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Existence and approximation of solutions to fractional order hybrid differential equations

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Abstract

We consider fractional hybrid differential equations involving the Caputo fractional derivative of order $0 < \alpha < 1$. Using fixed point theorems developed by Dhage *et al.* in Applied Mathematics Letters **34**, 76-80 (2014), we prove the existence and approximation of mild solutions. In addition, we provide a numerical example to illustrate the results obtained.

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1 Introduction

Fractional differential equations are of interest in many areas of applications, such as economics, signal identification and image processing, optical systems, aerodynamics, biophysics, thermal system materials and mechanical systems, control theory (see [2–6]). There are several results that investigate the existence of solutions of various classes of fractional differential equations. Much attention has been focused on the study of the existence and multiplicity of solutions as well as positive solutions for boundary value problem of fractional differential equations [7–9]. The main techniques used in these studies are fixed point techniques, Leray-Schauder theory, or upper and lower solutions methods (see, for example, [10–12] and the above references).

Recently, Dhage and Jahav [13] studied the existence and uniqueness of solutions of the first order ordinary differential equation which involves a perturbation of the addition or subtraction term given by

$$\begin{aligned} &\frac{d}{dt}\bigl[x(t) - f\bigl(t, x(t)\bigr)\bigr] = g\bigl(t, x(t)\bigr), \quad \text{a.e. } t \in [t_0, t_0 + a], \\ &x(t_0) = x_0. \end{aligned}$$

This type of equation has been called the hybrid differential equation. The existence result for solutions has been generalized to the fractional order hybrid differential equation

$$\begin{aligned} &\frac{d^\alpha}{dt^\alpha}\bigl[x(t)-f\bigl(t,x(t)\bigr)\bigr]=g\bigl(t,x(t)\bigr) \quad \text{a.e. } t\in[t_0,t_0+a], \\ &x(t_0)=x_0, \end{aligned} \tag{1.1}$$

by Lu *et al.* [14] using the Riemann-Liouville fractional derivative. Later, Herzallah and Beleanu [15] discussed the existence of mild solutions for the above fractional order hybrid differential equation (1.1) using the Caputo fractional derivative instead of taking it in the Riemann-Liouville sense. The importance of the investigation of the above hybrid differential equations is that these equations are a perturbation of nonlinear equations which generalizes various dynamic systems as a special case (see [16]).

Apart from the study of the existence of solutions, an approximation of the solution is also of interest. Dhange *et al.* in [1] imposed the concept of partial continuity and partial compactness to generalize the approach of Kranoselskii fixed point theorem and obtained the approximation of solutions to hybrid differential equation. Other results on the approximation of solutions to various types of equations can be found, for example, in [17, 18].

The main objective of this work is to extend the existence results in Herzallah and Beleanu [15] by following the approach in [17] to construct an iterative sequence that approximates the solution based on some fixed point theorem. We note that in [15], only the existence of a solution is proved. Our result gives both the existence and approximation of solutions to Caputo fractional order hybrid differential equations and also extends the existence results for ordinary hybrid differential equations. Moreover, the procedure in this paper allows us to approximate the solutions numerically.

This paper is organized as follows. In the next section, we introduce the notation and concepts of fractional order hybrid differential equations and discuss the frameworks of our problem. Section 3 is devoted to a proof of the existence and approximation of mild solutions of fractional order hybrid differential equations (1.1). Finally, in Section 4, we provide numerical example to illustrate the obtained results.

2 Preliminaries and framework

In this work, let $J=[t_0,t_0+a]$ be a closed and bounded interval in $\mathbb{R}$, where $t_0\geq 0$ and $a>0$. We denote the function space $C(J,\mathbb{R})$ for the space of continuous functions $x:J\to\mathbb{R}$. The space $C(J,\mathbb{R})$ is a Banach space when equipped with the supremum norm $\|\cdot\|$ given by

$$\|x\|=\sup_{t\in J}\bigl|x(t)\bigr|$$

for $x\in C(J,\mathbb{R})$.

We consider the fractional order nonlinear hybrid ordinary differential equation with initial value problem given by (1.1), where $f\in C(J\times\mathbb{R},\mathbb{R}), g\in C(J\times\mathbb{R},\mathbb{R})$ and the initial data $x_0\in\mathbb{R}$. The fractional order derivative used in this paper is taken in the sense of Caputo, which is defined as follows.

Definition 2.1 For $\alpha > 0$, the left Caputo fractional derivative of order α is defined by

$$\frac{d^\alpha}{dt^\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}D^n f(\tau)\,d\tau,$$

where n is a natural number such that $n-1 < \alpha < n$ and $D = d/d\tau$.

Note that the integrability of the nth order derivative of f is required for the Caputo fractional derivative.

We shall study the existence and approximation of mild solutions in the following sense.

Definition 2.2 The function $x \in C(J,\mathbb{R})$ is called a mild solution of the fractional nonlinear hybrid ordinary differential equation (1.1) if it satisfies the integral equation

$$x(t) = f\bigl(t,x(t)\bigr) + x_0 - f(t_0,x_0) + \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s,x(s)\bigr)\,ds, \quad t \in J. \tag{2.1}$$

We remark that the mild solution given in (2.1) can be obtained from (1.1) by applying the fractional integral I^α defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau$$

to both sides (see Lemma 11 in [15]).

In this paper, we consider the Banach space $C(J,\mathbb{R})$ together with a partial order relation. For any $x, y \in C(J,\mathbb{R})$, it is well known that the order relation $x \leq y$ given by $x(t) \leq y(t)$ for all $t \in J$ gives a partial ordering in $C(J,\mathbb{R})$. We shall mention the following important properties of the partially ordered Banach space $(C(J,\mathbb{R}),\leq)$, necessary for our study, from the work of [1, 18–20].

Let $E = (E, \preceq, \|\cdot\|)$ be a normed linear space equipped with a partial order relation $\preceq$. The space E is said to be regular if, for any nondecreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in E such that $x_n \to x^*$ as $n \to \infty$, we have $x_n \preceq x^*$ for all $n \in \mathbb{N}$. In particular, the space $C(J,\mathbb{R})$ is regular [1].

Definition 2.3 ([19]) An operator $T : E \to E$ is called nondecreasing if the order relation is preserved under T, that is, for any $x, y \in E$ such that $x \preceq y$, we have $Tx \preceq Ty$.

Definition 2.4 ([20]) An operator $T : E \to E$ is called partially continuous at $a \in E$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|Tx - Ta\| < \varepsilon$ for all x comparable to a in E with $\|x - a\| < \delta$. T is called partially continuous on E if it is partially continuous at every $a \in E$. In particular, if T is partially continuous on E, then it is continuous on every chain $\mathcal{C}$ in E. An operator T is called partially bounded if $T(\mathcal{C})$ is bounded for every chain $\mathcal{C}$ in E. An operator T is said to be uniformly partially bounded if all chains $T(\mathcal{C})$ in E are bounded by the same constant.

Definition 2.5 ([20]) An operator $T : E \to E$ is called partially compact if for any chains $\mathcal{C}$ in E, the set $T(\mathcal{C})$ is a relatively compact subset of E. An operator T is said to be partially totally bounded if for any totally ordered and bounded subset $\mathcal{C}$ of E, the set $T(\mathcal{C})$ is a relatively compact subset of E. If T is partially continuous and partially totally bounded, then we call it a partially completely continuous operator on E.

Definition 2.6 ([19]) Let E be a nonempty set equipped with an order relation $\preceq$ and a metric d. We say that the order relation $\preceq$ and the metric d are compatible if the following property is satisfied: if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone sequence in E for which a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*, then the whole sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, if $(E,\preceq,\|\cdot\|)$ is a partially ordered normed linear space, we say that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible whenever the order relation $\preceq$ and the metric d induced by the norm $\|\cdot\|$ are compatible.

We point out that the order relations and norms of $(\mathbb{R},\leq,|\cdot|)$ and $(C(J,\mathbb{R}),\leq,\|\cdot\|)$ are compatible.

Definition 2.7 ([19]) An upper semi-continuous and nondecreasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is called a $\mathcal{D}$-function if $\psi(0)=0$.

Definition 2.8 ([20]) Let $(E,\preceq,\|\cdot\|)$ be a normed linear space equipped with a partial order relation $\preceq$. A mapping $T:E\to E$ is called a partially nonlinear $\mathcal{D}$-Lipschitz if there is a $\mathcal{D}$-function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\|Tx-Ty\|\leq\psi\bigl(\|x-y\|\bigr)$$

for all comparable points $x,y\in E$. If $\psi(r)=kr$ for some positive constant k, then T is called a partially Lipschitz with a Lipschitz constant k. If $k<1$, we say that T is a partial contraction with contraction constant k. Moreover, T is said to be a nonlinear $\mathcal{D}$-contraction if it is nonlinear $\mathcal{D}$-Lipschitz with $\psi(j)<j$ for all $j>0$.

The following hybrid fixed point result of [19] is often applied to establish the existence and approximation of solutions of various differential and integral equations.

Theorem 2.1 ([20]) *Let $(E,\preceq,\|\cdot\|)$ be a regular partially ordered complete normed linear space. Suppose that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible. Let $\mathcal{P}:E\to E$ and $\mathcal{Q}:E\to E$ be two nondecreasing operators such that*:

(a) *$\mathcal{P}$ is a partially nonlinear $\mathcal{D}$-contraction.*
(b) *$\mathcal{Q}$ partially continuous and partially compact.*
(c) *There exists an element $x_0\in E$ such that $x_0\preceq\mathcal{P}x_0+\mathcal{Q}x_0$.*

Then there exists a solution x^* in E of the operator equation $\mathcal{P}x+\mathcal{Q}x=x$. In addition, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations given by

$$x_{n+1}=\mathcal{P}x_n+\mathcal{Q}x_n,\quad n=0,1,\ldots,$$

converges monotonically to x^*.

We shall state the framework and assumptions for our study now.

Assumption 1 We assume the following conditions.

(A0) The functions $f : J \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ are continuous.
(A1) f is nondecreasing in x for each $t \in J$ and $x \in \mathbb{R}$.
(A2) There exists a constant $M_f > 0$ such that $0 \leq |f(t,x)| \leq M_f$ for all $t \in J$ and $x \in \mathbb{R}$.
(A3) There exists a $\mathcal{D}$-contraction ϕ such that $0 \leq f(t,x) - f(t,y) \leq \phi(x-y)$ for $t \in J$, and $x, y \in \mathbb{R}$ with $x \geq y$.
(B1) g is nondecreasing in x for each $t \in J$ and $x \in \mathbb{R}$.
(B2) There exists a constant $M_g > 0$ such that $0 \leq |g(t,x)| \leq M_g$ for all $t \in J$ and $x \in \mathbb{R}$.
(B3) There exists a function $u \in C(J,\mathbb{R})$ such that u is a lower solution the problem (1.1), that is,

$$\begin{aligned} &\frac{d^\alpha}{dt^\alpha}\big[u(t) - f\big(t,u(t)\big)\big] \leq g\big(t,u(t)\big), \quad t \in J = [t_0, t_0 + a], \\ &u(t_0) \leq x_0 \in \mathbb{R}. \end{aligned} \tag{2.2}$$

3 Existence and approximation of mild solutions

This section is devoted to a proof of our main result on the existence and approximation of mild solutions of fractional order hybrid differential equations.

Theorem 3.1 *Suppose that the hypotheses* (A0)-(A3) *and* (B1)-(B3) *are satisfied. Then the initial value problem* (1.1) *has a mild solution* $x^* : J \to \mathbb{R}$ *and the sequence of successive approximations* x_n, $n = 1, 2, \ldots$, *defined by*

$$\begin{aligned} &x_{n+1}(t) = f\big(t, x_n(t)\big) + x_0 - f(t_0, x_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\big(s, x_n(s)\big)\, ds, \\ &x_1(t) = u(t), \end{aligned}$$

converges monotonically to x^*.

Proof We take the partially ordered Banach space $E = C(J,\mathbb{R})$. We prove the existence of a solution to problem (1.1) by considering the equivalent operator equation

$$\mathcal{P}x(t) + \mathcal{Q}x(t) = x(t),$$

where

$$\begin{aligned} &\mathcal{Q}x(t) = x_0 - f(t_0, x_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\big(s, x(s)\big)\, ds, \\ &\mathcal{P}x(t) = f\big(t, x(t)\big), \end{aligned}$$

for $t \in J$. We shall show that the operators $\mathcal{P}$ and $\mathcal{Q}$ satisfy all the conditions in Theorem 2.1.

Step I: First of all, we prove that $\mathcal{P}$ and $\mathcal{Q}$ are nondecreasing operators. For any $x, y \in E$ with $x \geq y$, we obtain from assumption (A3)

$$\mathcal{P}x(t) = f\big(t, x(t)\big) \geq f\big(t, y(t)\big) = \mathcal{P}y(t).$$

This means $\mathcal{P}$ is nondecreasing. For $\mathcal{Q}$, we have from assumption (B1)

$$\mathcal{Q}x(t) - \mathcal{Q}y(t) = \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big[g\big(s, x(s)\big) - g\big(s, y(s)\big)\big]\, ds \geq 0,$$

for any $x \geq y$ in E. Therefore, the operator $\mathcal{Q}$ is also nondecreasing.

Step II: In this step, we show that the operator $\mathcal{P}$ satisfies condition (a) in Theorem 2.1, that is, $\mathcal{P}$ is a partially bounded and partially nonlinear $\mathcal{D}$-contraction on E. For this purpose, let $x \in E$ be arbitrary. By the boundedness of f in condition (A2), we see that

$$\left|\mathcal{P}x(t)\right| = \left|f\bigl(t, x(t)\bigr)\right| \leq M_f,$$

for all $t \in J$. Therefore, we get $\|\mathcal{P}x\| \leq M_f$, which shows that $\mathcal{P}$ is bounded on E and so $\mathcal{P}$ is partially bounded. Moreover, for any $x, y \in E$ such that $x \geq y$, we see from assumption (A3) that

$$\left|\mathcal{P}x(t) - \mathcal{P}y(t)\right| = \left|f\bigl(t, x(t)\bigr) - f\bigl(t, y(t)\bigr)\right| \leq \phi\bigl(\left|x(t) - y(t)\right|\bigr) \leq \phi\bigl(\|x - y\|\bigr)$$

for each $t \in J$, where the last inequality is obtained from the condition that ϕ is nondecreasing. Hence, we have $\|\mathcal{P}x - \mathcal{P}y\| \leq \phi(\|x - y\|)$ for all $x, y \in E$ with $x \geq y$. This means that $\mathcal{P}$ is a partially nonlinear $\mathcal{D}$-contraction on E and, thus, partially continuous.

Step III: We verify the first property of $\mathcal{Q}$ in condition (b) of Theorem 2.1, that is, we prove that $\mathcal{Q}$ is partially continuous on E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain $\mathcal{C}$ in E satisfying $x_n \to x$ as $n \to \infty$. We obtain from the boundedness of g in (B2), the continuity of g in (A0), and the dominated convergence theorem

$$\begin{aligned}
\lim_{n\to\infty} (\mathcal{Q}x_n)(t) &= \lim_{n\to\infty} \left(x_0 - f(t_0, x_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x_n(s)\bigr)\, ds \right) \\
&= \lim_{n\to\infty} x_0 - \lim_{n\to\infty} f(t_0, x_0) + \lim_{n\to\infty} \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x_n(s)\bigr)\, ds \\
&= x_0 - f(t_0, x_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \lim_{n\to\infty} g\bigl(s, x_n(s)\bigr)\, ds \\
&= x_0 - f(t_0, x_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x(s)\bigr)\, ds \\
&= (\mathcal{Q}x)(t)
\end{aligned}$$

for each $t \in J$. This implies that $\mathcal{Q}x_n$ converges to $\mathcal{Q}x$ pointwise on J and the convergence is monotonic by the property of g. Next, we show that $\{\mathcal{Q}x_n\}_{n\in\mathbb{N}}$ is equicontinuous in E. Let $t_1, t_2 \in J = [t_0, t_0 + a]$ with $t_1 < t_2$. We have

$$\begin{aligned}
&\bigl|(\mathcal{Q}x_n)(t_2) - (\mathcal{Q}x_n)(t_1)\bigr| \\
&\quad = \left| \int_{t_0}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x_n(s)\bigr)\, ds - \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x_n(s)\, ds\bigr) \right| \\
&\quad \leq \left| \int_{t_0}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x_n(s)\bigr)\, ds - \int_{t_0}^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x_n(s)\bigr)\, ds \right| \\
&\qquad + \left| \int_{t_0}^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x_n(s)\bigr)\, ds - \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x_n(s)\bigr)\, ds \right| \\
&\quad = \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x_n(s)\bigr)\, ds \right|
\end{aligned}$$

$$
\begin{aligned}
&+ \left| \frac{1}{\Gamma(a)} \int_{t_0}^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right] g\bigl(s, x_n(s)\bigr)\, ds \right| \\
&\le \frac{M_g}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left|(t_2 - s)^{\alpha-1}\right| ds + \frac{M_g}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right| ds \\
&= \frac{M_g}{\Gamma(\alpha)} a^{\alpha-1}(t_2 - t_1) + \frac{M_g}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right| ds \\
&\to 0,
\end{aligned}
$$

as $t_2 - t_1 \to 0$ uniformly for all $n \in \mathbb{N}$, where we use the dominated convergence theorem for the limit in the second term above. This implies that $\mathcal{Q}x_n \to \mathcal{Q}x$ uniformly. Therefore, $\mathcal{Q}$ is partially continuous on E.

Step IV: Next we need to prove the remaining condition of operator $\mathcal{Q}$ in Theorem 2.1, that is, $\mathcal{Q}$ is partially compact. Let $\mathcal{C}$ be a chain in E. We shall show that $\mathcal{Q}(\mathcal{C})$ is uniformly bounded and equicontinuous in E. Let $y \in \mathcal{Q}(\mathcal{C})$ be arbitrary. We have $y = \mathcal{Q}(x)$ for some $x \in \mathcal{C}$. By hypothesis (B2), we see that

$$
\begin{aligned}
\left|y(t)\right| = \left|(\mathcal{Q})x(t)\right| &= \left| x_0 - f(t_0, x_0) + \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x_n(s)\bigr)\, ds \right| \\
&\le \left|x_0 - f(t_0, x_0)\right| + \frac{M_g}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left|(t_1 - s)^{\alpha-1}\right| ds \\
&\le \left|x_0 - f(t_0, x_0)\right| + \frac{M_g}{\alpha\Gamma(\alpha)} (t_1 - t_0)^{\alpha} \\
&= \left|x_0 - f(t_0, x_0)\right| + \frac{M_g}{\alpha\Gamma(\alpha)} a^{\alpha} =: K
\end{aligned}
$$

for all $t \in J$. Hence, we obtain $\|y(t)\| = \|(\mathcal{Q})x\| \le K$ for all $y \in \mathcal{Q}(\mathcal{C})$. This means $\mathcal{Q}(\mathcal{C})$ is uniformly bounded. We next show that $\mathcal{Q}(\mathcal{C})$ is equicontinuous. Let $y \in \mathcal{Q}(\mathcal{C})$ be arbitrary and take $t_1, t_2 \in J$ with $t_1 < t_2$. We have

$$
\begin{aligned}
&\left|(\mathcal{Q}x)(t_2) - (\mathcal{Q}x)(t_1)\right| \\
&\quad = \left| \int_{t_0}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x(s)\bigr)\, ds - \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x(s)\, ds\bigr) \right| \\
&\quad \le \left| \int_{t_0}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x(s)\bigr)\, ds - \int_{t_0}^{t_1} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x(s)\bigr)\, ds \right| \\
&\qquad + \left| \int_{t_0}^{t_1} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x(s)\bigr)\, ds - \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x(s)\bigr)\, ds \right| \\
&\quad = \left| \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x(s)\bigr)\, ds \right| + \left| \frac{1}{\Gamma(a)} \int_{t_0}^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right] g\bigl(s, x(s)\bigr)\, ds \right| \\
&\quad \le \frac{M_g}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left|(t_2 - s)^{\alpha-1}\right| ds + \frac{M_g}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right| ds \\
&\quad = \frac{M_g}{\Gamma(\alpha)} a^{\alpha-1}(t_2 - t_1) + \frac{M_g}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right| ds \\
&\quad \to 0
\end{aligned}
$$

as $t_2 - t_1 \to 0$ uniformly for $y \in \mathcal{Q}(\mathcal{C})$. This means $\mathcal{Q}(\mathcal{C})$ is equicontinuous. It follows that $\mathcal{Q}(\mathcal{C})$ is relatively compact. Hence, $\mathcal{Q}$ is partially compact.

Step V: By hypothesis (B3), the fractional hybrid equation (1.1) has a lower solution u defined on J, that is,

$$\frac{d^\alpha}{dt^\alpha}\bigl[u(t) - f\bigl(t, u(t)\bigr)\bigr] = g\bigl(t, u(t)\bigr), \quad t \in J,$$
$$u(t_0) \le x_0.$$

By formulating a mild solution, we see that

$$u(t) \le f\bigl(t, u(t)\bigr) + x_0 - f(t_0, x_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, u(s)\bigr)\, ds, \tag{3.1}$$

for $t \in J$. It follows that u satisfies the operator inequality $u \le \mathcal{P}u + \mathcal{Q}u$.

Thus, we conclude that the operators $\mathcal{P}$ and $\mathcal{Q}$ satisfy all conditions in Theorem 2.1. Then the operator equation $\mathcal{P}x + \mathcal{Q}x = x$ has a solution. Moreover, we have the approximation of solutions x_n as $n = 1, 2, \ldots$ for equation (1.1). □

4 Numerical examples

In this section, we give an example of hybrid fractional differential equation and show that our main result can be applied to construct an approximate sequence for a solution. We also illustrate it by showing a numerical result.

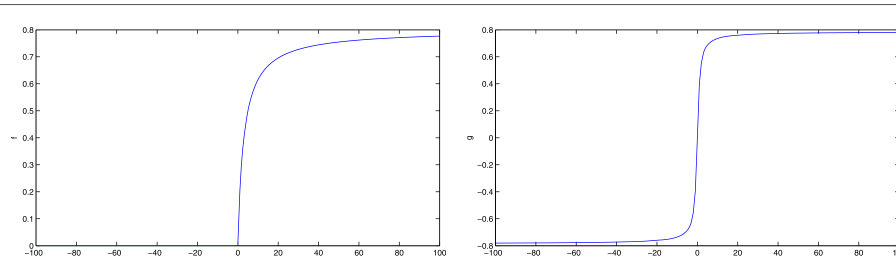
Example 4.1 Consider the following hybrid fractional differential equation:

$$\frac{d^\alpha}{dt^\alpha}\bigl[x(t) - f\bigl(t, x(t)\bigr)\bigr] = \frac{1}{2}\tan^{-1} x(t), \quad t \in J = [0,1], x(0) = 1, \tag{4.1}$$

where

$$f(x) = \begin{cases} \frac{4}{5}(\frac{x}{x+3}) & x \ge 0, \\ 0, & x < 0. \end{cases}$$

We have $g(t,x) = \frac{1}{2}\tan^{-1} x$. The graphs of these two functions are shown in Figure 1.



(a) Graph of function $f(t,x)$ (b) Graph of function $g(t,x)$

Figure 1 Graph of the functions $f(t,x)$ and $g(t,x)$.

It is clear that f and g are continuous functions on $J\times\mathbb{R}$. The assumption (A0) is satisfied. Moreover, both functions f and g are nondecreasing. This verifies assumption (A1) and (B1). The conditions (A2) and (B2) are also true since the function f is bounded by $M_f = \frac{4}{5}$, that is,

$$0 \le \bigl|f(t,x)\bigr| \le \frac{4}{5}\left|\frac{x}{x+3}\right| \le \frac{4}{5}$$

and the function g is bounded by $M_g = \frac{\pi}{4}$, that is,

$$0 \le \bigl|g(t,x)\bigr| = \frac{1}{2}\bigl|\tan^{-1}\bigr| \le M_g = \frac{\pi}{4}$$

for all $t\ x \in \mathbb{R}$.

To verify assumption (A3), we show that there exists a $\mathcal{D}$-contraction $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\phi(t) = \frac{4}{5}t$ for all $t > 0$ such that $0 \le f(t,x) - f(t,y) \le \phi(x-y)$ for all $t \in [0,1]$ and $x,y \in \mathbb{R}$ with $x \ge y$. First consider $x \ge y \ge 0$, we see that

$$\begin{aligned}
0 \le f(t,x) - f(t,y) &= \frac{4}{5}\left(\frac{x}{x+3} - \frac{y}{y+3}\right) \\
&\le \frac{4}{5}\left(\frac{(x-y)+y}{(x-y)+y+3} - \frac{y}{(x-y)+y+3}\right) \\
&= \frac{4}{5}\left(\frac{x-y}{(x-y)+y+3}\right) \\
&\le \frac{4}{5}\left(\frac{x-y}{(x-y)+3}\right) \\
&\le \frac{4}{5}|x-y| \\
&= \phi(x-y)
\end{aligned}$$

for all $t \in [0,1]$. It is easy to see that $0 \le f(t,x) - f(t,y) \le \phi(x-y)$ for all $t \in [0,1]$ also holds for $0 > x \ge y$ and $x \ge 0 > y$. Hence, (A3) is satisfied.

Finally, for assumption (B3), we see that $u(t) = 0.5$ for all $t \in [0,1]$ is a lower solution of (3.1). This can be seen from

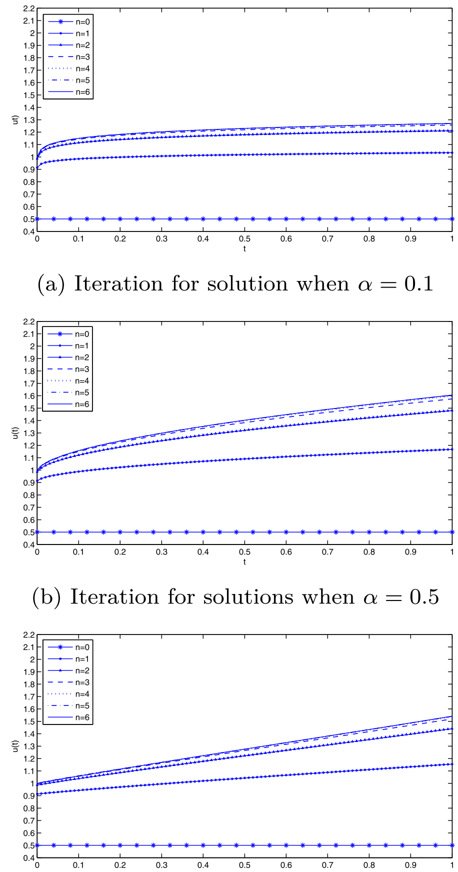
$$\begin{aligned}
&f\bigl(t,u(t)\bigr) + x_0 - f(0,x_0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s,u(s)\bigr)\,ds \\
&\quad= \frac{4}{5}\left(\frac{0.5}{0.5+3}\right) + 1 - \frac{4}{5}\left(\frac{1}{1+3}\right) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{2}\tan^{-1}(0.5)\,ds \\
&\quad= \frac{4}{35} + 1 - \frac{1}{5} + \frac{t^\alpha}{2\Gamma(\alpha+1)}\tan^{-1}(0.5)
\end{aligned}$$

for $t \in [0,1]$. This means

$$0.5 = u(t) \le f\bigl(t,u(t)\bigr) + x_0 - f(0,x_0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s,u(s)\bigr)\,ds$$

for $t \in [0,1]$ and $u(t) = 0.5$ is a lower solution. Since all assumptions are satisfied, we conclude from our main result in Theorem 3.1 that (4.1) has a solution $u^* : [0,1] \to \mathbb{R}$, which

Figure 2 Approximating sequence u_n for the solution when $\alpha = 0.1, 0.5, 0.9$.



(a) Iteration for solution when $\alpha = 0.1$

(b) Iteration for solutions when $\alpha = 0.5$

(c) Iteration for solutions when $\alpha = 0.9$

is a limit of the monotone sequence u_n, $n = 0, 1, 2, \ldots$, defined by

$$u_{n+1}(t) = f\bigl(t, u_n(t)\bigr) + 1 - \frac{4}{5}\left(\frac{1}{1+3}\right) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{2} \tan^{-1} u_n(s)\, ds \tag{4.2}$$

for all $t \in [0,1]$, where $u_0(t) = 0.5$ for $t \in [0,1]$.

The iterative sequence for the solution of (4.1) is numerically illustrated in Figure 2 for the fractional order derivative $\alpha = 0.1, 0.5$, and $\alpha = 0.9$. In the above iteration scheme for the sequence u_n defined by (4.2), we apply the trapezoidal rule for a numerical integration with step size 0.002. Since the exact solution is not explicitly known, we use the relative error between two iterates $\|u_n - u_{n-1}\|$ as a criterion to stop the iteration when its value is less than 0.002. In our example, the relative errors between two iterates $\|u_6 - u_5\|$ are 1.67×10^{-3}, 1.04×10^{-3}, and 6.62×10^{-4} for the case of $\alpha = 0.1, 0.5$, and $\alpha = 0.9$, respectively. The results show that the sequence of approximate solutions u_n converges monotonically.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

The main idea of this paper was proposed and mainly proved by P. S-N, while D.S. performed some proofs and provided a numerical example. All authors read and approved the final manuscript.

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