z^\dagger(p_1) \cdots Z^\dagger(-p_n) Z^\dagger(p_n) |0\rangle. \tag{7}$$

It should be stressed, though, that this formula is a particular case of the more general form (5). But even with this simplification, to specify the initial state $|\psi_0\rangle$ one still needs an infinite number of amplitudes $\mathcal{K}_{2n}(p_1, \ldots, p_n)$. The great technical advantage of the squeezed coherent states, whose concise expression is given by

$$|\psi_0\rangle \sim \exp\left(\int \mathrm{d}p \, K(p) Z^\dagger(p) Z^\dagger(-p) \right) |0\rangle, \tag{8}$$

then becomes evident. In this case, in fact, all the multi-particle amplitudes $\mathcal{K}_{2n}(p_1, \ldots, p_n)$ can be expressed in terms of products of the single amplitude $K(p)$ entering equation (8), therefore greatly simplifying the problem.

Squeezed coherent states naturally appear in two contexts: (i) in the purely boundary integrable field theories considered by Ghoshal and Zamolodchikov [36], where the amplitude $K(p)$ also satisfies additional conditions (boundary unitarity and crossing symmetry) and (ii) in quench processes in free theories, both bosonic and fermionic. In the latter case, it is worth noticing that the commutation or anti-commutation relations of the annihilation and creation operators $Z(p)$ and $Z^\dagger(p)$ of these theories can be cast in the form of ZF algebra (3) with $S = 1$ for the boson and $S = -1$ for the fermion. The only parameter entering these theories is in this case the mass of their excitation and, as shown in detail in appendix A, its sudden change can be taken into account by a Bogoliubov transformation of the annihilation and creation operators. Since the Bogoliubov transformations leave the commutation or anti-commutation relations invariant, in turn they can be seen as the transformations which leave invariant the ZF algebra of free theories. This observation leads us to investigate a more general class of transformations of the ZF operators in interacting integrable field theories which leave their algebra invariant.

3. Quenches in integrable systems

In this section we analyze the quantum quenches in systems which are integrable before and after the sudden change of one parameter Q, which can be for instance the mass of the particle or the coupling constant of the theory. One of the main tasks of this problem is to write down the pre-quench state (usually the vacuum, annihilated by the pre-quench particle operators) in terms of the post-quench particle basis. This task involves in principle the computation of an infinite number of inner products, an operation usually difficult to fulfill (for a discussion of related numerical issues see for example [16, 37]). Therefore, it would be useful to have a different approach. In principle, a possible way to determine the initial state $|\psi_0\rangle$ in terms of the post-quench creation–annihilation operators $Z, Z^\dagger$ is to implement the following program:

- for an arbitrary value of the parameter Q, find initially the relation between the ZF Z operators of the theory and the physical field operator ϕ, i.e. $\phi = f(Q; Z_Q)$;
- use the continuity of the field as a boundary condition in the quench process $Q_0 \to Q$
$$f(Q_0; Z_{Q_0}) = f(Q; Z_Q) \tag{9}$$
for deriving the relation between the old and the new ZF operators
$$Z_{Q_0} = F(Q_0, Q; Z_Q) = f^{-1}(Q_0; f(Q; Z_Q)); \tag{10}$$
- write the initial state $|\Omega_0\rangle$, which is known in the pre-quench ZF basis (and is typically the ground state defined by $Z_{Q_0}|\Omega_0\rangle = 0$), in the new basis using the above relation.

If this program can be realized, the time evolution of the initial state in the new basis can be computed easily. Going into more detail, the first step of this program consists in expanding the physical field operator as a series in the ZF operators using all of its form factors, i.e. the matrix elements of the field $\phi(x)$, in the asymptotic states. The second step involves the inversion of this series; this might require an ingenious ansatz for the function F. The third step requires us to deal with the most general expansion of a state in the post-quench basis and to determine the coefficients of $|\Omega_0\rangle$ term by term from the equation $F(Q_0, Q; Z_Q)|\Omega_0\rangle = 0$.

While the first step is essentially a re-expression of the body of information obtained by the form factors program, the other steps are in general highly non-trivial. In order to partially circumvent these difficulties, in the following we will exploit some general properties of integrable field theories. As previously stated, in free theories the relation between the new and the old creation/annihilation operators is of Bogoliubov type, fixed by the condition of leaving invariant the (trivial) ZF algebra of these theories. Analogously, for generic integrable theories, the transformation between the pre-quench and the post-quench ZF operators must respect the algebra. This leads us to investigate under which conditions this requirement is satisfied. Of course this is quite an abstract point of view: knowing that a certain transformation respects the algebra does not necessarily clarify the physical nature of the quench protocol. Nevertheless, it is surely important to understand what are the possible algebra-preserving transformations and whether their form is restrictive enough for making predictions about the initial state. In section 4, we integrate this analysis with a perturbative study of a typical integrable model, the sinh–Gordon model, and its non-relativistic counterpart, i.e. the Lieb–Liniger model.

3.1. Conditions required for transformations of the ZF algebra operators

We are looking for transformations of the creation–annihilation operators $Z, Z^\dagger$ that respect the ZF algebra. We also require that the transformations respect the translational invariance of the theory, since we are considering only homogeneous systems, both before and after the quench. For this reason we will write the ZF algebra in a momentum representation which, as we will see soon, better meets this requirement:

$$Z_{p_1} Z_{p_2} = S(p_1, p_2) Z_{p_2} Z_{p_1} \tag{11}$$
$$Z_{p_1} Z^\dagger_{p_2} = S(p_2, p_1) Z^\dagger_{p_2} Z_{p_1} + \delta(p_1 - p_2), \tag{12}$$

along with the standard properties of the S-matrix

$$S(p_1,p_2)^{-1} = S(p_1,p_2)^* = S(p_2,p_1) = S(-p_1,-p_2). \tag{13}$$

Notice that in comparison with the form of the ZF algebra in the rapidity representation (3), we have redefined[7] the operators as $Z_p \equiv Z(p) = Z(\theta(p))/\sqrt{E(p)}$ since $\delta(\theta(p)) = E(p)\delta(p)$.

We focus our attention on infinitesimal transformations, assuming that a finite transformation can be built up by repetitive action of the infinitesimal ones. We also allow for infinitesimal changes of the S-matrix[8]. Since we demand that the transformations commute with the momentum operator, the new operator must carry the same momentum as the old one but not necessarily the same rapidity, as the quench may involve a change of the mass of particles (this is why the momentum representation suits our problem better). Therefore both the transformed operator and the S-matrix are in general expressed as

$$Z'_p = Z_p + \epsilon W_p \tag{14}$$

$$S'(p_1,p_2) = S(p_1,p_2) + \epsilon T(p_1,p_2) \tag{15}$$

where ϵ is a small quantity, a function of the infinitesimal change δQ of the quench parameter. In order to satisfy the ZF algebra, they must fulfil the conditions

$$W_{p_1}Z_{p_2} + Z_{p_1}W_{p_2} = T(p_1,p_2)Z_{p_2}Z_{p_1} + S(p_1,p_2)(Z_{p_2}W_{p_1} + W_{p_2}Z_{p_1}) \tag{16}$$

$$W_{p_1}Z^\dagger_{p_2} + Z_{p_1}W^\dagger_{p_2} = T(p_2,p_1)Z^\dagger_{p_2}Z_{p_1} + S(p_2,p_1)(Z^\dagger_{p_2}W_{p_1} + W^\dagger_{p_2}Z_{p_1}) \tag{17}$$

for all values of p, p', along with the following conditions for T:

$$T(p_1,p_2)^* = T(p_2,p_1) = T(-p_1,-p_2) = -T(p_1,p_2)S^{-2}(p_1,p_2) \tag{18}$$

coming from the unitarity of the S-matrix.

The operator W_p can generally be written as an expansion in the operators $Z, Z^\dagger$:

$$W_p = \sum_{n,m=1}^{\infty} \delta\left(p + \sum_{i=0}^{n} q_i - \sum_{j=0}^{m} p_j\right) a_{n,m}(\{q_i\},\{p_j\}) \prod_{i=0}^{n} Z^\dagger_{q_i} \prod_{j=0}^{m} Z_{p_j}. \tag{19}$$

The above conditions are then translated into a sequence of relations between the coefficients $a_{n,m}$ of different orders. Below we construct and study several simple classes of solutions in which the above expansion terminates after a few terms.

[7] This definition is also tailored to our purposes, since the energy of particles may change under a quantum quench and we would like to absorb all changes into the transformation of the operators. In addition it is consistent with the usual normalization of energy eigenstates in the free limit which will be useful later.

[8] Note that this assumption may exclude the special case of free bosons. This is because for all integrable field theories except for that for free bosons, the S-matrix at zero momentum is $S(0) \equiv S(p,p) = -1$. Therefore the transition from a free bosonic point of an integrable theory to another point that does not correspond to free bosons is always discontinuous as far as the S-matrix is concerned. In the following we sometimes make use of the property $S(0) = -1$, in which case we mention it explicitly.

3.2. A first trial: linear Bogoliubov transformations

Let us initially assume that W corresponds to a linear Bogoliubov transformation which, in the infinitesimal form, means $W_p = a_p Z^\dagger_{-p}$. In this case it is easy to see that the previous conditions become

$$S(-p,p') = S(p,-p') = S(p',p) \qquad a_p/a_{-p} = S(p,p) \equiv S(0) \qquad T(p,p') = 0$$

for all values of p,p'. The first of these equations implies that $S(p,p')^2 = 1$, i.e. $S(p,p') = \pm 1$. We therefore arrive at the interesting result that *the linear Bogoliubov transformation is a symmetry of the algebra only in the trivial case of free fields, bosons or fermions*. Moreover, it is easy to show that any other linear combination of the operators is inconsistent with the general conditions (16) and (17).

3.3. Generators of S-matrix changes

We will now construct a transformation that induces a nonzero change T in the S-matrix and show that this transformation is unique, in the sense that any infinitesimal transformation that has the same effect must necessarily involve this one. First, observe that, since linear transformations are already excluded, the TZZ term in (16) can only be produced as a δ-function by-product of the commutation of higher order terms in W. More precisely, W must be of third order and must contain one $Z^\dagger$ operator and two Z operators, so that the commutation of W_p with $Z_{p'}$ produces a $ZZ\delta$ term. Furthermore, from equation (16) we see that the two Z operators in the residual $ZZ\delta$ term, which come originally from W_p, must carry momenta p,p' (the same as for W_p and $Z_{p'}$) and therefore the $Z^\dagger$ operator in W_p must carry momentum p' to ensure that W's total momentum is p. Thus W_p is of the form $Z^\dagger_{p'} Z_{p'} Z_p$. But this must be true for arbitrary p', so W should necessarily consist of a linear combination of all such terms. All of this leads to the ansatz

$$W_p = \left(\sum_q \alpha_{p,q} Z^\dagger_q Z_q \right) Z_p. \tag{20}$$

Let us verify it explicitly by substituting into the required conditions (16) and (17), the first of which gives

$$T(p_1,p_2) = S(p_1,p_2)(\alpha_{p_2,p_1} - \alpha_{p_1,p_2}), \tag{21}$$

while the second gives

$$T(p_2,p_1) = S(p_2,p_1)(\alpha^*_{p_2,p_1} + \alpha_{p_1,p_2}), \tag{22}$$

together with

$$\alpha_{p,q} + \alpha^*_{p,q} = 0. \tag{23}$$

Remarkably all of these requirements are simultaneously satisfied as long as condition (23) holds, i.e. if the coefficients $\alpha_{p,q}$ are purely imaginary. Hence, from now on we set $\alpha_{p,q} = \mathrm{i}a_{p,q}$ where $a_{p,q}$ are real functions. Notice that this solution also ensures that the S-matrix remains unitary, as expressed by the conditions (18) for T.

Studying these transformations in more detail, one realizes that

$$Z'^{\dagger}_p Z'_p = Z^{\dagger}_p Z_p + \mathrm{i}\epsilon Z^{\dagger}_p \left[\sum_q (a_{p,q} + a^*_{p,q}) Z^{\dagger}_q Z_q \right] Z_p = Z^{\dagger}_p Z_p, \tag{24}$$

i.e. the conserved charges

$$\hat{\mathcal{Q}}_s = q_s \int \mathrm{d}\theta \, \mathrm{e}^{s\theta} Z^{\dagger}_\theta Z_\theta \tag{25}$$

remain invariant (unless the factors q_s depend explicitly on the physical parameters whose infinitesimal change leads to this transformation). This allows us to easily derive the corresponding finite transformation[9]

$$Z'_p = \mathcal{P} \left\{ \exp \left(\mathrm{i} \int \hat{\mathcal{I}}_p(s) \, \mathrm{d}s \right) \right\} Z_p \qquad \text{where } \hat{\mathcal{I}}_p(s) = \sum_q a_{p,q}(s) Z^{\dagger}_q Z_q \tag{26}$$

which changes the S-matrix as follows:

$$S'(p,q) = \exp \left[\mathrm{i} \int (a_{q,p}(s) - a_{p,q}(s)) \, \mathrm{d}s \right] S(p,q). \tag{27}$$

Even though we have constructed this infinitesimal transformation heuristically, it is easy to show that this transformation is the only one that changes the S-matrix. Any other transformation that changes the S-matrix must necessarily be a linear combination of this one along with some other part that does not change it. Indeed if there was another transformation W' that also shifted S to the same $S + \epsilon T$, then from (16) and (17) their difference $W - W'$ would not change the S-matrix. We can therefore decompose any transformation that respects the ZF algebra (infinitesimal or finite) into two parts, one of which is of the above form and performs the shift of the S-matrix to the desired value, while the other leaves it invariant. In this way we have reduced the problem of finding the symmetries of the ZF algebra to the task of identifying those transformations which do not alter the S-matrix and which satisfy (16) and (17) with $T = 0$, i.e.

$$W_{p_1} Z_{p_2} + Z_{p_1} W_{p_2} = S(p_1, p_2)(Z_{p_2} W_{p_1} + W_{p_2} Z_{p_1}) \tag{28}$$

$$W_{p_1} Z^{\dagger}_{p_2} + Z_{p_1} W^{\dagger}_{p_2} = S(p_2, p_1)(Z^{\dagger}_{p_2} W_{p_1} + W^{\dagger}_{p_2} Z_{p_1}). \tag{29}$$

3.4. Other classes of transformations

Having reduced the problem to identifying the transformations of the ZF operator which do not alter the S-matrix, we will now consider more general classes of symmetry transformations of the ZF algebra. Let us assume initially that W is simply a single product of $Z, Z^{\dagger}$ operators. In order to check the condition (28), we have to consider how W_p commutes with $Z_{p'}$: when we swap $Z_{p'}$ with each of the operators in W one by one, this operation gives as output, for each of these terms, multiplicative S-matrix factors as well as additive δ-functions for each $Z^{\dagger}$, which are lower order products. One obvious way

[9] $\mathcal{P}\{\cdots\}$ denotes a path-ordering integration and s is a continuous parameter along some path.

to satisfy (28) is then to choose W_p in such a way that (a) the overall S-matrix factor is simply equal to $S(p,p')$ and (b) the residual lower order terms vanish.

Let us firstly focus our discussion on the point (a). We assume that W_p consists of $nZ^\dagger$ operators and mZ operators (in some ordering that is not relevant for the moment), i.e.

$$W_p = \prod_{i=1}^{n} Z^\dagger_{q_i} \prod_{j=1}^{m} Z_{r_j} \qquad \text{with } \sum_{j=1}^{m} r_j - \sum_{i=1}^{n} q_i = p. \tag{30}$$

Then we have

$$W_p Z_{p'} \simeq \left(\prod_{i=1}^{n} S(p',q_i) \prod_{j=1}^{m} S(r_j,p') \right) Z_{p'} W_p, \tag{31}$$

where we use the symbol $\simeq$ to denote equality for the highest order terms only (i.e. we ignore for now all residual lower order terms). To satisfy (28) we then require that the equation

$$\prod_{i=1}^{n} S(p',q_i) \prod_{j=1}^{m} S(r_j,p') = S(p,p') \tag{32}$$

holds[10]. In order to satisfy this relation for all p, p' and also independently of the specific functional form of the S-matrix, we have to exploit its general properties. In particular, taking into account that $S(p,p') = S^{-1}(p',p)$ we see that if

$$m = n+1 \qquad \text{and} \qquad q_i = r_j \qquad \text{for all } i = j = 1 \cdots n \quad \text{and} \quad r_{n+1} = p \tag{33}$$

then equation (32) becomes an identity.

As for the second condition (29), we have

$$W_p Z^\dagger_{p'} \simeq \left(\prod_{i=1}^{n} S(q_i,p') \prod_{j=1}^{m} S(p',r_j) \right) Z^\dagger_{p'} W_p \qquad , \tag{34}$$

and

$$Z_p W^\dagger_{p'} \simeq \left(\prod_{j=1}^{m} S(r'_j,p) \prod_{i=1}^{n} S(p,q'_i) \right) W^\dagger_{p'} Z_p, \tag{35}$$

and so we would similarly require

$$\prod_{i=1}^{n} S(q_i,p') \prod_{j=1}^{m} S(p',r_j) = \prod_{j=1}^{m} S(r'_j,p) \prod_{i=1}^{n} S(p,q'_i) = S(p',p). \tag{36}$$

Remarkably this condition is essentially identical to that of equation (32), i.e. our solution (33) of (32) automatically satisfies this one too. Hence, there exists a solution to these equations for arbitrarily high order $n+m$.

However this is not the end of the story, since one has also to check the point (b), namely that the residual terms vanish. In order to ensure this condition for all p, p',

[10] Of course this is not the only way to meet the condition (28) under (30) but, as the more detailed discussion presented in appendix B shows, the other options lead, at the end, to the same form.

instead of considering a single product (30), one has to look at a linear combination of such terms for all momenta q_i and choose their coefficients so that the residual terms cancel each other. If this could not be realized, one would still have the option to introduce into W_p suitable lower order terms and cancel the residual terms order by order. In this way the coefficients of terms of order n depend on those of order $n+2$ and we see that the construction of the transformation can be carried out recursively.

Provided that all of these requirements are met, the resulting transformation is of the form

$$W_p = \sum_{\{q_i\}} \alpha(p, \{q_i\}) \left(\prod_{i=1}^{n} Z^\dagger_{q_i} Z_{q_i} \right) Z_p + \text{suitable lower order terms.} \tag{37}$$

Here we only report the lowest order members of this family of transformations. The first one corresponds to $n = 1$ and is the one that we have found already in section 3.3 (in this case the residual terms result in a nonzero T, as we saw):

$$W_p = \mathrm{i} \sum_q a_{p,q} Z^\dagger_q Z_q Z_p. \tag{38}$$

For $n = 2$ we find that the coefficient must be simply an imaginary constant:

$$W_p = \mathrm{i} \sum_{q,r} Z^\dagger_q Z_q Z^\dagger_r Z_r Z_p. \tag{39}$$

Our study started by assuming that W_p is a single product of $Z, Z^\dagger$ operators, a monomial (even though later we had to generalize our assumption by considering linear combinations of similar terms and lower order ones). However this is obviously not the only possibility. Another possibility is investigated in appendix B, which starts from a binomial and leads to the discovery of transformations of another interesting type:

$$\begin{aligned} W_p &= \sum_q b_q (S_{p,q} S_{p,-q} - 1) Z_p Z^\dagger_q Z^\dagger_{-q} + \sum_q b^*_q (S_{p,q} S_{p,-q} - 1) Z_{-q} Z_q Z_p + 2 b_{-p} Z^\dagger_{-p} \\ &= \sum_q b_q (1 - S_{q,p} S_{-q,p}) Z^\dagger_q Z^\dagger_{-q} Z_p + \sum_q b^*_q (S_{p,q} S_{p,-q} - 1) Z_{-q} Z_q Z_p - 2 b_p Z^\dagger_{-p}, \end{aligned} \tag{40}$$

where b_q has been chosen to satisfy $b_{-q} = b_q S_{q,-q}$ and we have assumed that the S-matrix satisfies $S(0) = S_{p,p} = -1$. As already mentioned, this excludes only the case of free bosons since for all other integrable models it is always true. Note that the last term cannot be absorbed by reordering the operators of the first one.

Let us remark that one may continue in a similar way and construct other more complex classes of transformations. In particular, one may even consider the infinite series of products (19) which, unlike all cases presented above, do not give rise to expressions that terminate at finite order. The study of such transformations will be discussed elsewhere.

3.5. Properties of the two simple classes of transformations

In the previous sections we have found mainly two distinct classes of symmetries of the ZF algebra which led to equations (20) and (40). Transformations of the first type (20) are ones that generate a change in the S-matrix. However transformations of this first type do not change the ground state of the theory since $Z'_p = Z_p + \epsilon W_p$ annihilates the same vacuum as Z_p. The reason is that this transformation, like the second member of the same class (39), always contains Z operators numbering one more than its $Z^\dagger$ operators. Finally, let us notice that it does not reduce to the Bogoliubov transformation in the free limit $S = \pm 1$ since it does not depend explicitly on S.

Transformations of the second type (40) have three important properties. Firstly, they do not change the S-matrix. Secondly, in the free limit where $S \to \pm 1$, their nonlinear terms (first and second) vanish, leaving only the linear term (last) $Z^\dagger_{-p}$, which corresponds to the Bogoliubov transformation. Thirdly and most importantly, they change the ground state, since the first and last terms in (40) contain $Z^\dagger$ operators numbering one more than their Z operators, which means that the new annihilation operator does not annihilate the old ground state. In particular, as we will show next, the infinitesimal change in the ground state can be described as the creation of a pair of excitations with opposite momenta.

Indeed, if we denote by $|\Omega\rangle$ the ground state corresponding to the pre-quench operator Z, by definition this state satisfies

$$Z_k|\Omega\rangle = 0 \tag{41}$$

for all k. This condition, expressed in the basis of the post-quench operator Z' (with corresponding ground state $|\Omega'\rangle$), reads

$$(Z'_k - \epsilon W_k)(1 + \epsilon X)|\Omega'\rangle = \epsilon(Z'_k X - W_k)|\Omega'\rangle = 0 \tag{42}$$

where X is a suitable operator to be determined. For W given by (40) we easily find by normal ordering that

$$X = -\sum_p b_p Z'^\dagger_p Z'^\dagger_{-p}. \tag{43}$$

Notice that this is the infinitesimal version of a squeezed coherent state. In fact we can go much further and show that for real b_p any finite transformation generated by equation (40) 'transforms' the initial ground state into a squeezed coherent state[11]. To prove this it is sufficient to show that any state $|\Psi\rangle$ of the squeezed form

$$|\Psi\rangle = \mathcal{N}(K_q) \exp\left(\sum_q K_q Z^\dagger_q Z^\dagger_{-q}\right)|\Omega\rangle \tag{44}$$

in the pre-quench basis preserves its squeezed form under the infinitesimal transformation (40), i.e. it is 'transformed' into a squeezed state in the post-quench basis:

$$|\Psi\rangle = \mathcal{N}(K'_q) \exp\left(\sum_q K'_q Z'^\dagger_q Z'^\dagger_{-q}\right)|\Omega'\rangle. \tag{45}$$

[11] By 'transformed' ground state, we mean the expansion of the ground state of the pre-quench operator Z in the basis of the post-quench operator Z'.

If this is true then, since the finite transformation can be built up by successive application of infinitesimal ones and the initial ground state is 'transformed' into a squeezed state (43) which remains of this form after every infinitesimal step of this procedure, we conclude by induction that the transformation (40) does indeed 'generate' squeezed states.

To prove this statement we can follow a path parallel to the corresponding free field calculation. For free bosons or fermions one should show that under an infinitesimal Bogoliubov transformation

$$Z'_p = Z_p + \epsilon a_p Z^\dagger_{-p} \qquad (a_{-p} = a_p) \tag{46}$$

any squeezed state of the form (44) remains squeezed as well. The easiest way to see this is to employ the following equivalent form of (44):

$$|\Psi\rangle = \exp\left[\sum_q \Lambda_q \left(Z^\dagger_q Z^\dagger_{-q} - Z_{-q} Z_q\right)\right] |\Omega\rangle \tag{47}$$

where Λ_q is a known function of K_q. In the above we have assumed that a_q, K_q and Λ_q are all real functions and all of the following are restricted to this case. The operator $(Z^\dagger_q Z^\dagger_{-q} - Z_{-q} Z_q)$ in the exponent remains invariant under the Bogoliubov transformation (46) and therefore the only change comes from the ground state

$$|\Omega\rangle = \left(1 - \epsilon \sum_q a_q Z'^\dagger_q Z'^\dagger_{-q}\right) |\Omega\rangle' \tag{48}$$

which can be absorbed in a shift of the coefficient Λ_q in the exponent:

$$|\Psi\rangle = \exp\left[\sum_q (\Lambda_q - \epsilon a_q) \left(Z'^\dagger_q Z'^\dagger_{-q} - Z'_{-q} Z'_q\right)\right] |\Omega'\rangle. \tag{49}$$

This is exactly what we wished to show. To verify that equation (47) can also be written in the form (44) one can normal order the squeezing operator $\exp[\sum_q \Lambda_q (Z^\dagger_q Z^\dagger_{-q} - Z_{-q} Z_q)]$. Alternatively one may first decompose the exponential of the sum over momenta in (47) into an infinite product of exponentials[12]. Then observe that for each pair of opposite momentum modes, the commutation relations of the operators $Z^\dagger_q Z^\dagger_{-q}$, $Z_{-q} Z_q$ and $\frac{1}{2}(Z^\dagger_q Z_q + Z^\dagger_{-q} Z_{-q} \pm 1)$ (+ for bosons/− for fermions) form a closed algebra ($SU(1,1)$ algebra for bosons/$SU(2)$ algebra for fermions) and therefore we can rewrite the exponential as

$$\mathrm{e}^{2\Lambda_q (Z^\dagger_q Z^\dagger_{-q} - Z_{-q} Z_q)} = \mathrm{e}^{2K_q(\Lambda_q) Z^\dagger_q Z^\dagger_{-q}} \mathrm{e}^{2L_q(\Lambda_q) Z_q Z_{-q}} \mathrm{e}^{M_q(\Lambda_q)(Z^\dagger_q Z_q + Z^\dagger_{-q} Z_{-q} \pm 1)} \tag{50}$$

for some appropriate coefficients K_q, L_q, M_q that it is not necessary to determine here. Applying this operator to $|\Omega\rangle$, we directly derive the result mentioned above.

Extending this computation to the general integrable case is almost straightforward. The first step is to check that even in this general case, squeezed states can be equivalently written in both forms (44) and (47). This is true, since the commutation relations of the operators involved in the computation are not crucially different from the free fermionic

[12] To avoid ordering problems in this product, first restrict the summation variable q to positive values only, as we are allowed to do.

ones. Indeed, if we define $Y_q^\dagger = Z_q^\dagger Z_{-q}^\dagger$, $N_q = Z_q^\dagger Z_q$ and $\bar{N}_q = Z_q Z_q^\dagger = -N_q + 1$, then

$$[Y_q^\dagger, Y_p^\dagger] = 0 \tag{51}$$

$$[Y_q^\dagger, Y_p] = \delta_{q,-p} S_{-p,p}(N_{-p} - \bar{N}_p) + \delta_{q,p}(N_p - \bar{N}_{-p}) \tag{52}$$

$$[Y_q^\dagger, N_p] = -(\delta_{q,-p} + \delta_{q,p}) Y_q^\dagger \tag{53}$$

and the only difference from the free fermion case is the factor $S_{-p,p}$ in (52). The next step is to check whether the operator $(Z_q^\dagger Z_{-q}^\dagger - Z_{-q} Z_q)$ in the exponent of (47) remains invariant under the transformation (40), which turns out, after some algebra, to be true if b_p is real. This completes the proof of our statement.

Let us now discuss another property of transformations of this second type: the change induced in the conserved charges of the theory. As we have already seen in (24), transformations of the first type that we studied leave all conserved charges invariant. In the present case the transformation of $Z_p^\dagger Z_p$ turns out to be

$$Z_p'^\dagger Z'_p = Z_p^\dagger Z_p - 2\epsilon(b_p Z_p^\dagger Z_{-p}^\dagger + b_p^* Z_{-p} Z_p), \tag{54}$$

that is the conserved charges $\hat{\mathcal{Q}}_s$ do change but they remain quadratic in the Z operators. In particular the Hamiltonian of the system which in the momentum representation and according to our normalization of the ZF operators is $H = \sum_p E_p Z_p^\dagger Z_p$ is transformed as

$$H' = H + \epsilon \sum_p \left[\frac{\partial E_p}{\partial \epsilon} Z_p^\dagger Z_p - 2E_p(b_p Z_p^\dagger Z_{-p}^\dagger + b_p^* Z_{-p} Z_p) \right] \tag{55}$$

where $\partial E_p / \partial \epsilon$ reflects the change in the dispersion relation induced by the transformation. Notice that this expression is reminiscent of the analogous one for Bogoliubov transformations in free theories.

From a physical point of view and in particular from the perspective of quantum quenches, the action of transformations of these two types consists in 'dressing' the initial particle of the theory with a pair of newly created particles with opposite momenta, as depicted in figure 1. Especially for the second type, if such a transformation describes a quantum quench, then the initial ground state is expressed as a squeezed coherent state in the post-quench basis, at least for real b_p. Finally, the perturbation introduced in the Hamiltonian after a small quantum quench of this type would be of quadratic form in the pre-quench ZF operators.

4. Examples from physical theories

In this section we will present some first examples of transformations of ZF operators in the context of two important integrable models: the sinh–Gordon and the Lieb–Liniger ones. The first is one of the simplest and most well-studied relativistic integrable models consisting of particles of a single type, while the second describes a system of non-relativistic interacting bosons in $(1+1)$D and models experimental cold atom setups. As has been recently shown in [38], the two models are closely related to each other, since the Lieb–Liniger model can be obtained as a suitable non-relativistic limit of the sinh–Gordon model.

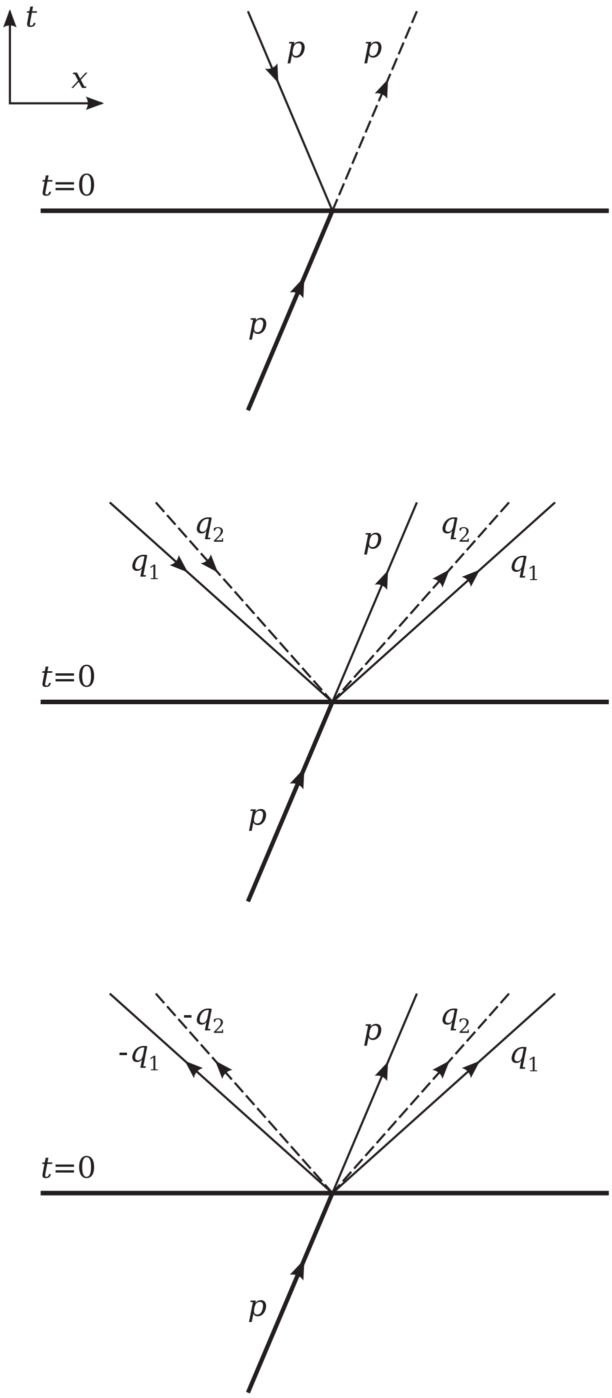

Figure 1. Diagrammatic representation of the transformations of the three types in the context of quantum quenches. The linear Bogoliubov transformation that works only in free systems corresponds to the transformation of an old particle into a new antiparticle of opposite momentum. The transformations of the two new types that work in the interacting case convert the old particle into a new particle of the same momentum accompanied (or 'dressed') by a particle–antiparticle (type I) or particle–particle (type II) pair with opposite momenta.

In the sinh–Gordon model, an example of infinitesimal transformation would correspond to a small quench of the mass m or the coupling constant g. Starting from arbitrary initial values, such a quench would introduce into the Hamiltonian perturbations ($\int \mathrm{d}x \cosh g\phi(x)$ and $\int \mathrm{d}x\,(2\cosh g\phi(x) + g\phi(x)\sinh g\phi(x))$ respectively) that correspond to an infinite series in terms of the $Z, Z^\dagger$ operators, as can be seen from their form factors [39]. An exception to this rule is when the initial model lies on a free point, $g = 0$. In this special case where a small g is abruptly switched on, each term of the expansion is smaller than the previous one by an amount of the order of g^2, due to the same property as holds for the form factors F^{ϕ}_{2n+1} of the physical field. Thus the derivation of the corresponding infinitesimal transformation is simpler and can be done by means of perturbation theory, which is what we do in the next section. In the Lieb–Liniger model on the other hand, we use the known relation [40] between the physical field operator and the ZF operators to find the infinitesimal transformation of the latter, again for the case where the interaction changes from zero to a small value. Using the non-relativistic limit mentioned above [38] we can verify the consistency of the results for the sinh–Gordon and Lieb–Liniger models.

Note that in both cases the transformation refers to a small change from a free bosonic to an interacting point of the theory and, according to a comment that we made in a footnote of 3.1, the S-matrix at zero momentum $S(0)$ is non-analytic (in fact discontinuous) in the coupling constant at such points. Therefore we anticipate this non-analyticity to become evident in our results and indeed it does this, as we will see below.

4.1. The sinh–Gordon model

The sinh–Gordon model is a relativistic field theory in (1+1)D defined by the Hamiltonian

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{m^2c^2}{g^2}(\cosh g\phi - 1) \tag{56}$$

where $\phi = \phi(x, t)$ is a real scalar field, m is a mass scale and c is the speed of light. In this integrable field theory there is only one type of particle with physical (renormalized) mass M given by

$$M^2 = m^2\frac{\sin\alpha\pi}{\alpha\pi} \tag{57}$$

where α is the dimensionless renormalized coupling constant

$$\alpha = \frac{cg^2}{8\pi + cg^2}. \tag{58}$$

Particle scattering is fully determined by the two-particle S-matrix given by

$$S_{\mathrm{sh-G}}(\theta, \alpha) = \frac{\sinh\theta - \mathrm{i}\sin\alpha\pi}{\sinh\theta + \mathrm{i}\sin\alpha\pi} \tag{59}$$

where θ is the rapidity difference between the particles.

To calculate the ZF operators from first-order perturbation theory we consider the ϕ^4 model

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{2}m^2c^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{60}$$

with coupling constant $\lambda = m^2c^2g^2$ (we set $c = 1$ from now on). We first define the auxiliary operators

$$B_+^\dagger(k) = \Omega_+ A^\dagger(k)\Omega_+^\dagger, \qquad B_-^\dagger(k) = \Omega_- A^\dagger(k)\Omega_-^\dagger \tag{61}$$

where $\Omega_\pm$ are the following evolution operators (Møller operators):

$$\Omega_\pm = \lim_{T\to\pm\infty} \mathrm{e}^{-\mathrm{i}\int_{-T}^0 \mathrm{d}t H_{\mathrm{int}}(t)}. \tag{62}$$

As is known from the general scattering theory, the operators $B_+^\dagger(k), B_-^\dagger(k)$ when acting on the vacuum state of the interacting theory $|\Omega\rangle$, create 'in' and 'out' states respectively. If we consider the interaction Hamiltonian H_{int} as normal-ordered, we have

$$\begin{aligned} B_\pm^\dagger(k) &= A^\dagger(k) - \mathrm{i}\frac{\lambda}{4!}\int_{\mp\infty}^0 \mathrm{d}t \int \mathrm{d}x\ \left[:\!\phi^4(x,t)\!:, A^\dagger(k)\right] \\ &= A^\dagger(k) - \mathrm{i}\frac{\lambda}{3!}\int_{\mp\infty}^0 \mathrm{d}t \int \mathrm{d}x \frac{\mathrm{e}^{-\mathrm{i}E_k t+\mathrm{i}kx}}{\sqrt{2E_k}} :\!\phi^3(x,t)\!: \end{aligned} \tag{63}$$

and expanding in terms of the free boson creation/annihilation operators $A(k), A^\dagger(k)$,

$$\begin{aligned} B_\pm^\dagger(k) = A^\dagger(k) - \mathrm{i}\frac{\lambda}{3!}\int_{\mp\infty}^0 \mathrm{d}t\, \frac{\mathrm{e}^{-\mathrm{i}E_k t}}{\sqrt{2E_k}} \int \frac{\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\, 2\pi\delta(k+\sum_i k_i)}{(2\pi)^3\sqrt{2^3 E_{k_1}E_{k_2}E_{k_3}}} \\ \times \big[A(k_1)A(k_2)A(k_3)\mathrm{e}^{-\mathrm{i}(E_{k_1}+E_{k_2}+E_{k_3})t} \\ + 3A^\dagger(-k_1)A(k_2)A(k_3)\mathrm{e}^{-\mathrm{i}(-E_{k_1}+E_{k_2}+E_{k_3})t} \\ + 3A^\dagger(-k_1)A^\dagger(-k_2)A(k_3)\mathrm{e}^{-\mathrm{i}(-E_{k_1}-E_{k_2}+E_{k_3})t} \\ + A^\dagger(-k_1)A^\dagger(-k_2)A^\dagger(-k_3)\mathrm{e}^{-\mathrm{i}(-E_{k_1}-E_{k_2}-E_{k_3})t}\big]. \end{aligned} \tag{64}$$

In scattering theory the t-integration is understood under the 'adiabatic switching' prescription which means introducing an $\mathrm{e}^{-\epsilon|t|}$ factor into the integrand. According to the identity

$$\int_{\pm\infty}^0 \mathrm{d}t\,\mathrm{e}^{-\epsilon|t|-\mathrm{i}\omega t} = \frac{\mathrm{i}}{\omega \mp \mathrm{i}\epsilon} \tag{65}$$

we then find

$$\begin{aligned} B_\pm^\dagger(k) = A^\dagger(k) + \frac{\lambda}{3!}\frac{1}{\sqrt{2E_k}} \int \frac{\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\, 2\pi\delta(k+\sum_i k_i)}{(2\pi)^3\sqrt{2^3 E_{k_1}E_{k_2}E_{k_3}}} \bigg[\frac{A(k_1)A(k_2)A(k_3)}{E_k+E_{k_1}+E_{k_2}+E_{k_3}} \\ + 3\frac{A^\dagger(-k_1)A(k_2)A(k_3)}{E_k-E_{k_1}+E_{k_2}+E_{k_3}} + 3\frac{A^\dagger(-k_1)A^\dagger(-k_2)A(k_3)}{E_k-E_{k_1}-E_{k_2}+E_{k_3}\pm\mathrm{i}\epsilon} \\ + \frac{A^\dagger(-k_1)A^\dagger(-k_2)A^\dagger(-k_3)}{E_k-E_{k_1}-E_{k_2}-E_{k_3}}\bigg]. \end{aligned} \tag{66}$$

Notice that we kept the $\pm\mathrm{i}\epsilon$ shift only in the third term since this is the only one that has a singularity (at $k_1 = -k, k_2 = -k_3$ or $k_2 = -k, k_1 = -k_3$). Using the formal identity

$$\lim_{\epsilon\to 0^+} \frac{1}{\omega \pm \mathrm{i}\epsilon} = \mathcal{P}\left(\frac{1}{\omega}\right) \mp \mathrm{i}\pi\,\delta(\omega) \tag{67}$$

we can rewrite the above as

$$B_{\pm}^{\dagger}(k) = B^{\dagger}(k) \mp \frac{\mathrm{i}\lambda}{8} \int \frac{\mathrm{d}q}{2\pi} \frac{1}{|kE_q - qE_k|} A^{\dagger}(k) A^{\dagger}(q) A(q) \tag{68}$$

where

$$\begin{aligned} B^{\dagger}(k) \equiv A^{\dagger}(k) + \frac{\lambda}{3!} \frac{1}{\sqrt{2E_k}} \int & \frac{\mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3\, 2\pi\delta(k + \sum_i k_i)}{(2\pi)^3 \sqrt{2^3 E_{k_1} E_{k_2} E_{k_3}}} \Bigg[\frac{A(k_1)A(k_2)A(k_3)}{E_k + E_{k_1} + E_{k_2} + E_{k_3}} \\ & + 3 \frac{A^{\dagger}(-k_1)A(k_2)A(k_3)}{E_k - E_{k_1} + E_{k_2} + E_{k_3}} + 3\, \mathcal{P.V.} \frac{A^{\dagger}(-k_1)A^{\dagger}(-k_2)A(k_3)}{E_k - E_{k_1} - E_{k_2} + E_{k_3}} \\ & + \frac{A^{\dagger}(-k_1)A^{\dagger}(-k_2)A^{\dagger}(-k_3)}{E_k - E_{k_1} - E_{k_2} - E_{k_3}} \Bigg]. \end{aligned} \tag{69}$$

From their definition (61), each of the two operators $B_{\pm}^{\dagger}(k)$ satisfy the same standard commutation relations as the free operators $A(k)$, $A^{\dagger}(k)$. Now let us define the operator

$$\begin{aligned} Z^{\dagger}(k) &\equiv B^{\dagger}(k) - \frac{\mathrm{i}\lambda}{8} \int \frac{\mathrm{d}q}{2\pi} \left(\frac{1}{kE_q - qE_k} \right) A^{\dagger}(k) A^{\dagger}(q) A(q) \\ &= B_{\pm}^{\dagger}(k) - \frac{\mathrm{i}\lambda}{8} \int \frac{\mathrm{d}q}{2\pi} \left(\frac{1}{kE_q - qE_k} \mp \frac{1}{|kE_q - qE_k|} \right) A^{\dagger}(k) A^{\dagger}(q) A(q). \end{aligned} \tag{70}$$

Since this is of the form (20), we automatically know that $Z^{\dagger}(k)$ and $Z(k)$ satisfy the ZF algebra with a non-trivial S-matrix given by

$$S(k,q) = 1 - \frac{\mathrm{i}\lambda}{4(kE_q - qE_k)} \tag{71}$$

which is indeed the correct first-order perturbation for the S-matrix (59):

$$S_{\mathrm{sh-G}}(\theta, \alpha) = 1 - cg^2 \frac{\mathrm{i}}{4 \sinh\theta} + \mathcal{O}(c^2 g^4). \tag{72}$$

Notice the infrared singularity in the coefficients of (70) and (72) when the momentum difference $k - q$ tends to zero. This reflects the non-analyticity of $S(0)$ as $g \to 0$ which we talked about in the introduction of this section.

Next we consider the states created by the action of $Z^{\dagger}(k)$ on the perturbed vacuum $|\Omega\rangle$, which by a calculation similar to the ones above, turns out to be

$$|\Omega\rangle = \Omega_{\pm}|0\rangle = \left(1 - \frac{\lambda}{4!} \int \prod_i^4 \mathrm{d}k_i \frac{2\pi\delta(\sum_i^4 k_i)}{(2\pi)^4 \sqrt{2^4 \prod_i^4 E_{k_i}}} \frac{A^{\dagger}(k_1)A^{\dagger}(k_2)A^{\dagger}(k_3)A^{\dagger}(k_4)}{\sum_i^4 E_{k_i}} \right) |0\rangle. \tag{73}$$

For the one-particle states it can be immediately seen that $Z^{\dagger}(k)|\Omega\rangle = B_{\pm}^{\dagger}(k)|\Omega\rangle$ always up to first order in λ. However, in order to verify that $Z^{\dagger}(k)$ plays the right role in creating in and out scattering states, we should check the two-particle states $Z^{\dagger}(k_1)Z^{\dagger}(k_2)|\Omega\rangle$. By normal ordering we find

$$\begin{aligned} Z^{\dagger}(k_1)Z^{\dagger}(k_2)|\Omega\rangle &= B_{\pm}^{\dagger}(k_1)B_{\pm}^{\dagger}(k_2)|\Omega\rangle \\ &- \frac{\mathrm{i}\lambda}{8} \left(\frac{1}{k_1 E_{k_2} - k_2 E_{k_1}} \mp \frac{1}{|k_1 E_{k_2} - k_2 E_{k_1}|} \right) A^{\dagger}(k_1) A^{\dagger}(k_2) |0\rangle. \end{aligned} \tag{74}$$

Observing that k/E_k is a monotonically increasing function of k, we can easily see that if $k_1 > k_2$, then

$$Z^\dagger(k_1)Z^\dagger(k_2)|\Omega\rangle = B^\dagger_+(k_1)B^\dagger_+(k_2)|\Omega\rangle, \tag{75}$$

i.e. it defines an in state, while if $k_1 < k_2$, then

$$Z^\dagger(k_1)Z^\dagger(k_2)|\Omega\rangle = B^\dagger_-(k_1)B^\dagger_-(k_2)|\Omega\rangle, \tag{76}$$

i.e. it defines an out state.

This example, apart from demonstrating how ZF operators emerge from the standard perturbation theory of scattering, also illustrates the concepts developed before and in particular the role of the first class of transformations (20) that we derived abstractly. We close our presentation of physical examples with the Lieb–Liniger model. We will also verify the consistency of our results for the two models, under the double non-relativistic limit that reduces the former to the latter.

4.2. The Lieb–Liniger model

The Lieb–Liniger model describes a $(1+1)$D system of non-relativistic bosons interacting with each other with a δ-function potential. Its Hamiltonian in second-quantized form is

$$H = \int_{-L}^{+L} \mathrm{d}x \, \left(\tfrac{1}{2}\partial_x\Psi^\dagger(x)\partial_x\Psi(x) + \lambda\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)\right), \tag{77}$$

where λ is now the interaction strength. The ground state energy for a system of N bosons as well as its thermodynamics can be exactly worked out by means of the Bethe ansatz [41]. The exact solution expresses the energy of the ground state and the excitation spectrum in terms of the dimensionless coupling constant $\gamma \equiv \lambda/\rho$ where $\rho = N/L$ is the density of bosons with $N, L \to \infty$.

In this model the relation between the bosonic field operator $\Psi(x)$ and the ZF operators $R_\lambda(k)$ that diagonalize the Hamiltonian for $L \to \infty$ has already been found using the inverse scattering method [40]:

$$\Psi(x) = \sum_{N=0}^{\infty} \int \prod_{i=1}^{N} \frac{\mathrm{d}p_i}{2\pi} \prod_{j=0}^{N} \frac{\mathrm{d}k_j}{2\pi} g_N(\{p\},\{k\};x) R^\dagger_\lambda(p_1)\cdots R^\dagger_\lambda(p_N) R_\lambda(k_N)\cdots R_\lambda(k_1)R_\lambda(k_0) \tag{78}$$

where

$$g_N(\{p\},\{k\};x) = \frac{(-\lambda)^N \exp[\mathrm{i}(\sum_{i=0}^N k_i - \sum_{i=1}^N p_i)x]}{\prod_{j=1}^N (p_j - k_j - \mathrm{i}\epsilon)(p_j - k_{j-1} - \mathrm{i}\epsilon)}. \tag{79}$$

Indeed it can be shown that the $R, R^\dagger$ operators diagonalize the Hamiltonian and satisfy the ZF algebra

$$[H, R^\dagger_\lambda(q)] = q^2 R^\dagger_\lambda(q), \tag{80}$$

$$R_\lambda(q)R_\lambda(q') = S_\lambda(q'-q)R_\lambda(q')R_\lambda(q), \tag{81}$$

$$R_\lambda(q)R^\dagger_\lambda(q') = S_\lambda(q-q')R^\dagger_\lambda(q')R_\lambda(q) + 2\pi\delta(q-q') \tag{82}$$

where the S-matrix is

$$S_\lambda(q) = \frac{q - \mathrm{i}\lambda}{q + \mathrm{i}\lambda}. \tag{83}$$

Let us consider the infinitesimal transformation from the free bosonic point $\lambda = 0$ to a small value λ. From (78) we have[13]

$$R_\lambda(k) = R_0(k) + \lambda \int \frac{\mathrm{d}q\,\mathrm{d}q'}{(2\pi)^2} \frac{R_0^\dagger(q+q'-k)R_0(q)R_0(q')}{(q-k-\mathrm{i}\epsilon)(q'-k-\mathrm{i}\epsilon)} + \mathcal{O}(\lambda^2). \tag{84}$$

The S-matrix is no longer the unit matrix but becomes instead

$$S_\lambda(p) = 1 - \lambda\frac{2\mathrm{i}}{p} + \mathcal{O}(\lambda^2) \tag{85}$$

and so, according to our previous findings, we expect the transformation to contain the generator of S-matrix shifts (20) with coefficients $a_{k,q} = \mathrm{i}/(q-k)$. Indeed, using the identity (67) we recognize that part of the infinitesimal transformation (84) has exactly the form of (20) with the right coefficient

$$\int \frac{\mathrm{d}q}{2\pi} \frac{\mathrm{i}}{q-k} R_0^\dagger(q)R_0(q)R_0(k), \tag{86}$$

while the remaining part does not affect the S-matrix. Once again, notice the infrared singularity in the coefficient of the above expression for $q = k$.

Lastly, we mention that the infinitesimal transformations (70) and (84) derived for the sinh–Gordon and Lieb–Liniger models, respectively, are consistent with each other under the double non-relativistic limit $c \to \infty$, $g \to 0$, gc : const, that leads from the former to the latter model. Following [38] we substitute the field ϕ in (63) as

$$\phi(x,t) = \frac{1}{\sqrt{2m}}(\psi(x,t)\mathrm{e}^{-\mathrm{i}mc^2 t} + \psi^\dagger(x,t)\mathrm{e}^{+\mathrm{i}mc^2 t}) \tag{87}$$

and keep only the non-oscillating terms, rewriting first all expressions with their c dependence explicit and taking into account that in the non-relativistic limit, $E_k = mc^2 + k^2/2m + \cdots$. After some algebra we verify that (70) reduces to (84).

5. Conclusions

In this paper we have investigated how the initial state of a quantum quench in an integrable model can be expressed, from first principles, in terms of ZF operators, without relying on the usual mapping to slab geometry and the associated boundary renormalization group arguments [11, 12, 24]. We show that this result can be achieved by deriving the relation between the pre-quench and the post-quench operators on the condition that such a relation respects the Zamolodchikov–Faddeev (ZF) algebra satisfied by integrable models.

[13] Note that, unlike in a relativistic free field theory where the creation–annihilation operators are linear combinations of the field ϕ and its conjugate momentum π, in a non-relativistic free field theory the creation–annihilation operators are the bosonic field itself $R_0(k) = \int \mathrm{d}x\,\mathrm{e}^{-\mathrm{i}kx}\Psi(x)$. Also the conjugate momentum is $\Pi = \mathrm{i}\Psi^\dagger$ and does not appear in the Hamiltonian.

Under the conditions that such transformations must satisfy the ZF algebra at the infinitesimal order, we initially showed that the usual linear Bogoliubov transformations do not respect the ZF algebra, apart from for the trivial cases of free bosons or fermions, a result that holds generally for finite transformations too.

We have then identified two important classes of transformations. Those in the first class change the S-matrix of the theory but preserve its ground state as well as its conserved charges. We also argued that any infinitesimal transformation can be decomposed into a part that induces the S-matrix shift and a remaining part that does not alter the S-matrix. Those in the second class belong to the latter subset of transformations, which can be regarded as a generalization of the Bogoliubov transformations for interacting theories, since they reduce to the usual Bogoliubov transformations whenever the integrable model reaches a free bosonic or fermionic point. Like in the free case, the ground state of the system becomes a squeezed state when expressed in the transformed ZF basis under such a generalized Bogoliubov transformation, at least when its coefficients are real. We have also shown that the change in the Hamiltonian (and in the other conserved charges of the theory) is of the same form as that for the Bogoliubov transformations. The net effect of transformations of this type is to 'dress' the initial quasiparticle with pairs of new particles with momenta opposite to each other.

We have also outlined how one could proceed further in this program to identify the transformations which preserve the ZF algebra, in particular pointing out the existence of transformations of higher complexity characterized by the fact that, even for infinitesimal quenches, they are associated with an *infinite* series of terms given by products of the initial creation/annihilation operators. From the quantum quench perspective, this means that even a small quench of the physical parameters of an integrable model may result in an infinite series which links the pre-quench and the post-quench operators. In this case, the calculation of the initial state made on first principles is rather difficult, unless a truncation or resummation of the series can be established on the grounds of a different argument. In such a case, for instance, it may be possible to reorganize the terms of the series on the basis of a small-density expansion, following the concepts developed in [25]. We hope that our work on ZF algebra transformations will stimulate further investigation of their structure and properties, from both pure and applied points of view.

Lastly we exemplified our approach in the context of the sinh–Gordon and Lieb–Liniger models. We restricted ourselves to perturbations about the free bosonic point of these models since in this case the transformations can be found relatively easily and contain only up to cubic terms in the ZF operators. We expect analogous simplification to occur near free points of other integrable models too and it would be interesting to explore some physical realization of such quench processes. As regards the sinh–Gordon model, an application of our results to the corresponding quantum quench problem of an abrupt switch-on of the interaction would give results comparable with those from earlier work [42]. For the Lieb–Liniger model, however, further manipulation is required, mainly due to the fact that the ground state is not the empty vacuum but contains a large number of particles, the number being proportional to the size of the system. This issue is discussed and a numerical approximation is developed in [37]. One recent numerical study of a special quantum quench in the Lieb–Liniger model is [17].

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Appendix A. Squeezed states in free quantum field theories

In this appendix we show that the squeezed coherent states in a quantum quench of free theories are a consequence of the Bogoliubov transformation of their operators.

Bosonic theory. Let us consider firstly the quench in a bosonic theory with Hamiltonian [12]

$$H = \tfrac{1}{2}\int \left[\pi^2 + (\nabla\varphi)^2 + m_0^2\varphi^2\right]\, \mathrm{d}x.$$

This system can be diagonalized in momentum space:

$$H = \int \Omega_k^0 : A_k^{0\dagger} A_k^0 :, \qquad (\Omega_k^0)^2 = m_0^2 + k^2,$$

$$A_k^0 = \frac{1}{\sqrt{2\Omega_k^0}}\left(\Omega_k^0\varphi_k + \mathrm{i}\pi_k\right), \qquad A_k^{0\dagger} = \frac{1}{\sqrt{2\Omega_k^0}}\left(\Omega_k^0\varphi_{-k} - \mathrm{i}\pi_{-k}\right),$$

with the ground state $|\Psi_0\rangle$ identified by the condition

$$A_k^0|\Psi_0\rangle = 0. \tag{A.1}$$

Imagine now that, after having prepared the system in its ground state, we quench the mass $m_0 \to m$. The relation between the pre-quench ladder operators $(A_k^0, A_k^{0\dagger})$ and the post-quench ones $(A_k, A_k^\dagger)$ is a Bogoliubov transformation:

$$\begin{aligned} A_k &= c_k A_k^0 + d_k A_{-k}^{0\dagger}, \qquad & A_k^\dagger &= c_k A_k^{0\dagger} + d_k A_{-k}^0 \\ A_k^0 &= c_k A_k - d_k A_{-k}^\dagger, \qquad & A_k^{0\dagger} &= c_k A_k^\dagger - d_k A_{-k}, \end{aligned}$$

where the coefficients are given by

$$c_k = \frac{\Omega_k + \Omega_k^0}{2\sqrt{\Omega_k\Omega_k^0}}, \qquad d_k = \frac{\Omega_k - \Omega_k^0}{2\sqrt{\Omega_k\Omega_k^0}}.$$

Substituting the expression for A_k^0 from the Bogoliubov transformation into equation (A.1), we see that, in terms of the new operators, the initial state satisfies the condition

$$[c_k A_k - d_k A_{-k}^\dagger]|\Psi_0\rangle = 0. \tag{A.2}$$

whose solution is given in terms of a squeezed coherent state:

$$|\Psi_0\rangle = \mathcal{N}\exp\left[\int_{-\infty}^{\infty} K_{\mathrm{boson}}(k) A_k^\dagger A_{-k}^\dagger\, \mathrm{d}k\right]|0\rangle, \tag{A.3}$$

where

$$K_{\mathrm{boson}}(k) = \frac{\Omega_k^0 - \Omega_k}{\Omega_k^0 + \Omega_k}. \tag{A.4}$$

This quantity can be written in a suggestive way by introducing the rapidities of the particle relative to the initial and final situations, i.e.

$$\begin{aligned} \Omega^0 &= m_0 \cosh\theta_0, \qquad k = m_0 \sinh\theta_0 \\ \Omega &= m\cosh\theta, \qquad k = m\sinh\theta. \end{aligned} \tag{A.5}$$

From the equality of the initial and final momenta, we have the relation which links the two rapidities:

$$m_0 \sinh\theta_0 = m\sinh\theta \Rightarrow \frac{m_0}{m} = \frac{\sinh\theta}{\sinh\theta_0} \tag{A.6}$$

and therefore, the amplitude $K_{\text{boson}}(k)$ of equation (A.4) can be neatly written as

$$\begin{aligned} K_{\text{boson}}(\theta,\theta_0) &= \frac{m_0\cosh\theta_0 - m\cosh\theta}{m_0\cosh\theta_0 + m\cosh\theta} = \frac{m_0/m\cosh\theta_0 - \cosh\theta}{m_0/m\cosh\theta_0 + \cosh\theta} \\ &= \frac{\sinh\theta\cosh\theta_0 - \sinh\theta_0\cosh\theta}{\sinh\theta\cosh\theta_0 + \sinh\theta_0\cosh\theta} = \frac{\sinh(\theta-\theta_0)}{\sinh(\theta+\theta_0)}. \end{aligned} \tag{A.7}$$

Fermionic theory. One can easily work out the Bogoliubov transformation relative to the quench of the mass of a free fermionic system [23]. Consider, in particular, a free Majorana fermion in $(1+1)$ dimensions, with the mode expansion of the two components of this field given by

$$\psi_1(x,t) = \int_{-\infty}^{+\infty} \mathrm{d}p \, \left[\alpha(p)A(p)\mathrm{e}^{-\mathrm{i}Et+\mathrm{i}px} + \bar{\alpha}(p)A^\dagger(p)\mathrm{e}^{\mathrm{i}Et-\mathrm{i}px}\right]$$

$$\psi_2(x,t) = \int_{-\infty}^{+\infty} \mathrm{d}p \, \left[\beta(p)A(p)\mathrm{e}^{-\mathrm{i}Et+\mathrm{i}px} + \bar{\beta}(p)A^\dagger(p)\mathrm{e}^{\mathrm{i}Et-\mathrm{i}px}\right]$$

where

$$\alpha(p) = \frac{\omega}{2\pi\sqrt{2}}\frac{\sqrt{E+p}}{E}, \qquad \overline{\alpha}(p) = \frac{\overline{\omega}}{2\pi\sqrt{2}}\frac{\sqrt{E+p}}{E}$$

$$\beta(p) = \frac{\bar{\omega}}{2\pi\sqrt{2}}\frac{\sqrt{E-p}}{E}, \qquad \bar{\beta}(p) = \frac{\omega}{2\pi\sqrt{2}}\frac{\sqrt{E-p}}{E}$$

with $\omega = \exp(\mathrm{i}\pi/4)$. At $t=0$, i.e. at the instant of the quench, we can extract the Fourier mode of each component of the Majorana field $\psi_i(x,0) = \int \mathrm{d}p \, \hat{\psi}_i(p)\mathrm{e}^{\mathrm{i}px}$, given by

$$\hat{\psi}_1(p) = \alpha(p)A(p) + \overline{\alpha}(-p)A^\dagger(-p) \qquad \hat{\psi}_2(p) = \beta(p)A(p) + \overline{\beta}(-p)A^\dagger(-p).$$

Suppose now that the mass of the field is changed from m_0 to m at $t=0$ and let us denote by $(A_0(p), A_0^\dagger(p))$ and $(A(p), A^\dagger(p))$ the sets of oscillators before and after the quench. The proper boundary condition associated with such a situation is the continuity of the field components before and after the quench, i.e. $\psi_i^0(x,t=0) = \psi_i(x,t=0)$, which implies $\hat{\psi}_i^0(p) = \hat{\psi}_i(p)$. This gives rise to the Bogoliubov transformation between the two sets of oscillators:

$$A_0(p) = u(p)A(p) + \mathrm{i}v(p)A^\dagger(-p) \qquad A_0^\dagger(p) = u(p)A^\dagger(p) - \mathrm{i}v(p)A(-p)$$

where

$$u(p) = \frac{1}{2E}\left[\sqrt{(E_0+p)(E+p)} + \sqrt{(E_0-p)(E-p)}\right]$$

$$v(p) = \frac{1}{2E}\left[\sqrt{(E_0-p)(E+p)} - \sqrt{(E_0+p)(E-p)}\right].$$

Notice that these functions satisfy the relations $u(p) = u(-p)$ and $v(p) = -v(-p)$ together with $u^2(p) + v^2(p) = E_0/E$, which refers to the normalization of the respective set of oscillators.

With the same procedure as we used in the bosonic case, it is easy to see that the boundary state corresponding to this quench can be written as

$$|B\rangle = \exp\left(\int_{-\infty}^{\infty} \mathrm{d}p\, K_{\text{fermion}}(p) A^\dagger(p) A^\dagger(-p)\right)|0\rangle,$$

where

$$K_{\text{fermion}}(p) = -K_{\text{fermion}}(-p) = \mathrm{i}\frac{\sqrt{(E_0 - p)(E + p)} - \sqrt{(E_0 + p)(E - p)}}{\sqrt{(E_0 + p)(E + p)} + \sqrt{(E_0 - p)(E - p)}}. \tag{A.8}$$

As in the bosonic case, this quantity can be expressed in a more concise form by introducing the rapidities of the particle before and after the quench, i.e.

$$E_0 \pm p = m_0 \mathrm{e}^{\pm\theta_0}, \qquad E \pm p = m\mathrm{e}^{\pm\theta}.$$

Substituting these expressions in (A.8), we get

$$K_{\text{fermion}}(\theta, \theta_0) = \mathrm{i}\frac{\sinh((\theta - \theta_0)/2)}{\cosh((\theta + \theta_0)/2)}. \tag{A.9}$$

In conclusion, the squeezed coherent form of the initial state in free theories comes from the fact that, in any quantum quench of these systems, the creation–annihilation operators before and after it are related by a Bogoliubov transformation. And this in turn is a consequence of the canonical commutation/anti-commutation relations satisfied by these fields[14].

Finally, notice that both the bosonic and fermionic amplitudes $K(p)$ do not satisfy, in general, a unitarity equation, in contrast to the amplitudes of squeezed coherent states in the purely boundary integrable theories studied by Ghoshal and Zamolodchikov [36]. The reason for this condition being required for purely boundary field theory but not for an arbitrary quench process is quite easy to understand. In purely boundary theory, all the degrees of freedom for $t < 0$ are completely frozen: the hard-wall boundary does not allow any process of transmission through the boundary and this ends up in the unitarity condition. However, for quantum quenches in free theories, there are degrees of freedom also for $t < 0$, which are related to the ones for $t > 0$ just by the Bogoliubov transformations. Stated another way, this says that the boundary condition in free theories allows for transmission, as is shown in figure A.1. The degrees of freedom present on both sides of the boundary prevent the bosonic and fermionic amplitudes (A.4) and (A.8) from satisfying a unitarity equation. Notice that the only case where they satisfy a unitarity equation is that where we freeze the degrees of freedom before the quench by taking the limit $m_0 \to \infty$: in this limit, in fact, we have a transmissionless boundary which implements in both theories the Dirichlet boundary condition, with $K_{\text{boson}} = 1$ and $K_{\text{fermion}} = \mathrm{i}\tanh(\theta/2)$ respectively.

[14] To be precise there can be exceptions to this rule since it is possible to construct generalized Bogoliubov transformations which satisfy the CCR/CAR but are nonlinear [43]–[45]. These however correspond to non-quadratic Hamiltonians which, even though they can be reduced to free ones, are uncommon in physically interesting cases.

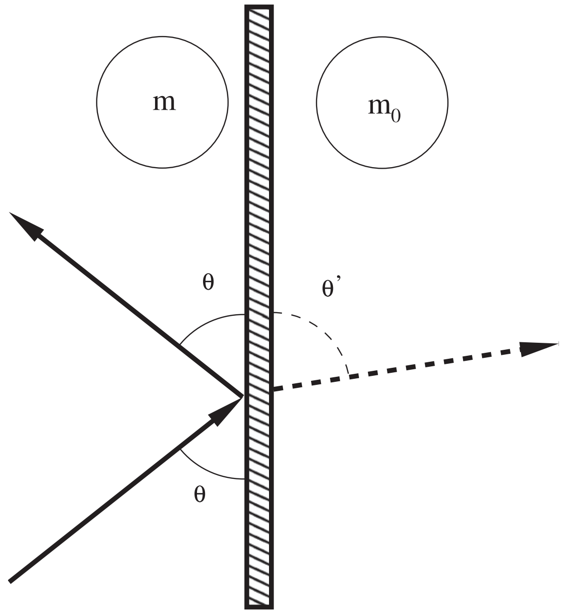

Figure A.1. The boundary condition for a mass quench allows for transmission. Notice that, for free theories, there is no particle production at the boundary and the transmitted particle always has the same momentum as the incoming one, since momentum is conserved at the boundary. However, since the masses are different at $x < 0$ and $x > 0$, the rapidity changes from θ to θ'.

Appendix B. Derivation of two classes of generators of ZF algebra transformations

In this appendix we present the derivation of the two simple classes of infinitesimal transformations of the ZF algebra that we introduced in the main text. As already mentioned, we can always write W as a linear combination of products of Z and $Z^\dagger$. The first class arises when we consider W to be a single such product:

$$W_p = \prod Z_i \prod Z_j^\dagger \qquad \text{with } p = \sum p_i - \sum p_j. \tag{B.1}$$

For brevity we use the notation $Z_i \equiv Z_{p_i}$ when there is no confusion about the meaning of the indices, and also the symbol '$\simeq$' that we defined in section 3.4. Then we find

$$W_1 Z_2 \simeq \lambda_{12} Z_2 W_1 \tag{B.2}$$

where $\lambda_{12} = \prod S_{i2} \prod S_{2j}$ (note the implicit dependence on p_1 through the momentum condition in (B.1)). Similarly we have

$$W_1 Z_2^\dagger \simeq \lambda_{12}^* Z_2^\dagger W_1. \tag{B.3}$$

From (16) and (17) we have the conditions

$$W_1 Z_2 + Z_1 W_2 \simeq S_{12}(Z_2 W_1 + W_2 Z_1) \tag{B.4}$$

$$W_1 Z_2^\dagger + Z_1 W_2^\dagger \simeq S_{21}(Z_2^\dagger W_1 + W_2^\dagger Z_1) \tag{B.5}$$

where we ignored T_{12} since the corresponding terms are not of highest order (unless W is a single operator instead of a product, which is the linear case that has already been excluded as not fulfilling the conditions). In the main text equation (32), we considered the simple choice $\lambda_{12} = S_{12}$ which automatically satisfies the above condition but may not be the only possibility. Indeed (B.4) is in general a weaker condition than this choice. Strictly speaking, the two conditions above require

$$(\lambda_{12} - S_{12})Z_2W_1 + (\lambda_{21}^* - S_{12})W_2Z_1 \simeq 0 \tag{B.6}$$

$$(\lambda_{12}^* - S_{21})Z_2^\dagger W_1 + (\lambda_{21} - S_{21})W_2^\dagger Z_1 \simeq 0 \tag{B.7}$$

which means either that $\lambda_{12} = S_{12}$ as before, or that W_2Z_1 and Z_2W_1 contain exactly the same operators and similarly for $W_2^\dagger Z_1$ and $Z_2^\dagger W_1$. In this last case W_p must contain Z_p, i.e. $W_p = Z_p\tilde{W}_0$ where $\tilde{W}_0$ has momentum zero, and, as can be easily deduced from the above equations, it must be independent of p and contain the same operators as its Hermitian conjugate $\tilde{W}_0^\dagger$, that is $\tilde{W}_0 \simeq \prod Z_i^\dagger Z_i$. But this is again exactly the same case as was considered before (33) and it corresponds to $\lambda_{12} = S_{12}$.

Another possibility arises when we consider W_p to be a linear combination of two products of operators of the previous form:

$$W_p = a_{+,p}W_{+,p} + a_{-,p}W_{-,p}. \tag{B.8}$$

The conditions are then

$$\begin{aligned} &a_{+1}W_{+1}Z_2 + a_{+2}Z_1W_{+2} + a_{-1}W_{-1}Z_2 + a_{-2}Z_1W_{-2} \\ &\qquad \simeq S_{12}(a_{+1}Z_2W_{+1} + a_{+2}W_{+2}Z_1 + a_{-1}Z_2W_{-1} + a_{-2}W_{-2}Z_1) \end{aligned} \tag{B.9}$$

$$\begin{aligned} &a_{+1}W_{+1}Z_2^\dagger + a_{+2}^*Z_1W_{+2}^\dagger + a_{-1}W_{-1}Z_2^\dagger + a_{-2}^*Z_1W_{-2}^\dagger \\ &\qquad \simeq S_{21}(a_{+1}Z_2^\dagger W_{+1} + a_{+2}^*W_{+2}^\dagger Z_1 + a_{-1}Z_2^\dagger W_{-1} + a_{-2}^*W_{-2}^\dagger Z_1) \end{aligned} \tag{B.10}$$

or

$$\begin{aligned} &(\lambda_{+12} - S_{12})a_{+1}Z_2W_{+1} + (\lambda_{+21}^* - S_{12})a_{+2}W_{+2}Z_1 \\ &\qquad + (\lambda_{-12} - S_{12})a_{-1}Z_2W_{-1} + (\lambda_{-21}^* - S_{12})a_{-2}W_{-2}Z_1 \simeq 0 \end{aligned} \tag{B.11}$$

$$\begin{aligned} &(\lambda_{+12}^* - S_{21})a_{+1}Z_2^\dagger W_{+1} + (\lambda_{+21} - S_{21})a_{+2}^*W_{+2}^\dagger Z_1 \\ &\qquad + (\lambda_{-12}^* - S_{21})a_{-1}Z_2^\dagger W_{-1} + (\lambda_{-21} - S_{21})a_{-2}^*W_{-2}^\dagger Z_1 \simeq 0. \end{aligned} \tag{B.12}$$

Like before, one obvious choice is $\lambda_{+12} = \lambda_{-12} = S_{12}$ which is trivial since each of W_+ and W_- must fall in the earlier discussed case, but unlike before there is an alternative that is not equally trivial. If Z_2W_{+1} and $W_{-2}Z_1$ contain the same operators and the same holds for $Z_2^\dagger W_{+1}$ and $W_{-2}^\dagger Z_1$ (and similarly for the other pairs), then $W_{+,p}$ and $W_{-,p}$ can be written in the form

$$W_{+,p} = Z_p\tilde{W}_0 \qquad \text{and} \qquad W_{-,p} = Z_p\tilde{W}_0^\dagger \tag{B.13}$$

where $\tilde{W}_0$ has momentum zero and must be independent of p. If we define

$$Z_p\tilde{W}_0 \simeq \mu_p\tilde{W}_0Z_p \tag{B.14}$$

then $\lambda_{+12} = \mu_2^* S_{12}$, $\lambda_{-12} = \mu_2 S_{12}$ and the above conditions are equivalent to the set of equations

$$\frac{a_{-2}}{a_{-1}} = \frac{\mu_2 - 1}{\mu_1 - 1} \tag{B.15}$$

$$\frac{a_{+2}}{a_{+1}} = \frac{\mu_2^* - 1}{\mu_1^* - 1} \tag{B.16}$$

$$\frac{a_{-2}^*}{a_{+1}} = -\frac{\mu_2^* - 1}{\mu_1^* - 1} \tag{B.17}$$

$$\frac{a_{+2}^*}{a_{-1}} = -\frac{\mu_2 - 1}{\mu_1 - 1} \tag{B.18}$$

which have to be valid for any choice of p_1, p_2. The solution is

$$a_{+,p} = b(\mu_p^* - 1) \tag{B.19}$$

$$a_{-,p} = -b^*(\mu_p - 1) \tag{B.20}$$

for some constant b.

A simple choice for $\tilde{W}_0$ is the product of two operators that create a pair of particles with opposite momenta $Z_q^\dagger Z_{-q}^\dagger$. Obviously a sum over all such pairs can be used, leading to the following expression for W_p:

$$W_p \simeq \sum_q b_q (S_{p,q} S_{p,-q} - 1) Z_p Z_q^\dagger Z_{-q}^\dagger + \sum_q b_q^* (S_{p,q} S_{p,-q} - 1) Z_{-q} Z_q Z_p \tag{B.21}$$

where by symmetry of the sums under $q \to -q$ we find that b_q can be chosen to satisfy $b_{-q} = b_q S_{q,-q}$. Substituting into the general conditions (28) and (29) we find out that an additional lower order term proportional to $Z_{-p}^\dagger$ is necessary in W_p in order to satisfy the full-form conditions. Overall the transformation is

$$W_p = \sum_q b_q (S_{p,q} S_{p,-q} - 1) Z_p Z_q^\dagger Z_{-q}^\dagger + \sum_q b_q^* (S_{p,q} S_{p,-q} - 1) Z_{-q} Z_q Z_p + 2 b_{-p} Z_{-p}^\dagger. \tag{B.22}$$

Note that the last term cannot be absorbed by reordering the operators of the first one.

Other choices for $\tilde{W}_0$ are still possible and may lead to more complex families of transformations. One may also consider linear combinations of more than two products and continue in a similar fashion.

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