Dynamics on spatial networks and the effect of distance coarse graining

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Recently, spatial networks have attracted much attention. The spatial network is constructed from a regular lattice by adding long-range edges (shortcuts) with probability $P(r) \sim r^{-\delta}$, where r is the geographical distance between the two ends of the edge. Also, a cost constraint on the total length of the additional edges is introduced ($\sum r = C$). It has been pointed out that such networks have optimal exponents δ for the average shortest path, traffic dynamics and navigation. However, when δ is large, too many generated long-range edges will be added to the network. In this scenario, the total cost constraint cannot be satisfied. In this paper, we propose a distance coarse graining procedure to solve this problem. We find that the optimal exponents δ for the traffic process, navigation and synchronization indeed result from the trade-off between the probability density function of long-range edges and the total cost constraint, but the optimal exponent δ for percolation is actually due to dissatisfying the total cost constraint. On the other hand, because the distance coarse graining procedure widely exists in the real world, our work is also meaningful in this aspect.

1. Introduction

The research on complex networks has been one of the most active fields not only in physics but also in other various disciplines of natural and social sciences [1–4]. In traditional statistical mechanics, interactions mainly exist between neighboring elements. By introducing the complex topology of the networks, the whole system can emerge with some new properties such as small-world [2], scale-free degree distribution [3] and community structures [4]. However, the spatial property is of great significance as well, which makes the interaction between nodes go beyond the neighboring effect but under the restraint of their underlying geographical site. This property matters much in lots of empirical networks including neural networks [5], communication networks [6], electric-power grids [7], transportation systems [8–11] and even social networks [12–16]. Generally, the geography information of the nodes and the distance between nodes in such a network would determine the characteristics of the network and play an important role in the dynamics happening in the network [17].

About the networks embedded in the geographical space, there are many works [16,18–23,26–28]. The first category focuses on the empirical spatial networks. Specifically, networks with strong geographical constraints, such as power grids or transport networks, are found with fractal scaling [18,19,29]. Besides, some researchers introduce some ways to model empirical geographical networks including epidemic spreading networks [16], urban street networks [20], facilities distribution networks [21], transportation networks and the Internet [22]. In the second category, researchers theoretically discuss the spatial properties on artificial models. For example, when supplementing long-range edges whose Euclidean

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lengths l are distributed according to $q(l) \propto l^{-\alpha}$ to D-dimensional lattices, Sen, Banerjee and Biswas conjecture that the two transition points from random networks and regular networks to the networks with small-world effects in any dimension are $\alpha = D$ and $\alpha = D + 1$ respectively [23], and related results have been analytically studied recently [24,25]. Also, Xulvi–Brunet and Sokolov construct a growing network model by $\prod_{i\to j} \propto k_i l_{ij}^{-\alpha}$, where l_{ij} is the Euclidean distance between i and j. Numerical simulations have shown that for $\alpha < 1$, the degree distribution is a power-law distribution, and for $\alpha > 1$, it is fitted by a stretched exponential distribution [26]. Additionally, by taking the degree and geographical distance into account, Xie et al. in Ref. [27] propose a growing network model based on an optimal policy involving both topological and geographical measures. In another work, Wang, Zhu and Gu model a kind of two-dimensional planes on which nodes move randomly with a triangular lattice and investigate critical global connectivity of these networks [28].

However, few of these former works in spatial networks are related to the total cost constraint. The total cost is defined as the total length of the edges in geographical network models. For a specific edge, the longer it is, the more it will cost. Very recently, some researchers take this aspect into account [15,30,31]. Because these network models are with spatial properties, the lattice (geographical) distance is used to measure the difference between geometric locations of different nodes. Specifically, on an n-dimensional lattice, the lattice distance between any pair of nodes i, j should be calculated along the underlying lattice. For example, on a 1-dimensional lattice (ring), any two neighboring nodes have a lattice distance 1. The distance between any pair of nodes i, j can be expressed by $r_{ij} = x + 1$, where x is the number of nodes between node i and j in the ring. The lattice lengths r of an edge between i and j equals the Manhattan distance r_{ij}. In [30], based on a 1-dimensional lattice (ring) and subjected to a limited cost C, long-range connections are added under the probability density function (PDF) $P(r) = ar^{-\delta}$, where r is the lattice length of these additional edges. Note that the addition of long-range connections to the system stops when their total length (cost), $\sum r$, reaches the given value C. Some basic topological properties of this network model with different values of δ are studied. It is found that the network has the minimum average shortest path when $\delta = 2$ in 1-dimensional spatial networks. In addition, the authors investigate a classic traffic model on these networks. It is found that $\delta = 1.5$ is the optimization value for the traffic process on the spatial networks. In Ref. [31], pairs of sites i, j in 2-dimensional lattices are randomly chosen to receive long-range connections with probability $P_{ij} \sim r_{ij}^{-\alpha}$, where r_{ij} is the lattice distance between sites i and j. With the total cost constraint, Li et al. find $\alpha = 3$ corresponds to the minimum average shortest path. Moreover, they claim that the optimal value for navigation is $\alpha = 3$ in 2-dimensional spatial networks and $\alpha = 2$ in 1-dimensional ones [33]. Very recently, the 1-dimensional scenario has been strictly proved by Li et al. [32].

In the spatial network, the probability distribution of these long-range connections is chosen as $P(r) = ar^{-\delta}$. When the controlling parameter δ is small, long-range connections are more likely with large lattice lengths r. When δ is large, most of the long-range connections are with small lattice lengths r. Due to the total length constraint, the number of long-range edges can be very large when their lengths r are small. Therefore, the network will have many (several) long-range connections when δ is large (small). One serious problem is when δ is large, too many long-range connections will be added to the underlying lattice. In this paper, we use a so-called distance coarse graining procedure to solve this problem. Actually, the distance coarse graining procedure commonly exists in many empirical researches, where the distances are estimated approximately [8–11]. For example, when studying the railway network of China, three stations in the city of Wuhan are merged into one node and four stations in the city of Shanghai are merged into one node [11]. This procedure named distance coarse graining is introduced to the spatial network models here. In fact, the coarse graining concept has been introduced to study networks since a long time ago [34–36]. All related works focus on how to reduce the size of the network and keep other properties such as degree distribution, cluster coefficient, degree correlation [34], random walks [35] and synchronizability [36] unchanged. Here, we will study how the distance coarse graining procedure affects the topology and dynamical process on the spatial network model.

2. Generating binary spatial networks with coarse grained distances

In this paper, we also use lattice distance to measure the geometric distance. In the following text, whenever "distance" is mentioned, it refers to the lattice distance r_{ij} between two nodes i, j. Likewise, whenever "length" is mentioned (excluding the shortest path length), it refers to the lattice length r of a long-range edge. An example can be seen in the upper picture of Fig. 1(b). Note that once defined, the r_{ij} of any two nodes will never be changed in the edge addition process.

The spatial network model is embedded in a 1-dimensional lattice. Long-range edges are generated and added to the underlying ring according to the probability $P(r) = ar^{-\delta}$, where r is the length of these additional edges. A total cost C is introduced to this network model. Actually, it is used to constrain the total length of the long-range connections, which means that the $\sum r$ should equal C after adding all these long-range connections. A typical spatial network model is shown in Fig. 1(a).

Obviously, there are two significant features of the spatial network: the power-law distribution of the length of these long-range edges and the restriction on the total cost. It is easy to tell that when the exponent δ is small, long-range edges are more likely with large lattice lengths r. When δ is large, most of the long-range edges are with small lattice lengths r. Because $\sum r$ should equal C, the number of long-range edges can be very large when their lengths r are small. Hence, many (several) long-range edges will be added to the underlying 1-dimensional lattice when δ is large (small). However, when δ is too large, there are so many long-range edges with small lengths that the underlying lattice cannot provide enough room for them. To solve this problem, a so-called distance coarse graining procedure is effective here. As mentioned above,

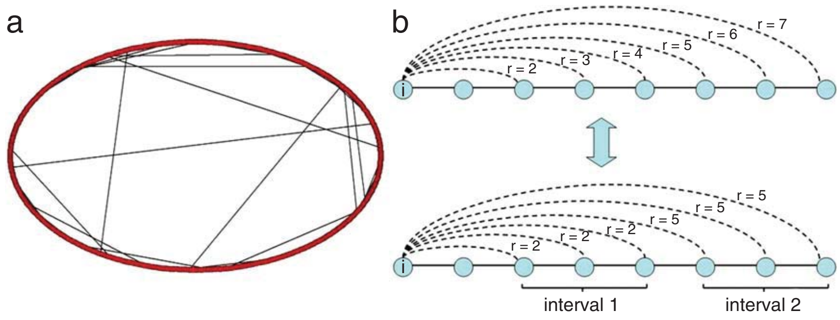

Fig. 1. (Color online) (a) A typical binary spatial network. $N = 500$, $C = c \times N = 10 \times 500 = 5000$ and $\delta = 2$. (b) The illustration for the mechanism of distance coarse graining procedure where the upper example is without distance coarse graining and the lower one is with distance coarse graining ($W = 3$). Suppose the node i is chosen, its long-range edges (dashed line) with $r = 2, 3, 4$ and $r = 5, 6, 7$ will be regarded as $r = 2$ and $r = 5$ respectively, so that the networks with distance coarse graining can provide more room for these long-range edges with small r.

researchers incline to use the distance coarse graining procedure to handle the geographic data in the real world [12–14,37,38]. For instance, geographic data are limited to the level of towns and cities in Ref. [14]. That is to say, researchers estimate the distances approximately by regarding a range of distance as a typical value. Likewise, in our spatial network model, we divide the value range of the distance into many continuous intervals in which all the distances are considered as one typical value as shown in the bottom picture of Fig. 1(b). With the distance coarse graining procedure, the spatial network is constructed as follows:

(1) N nodes are arranged in a 1-dimensional lattice. Every node is connected with its nearest neighbors, which can keep every node reachable. Additionally, between any pair of nodes, there is a well defined lattice distance r_{ij}.
(2) Set the interval length W of the coarse graining procedure. For each node, divide the distance between it and other nodes into $\frac{N_{\max}-1}{W}$ intervals like the lower picture in Fig. 1(b) ($N_{\max}$ is the largest lattice distance between any nodes in the underlying lattice). The mth interval includes the distance from $(m-1)W+2$ to $mW+1$.
(3) A node i is chosen randomly, and a certain long-range edge with length $r (2 \le r \le N_{\max})$ is generated with probability $P(r) = ar^{-\delta}$, where a is determined from the normalization condition $\sum_{r=2}^{N_{\max}} P(r) = 1$.
(4) Find the interval that r belongs to, say the mth interval here. Then, one of the nodes in this interval is picked randomly, called j. Connect nodes i and j if there exists no edge between them yet.
(5) After Step 4, a certain cost r_{ij} is generated. Repeat Steps 3 and 4 until the total cost reaches C.

Firstly, we should determine how to choose an appropriate value of the controlling parameter W under different total costs C. Clearly, when $W = 1$, there is no distance coarse graining at all. If W is too small, the spatial network model still cannot provide enough room for long-range connections with small lattice lengths. On the contrary, because all the nodes in one distance coarse graining interval are regarded as the same, if W is too large, too many nodes are considered in one distance coarse graining interval so that the power-law distance distribution will be destroyed significantly.

In order to deduce the formula of appropriate values of W, we consider an extreme condition with δ equal to a very large positive value. Under this circumstance, all long-range edges are with a short length and a large number of these long-range edges will be generated. Due to their short lengths r, all of them will be located in the first distance coarse graining interval of each node. Consequently, the total cost is $C = (2+3+4+\cdots+W)N$. For $C = cN$, $2+3+4+\cdots+W = c$, where c is the average cost on each node. After simplification, $W^2 + W - 2 - 2c = 0$. So we can get $W = \frac{\sqrt{9+8c}-1}{2}$. Generally, W only has to obey $W \ge [\frac{\sqrt{9+8c}-1}{2}]_{floor}$, where $[\cdot]_{floor}$ represents the operation of rounding downward. In this paper, we choose $W = 4$ when $c = 10$, $W = 7$ when $c = 30$ and $W = 9$ when $c = 50$ as examples.

As mentioned above, the binary (unweighted) spatial network cannot be generated without distance coarse graining. In Ref. [30], to get the binary spatial network, authors firstly generate weighted spatial networks by allowing reconnection between any two nodes. Thus, there is infinite room for long-range edges even when the exponent δ equals a large value. Then they project the weighted spatial network into an unweighted one by imposing all the weights of the existing edges to 1. This will lead to dissatisfying the total cost constraint. On the contrary, through the distance coarse graining procedure, the network can provide enough room for these long-range edges. Under this circumstance, the binary spatial network can be generated independent from the weighted one. In Fig. 2, we compare the total cost of these two different binary spatial networks. Clearly, only the network with coarse grained distances can satisfy the total cost limit as the standard value while the projected spatial networks cannot.

Then, we will discuss the length distribution of the long-range edges in the spatial networks with coarse grained distances. Intuitively, the power-law length distribution will get destroyed when the distance is coarse grained. Actually, the distance coarse graining procedure can preserve the power-law length distribution effectively. The results are reported in Fig. 3. We can see that though the length distribution in projected binary spatial networks has the same slope as the standard length power-law distribution, there is still a gap between them. On the contrary, length distribution of the long-range edges in the networks with coarse grained distances and the standard distance power-law distribution are similar except for the

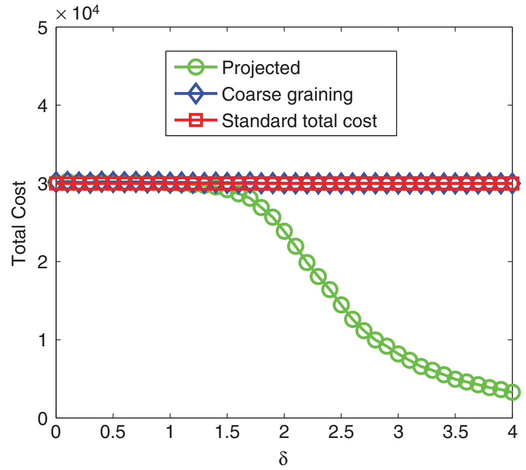

Fig. 2. (Color online) Total costs of two different binary spatial networks. The cycle represents the projected binary spatial networks in Ref. [11]. The lozenge stands for the binary spatial network with coarse grained distances with $W = 7$. The square is the standard total cost which equals $C = N \times c = 1000 \times 30 = 30\,000$. The results are under 100 independent realizations.

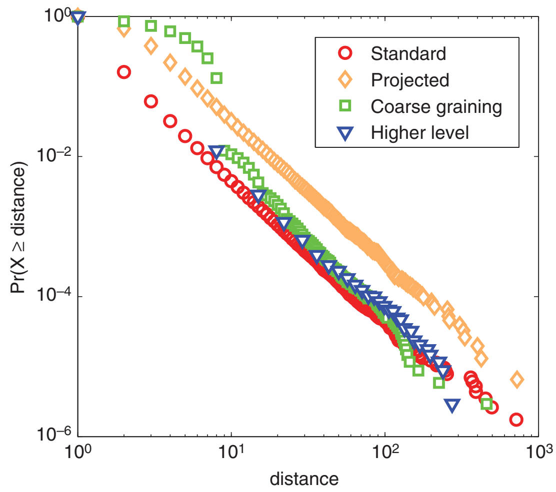

Fig. 3. (Color online) Distance distribution of two different spatial networks. The lozenge represents the result of the projected binary spatial networks in Ref. [11]. The square stands for the result of the binary spatial network with coarse grained distances ($W = 7$). The triangle is the distance distribution under the higher scale (see the text) of the coarse grained network. The cycle is the standard distance power-law distribution with given $N = 10\,000$, $c = 30$ and $\delta = 3$. The results are under 10 independent realizations.

top part. This is reasonable, because the nodes in each coarse grained interval are regarded homogeneous. Consequently, the distribution plot is also divided into some continuous intervals in which the length distribution does not obey power-law. This effect appears to be more serious in the top of the curve(the top of the curve is exaggerated in log scale). However, when we estimate the length with a higher scale, which means marking all the lengths in one coarse graining interval as a typical value (lengths from $(m-1)W+2$ to $mW+1$ are regarded as $(m-1)W+2$), the length distribution of the long-range edges becomes exactly the same with the standard power-law. See the line with triangle markers in Fig. 3.

One of the most important results in the former works about spatial networks is that the minimum average shortest path length happens at $\delta = 2$ regardless of the total cost [30,31]. The shortest path here means the traditional topological shortest path which is defined as a path between two nodes such that the number of the edges is minimized. The shortest path length is the number of edges in the shortest path. Here, we investigate how the optimal δ changes as the distance coarse graining procedure. Fig. 4 shows the result for three different levels of distance coarse graining (with different W). Obviously, the larger W we choose to coarse grain the distance, the more severely the optimal point shifts to a lower value. However, as the size of the network gets bigger, the shifting effect of the distance coarse graining becomes less significant. That is to say, notwithstanding the coarse graining in distance, the value of δ for the minimum shortest path length still equals 2 in large networks.

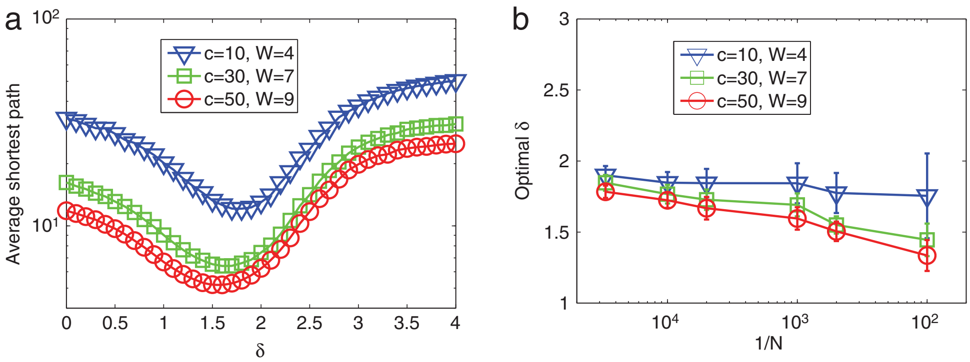

Fig. 4. (Color online) Results for how the optimal δ changes through the distance coarse graining procedure. (a) is the average shortest path length for different δ under different coarse graining parameters W, here $N = 1000$. (b) is how the value of δ for minimum average shortest path changes when the network gets larger. The results are under 100 independent realizations.

3. Dynamics on spatial networks and effects of distance coarse graining

Dynamics on spatial networks is always an interesting topic. For example, it can help us design the principle for the optimal transportation networks [31]. In this section, some main dynamical processes will be studied on spatial networks. Furthermore, people tend to coarse grain the distance when designing real networks most of the time, especially these transport networks, so understanding how the distance coarse graining affects the function of the network is not only interesting but also useful. Previous works [30,31] pointed out that the spatial properties of the network will result in the optimal value of δ for navigation and traffic processes, respectively. Firstly, how the optimal δ for navigation and traffic processes shifts with the distance coarse graining procedure will be investigated. Secondly, we will study two extra dynamics named synchronization and percolation in this so-called binary spatial network with coarse grained distances.

3.1. The effect on the traffic process

In the traffic process [39], all the nodes embedded on the spatial network are treated as both hosts and routers. Every node can deliver at most D packets within one step toward their destinations. At each time step, there are R packets generated homogeneously on the nodes in the system. The packets are delivered from their own original nodes to destination nodes by a special routing strategy. There is a critical value R_c, which can best reflect the maximum capability for a system to handle its traffic. In particular, for $R < R_c$, the numbers of created and delivered packets are balanced, which leads to a steady free traffic flow. For $R > R_c$, traffic congestion occurs as the number of accumulated packets increases with time, simply because the capacities of the nodes for delivering packets are limited.

In fact, the whole traffic dynamics can be represented by analyzing the largest betweenness of the network [40]. The betweenness of a node is the number of shortest paths passing through this node. Note that with the increasing of parameter R (number of packets generated in every step), the system undergoes a continuous phase transition to a congested phase. Below the critical value R_c, there is no accumulation at any node in the network, and the number of packets that arrive at node i is $Rg_i/N(N-1)$ on average. Therefore, a particular node will collapse when $Rg_i/N(N-1) > D_i$, where g_i is the betweenness coefficient and D_i is the transferring capacity of node i. Therefore, congestion occurs at the node with the largest betweenness. Thus, R_c can be estimated as $R_c = D_i N(N-1)/g_{\max}$, where $g_{\max}$ is the largest betweenness coefficient of the network and $D_i = 1$.

Here, we study the traffic dynamics in the spatial network with coarse grained distances. The results are given in Fig. 5(a) and (b). In Fig. 5(a), it is quite obvious that there exists an optimal δ for R_c, which means that the transport capacity reaches its maximum in this spatial network. From Fig. 5(b), we can clearly see that this optimal δ gets closer to 1.5 gradually as the size of the network becomes larger. Actually, even dissatisfying the total cost, the projected spatial network also has the optimal δ for traffic process equaling 1.5 in Ref. [30].

3.2. Effects on navigation

Another aspect that we are going to investigate here is navigation. In fact, the navigation process in networks is based on local information, which is different from the shortest path with global information [25,41]. Hence, the navigation reflects another ability of the networks. According to Ref. [31], we choose the navigation strategy as the greedy algorithm [42] in this paper. In previous works, Kleinberg found that $\alpha = 2$ for $P_{ij} \sim r_{ij}^{-\alpha}$ was the optimal value in the navigation with the greedy algorithm in 2-dimensional spatial networks without total cost limit [42]. When adding the cost restriction, Li et al. found that the optimal value was $\alpha = 3$ in 2-dimensional spatial networks and $\alpha = 2$ in 1-dimensional ones [31].

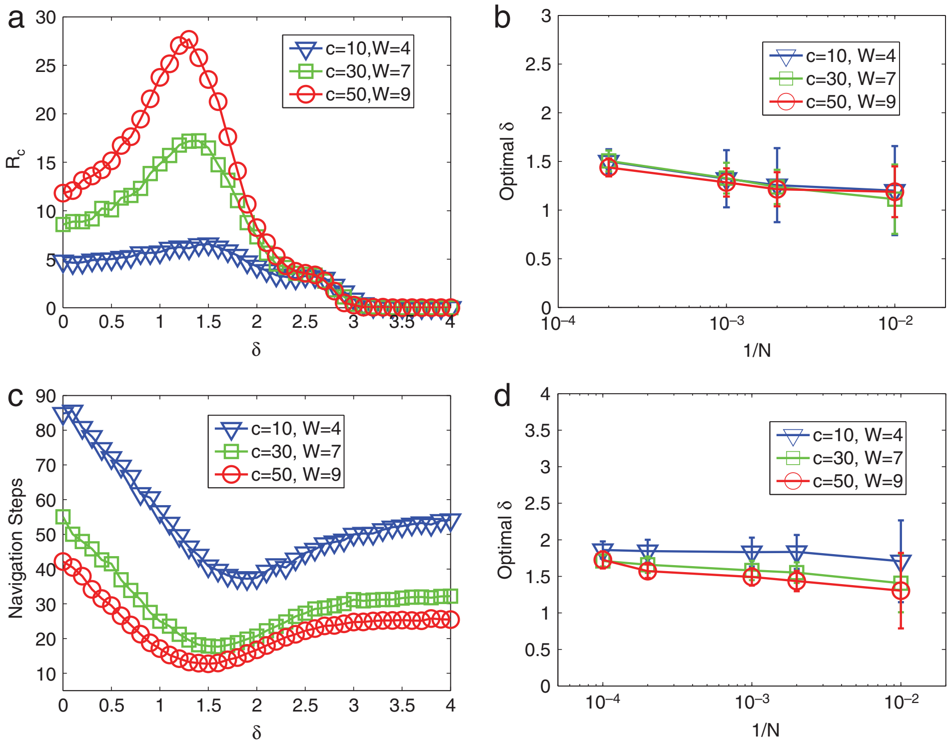

Fig. 5. (Color online) Effects of distance coarse graining on the functions of the binary spatial networks. (a) is, the critical value R_c in traffic process for different δ under different coarse graining parameters W, where $N = 1000$. (b) shows, how the optimal δ for traffic process changes when the network gets larger. (c) is the navigation steps for different δ under different coarse graining parameters W, where $N = 1000$. (d) shows how the optimal δ for navigation changes when the network gets larger. The results are under 50 independent realizations.

Here, though we choose the probability density function (PDF) of the distance distribution as $P(r) = ar^{-\delta}$, δ equals α in one-dimensional space [33]. What interests us most is how this optimal value performs when the distance is coarse grained. In Fig. 5(c) and (d), the results are reported. The optimal δ shifts to a smaller value as the coarse grained interval length W gets larger. However, when the size of the network is large enough, the optimal δ will come back to 2 even if the distances are coarse grained, as shown in Fig. 5(d).

3.3. *Effects on synchronization*

Furthermore, we will study synchronization in this so-called binary spatial network with distance coarse graining. Synchronization is a universal phenomenon that emerges from a population of dynamically interacting units. It plays an important role from physics to biology and has attracted much attention for hundreds of years [43,44].

In former works, the analysis of Master Stability Function (MSF) allows us to use the eigenratio λ_N/λ_2 of the Laplacian matrix to represent the synchronizability of a network, where λ_N is the largest and λ_2 is the smallest nonzero eigenvalue of the Laplacian matrix [44]. Hence, we can calculate the synchronizability λ_N/λ_2 under different values of δ. In Fig. 6, synchronizability of the spatial network is enhanced for some special values of δ. Likewise, there is also an optimal δ for synchronizability, and it is 1.5 approximately when the network is large enough.

3.4. *Effects on percolation*

As discussed in the former section, there are two different ways to generate binary spatial networks. The first way is to project the weighted spatial network to a binary one. The second way is to generate the binary spatial networks directly by the distance coarse graining procedure. We compared the percolation performances between these two kinds of networks. In percolation, we consider what will happen with a network if a random fraction $1-p$ of its edges are removed. In this bond percolation problem, the giant connected component plays the role of the percolation cluster, which may be destroyed by decreasing p [45]. Obviously, different δ in spatial networks will result in different critical parameters p_c. The smaller the critical parameter p_c is, the better the networks perform in percolation. The percolation performances of these two kinds of binary spatial networks are shown in Fig. 7.

From Fig. 7, we can see that the projected binary spatial networks have an optimal δ for percolation while coarse grained binary spatial networks do not have this. Actually, when projecting the weighted spatial networks, the total cost constraint

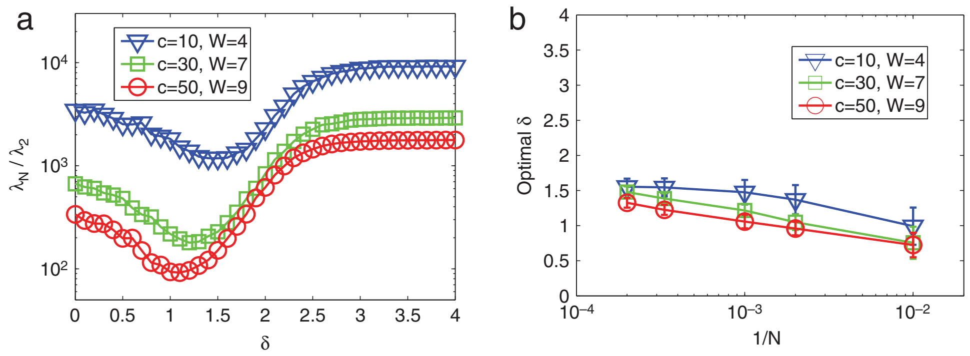

Fig. 6. (Color online) The effect of distance coarse graining on the synchronizability of the binary spatial networks. (a) is the synchronizability for different δ under different coarse graining parameters W, where $N = 1000$. (b) shows how the optimal δ for synchronizability changes when the network gets larger. The results are under 50 independent realizations.

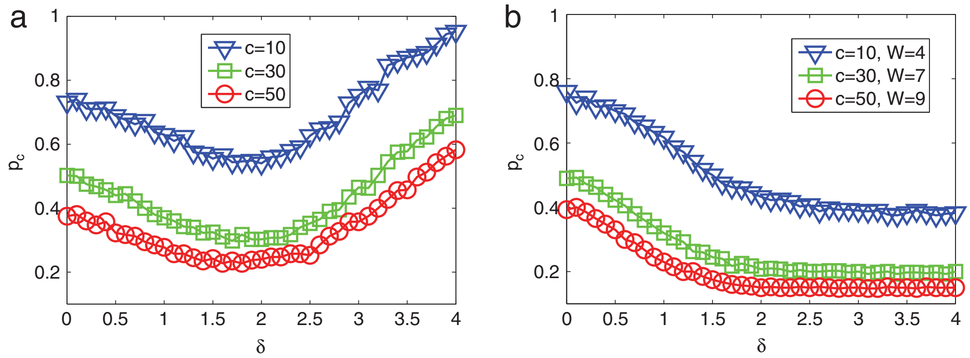

Fig. 7. (Color online) Results for the percolation performance of (a) projected binary spatial networks and (b) coarse grained binary spatial networks. Here, the network size is $N = 500$. The results are under 50 independent realizations.

no longer gets satisfied, so in the projected binary spatial network, the number of total edges becomes smaller than the expected value as δ gets larger [30]. This is why the projected binary spatial networks seem to have an optimal δ for percolation. Actually, when the binary spatial networks are designed, they are supposed to be constrained by the total cost. We show that after the total cost constraint is added by the distance coarse graining procedure, the spatial network may perform entirely differently in some functions such as percolation. It indicates that the distance coarse graining is necessary when we analyze some dynamics on spatial networks.

4. Conclusion

Complex networks have been a hot topic in science for more than ten years. Works in this field are based on the topology of the networks. So far, many empirical works claim that the distance distributions of real networks obey a power-law distribution. In the aspect of theoretical modeling, researchers begin to pay attention to these spatial networks. Very recently, these spatial networks were constructed with power-law distance distributions and under total cost restrictions. These kind of spatial networks can reflect the trade-off of real networks between the efficiency (distance power-law distribution) and the total cost.

In order to satisfy the total cost constraint, we introduce a distance coarse graining procedure to the spatial network models. In previous works, researchers find that there exist stable optimal power-law exponents δ for minimum shortest path, traffic process and navigation. In this paper, we study some main dynamics in such binary spatial networks with coarse grained distances. Specifically, the distance coarse graining procedure will make the optimal exponent δ in power-law distance distribution shift to smaller values for all of average shortest path, traffic process and navigation. Interestingly, when the network is large enough, the effect of distance coarse graining can be ignored. Hence, the optimal exponents δ for the traffic process, navigation and synchronization indeed result from the spatial structure. On the contrary, though there seems to be an optimal value of δ for percolation in the projected spatial networks, it actually results from dissatisfying the total cost constraint. Given the total cost constraint by the distance coarse graining procedure, the optimal exponent δ for percolation disappears. Hence, the distance coarse graining is very important in the study of spatial networks.

Additionally, in most cases of real lives and empirical researches, the distances are estimated approximately. In other words, people incline to regard a range of distances as a typical distance, so the study of distance coarse graining is also meaningful in this aspect. Also, investigation of functions of other spatial related networks with coarse grained distances can be an interesting extension.

Acknowledgments

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