Published in "Physical Review E - statistical, non linear and soft matter physics
83(5): 056123, 2011"
which should be cited to refer to this work.

Coarse graining for synchronization in directed networks

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Coarse-graining model is a promising way to analyze and visualize large-scale networks. The coarse-grained networks are required to preserve statistical properties as well as the dynamic behaviors of the initial networks. Some methods have been proposed and found effective in undirected networks, while the study on coarse-graining directed networks lacks of consideration. In this paper we proposed a path-based coarse-graining (PCG) method to coarse grain the directed networks. Performing the linear stability analysis of synchronization and numerical simulation of the Kuramoto model on four kinds of directed networks, including tree networks and variants of Barabási-Albert networks, Watts-Strogatz networks, and Erdös-Rényi networks, we find our method can effectively preserve the network synchronizability.

I. INTRODUCTION

Complex networks have become a key approach to understanding many social, biological, chemical, physical, and information systems where nodes represent individuals and links denote the relations or interactions between nodes. In this sense to study the dynamics of such systems is actually to investigate the dynamical behaviors on the networks. In particular, the network synchronization as an important emerging phenomenon of a population of dynamically interacting units in various fields of science has attracted much attention [1–12]. Most works focused on studying the relation between network topology and the synchronization [3–7], enhancing the synchronizability by designing the weighting strategies [8–12]. Moveover, some efforts have been made to study the synchronization in directed networks [13–17]. It has been pointed out that the optimal structure for synchronizability is a directed tree [13,14] and the convergence time is strongly related to the depth of the tree [15,16]. Most of the experiments on investigating the dynamic behaviors are implemented on small-size networks. However, when the networks contain very large number of nodes, it becomes sometimes impossible to model the dynamic process. For example, to investigate the synchronization, extrapolating the coupled differential equations model of a single node to this large system is too complicated to be carried out.

A promising way to address this problem is to coarse grain the network, namely to reduce the network complexity by means of mapping the large network into a smaller one. The coarse-graining techniques have been successfully applied to model large genetic networks [18] and extract the slowest motions in protein networks [19]. Essentially, the coarse-graining process is very similar to the problem of cluster finding or community detection in networks (see Refs. [20–23] for some popular methods). The coarse-grained network is obtained by merging the nodes in the same cluster or community. However, the coarse-graining model is far beyond the clustering techniques, since it requires the coarse-grained networks keep the same topological properties or dynamic behaviors as the initial networks, such as preserving the degree distribution, cluster coefficient, assortativity correlation [24], the properties of random walk on the network [25], synchronization [26], and critical phenomena [27]. Most of the former works on coarse graining consider undirected networks. However, in many real systems, the interactions between individuals are not reciprocal. For example, the food web, gene regulation system, metabolic system, and neural system are usually represented by directed networks where the nodes are affected by their upstream nodes. In directed networks of course we can ignore link directions and apply methods developed for undirected networks, but this approach discarding potentially useful information contained in the link directions may lead to dramatic change of the key organizational features when coarse graining the networks [28]. In addition, some prominent methods may confront problems when applied to directed networks. Among all these existing coarse-graining methods for undirected networks, the spectral coarse-graining (SCG) method is a very general method which can be applied in many dynamic processes such as synchronization, random walk, and epidemic spread [25,26]. In order to preserve a typical eigenvalue, the SCG method coarse grains the nodes with similar elements in the corresponding eigenvectors. For different dynamic processes, different eigenvalues should be considered. Therefore the choice of the eigenvectors is indeed problem dependent. As the synchronizability is mainly related to the largest and smallest nonzero eigenvalue, the SCG method for synchronization takes the p_2 and p_N into consideration (p_2 and p_N are, respectively, the eigenvectors for the smallest nonzero and the largest eigenvalue). However, this method may not provide good performance in directed networks since the eigenvector elements cannot successfully characterize the nodes' dynamic role. For instance, the nodes in different layers may have exactly the same eigenvector elements in directed acyclic networks while the nodes with exactly the same topology may have totally different eigenvector elements in directed networks with cycles. In a word, to design an effective coarse-graining method for directed networks is still challenging.

In this paper we propose a path-based coarse-graining (PCG) method to coarse-grain directed networks for

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synchronization. The basic idea is that the nodes who obtain the same impacts from other nodes are similar to each other, and thus can be merged. The impacts that one node receives from other nodes are calculated via tracing the origin of the source in the directed networks (i.e., along the opposite direction of links). It has been pointed out that the dynamical correlation can be predicted from such topological similarity [29]. Therefore, coarse graining in this way will most naturally merge the nodes according to their functional performance and likely preserve the dynamical properties. The linear stability analysis of synchronization and numerical simulation of the Kuramoto model on four kinds of directed networks, including tree networks and variants of Barabási-Albert networks, Watts-Strogatz networks, and Erdös-Rényi networks, show that our method can effectively preserve the synchronizability of the initial directed networks. Additionally, we find the far sources play more important roles when identifying the nodes' roles in directed networks with obvious hierarchy structure, while the near sources are more important in the directed networks with many loops.

II. PATH-BASED COARSE-GRAINING (PCG) METHOD

A. Definition of node's impact vector

Many structural-based similarity indices have been proposed to quantify the nodes' similarity [30–32], most of which only work for undirected networks. How to define the nodes' similarity in directed networks is still a challenge. Here we propose a method via tracing the origin of impacts in directed network. The basic assumption is that two nodes are structural similar if they obtain the same impacts from other nodes, and thus they are more likely to be merged during the coarse-graining process. Given a directed network $G(V,E)$, where V and E denote the set of nodes and directed links, respectively. Multiple links and self-connections are not allowed. The impact of node x on node y is defined by summing over the collection of directed paths from x to y with exponential weights by length. The mathematical expression reads

$$f_{x\to y} = \sum_{l=1}^{l_{\max}} \beta^l |\mathcal{P}_{x\to y}^{\langle l\rangle}|, \tag{1}$$

where $|\mathcal{P}_{x\to y}^{\langle l\rangle}|$ is the set of all directed paths with length l starting from node x to node y. Mathematically, $|\mathcal{P}_{x\to y}^{\langle l\rangle}| = (A^l)_{xy}$, where A is the adjacency matrix: if x points to y $A_{xy} = 1$, otherwise $A_{xy} = 0$. β is a free parameter that controls the weights of the paths. Smaller β indicates assigning more weights on the short paths, and vice versa. Here the paths whose lengths are not larger than $l_{\max}$ are considered. If $l_{\max} = \infty$, namely considering all directed paths from x to y, Eq. (1) is similar to the *Katz* index [33]. However, the significant difference is twofold: on one hand the adjacency matrix in Katz index is symmetrical while asymmetrical in Eq. (1) and on the other hand the parameter β is usually smaller than unit in Katz index, namely assigning more weights to the short paths, while in Eq. (1) β has no limitation. Since counting all paths between every pair of nodes is very time consuming especially in large networks, here we set $l_{\max}$ equal to the length of the longest path among all the shortest paths between two nodes. Note that when $l_{\max} = \infty$ and β is smaller than the reciprocal of the largest eigenvalue of A (i.e., ensure the convergence), the impact matrix F with element f_{xy} defined in Eq. (1) can be directly calculated by $F = (I - \beta A)^{-1} - I$.

B. Group partition via *k*-means clustering

We assign each node x a N dimensional impact vector which is equal to the xth column of matrix F, namely $\vec{f}_x = (f_{1x}, f_{2x}, f_{3x}, \ldots, f_{Nx})^T$, where $N = |V|$ is the number of nodes. Clearly if two nodes receive the same impacts from their ancestors (i.e., upstream nodes), they tend to have the same phase in synchronization, and thus are more likely to be merged during coarse graining. Suppose we are going to coarse grain a network containing N nodes to a smaller one with K nodes. We adopt the k-means clustering method [34] to partition the N nodes into K groups. The nodes in the same group will be merged. The k-means clustering technique aims at minimizing the within-cluster sum of squares:

$$E = \sum_{i=1}^{K} \sum_{x\in V(i)} ||\vec{f}_x - \vec{c}(i)||^2, \tag{2}$$

where $V(i)$ is the set of nodes in cluster i ($i = 1,2,\ldots,K$) and $\vec{c}(i)$ is the centroid of cluster i which is equal to the mean of points in cluster i, namely

$$c_k(i) = \frac{1}{|V(i)|} \sum_{x\in V(i)} f_{kx}. \tag{3}$$

The detailed steps of k-means clustering are shown as follows. (i) Choose K vectors as the initial centroid of each cluster. (ii) Randomly choose a node x from the set V. This node will belong to the cluster i if the distance between its vector $\vec{f}_x$ and the centroid of cluster i, namely $\vec{c}(i)$, is the minimum among all the centroids of K clusters. (iii) Update the centroid of each cluster according to Eq. (3). (iv) Repeat steps (ii) and (iii) until all the centroids cannot be modified. Note that for a given K, clusters will depend heavily on the initial configuration of the set of centroids, thus making interpretation of the clusters rather uncertain. Different initialization may lead to different solutions which may be trapped in the local minimum. Clusters should be, as much as possible, compact, well separated, and interpretable, possibly with the help of some additional variables, such as the F statistic. Here we only focus on whether the clusters are compact, namely the vectors (nodes) within one cluster are close (similar) enough, while neglect if the clusters are well separated. Therefore we will finally choose the clustering result subject to the lowest E among L-possible solutions (we set $L = 20$ in this paper).

C. Weighting strategy for the coarse-grained networks

Another crucial problem in the process of coarse graining is how to update the links' weights after merging the nodes so that the resulting network is truly representative of the initial one. An effective weighting strategy was proposed by Gfeller *et al.* [26]. Here we apply it to directed networks. Specifically, when we merge the nodes in cluster i to form a new node labeled by m_i, the weights of the merged links update according to the following principle:

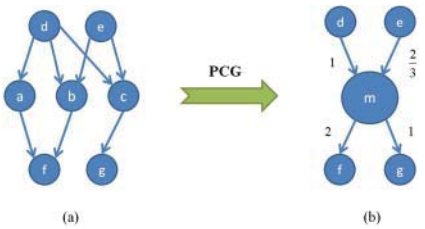

FIG. 1. (Color online) A simple illustration of how to update the links' weights in the coarse-graining process. (a) The initial network consisting of seven nodes and eight directed links. (b) The reduced network after merging the three nodes a, b, and c. Numbers on the links indicate the new weights of the links.

$$w_{x\to m_i} = \frac{\sum_{y\in V(i)} w_{x\to y}}{|V(i)|} \quad \text{for the in-links of } m_i,$$
$$w_{m_i\to x} = \sum_{y\in V(i)} w_{x\to y} \quad \text{for the out-links of } m_i, \qquad (4)$$

where $w_{x\to y}$ indicates the weight of the directed link from x to y, which can also be interpreted as the coupling strength. A simple illustration is shown in Fig. 1. The initial network as shown in Fig. 1(a) consists of seven nodes and eight directed links. Assuming the initial links' weights are all equal to unit. After merging the three nodes a, b, and c, a new node m is generated, and according to Eq. (4) the links weights in the reduced network are drawn as in Fig. 1(b). Indeed, since the three nodes in total receive three in-links from node d, while two from node e, the weights of m's two in-links are, respectively, $w_{d\to m} = 3/3 = 1$ and $w_{e\to m} = 2/3$. For m's out-links, since the three nodes have two out-links to node f and one to g, the weights are, respectively, $w_{m\to f} = 1+1 = 2$ and $w_{m\to g} = 1$.

Under the framework of master stability analysis, the synchronizability of an undirected network can be quantified by the ratio between the largest and the smallest nonzero eigenvalues of the Laplacian matrix of this network, namely $R = \lambda_N/\lambda_2$, where λ_N and λ_2 are, respectively, the largest and the smallest nonzero eigenvalues of the Laplacian matrix [35–37]. In directed networks, since the Laplacian matrix defined as $L_{ij} = k_i^{\text{in}}\delta_{ij} - a_{ij}$ is asymmetric with zero rowsum, it has complex eigenvalues. In order to achieve the synchronization condition, every eigenvalue is entirely contained in the region of negative Lyapunov exponent for the particular master stability function. Once the stability zone is bounded and the imaginary part of complex eigenvalue is small enough, the network synchronizability can be approximately measured by the real part of eigenvalue ratio $R = \lambda_N^r/\lambda_2^r$, where λ_N^r and λ_2^r are, respectively, the largest and the second smallest real parts of eigenvalues [11,38,39]. Generally speaking, the stronger the synchronizability, the smaller the ratio R. Note that an accurate index for measuring the synchronizability in directed networks has not yet been proposed and asks for further studies. Here we use the approximate index $R = \lambda_N^r/\lambda_2^r$ as an indicator to see whether the synchronizability of a directed network changes after coarse graining. Usually λ_N is proportional to the largest degree $k_{\max}$ (i.e., largest node's strength in weighted network) of the network and λ_2 corresponds to the lowest degree $k_{\min}$ (i.e., lowest node's strength in weighted network) [1]. Therefore, keeping the $k_{\max}$ and $k_{\min}$ unchanged can effectively help to maintain the synchronizability after coarse graining. Thus, in the coarse-graining process the nodes with largest and smallest in-degrees can only be merged if the $k_{\max}$ and $k_{\min}$ of the coarse-grained network are respectively equal to that of the initial network. Otherwise, we will randomly selected two nodes, one with largest in-degree and the other with the smallest in-degree, before group partition. Then the rest $N-2$ nodes will be classified into $K-2$ groups according to k-means clustering. Note that, unless stated otherwise, k always refers to the in-degree. In the Appendix we further discuss the effect of the constraint of keeping $k_{\max}$ and $k_{\min}$ on coarse-graining results. It shows that the eigenvalue ratio R is sensitive to $k_{\max}$ and $k_{\min}$, while the order parameter of Kuramoto model does not.

Finally, for the aspect of computational complexity, the k-means clustering algorithm is of $O(N^2)$, and the time complexity of calculating the impact-vector F is $O(N^3)$. Likewise, the calculation of eigenvectors in SCG method also takes $O(N^3)$. However, with the development of computing techniques, lots of fast calculation algorithms can help to reduce the computational complexity and make our method be able to deal with large networks. For example, the computational complexity of Katz index (i.e., the case for $\beta < 1$) can be reduced to $O(N+M)$, where M is the number of edges in the network [40].

III. RESULTS

A. Coarse graining on modeled networks

We apply the path-based coarse-graining (PCG) approach to four kinds of directed networks. (i) Directed tree network. A tree with N nodes and L layers is generated starting from a directed train with length L, in which each node represents a layer. Then rest $N-L$ nodes are added one by one. Each new added node is connected by a directed link starting from one of its ancestors which are not located in the layer L. (ii) A variant of Barabási-Albert networks [41]: Directed BA network. An acyclic directed BA network is generated by using the mechanism for undirected BA network and assuming the link direction can only be from older node to younger node. (iii) A variant of Watts-Strogatz networks [42]: Directed WS network. The model starts from a completely regular network with identical degree and clockwise links. Each link will be rewired with two randomly selected nodes with probability q [$\in (0,1)$]. (iv) A variant of Erdös-Rényi networks [43]: Directed ER random network. The directed ER random networks can be generated by setting $q = 1$.

First we investigate the performance of PCG on the four kinds of networks. The synchronizability of the coarse-grained network R in the (β, K) plane is shown in Fig. 2, where K is the size of the coarse-grained network. Interestingly, we find that in tree network and acyclic BA network larger β in average provides better results than smaller β. In particular, in tree network there is an obvious line at $\beta \approx 1$. In the BA network with $\beta > 1$ the coarse-grained network can keep the synchronizability exactly the same as the initial network. In the cyclic WS network the β that best preserves

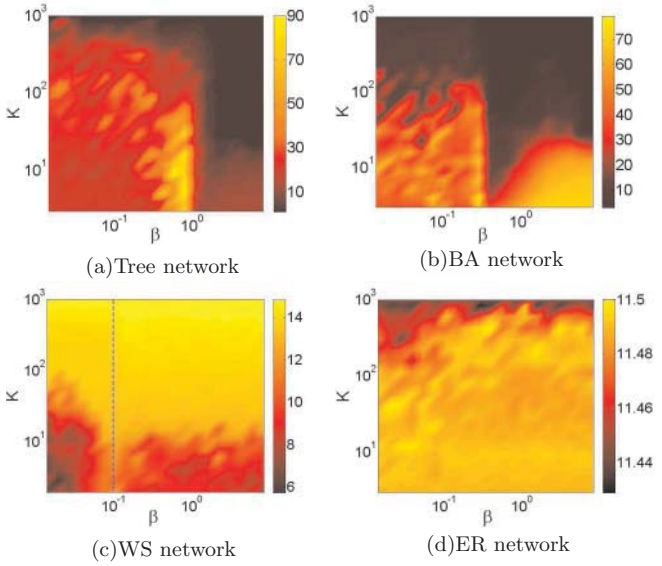

FIG. 2. (Color online) The synchronizability R in the (β,K) plane for (a) directed tree networks ($N = 1000$, $L = 20$), (b) directed BA network ($N = 1000$, $\bar{k} = 3$), (c) directed WS network ($N = 1000$, $\bar{k} = 10$, $q = 0.1$), and (d) directed ER network ($N = 1000$, $\bar{k} = 10$).

the synchronizability is around 0.1. It seems that networks with more loops tend to obtain better coarse graining with smaller β (see Sec. III C for detailed discussion of the relationship between the optimal parameter β^* and the number of loops in network). The result in Fig. 2(d) shows that the synchronizability of the coarse-grained ER network is not sensitive to β regardless of K, since the total fluctuation is smaller than 0.07.

We compare the PCG method with other two methods, namely random coarse graining (RCG) and spectral coarse graining (SCG) [26]. In RCG the N elements of each node's vector are randomly selected in the range of (0,1). Then the nodes will be classified into K groups by using k-means clustering. In directed networks the egeinvalues and egeinvectors of their Laplacian matrixes have complex values. When we apply SCG to directed networks, we consider only the real parts of the values in this paper. In practice, we define I equally distributed intervals between the maximum and minimum of p_2^r (p_N^r), where p_2^r and p_N^r are the egeinvectors corresponding to the second smallest and the largest real-part egeinvalues of the Laplacian matrix, respectively. The nodes whose eigenvector components in p_2^r (p_N^r) fall in the same interval are merged. Specifically, if the elements in both p_2^r and p_N^r are identical, we will randomly divide the nodes into K groups. Actually, the relation between I and K strongly depends on the network structure (i.e., the distribution of the elements in p_2^r and p_N^r). For instance, considering the initial WS and ER network shown in Fig. 2, when $I = 800$ the size of the coarse-grained WS network is 951, while the reduced ER network only contains 281 clusters. Note that there are many potential ways to apply the SCG method to directed networks making use of the imaginary part of the elements in eigenvectors. For example, the vectors p_2 (p_N) can be generated by combining the real part and the imaginary part [such as $\sqrt{(p_2^r)^2+(p_2^i)^2}$, $p_2^r+p_2^i$, $p_2^r p_2^i$, *et al.*].

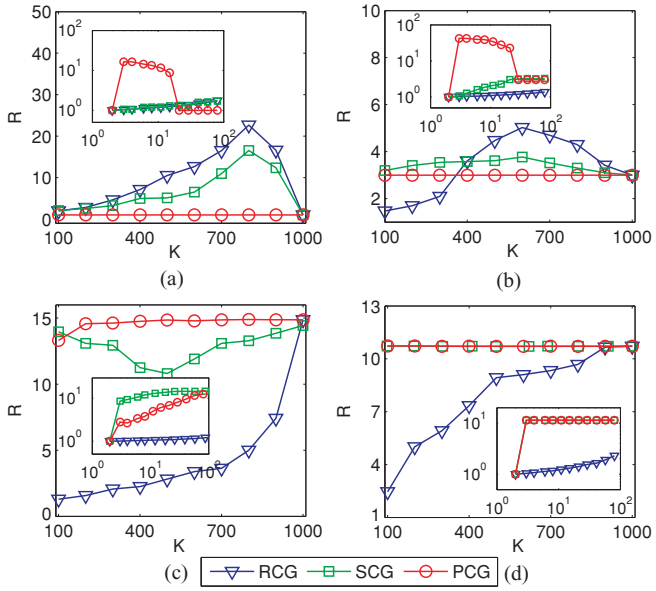

FIG. 3. (Color online) The evolution of the ratio $R = \lambda_N^r/\lambda_2^r$ as a function of the size of the coarse-grained network K. The initial networks are the same as the ones in Fig. 2. We use the typical parameter $\beta = 5$ for (a) directed tree networks and (b) directed BA networks, and $\beta = 0.1$ for (c) directed WS networks and (d) directed ER networks. The results for RCG and PCG are obtained by averaging over 100 independent network realizations. Insets show the results for $K \in [2,100]$.

Another way is grouping the nodes according to four vectors, namely $(p_2^r, p_2^i, p_N^r, p_N^i)$. If the imaginary parts are appropriately considered, the performance of SCG can be improved. However, how to find the right way to make use of the imaginary parts is a tough problem and inappropriately involving the imaginary parts in SCG method may lead to even worse results.

Figure 3 shows how the indicator R changes with K on the four kinds of directed networks with typical β. Overall speaking, PCG outperforms SCG, and RCG performs worst. As shown in Figs. 3(a)–3(c), with SCG and RCG, the synchronizability changes even only if a few nodes have been merged, while the PCG displays a large stable range. In a directed tree network, most of the elements in eigenvectors corresponding to the smallest and the largest eigenvalues are identical. Thus it is impossible to distinguish the role of nodes by the analysis on the eigenvectors as suggested in Ref. [26]. In the tree networks, the most effective coarse-graining strategy is to merge the nodes in the same layer. PCG can indeed well identify the nodes in different layers by using a larger parameter β (> 1). In this sense, PCG is very effective in acyclic directed tree network. From Fig. 3(a) one can see that when $K > L$ PCG can keep the synchronizability exactly the same as the initial network. When $K = L$ the tree will be reduced to a train with length L, namely all the nodes in the same layer are merged. When $K < L$ there exist a sudden jump of R [see inset of Fig. 3(a)]. This is caused by merging the nodes in different layers and thus leading to a smaller $k_{\min}$ according to the weighting strategy in Eq. (4). In this case, if we artificially set $k_{\min}$ of the reduced network equal to that of the initial network, the synchronizability can be well preserved

(exactly equal to 1). Similar phenomenon exists in acyclic BA network where the hierarchical structure is clear.

It has been demonstrated that the synchronizability of the directed BA network with average in-degree $\bar{k} = 3$ is exactly 3 [11]. Figure 3(b) shows that PCG with parameter $\beta = 5$ can guarantee $R = 3$ by keeping the network acyclic and $k_{\max}$ and $k_{\min}$ unchanged, even the network is reduced to 30 clusters (i.e., $K = 30$). When $K < 30$ merging may generate some loops and decrease $k_{\min}$, and thus lead to a sharp increase of R. It cannot be perfectly avoided by artificially keeping $k_{\min}$ as what we did in the tree network, instead R can effectively reduce to around 3, since here the loops also play a role. On the contrary, the SCG method may induce loops even merging a few nodes (i.e., for a larger K). For example, when $K = 600$, the synchronizability of the reduced SCG network is $R = 3.77$, while synchronizability of the reduced PCG network is exactly equal to 3.

In the networks with cycles including directed WS networks and directed ER networks, there are no clear hierarchical structures, thus the local information (i.e., short paths) plays a more important role to quantify the node's impact during the coarse-graining process, and thus a relative small β is required. Here we use $\beta = 0.1$. The results show that the PCG method performs as well as the SCG method in directed ER networks while much better than SCG and RCG in directed WS networks.

In addition, we point out that grouping the nodes aiming at preserving the dynamics cannot maintain the topological properties at the same time, although the grouping is according to the topological similarity. Generally the average degree of the coarse-grained network is larger than that of the initial network. For comparison, we generate a group of modeled networks which have the same topological properties as the initial network and same size as the coarse-grained network. It is shown that the average number of reachable nodes and loop number of the coarse-grained networks are always higher than that of the modeled networks, while the average shortest distance of the coarse-grained networks is always smaller than the modeled networks. Moreover, the coarse-graining procedure may change the degree distribution of the initial networks. However, the topological properties of the PCG networks are relatively closer to the initial networks than the SCG networks especially in the acyclic networks (not so obvious in directed networks with cycles). For example, the PCG method can prevent the producing of loops and keep the coarse-grained networks still partially reachable.

B. Kuramoto model on coarse-grained networks

Since the Laplacian matrices for directed networks are asymmetric, the egeinvalues λ_2 and λ_N are complex. In this case the indicator R cannot exactly represent the synchronizability of a network. Hence we further test our method with the Kuramoto model [44,45], which is a classical model to investigate the phase synchronization phenomena. The coupled Kuramoto model in the network can be written as

$$\dot{\theta}_i = \omega_i + \sigma \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i), \quad i = 1,2,\ldots,N, \qquad (5)$$

where ω_i and θ_i are the natural frequency and the phase of oscillator i, respectively, and A is the adjacency matrix. The collective dynamics of the whole population is measured by the macroscopic complex order parameter,

$$r(t)e^{i\phi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}, \qquad (6)$$

where the modulus $r(t)$ ($\in [0,1]$) measures the phase coherence of the population and $\phi(t)$ is the average phase. $r(t) \simeq 1$ and $r(t) \simeq 0$ describe the limits in which all oscillators are, respectively, phase locked and moving incoherently. By studying the behavior of the order parameter $r(t)$, we are able to investigate whether the topology of the coarse-grained network is representative of the initial one. The initial network is a tree network as shown in Fig. 4(a), which contains 100 nodes and has 10 layers. After the PCG procedure we obtain a train-like network with depth equal to 10 [see Fig. 4(b)]. With the SCG method a cyclic network will be generated as shown in Fig. 4(c). The result of how the order parameter $r(t)$ of the Kuramoto model performs in these three networks is shown in Fig. 4(d). It is obvious that $r(t)$ of the PCG network converges with almost the same speed as the initial one, while

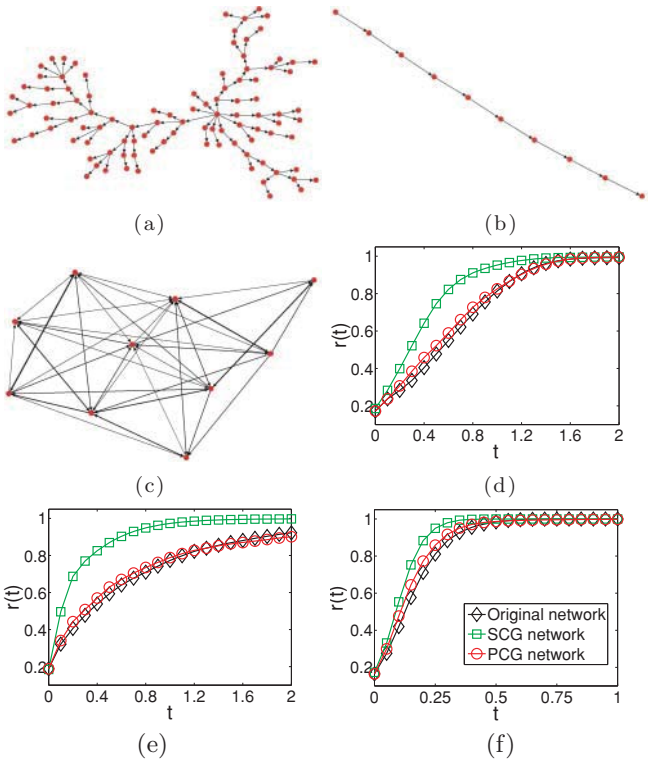

FIG. 4. (Color online) Given a specific directed tree network with $N = 100$ nodes and $L = 10$ layers as shown in (a), the coarse-grained networks through PCG and SCG are, respectively, presented in (b) and (c) which consist of $K = 10$ clusters. (d) The performance of the Kuramoto model on these three networks, namely (a) original network, (b) PCG network, and (c) SCG network. (e) and (f) The results of WS network (with $N = 100, \bar{k} = 4, q = 0.1$) and BA network (with $N = 100$ and $\bar{k} = 3$), respectively. Their coarse-grained networks all contain $K = 25$ clusters. The coupling strength is $\sigma = 10$ and w_i is randomly selected in the range of $(-0.5,0.5)$. Initially θ_i is randomly chosen in $(-\pi,\pi)$.

in the SCG network it converges faster. Moreover, the results of directed WS network and BA network are, respectively, shown in Figs. 4(e) and 4(f). Their coarse-grained networks all contain 25 clusters. It is clearly that the PCG method can preserve the synchronizability more effectively than SCG.

C. The optimal parameter β^* for different networks

In different networks the optimal parameter β^* corresponding to the best performance on coarse graining are different. Empirically the β^* of acyclic networks is larger than those containing loops. To investigate whether β^* is affected by the cycles in networks, we carry out an experiment based on directed BA networks, on which loops are generated by reshuffling some links. Specifically, we randomly select two directed links from the network, for example, one is from node A to B and the other is from node C to D. Then we rewired these two links as A to D and C to B. In this way the degree of these nodes will not be changed during the reshuffling procedure. On average, reshuffling more links leads to more loops, see an example in Fig. 5(a) where the numbers of loops with length 3, 4, and 5 all increase with the increasing of reshuffling steps. Now we would like to find the optimal parameters for the reshuffled networks. For a given network, β^* might be different with different K as we have shown in Fig. 2. However, in practice, checking the optimal parameter for different K in advance is sometimes impossible. Thus here we ignore the relationship between β and K, and consider the general performances of one parameter on the coarse-grained networks with the possible sizes we concerned. The β^* is thus corresponding to the β that yields the minimum synchronizability difference between the coarse-grained networks and the initial networks, which can be mathematically expressed by

$$d = \sum_{K=n}^{N} |R_K - R_0|, \tag{7}$$

where R_K is the synchronizability of the coarse-grained network with K nodes, R_0 is the synchronizability of the initial network, and n is the minimum size of the coarse-grained network that we considered. Since too small K may lead to dramatic change of R, here we choose $n = 10$ in the example shown in Fig. 5. We obtain β^* subject to the minimum d. The dependence of β^* on the number of reshuffling steps is shown in Fig. 5(b). Instead of considering all possible β which is very time consuming, we test the parameter β in the range of [0.01,10] with steps 0.01, 0.1, and 1, respectively, in [0.01,0.1), [0.1,1), and [1,10]. It is clear that β^* decreases with the increasing of reshuffling steps. Actually, if directed networks have obvious hierarchical structure and rare loops, PCG can perform better with a relatively large β since it emphasizes on long paths to detect the hierarchical structure. However, in directed networks with many loops, the hierarchical structure is not clear. As a path involved in loops can be regarded as an infinite long path, its effect on the impact vector will be enormously amplified with a large β, and thus leading to noise when characterizing the dynamic role of a node. In this case it is better to pay more attention to the impacts from local structure, namely emphasize the effects of short paths by using small β.

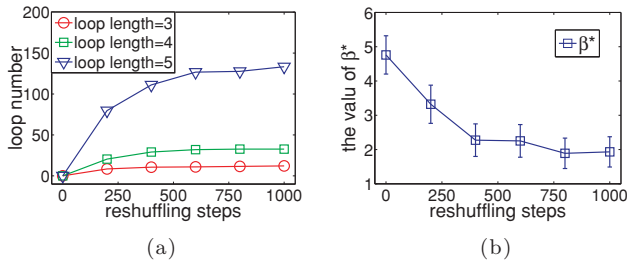

FIG. 5. (Color online) (a) The dependence of the number of loops with different length on the reshuffling steps in directed BA network ($N = 100$, $\bar{k} = 5$). (b) The β^* as function of the reshuffling steps. Each point is obtained by averaging over 100 independent network realizations.

IV. CONCLUSION

Coarse graining is an effective way to analyze and visualize large networks. Many methods and models have been proposed to reduce the size of the networks and preserve main properties such as degree distribution, cluster coefficient, degree correlation, as well as some dynamic behaviors such as random walks, synchronizability, and critical phenomena. However, most of these works take into account the undirected networks, while the study on coarse graining of directed networks lacks attention. In this paper we introduce a path-based coarse-graining (PCG) method which assumes that two nodes are structural similar if they obtain the same impacts from other nodes, and thus they are more likely to be merged during the coarse-graining process. The impacts that a node obtained from other nodes are calculated via tracing the origin of impacts in directed network. Specifically, the impact of node x on node y is defined by summing over the collection

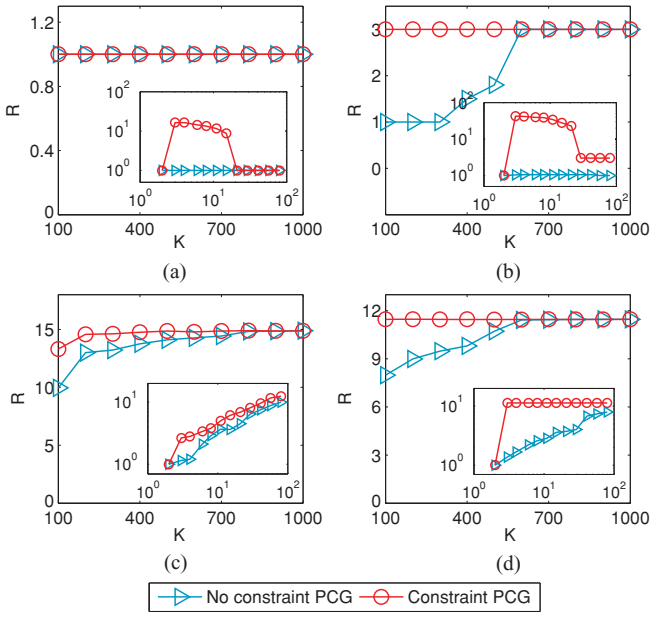

FIG. 6. (Color online) Comparison of the PCG with and without the constraint of $k_{\max}$ and $k_{\min}$. All the parameters in this figure are the same to the ones in Fig. 2.

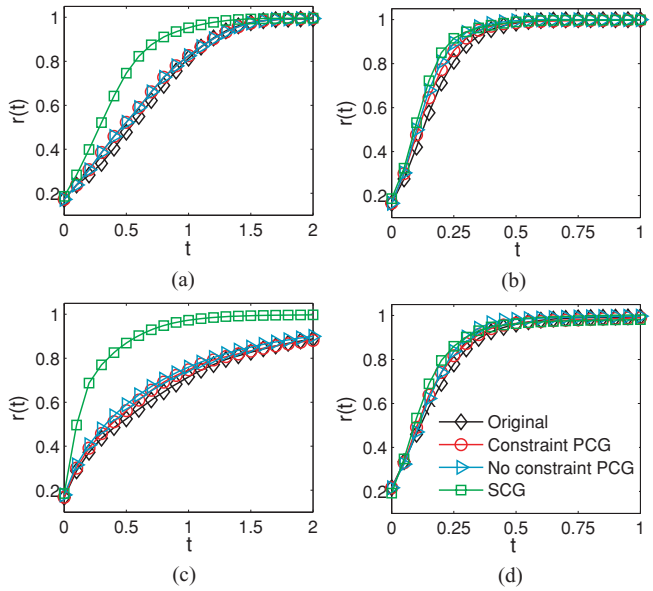

FIG. 7. (Color online) The order parameter of the Kuramoto model on four networks. All the parameters in this figure are the same to the ones in Fig. 4.

of directed paths from x to y with exponential weights by length, which are controlled by a parameter β. Larger β indicates the long paths are more important (i.e., assign more weights to the long paths). Numerical analysis on four kinds of directed networks, including tree-like networks and variants of Barabási-Albert networks, Watts-Strogatz networks, and Erdös-Rényi networks, shows that our method can effectively preserve the synchronizability during the coarse-graining process. This result is further demonstrated by the Kuramoto model. In addition, we find that the long paths play more important roles on the coarse graining in the tree-like networks, while in the cyclic networks the long paths that involve the loops usually have negative effects on quantifying the impacts of one node on the other nodes during the coarse-graining process, and thus a smaller parameter β gives better performance.

Finally, we claim that the idea for merging nodes which receive the same impacts from the network is quite general for coarse-graining directed networks. For example, for random walk two nodes in a directed network having exactly the same upstream neighbors should be grouped together since their random walker probabilities come from the same sources. In this sense, coarse-graining directed networks for other dynamics can be interesting extensions.

ACKNOWLEDGMENTS

We thank Yi-Cheng Zhang, Tao Zhou, and Jie Ren for their helpful suggestions. This work is supported by the Swiss National Science Foundation under Grant No. (200020-132253).

APPENDIX: PCG METHOD WITHOUT DEGREE CONSTRAINT

In the paper we assumed that the nodes with largest and smallest in-degrees can only be merged if the $k_{\max}$ and $k_{\min}$ of the coarse-grained network are respectively equal to that of the initial network. In order to investigate the effect of keeping the maximum and minimum in-degree on the coarse-graining result, we remove the constraint of $k_{\max}$ and $k_{\min}$ in PCG and see the performance of the modified PCG method. As we mainly consider synchronization, the indicator R is shown to be sensitive to the $k_{\max}$ and $k_{\min}$ (see Fig. 6). It is obvious that the PCG with constraint performs better than that without constraint. However, as shown in Fig. 7, the order parameter $r(t)$ of the Kuramoto model does not show obvious differences. Moreover, we compared the RCG with and without the in-degree constraint. The result shows that the degree constraint cannot prominently improve the performance of RCG.

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