Wang et al. *Journal of Inequalities and Applications* (2016) 2016:244
DOI 10.1186/s13660-016-1189-7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

A Bernstein type inequality associated with wavelet bi-frame decomposition

Kai-Cheng Wang[1*], Kuei-Fang Chang[2], Chi-I Yang[1] and Chiou-Yueh Gun[3]

*Correspondence:
gtotony98@gmail.com
[1]PhD Program in Mechanical and Aeronautical Engineering, Feng-Chia University, Wenhwa Road, Taichung, 40724, Taiwan
Full list of author information is available at the end of the article

Abstract

Bernstein inequality is an essential inequality for Besov spaces. Smoothness based approaches are widely used in establishing the inequality. Yet, despite numerous studies over the last two decades, there is still little research focusing on decay-based approaches. However, motivating authors to establish inequality poses challenges, many of which can be overcome by means of the completeness of wavelet bi-frames in Lebesgue spaces and the stability of wavelet coefficients. The research has shown how wavelets with decay conditions enable descriptions of Lebesgue spaces, and in particular, the Bernstein inequality.

MSC: 41A17; 41A46; 42C40; 42C15

Keywords: Calderón-Zygmund decomposition theorem; Bernstein inequality; bi-frame; unconditional; wavelet

1 Introduction

It is well known that the Jackson and Bernstein inequalities reflect relations between quality and size of the approximation for a certain pair spaces [1], p.201, and they are used to characterize the approximation spaces by means of the interpolation spaces [1], p.235, Theorem 9.1. While the Bernstein inequality has a variety of applications in approximation theory, it is usually used to prove the inverse theorems. Wavelet decomposition into this area has shown how wavelets convey their contribution across various types. In 1993 an article [2] about Bernstein inequality was publish by Jia that has been the subject of much discussion ever since. References [3–5] seem to agree with [2], although no further improvement has been made. As shown in the above, the existing research as regards the Bernstein inequality has emphasized the necessity of smoothness, and it ignored the feasibility of decay (only-) based approaches. Hence, in order to help fill this gap in our knowledge, we establish the feasibility of decay-based approaches to the Bernstein inequality. The aim has been accomplished by means of completeness of wavelet bi-frames in Lebesgue spaces and stability of wavelet coefficients.

The completeness of wavelet bi-frames in Lebesgue spaces probably ranges between two types of methodologies: the use of Calderón-Zygmund operators (CZOs) and the use of the Calderón-Zygmund decomposition theorem (CZD). On the one hand, the use of Calderón-Zygmund operators is a way of gaining L^p-boundedness for certain operators by providing smoothness and the well-known theorem. This kind of gaining L^p-boundedness is used widely. References [6], p.856, Theorem 3.3, [3, 4, 7–9], [10], p.295, Theorems 9.1.5-9.1.6, [11], [12], Chapter 5, Theorems 6.14, 6.23, [13–16], [17], Lemma 3.1, [18], Chapter 6, and [19], Section 7.3, Theorem 1, and [20] ensure sufficiently smooth wavelet frames are complete/unconditional bases for Lebesgue spaces. The use of CZD, on the other hand, is a way of gaining L^p-boundedness for certain operators that for one reason or another does not involve smoothness. While all seem to agree that this kind of gaining L^p-boundedness cannot be entirely separate from complicated estimations, other researchers, for instance, Gripenberg [11] and Wojtaszczyk [20] insist on the use of CZD. They argue that it seems strange to posit smoothness on wavelets without specifying certain function spaces' nature. The present authors also contend that decay on wavelets should be enough to answer completeness problems in Lebesgue spaces, and thus, the Bernstein inequality.

This study is divided into four main sections. Section 2 is a review of preparatory facts, addressing both theoretical aspects and Lemma 2.3. Section 3 describes the methodology and procedures for completeness, l^p-stability of wavelet bi-frames in Lebesgue spaces. Finally, completeness of bi-frames in Lebesgue spaces is presented, and, in particular, the Bernstein inequality.

2 Preliminaries and notations

Throughout this paper, we use the following notations. We write $A \lesssim B$ if $A \leq CB$ for some positive constant C. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Let X, Y be two (quasi-)normed spaces. $X \hookrightarrow Y$ means that X is continuously embedded in Y. In other words, one has $X \subset Y$ and $\|\cdot\|_Y \lesssim \|\cdot\|_X$.

For $0 < p < \infty$, $\|f\|_p := (\int_{\mathbb{R}} |f|^p)^{\frac{1}{p}}$ is the norm for the Lebesgue space $L^p(\mathbb{R})$, and $\|f\|_\infty :=$ ess sup $|f|$. For $0 < p < \infty$, $l^p(\mathbb{Z})$ denotes the space of all sequences a such that $\|a\|_{l^p(\mathbb{Z})} < \infty$, where $\|a\|_{l^p(\mathbb{Z})} := (\sum_{i\in\mathbb{Z}} |a_i|^p)^{\frac{1}{p}}$; $\|a\|_{l^\infty(\mathbb{Z})} := \sup_{i\in\mathbb{Z}} |a_i|$. $\{f_i\}_{i\in\mathbb{Z}} \in L^p(\mathbb{R})$ is l^p-stable if $\|\sum_{i\in\mathbb{Z}} a_i f_i\|_p \sim \|a\|_{l^p(\mathbb{Z})}$, $1 < p < \infty$ for some $a := \{a_i\}_{i\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$.

For $f : \mathbb{R} \to \mathbb{C}$, $h \in \mathbb{R}$, the difference of r-order ($r \in \mathbb{N}$) [1], p.44, is given by

$$\triangle_h^r f(\cdot) := \triangle_h^1 \triangle_h^{r-1} f(\cdot) = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} f(\cdot + ih),$$

where $\triangle_h^1 f(\cdot) := f(\cdot + h) - f(\cdot)$. The rth modulus of smoothness of $f \in L^p(\mathbb{R})$, $0 < p \leq \infty$ is defined by

$$w_r(f,t)_p := \sup_{0<h\leq t} \left\| \triangle_h^r f \right\|_p.$$

We see that the norm of operator $\triangle_h^r$ in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, does not exceed 2^r.

Let $\alpha > 0$ be given and let $r := [\![\alpha]\!] + 1$. For $0 < p, \tau \leq \infty$, the Besov space $\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))$ [1], p.54, is the collection of all functions $f \in L^p(\mathbb{R})$ such that

$$|f|_{\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))} := \begin{cases} [\int_0^\infty (t^{-\alpha} w_r(f,t)_p)^\tau \frac{dt}{t}]^{\frac{1}{\tau}}, & 0 < \tau < \infty, \\ \sup_{t>0} t^{-\alpha} w_r(f,t)_p, & \tau = \infty, \end{cases}$$

is finite. The quantity is a seminorm if $1 \leq p, \tau \leq \infty$ (a quasi-seminorm in the other case). The norm for $\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))$ is $\|f\|_{\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))} := |f|_{\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))} + \|f\|_p$. Of special interest are the spaces $\mathcal{B}^\alpha := \mathcal{B}_\tau^\alpha(L^\tau(\mathbb{R}))$ where $\frac{1}{\tau} = \alpha + \frac{1}{p}$ and for fixed $p \in (1, \infty)$. There is another equivalent seminorm for $\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))$, *i.e.*,

$$|f|_{\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))} \sim \begin{cases} [\sum_{l\in\mathbb{Z}}(2^{l\alpha}w_r(f, 2^{-l})_p)^\tau]^{\frac{1}{\tau}}, & 0 < \tau < \infty, \\ \sup_{l\geq 0} 2^{l\alpha} w_r(f, 2^{-l})_p, & \tau = \infty. \end{cases}$$

The bases in this study are Schauder bases. Unconditionality for a sequence $\{x_n\}$ can be characterized as follows.

Theorem 2.1 ([21], Theorem 2.8) *Given a sequence $\{x_n\}$ in a Banach space. The following statements are equivalent.*

(1) *$\sum x_n$ converges unconditionally.*
(2) *$\sum \epsilon_n x_n$ converges for every choice of signs $\epsilon_n = \pm 1$.*
(3) *$\sum \lambda_n x_n$ converges for every bounded sequence of scalars $\{\lambda_n\}$.*

A sequence $\{\psi_i : i \in \mathbb{Z}\}$ (not necessarily linearly independent) in $L^2(\mathbb{R})$ is called a frame for $L^2(\mathbb{R})$ if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in\mathbb{Z}} |\langle f, \psi_i\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in L^2(\mathbb{R}).$$

The numbers A, B are called frame bounds and if $A = B$, we call this a tight frame. We define the frame operator S of $\{\psi_i : i \in \mathbb{Z}\}$,

$$Sf := \sum_{i\in\mathbb{Z}} \langle f, \psi_i\rangle \psi_i, \tag{2.1}$$

and each $f \in L^2(\mathbb{R})$ has the decomposition

$$f = \sum_{i\in\mathbb{Z}} \langle f, S'\psi_i\rangle \psi_i = \sum_{i\in\mathbb{Z}} \langle f, \psi_i\rangle S'\psi_i.$$

The series converges unconditionally in $L^2(\mathbb{R})$ [22], pp.90-91. $\{S'\psi_i : i \in \mathbb{Z}\}$ is called the canonical dual of $\{\psi_i : i \in \mathbb{Z}\}$, and every frame has its own canonical dual frame for $L^2(\mathbb{R})$. T is called the pre-frame operator or the synthesis operator; it is bounded. We have

$$T : l^2(\mathbb{N}) \to \mathcal{H}, \qquad T\{c_i\}_{i\in\mathbb{Z}} := \sum_{i\in\mathbb{Z}} c_i\psi_i,$$

$S = TT^*$. Both S and S' are of type $(2,2)$, bounded, invertible ($SS' = S'S = I$ on $L^2(\mathbb{R})$), self-adjoint, and positive. $\{S'\psi_i : i \in \mathbb{Z}\}$ is also a frame for $L^2(\mathbb{R})$ and its frame operator is S'. The canonical dual frame $\{S'\psi_i : i \in \mathbb{Z}\}$ of a tight frame is simply $\{\frac{1}{A}\psi_i : i \in \mathbb{Z}\}$.

A frame which is not a basis, is said to be overcomplete and thus, for overcomplete frames, the coefficients in the series expansion of a $f \in L^2(\mathbb{R})$ are not unique. It means that there exists a frame $\{\eta_i\} \neq \{S'\psi_i\}$ for which

$$f = \sum_{i\in\mathbb{Z}} \langle f, \eta_i\rangle \psi_i,$$

and $\{\eta_i\}$ is called the alternate dual frame of frame $\{\psi_i\}$. The affine wavelet frame system for $L^2(\mathbb{R})$ generated by ψ is defined as $\{\psi_{j,k}\} := \{\psi_{j,k} : j,k \in \mathbb{Z}\}$, $\psi_{j,k}(x) := 2^{j/2}\psi(2^j x + k)$. $\{\psi_{j,k}^{(p)}\} := \{\psi_{j,k}^{(p)} : j,k \in \mathbb{Z}\}$ is the primal wavelet frame of $\{\psi_{j,k}\}$ where $\psi_{j,k}^{(p)} := 2^{j/p}\psi(2^j x + k)$. In general [22], p.276,

$$\begin{aligned} &S' D^j T_k \psi = D^j S' T_k \psi, \\ &D^j S' T_k \psi \neq D^j T_k S' \psi, \end{aligned}$$

where $D^j(\cdot)(x) := 2^{j/2}(\cdot)(2^j x)$, $T_k(\cdot)(x) := (\cdot)(x + k)$. In this case, we say that the canonical dual frame $\{S'\psi_{j,k}\}$ of $\{\psi_{j,k}\}$ does not have a wavelet structure.

We say that $\{\psi_{j,k}\}$ and $\{\widetilde{\psi}_{j,k}\}$ are a pair of dual wavelet frames (bi-frames/sibling frames [22], p.277, [23, 24]) if both are frames for $L^2(\mathbb{R})$ and

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \widetilde{\psi}_{j,k}\rangle \psi_{j,k},$$

for all $f \in L^2(\mathbb{R})$. We say that ψ satisfies the condition $\mathcal{M}^p$ ($\psi \in \mathcal{M}^p$) if

$$\int_0^\infty \bigl[\log_2(1 + x)\bigr]\Psi^p(x)\,dx < \infty, \tag{2.2}$$

where $0 < p < \infty$ and $\Psi(x) := \sup_{x\le|y|} |\psi|(y)$, $x \geq 0$. L_ψ is the best lower bound for ψ satisfying $\mathcal{M}^p$ for all $p \in (L_\psi, \infty)$ and $\psi \notin \mathcal{M}^p$ for all $p \in [0, L_\psi]$.

Remark 2.2 The aim of this remark is to gain a better understanding of condition $\mathcal{M}$ as an essential feature in this study.

1. The case of wavelets satisfying $\mathcal{M}^1$ is enough for most cases in this study. In contrast, we consider a stronger condition $\mathcal{M}^p$, $0 < p < 1$ for the Bernstein inequality.
2. We need to exercise caution as regards $\mathcal{M}^p$. In general, given a wavelet ψ, there does exist a lower bound for p. One possible reason for p there having a lower bound may lie in

   $$\sum_{s\in\mathbb{N}} [\![\log_2 s]\!]|\psi|^p(s) \sim \int_1^\infty \bigl[\log_2(1 + x)\bigr]\Psi^p(x)\,dx.$$

   They either both converge or both diverge. Let ψ be a wavelet with a polynomial majorant, *i.e.*,

   $$|\psi|(x) \lesssim \bigl(1 + |x|\bigr)^{-\beta}, \quad \beta > 1.$$

   So $L_\psi := \beta^{-1}$, and it yields $\psi \in \mathcal{M}^p$. Indeed,

   $$\int_1^\infty \bigl[\log_2(1 + x)\bigr]\Psi^p(x)\,dx \lesssim \int_1^\infty \frac{\log_2(1 + x)}{(1 + x)^{p\beta}}\,dx < \infty.$$

$\sum_{s\in\mathbb{N}} [\![\log_2 s]\!]|\psi|^p(s)$ may diverge if $\beta \leq 1$. It is worth pointing out that $L_\psi := 0$ if ψ has exponential decay or is compactly supported.

We denote by $\mathcal{F}_\psi := \{\psi_{j,k} : j,k \in \mathbb{Z}\}$ a frame for $L^2(\mathbb{R})$, and $\mathcal{F}_{\psi,\mathcal{M}^p} := \{\psi_{j,k} : \psi \in \mathcal{M}^p, j,k \in \mathbb{Z}, p \in (L_\psi, \infty)\}$ is a frame for $L^2(\mathbb{R})$.

The following is the CZD [25]. For all $f \in L^1 \cap L^2(\mathbb{R})$ and $\alpha > 0$, there exists a collection $\Omega \subset \mathbb{Z}^2$ such that the intervals $\{I_{m,n}\}_{(m,n)\in\Omega}$ are disjoint, $I_{m,n} := [2^{-m}n, 2^{-m}(n+1))$,

$$\bigl|f(x)\bigr| \le \alpha,$$

almost everywhere on $F := \mathbb{R} \setminus \bigcup_{(m,n)\in\Omega} I_{m,n}$, and also, for all $(m,n) \in \Omega$,

$$\alpha < 2^m \int_{I_{m,n}} |f| \le 2\alpha.$$

Therefore,

$$\sum_{(m,n)\in\Omega} \alpha 2^{-m} < \|f\|_1.$$

Lemma 2.3 is generalized from [26], Lemma 3.1. (1) and (2) can be found in our early work [26], Lemma 3.1, and Section 4. (4) is a direct consequence from (3). The proof for Lemma 2.3(3) is similar to [26], Lemma 3.1. To be rigorous, we include a proof.

Lemma 2.3 *Let ψ satisfy condition $\mathcal{M}^p$, $p \in (L_\psi, 1]$, $0 \le L_\psi < 1$. Then*:
(1) $\|\psi\|_{W(L^\infty,l^1)} := \sum_{k\in\mathbb{Z}} \sup_{[0,1)} |\psi|(x+k) < \infty$, *and it leads to $\psi \in L^r(\mathbb{R})$, for all* $1 \le r \le \infty$.
(2) Ψ *is a radial decreasing L^1-majorant of ψ.*
(3) $\sum_{j,k\in\mathbb{Z}} [\sup_{x\in\mathbb{R}} |\psi|(2^j x + k)]^q < \infty$, *for all $p \le q < \infty$.*
(4) $\sum_{j,k\in\mathbb{Z}} \sup_{x\in\mathbb{R}} |\psi|^q(2^j x + k) < \infty$, *for all $p \le q < \infty$.*

Proof of Lemma 2.3(3) (3) can be obtained by providing the equivalence as follows:

$$\begin{aligned}
&\sum_{j,k\in\mathbb{Z}} \Bigl[\sup_{[0,1)} |\psi|\bigl(2^j x + k\bigr)\Bigr]^p < \infty, \\
&\sum_{j,k\in\mathbb{Z}} \Bigl[\sup_{x\in\mathbb{R}} |\psi|\bigl(2^j x + k\bigr)\Bigr]^p < \infty, \quad \text{and} \\
&\sum_{j,k\in\mathbb{Z}} \Bigl[\sup_{I_{m,n}} |\psi|\bigl(2^j x + k\bigr)\Bigr]^p < \infty.
\end{aligned} \tag{2.3}$$

The finiteness does not depend on m or n.

$\int_0^\infty \Psi^p(x)\,dx < \infty$ and the part of 'equivalence' is obvious. Let

$$I^*_{m,n} := \bigl[2^{-m}(n-1), 2^{-m}(n+2)\bigr).$$

For any fixed $(m,n) \in \Omega$,

$$\begin{aligned}
&\Biggl\{\sum_{\substack{j,k\in\mathbb{Z}\\ j\ge m}} \Bigl[\sup_{2^j I_{m,n}+k} |\psi|(x)\Bigr]^p\Biggr\} \int_{\mathbb{R}\setminus I^*_{m,n}} 2^{j/2}|\psi_{j,k}| \\
&\quad \le \Biggl\{\sum_{\substack{j,k\in\mathbb{Z}\\ j\ge m}} \Bigl[\sup_{2^j I_{m,n}+k} |\psi|(x)\Bigr]^p\Biggr\} \biggl(\int_{-\infty}^{2^{-m}(n-1)} + \int_{2^{-m}(n+2)}^{\infty}\biggr) 2^{j/2}|\psi_{j,k}|
\end{aligned}$$

$$\leq \left\{\sum_{\substack{j,k\in\mathbb{Z}\\ j\geq m}}\Big[\sup_{2^j I_{m,n}+k}|\psi|(x)\Big]^p\right\}\left(\int_{-\infty}^{2^{j-m}(n-1)+k}+\int_{2^{j-m}(n+2)+k}^{\infty}\right)|\psi|$$

$$= \left\{\sum_{\substack{j,k\in\mathbb{Z}\\ j\geq m}}\Big[\sup_{[k,k+2^{j-m})}|\psi|(x)\Big]^p\right\}\left(\int_{-\infty}^{k-2^{j-m}}+\int_{2^{j-m+1}+k}^{\infty}\right)|\psi|<\infty. \tag{2.4}$$

The finiteness does not depend on (m,n) and can be proven by breaking k into parts. First,

$$\sum_{j=m}^{\infty}\left(\sum_{k\leq -2^{j-m+1}}+\sum_{k>2^{j-m-1}}\right)\Big[\sup_{[k,k+2^{j-m})}|\psi|(x)\Big]^p\left(\int_{-\infty}^{k-2^{j-m}}+\int_{2^{j-m+1}+k}^{\infty}\right)|\psi|$$

$$\leq 2\|\psi\|_1\sum_{j=m}^{\infty}\left(\sum_{-\infty}^{k=-2\cdot 2^{j-m}}\Psi^p\big(-k-2^{j-m}\big)+\sum_{k=[\frac{1}{2}2^{j-m}]+1}^{\infty}\Psi^p(k)\right)$$

$$\leq 4\|\psi\|_1\sum_{j=m}^{\infty}\sum_{k=[\frac{1}{2}2^{j-m}]+1}^{\infty}\Psi^p(k)\leq 4\|\psi\|_1\sum_{p=1}^{\infty}\big([\log_2 p]+2\big)\Psi^p(p)$$

$$\leq 4\|\psi\|_1\int_1^{\infty}(2+\log_2 x)\Psi^p(x)\,dx<\infty.$$

Second, applying the Dirichlet test,

$$\sum_{j=m}^{\infty}\left(\sum_{-2^{j-m+1}<k\leq 2^{j-m-1}}\right)\Big[\sup_{[k,k+2^{j-m})}|\psi|(x)\Big]^p\left(\int_{-\infty}^{k-2^{j-m}}+\int_{2^{j-m+1}+k}^{\infty}\right)|\psi|$$

$$\leq \sum_{j=m}^{\infty}\left(\sum_{-2^{j-m+1}<k\leq 2^{j-m-1}}\right)\Big[\sup_{[k,k+2^{j-m})}|\psi|(x)\Big]^p\left(\int_{-\infty}^{-2^{j-m-1}}+\int_{2^{j-m+1}+k}^{\infty}\right)|\psi|<\infty. \tag{2.5}$$

Because the double summation of (2.6) is finite, (2.5) is finite. We have

$$\sum_{j>m}\sum_{k}\Big[\sup_{[k,k+2^{j-m})}|\psi|(x)\Big]^p<\infty. \tag{2.6}$$

We will prove (2.6) by using a pedagogical argument. Assuming that (2.6) does not hold, and given $V>0$, there exist $j_0>m$, $k_0\in\mathbb{Z}$, $q_1,q_2\in\mathbb{N}$, such that

$$V<\sum_{j=j_0}^{j_0+q_1-1}\sum_{k=k_0}^{k_0+q_2-1}\Big[\sup_{[k,k+2^{j-m})}|\psi|(x)\Big]^p\leq q_1q_2\Big[\sup_{[k_1,k_1+2^{j_1-m})}|\psi|(x)\Big]^p,$$

where $j_1\in\{j_0,j_0+1,\ldots,j_0+q_1-1\}$, $k_1\in\{k_0,k_0+1,\ldots,k_0+q_2-1\}$. There also exist x_0 and $n_0\in\mathbb{Z}$ with $x_0\in[k_1,k_1+2^{j_1-m})$ such that

$$\begin{aligned}\left(\frac{V}{q_1q_2}\right)^{\frac{1}{p}}&<|\psi|(x_0)\leq\sup_{[n_0,n_0+1)}|\psi|(x)\\ &\leq\Psi(t)\chi_{[0,n_1]}(t)\\ &\leq\Psi(t)\chi_{[0,n_1]}(t)+\Psi(t)\chi_{[n_1,n_1+q_1q_2]}(t),\end{aligned}$$

where $n_1 := \min\{|n_0|, |n_0+1|\}$. Performing integration on both sides,

$$V < \int_0^{n_1+q_1q_2} \frac{V}{q_1q_2}\,dt < \int_0^{n_1} \Psi^p(t)\,dt + \int_{n_1}^{n_1+q_1q_2} \Psi^p(t)\,dt.$$

This is a contradiction to $\int_0^\infty \Psi^p(t)\,dt < \infty$.

Putting $(m,n) = (0,1)$ into (2.4), we obtain

$$\sum_{\substack{j,k\in\mathbb{Z}\\ j\geq 0}} \Bigl[\sup_{2^j[0,1)} |\psi|(x+k)\Bigr]^p < \infty,$$

and this leads to

$$\sum_{\substack{j',k'\in\mathbb{Z}\\ j'<0}} \Bigl[\sup_{2^{j'}[0,1)} |\psi|\bigl(x+k'\bigr)\Bigr]^p \leq \sum_{\substack{j,k\in\mathbb{Z}\\ j\geq 0}} \Bigl[\sup_{2^j[0,1)} |\psi|(x+k)\Bigr]^p < \infty.$$

So (2.3) holds. □

3 L^p-Boundedness for affine operators

This section is designed to establish the Bernstein inequality. We investigate this question by constructing and administering a specially designed version of affine operator. By using CZD, we conclude that the operator is of type (p,p), $1 < p < \infty$. Finally, stability of wavelet bi-frames was also characterized to convey their contribution to the Bernstein inequality. Special attention is given to the following.

Remark 3.1 The use of bi-frames was motivated by two facts.

1. Dual frames of frames are not unique but infinite numbers [27]. Their structures are quite complicated and may not have a wavelet structure. Therefore, it will be more efficient to represent elements in Banach spaces via bi-frames compared to other types of frames/bases.
2. Dual frames having no wavelet structure would make much impact on completeness problems in Lebesgue spaces [18], pp.130, 136, [28]. More details can be found in [26], Remark 2.2.

Let $\{\phi_{j,k} : \phi \in \mathcal{M}^1, j,k \in \mathbb{Z}\}$ and $\{\varphi_{j,k} : \varphi \in \mathcal{M}^1, j,k \in \mathbb{Z}\}$ be two Bessel sequences for $L^2(\mathbb{R})$. We define the affine operator Q by

$$Q(\cdot) := \sum_{j,k\in\mathbb{Z}} \theta_{j,k} \langle \cdot, \phi_{j,k}\rangle \varphi_{j,k}, \tag{3.1}$$

where $\Theta := \{\theta_{j,k} : j,k \in \mathbb{Z}\} \in l^\infty$. Q is of type $(2,2)$ since

$$\bigl\|Q(\cdot)\bigr\|_2^2 = \Bigl\|\sum_{j,k\in\mathbb{Z}} \theta_{j,k}\langle \cdot, \phi_{j,k}\rangle\varphi_{j,k}\Bigr\|_2^2 \lesssim \sum_{j,k\in\mathbb{Z}} \bigl|\langle \cdot, \phi_{j,k}\rangle\bigr|^2 \lesssim \|\cdot\|_2^2.$$

Let $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ be a pair dual wavelet frames. We denote by P_m the projection from $L^2(\mathbb{R})$ onto a subspace of

$$\mathcal{V}_m := \overline{\operatorname{span}}\{\widetilde{\psi}_{j,k} : j,k \in \mathbb{Z}, j < m\},$$

$$P_m(f) := \sum_{\substack{j,k\in\mathbb{Z}\\ j<m}} \langle f, \psi_{j,k}\rangle \widetilde{\psi}_{j,k}.$$

We set

$$\begin{aligned}
g &:= f\chi_F + \sum_{(m,n)\in\Omega} P_m(f\chi_{I_{m,n}}),\\
h &:= f - g = \sum_{(m,n)\in\Omega} \bigl[f\chi_{I_{m,n}} - P_m(f\chi_{I_{m,n}})\bigr]\\
&= \sum_{(m,n)\in\Omega} \biggl[\sum_{j,k\in\mathbb{Z}} \langle f\chi_{I_{m,n}}, \psi_{j,k}\rangle \widetilde{\psi}_{j,k} - P_m(f\chi_{I_{m,n}})\biggr]\\
&= \sum_{(m,n)\in\Omega} \sum_{\substack{j,k\in\mathbb{Z}\\ j\geq m}} \langle f\chi_{I_{m,n}}, \psi_{j,k}\rangle \widetilde{\psi}_{j,k},\\
Qh &= \sum_{j,k\in\mathbb{Z}} \theta_{j,k} \biggl\langle \sum_{(m,n)\in\Omega} \sum_{\substack{j',k'\in\mathbb{Z}\\ j'\geq m}} \langle f\chi_{I_{m,n}}, \psi_{j',k'}\rangle \widetilde{\psi}_{j',k'}, \phi_{j,k}\biggr\rangle \varphi_{j,k}\\
&= \sum_{(m,n)\in\Omega} \sum_{j,k\in\mathbb{Z}} \sum_{\substack{j',k'\in\mathbb{Z}\\ j'\geq m}} \theta_{j,k} \langle f\chi_{I_{m,n}}, \psi_{j',k'}\rangle \langle \widetilde{\psi}_{j',k'}, \phi_{j,k}\rangle \varphi_{j,k}.
\end{aligned}$$

Theorem 3.2 is significantly superior to our early work [26] and [8, 9, 17]. It is not only so that the hypotheses are more flexible, but also the estimation is more complicated in comparison to [26]. Mainly, $\{\widetilde{\psi}_{j,k}\}$ and $\{\phi_{j,k}\}$ may not be biorthogonal in $L^2(\mathbb{R})$. Theorem 3.2 motivates us to believe that linear independence has no significant effect on the boundedness for an affine operator.

Theorem 3.2 *Under the hypotheses given above, the operator Q is of weak type* $(1,1)$ *and of type* (p,p), *for all* $1 < p < \infty$.

Proof for Theorem 3.2 Q is of weak type $(1,1)$ is based on two inequalities:

$$\|g\|_2^2 \leq \alpha A_1 \|f\|_1 \quad \text{and} \tag{3.2}$$

$$m\bigl\{x : |Qh| > \alpha/2\bigr\} \leq \frac{B_1}{\alpha}\|f\|_1, \tag{3.3}$$

where A_1 and B_1 do not depend on f or α. Indeed, for all $f \in L^1 \cap L^2(\mathbb{R})$ and $\alpha > 0$,

$$\begin{aligned}
m\bigl\{x : |Qf| > \alpha\bigr\} &\leq m\bigl\{x : |Qg| > \alpha/2\bigr\} + m\bigl\{x : |Qh| > \alpha/2\bigr\}\\
&\leq m\bigl\{x : |Qg|^2 > \alpha^2/4\bigr\} + \frac{B_1}{\alpha}\|f\|_1\\
&\leq \bigl[(4A_1 + B_1)/\alpha\bigr]\|f\|_1.
\end{aligned}$$

Suppose for the moment that we know, and then we will immediately see from the Marcinkiewicz interpolation theorem, that Q is of type (p,p), $1 < p < 2$, and thus by duality for all p, $2 < p < \infty$.

First, we claim that the first inequality holds. By Lemma 2.3 and the Dirichlet test, for any $(m,n),(m',n')\in\Omega$ with $m\le m'$, there exist constants M_1, M_2, and M_3 which do not depend on m or n, such that

$$M_1 := 4\|\widetilde{\psi}\|_2^2\Bigg[\sum_{\substack{j',k'\in\mathbb{Z}\\ j'<m'}} 2^{\frac{j'-m'}{2}}\sup_{I_{m',n'}}|\psi|\big(2^{j'}x+k'\big)\Bigg]<\infty,$$

$$M_3 := \Bigg[\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}} 2^{j-m'}\sup_{I_{m',n'}}|\psi|\big(2^{j}x+k\big)\Bigg]^{-1}\Bigg[1+\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}} 2^{\frac{j-m'}{2}}\sup_{I_{m',n'}}|\psi|\big(2^{j}x+k\big)\Bigg]<\infty,$$

$$\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}} 2^{\frac{j-m'}{2}}\sup_{I_{m,n}}|\psi|\big(2^{j}x+k\big)\le M_2\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}} 2^{\frac{j-m'}{2}}\sup_{I_{m',n'}}|\psi|\big(2^{j}x+k\big)<\infty,$$

$$\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}} 2^{\frac{j-m'}{2}}\sup_{I_{m',n'}}|\psi|\big(2^{j}x+k\big)\le M_3\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}} 2^{j-m'}\sup_{I_{m',n'}}|\psi|\big(2^{j}x+k\big)<\infty,$$

$$\begin{aligned}
&\Bigg\|\sum_{(m,n)\in\Omega} P_m(f\chi_{I_{m,n}})\Bigg\|_2^2\\
&\quad=\int_{\mathbb{R}}\sum_{(m,n)\in\Omega}\sum_{(m',n')\in\Omega} P_m(f\chi_{I_{m,n}})\overline{P_{m'}(f\chi_{I_{m',n'}})}\\
&\quad\le\sum_{(m,n)\in\Omega}\sum_{(m',n')\in\Omega}\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}}\sum_{\substack{j',k'\in\mathbb{Z}\\ j'<m'}}\big|\langle f\chi_{I_{m,n}},\psi_{j,k}\rangle\langle f\chi_{I_{m',n'}},\psi_{j',k'}\rangle\langle\widetilde{\psi}_{j,k},\widetilde{\psi}_{j',k'}\rangle\big|\\
&\quad\le 2\|\widetilde{\psi}\|_2^2\sum_{(m,n)\in\Omega}\sum_{\substack{(m',n')\in\Omega\\ m\le m',n\le n'}}\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}}\sum_{\substack{j',k'\in\mathbb{Z}\\ j'<m'}}\bigg(\int_{I_{m,n}}|f|\bigg)\Big(2^{\frac{j}{2}}\sup_{I_{m,n}}|\psi|\big(2^{j}x+k\big)\Big)\\
&\qquad\times\Big(\big(2\cdot2^{-m'}\alpha\big)2^{\frac{j'}{2}}\sup_{I_{m',n'}}|\psi|\big(2^{j'}x+k'\big)\Big)\\
&\quad\le 4\alpha\|\widetilde{\psi}\|_2^2\bigg(\sum_{(m,n)\in\Omega}\int_{I_{m,n}}|f|\bigg)\sum_{\substack{(m',n')\in\Omega\\ m\le m'}}\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}}\Big(2^{\frac{j-m'}{2}}\sup_{I_{m,n}}|\psi|\big(2^{j}x+k\big)\Big)\\
&\qquad\times\sum_{\substack{j',k'\in\mathbb{Z}\\ j'<m'}}\Big(2^{\frac{j'-m'}{2}}\sup_{I_{m',n'}}|\psi|\big(2^{j'}x+k'\big)\Big)\\
&\quad\le\alpha M_1M_2M_3\bigg(\sum_{(m,n)\in\Omega}\int_{I_{m,n}}|f|\bigg)\sum_{\substack{(m',n')\in\Omega\\ m\le m'}}\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}}2^{j-m'}\sup_{I_{m',n'}}|\psi|\big(2^{j}x+k\big)\\
&\quad\le\alpha M_1M_2M_3\bigg(\sum_{(m,n)\in\Omega}\int_{I_{m,n}}|f|\bigg)\Bigg(\sum_{p\in\mathbb{Z}}\sum_{m'=m}^{\infty}\sum_{n'=p2^{m'-m}}^{(p+1)2^{m'-m}+1}\Bigg)\\
&\qquad\times\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}}2^{j-m'}\sup_{I_{m',n'}}|\psi|\big(2^{j}x+k\big)\\
&\quad\le\alpha M_1M_2M_3\bigg(\sum_{(m,n)\in\Omega}\int_{I_{m,n}}|f|\bigg)\sum_{p\in\mathbb{Z}}\sum_{\substack{j,k\in\mathbb{Z}\\ j<m}}2^{j-m}\sup_{I_{m,p}}|\psi|\big(2^{j}x+k\big)\le\alpha N\|f\|_1,
\end{aligned}$$

where, by the Dirichlet test,

$$N := M_1 M_2 M_3 \sum_{p\in\mathbb{Z}} \sum_{\substack{j,k\in\mathbb{Z}\\ j<m}} 2^{j-m} \sup_{I_{m,p}} |\psi|\bigl(2^j x + k\bigr) < \infty.$$

The finiteness of N derives from the following.

1. We recall that $\{I_{m',n'}\}_{(m',n')\in\Omega}$ are disjoint. Under $m \le m'$ and $p2^{m'-m} \le n' < (p+1)2^{m'-m}$, we have

$$I_{m',n'} \subset I_{m,p}.$$

   It shows that, for each p,

$$\sum_{m'=m}^{\infty} \sum_{n'=p2^{m'-m}}^{(p+1)2^{m'-m}-1} 2^{-m'} < 2^{-m}.$$

2. For any finite terms of p, j, and k,

$$\sum_{\substack{p,j,k\\ j<m}} \sup_{I_{m,p}} |\psi|\bigl(2^j x + k\bigr) \le \sum_{\text{finite } k'} \sup_{[0,1)} |\psi|\bigl(x + k'\bigr) < \|\psi\|_{W(L^\infty,\ell^1)} < \infty.$$

Next,

$$\int_{\mathbb{R}} |f\chi_F|^2 \le \alpha \int_F |f| \le \alpha \|f\|_1,$$

and it leads to $\|g\|_2^2 \le (N+1)\alpha\|f\|_1$.

For the second inequality, we set

$$I^*_{m,n} := \bigl[2^{-m}(n-1), 2^{-m}(n+2)\bigr), \qquad F^* := \mathbb{R} \setminus \bigcup_{(m,n)\in\Omega} I^*_{m,n},$$

$$B_1' := \|\Theta\|_\infty \|\widetilde{\psi}\|_1 \|\varphi\|_1 \Biggl[\sum_{\substack{j',k'\in\mathbb{Z}\\ j'\ge m}} \sup_{I_{m,n}} |\psi|\bigl(2^{j'}x + k'\bigr)\Biggr] \sum_{j,k\in\mathbb{Z}} \sup_{x\in\mathbb{R}} |\phi|\bigl(2^j x + k\bigr),$$

$$\begin{aligned}
\int_{F^*} |Qh| &\le \int_{\mathbb{R}\setminus I^*_{m,n}} |Qh| \\
&\le \|\Theta\|_\infty \sum_{(m,n)\in\Omega} \sum_{j,k\in\mathbb{Z}} \sum_{\substack{j',k'\in\mathbb{Z}\\ j'\ge m}} \bigl|\langle f\chi_{I_{m,n}}, \psi_{j',k'}\rangle \langle \widetilde{\psi}_{j',k'}, \phi_{j,k}\rangle\bigr| \biggl(\int_{\mathbb{R}\setminus I^*_{m,n}} |\varphi_{j,k}|\biggr) \\
&\le \|\Theta\|_\infty \|\widetilde{\psi}\|_1 \|\varphi\|_1 \biggl(\sum_{(m,n)\in S} \int_{I_{m,n}} |f|\biggr) \Biggl[\sum_{\substack{j',k'\in\mathbb{Z}\\ j'\ge m}} \sup_{I_{m,n}} |\psi|\bigl(2^{j'}x + k'\bigr)\Biggr] \\
&\quad \times \Biggl[\sum_{j,k\in\mathbb{Z}} \sup_{x\in\mathbb{R}} |\phi|\bigl(2^j x + k\bigr)\Biggr] \le B_1' \|f\|_1.
\end{aligned}$$

Finally, we note that $m\{\mathbb{R} \setminus F^*\} < 3/\alpha\|f\|_1$ and thus

$$m\{x : |Sh| > \alpha/2\} \leq \left[\left(2B_1' + 3\right)/\alpha\right]\|f\|_1. \qquad \square$$

Next, we will characterize Lebesgue spaces by wavelet coefficients. There is considerable scattered literature about characterizing function spaces that depends only on wavelet coefficients: how the wavelet coefficients actually function, how they process information or are affected by individual's aspects. For example, Borup *et al.* [3, 4] have written about the boundedness of synthesis operators based on the stability of wavelet coefficients. Daubechies [10], Section 9.2, Li and Sun [17], Proposition 3.2, and Meyer [18], Section 6.2, Theorem 1, have discussed L^p norms versus l^p norms of wavelet coefficients. Canuto and Tabacco [29], Härdle *et al.* [30], Proposition 8.3, and Wojtaszczyk [31], Section 8.1, have investigated l^p-stability for multilevel decompositions. Theorem 3.3 is done in the hope that it may provide an alternative solution to aspects mentioned. Additionally, Theorem 3.3 could have a considerable impact on establishing the Bernstein inequality and simplifying the analysis procedure.

Theorem 3.3 *Let $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ be a pair of dual wavelet frames for $L^2(\mathbb{R})$ and $f = \sum_{j,k\in\mathbb{Z}} a_{j,k}\psi_{j,k} \in L^p(\mathbb{R})$, $\{a_{j,k}\} \in l^\infty(\mathbb{Z}\times\mathbb{Z})$, $1 < p < \infty$. Then:*

(1) $\|f\|_p \sim \|(\sum_{j,k\in\mathbb{Z}} |a_{j,k}|^2|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p$.

(2) *The primal wavelet frame $\{\psi_{j,k}^{(p)}\}$ is unconditional and l^p-stable in $L^p(\mathbb{R})$, for all $1 < p < \infty$.*

Proof of Theorem 3.3 From Theorem 3.2, R has L^p-boundedness and $\{\psi_{j,k}^{(p)}\}$ is unconditional in $L^p(\mathbb{R})$. We have

$$R(\cdot) := \sum_{j,k\in\mathbb{Z}} \epsilon_{j,k}\langle\cdot, \widetilde{\psi}_{j,k}^{(p')}\rangle\psi_{j,k}^{(p)}, \quad \epsilon_{j,k} = \pm 1, \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.4}$$

Let $\psi_{j,k}^* := |\psi_{j,k}^{(p)}|^{\frac{2}{p}}(\sum_{j,k\in\mathbb{Z}} |\psi_{j,k}^{(p)}|^2)^{\frac{1}{2}-\frac{1}{p}}$. By following the lines of the proof in [32], Theorem 2, (1) can be obtained. From [32], Theorems 4 and 5, l^p-stability can be obtained by providing:

1. $\{\psi_{j,k}^{(p)}\}$ is unconditional in $L^p(\mathbb{R})$;
2. $\inf_{j,k\in\mathbb{Z}} \|\psi_{j,k}^*\|_p > 0$ for all $1 < p < 2$, and
3. $\sup_{j,k\in\mathbb{Z}} \|\psi_{j,k}^*\|_p < \infty$ for all $2 \leq p < \infty$.

Indeed,

$$\begin{aligned} \|\psi_{j,k}^*\|_p^p &= \int_{\mathbb{R}} |\psi_{j,k}^{(p)}|^2\left(\sum_{j,k\in\mathbb{Z}} |\psi_{j,k}^{(p)}|^2\right)^{\frac{p}{2}-1} \\ &= \int_{\mathbb{R}} 2^j|\psi|^2\left(2^jx+k\right)\left(\sum_{j,k\in\mathbb{Z}} |\psi|^2\left(2^jx+k\right)\right)^{\frac{p}{2}-1} dx \\ &\begin{cases} \leq \|\psi\|_\infty^{\frac{p}{2}} \int_{\mathbb{R}} 2^j|\psi|(2^jx+k)(\sum_{j,k\in\mathbb{Z}} |\psi|(2^jx+k))^{\frac{p}{2}-1} dx, & 2 \leq p < \infty \\ \geq \int_{\mathbb{R}} 2^j|\psi|^2(2^jx+k)[\sum_{j,k\in\mathbb{Z}}(\sup_{x\in\mathbb{R}} |\psi|(2^jx+k))^2]^{\frac{p}{2}-1} dx, & 1 < p < 2 \end{cases} \\ &\begin{cases} \leq \|\psi\|_\infty^{\frac{p}{2}} \|\psi\|_1(\sum_{j,k\in\mathbb{Z}} \sup_{x\in\mathbb{R}} |\psi|(2^jx+k))^{\frac{p}{2}-1} < \infty, & 2 \leq p < \infty \\ = \|\psi\|_2^2[\sum_{j,k\in\mathbb{Z}}(\sup_{x\in\mathbb{R}} |\psi|(2^jx+k))^2]^{\frac{p}{2}-1} > 0, & 1 < p < 2. \end{cases} \end{aligned} \qquad \square$$

4 Completeness for bi-frames in Lebesgue spaces

The findings in this section should contribute to a better understanding of how decay-based approaches affect, manage, and are of usage to problems we are concerned with. This section also demonstrates the feasibility of using decay (only-) based approaches in the Bernstein inequality.

Theorem 4.1 *Let $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ be a pair of dual wavelet frames for $L^2(\mathbb{R})$. Then*:

(1) *The operator S associated $\mathcal{F}_{\psi,\mathcal{M}^1}$ has L^p-boundedness, for all $1 < p < \infty$.*

(2) *Both $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ are complete in $L^p(\mathbb{R})$, $1 < p < \infty$.*

Proof of Theorem 4.1 From Theorem 3.2, the operator S, R_1 and R_2 associated $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ have L^p-boundedness, for all $1 < p < \infty$. We have

$$R_1(\cdot) := \sum_{j,k\in\mathbb{Z}} \epsilon_{j,k} \langle \cdot, \widetilde{\psi}_{j,k} \rangle \psi_{j,k}, \quad \epsilon_{j,k} = \pm 1,$$

$$R_2(\cdot) := \sum_{j,k\in\mathbb{Z}} \epsilon_{j,k} \langle \cdot, \psi_{j,k} \rangle \widetilde{\psi}_{j,k}, \quad \epsilon_{j,k} = \pm 1.$$

Since $L^2 \cap L^p(\mathbb{R})$ is a dense subset of $L^p(\mathbb{R})$, $1 < p < \infty$, we have

$$\sum_{j,k\in\mathbb{Z}} \langle f, \widetilde{\psi}_{j,k} \rangle \psi_{j,k} = f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle \widetilde{\psi}_{j,k} \in L^p(\mathbb{R}),$$

for any $f \in L^2 \cap L^p(\mathbb{R})$. Thus, for each $f \in L^p(\mathbb{R})$, the series $\sum_{j,k\in\mathbb{Z}} \langle f, \widetilde{\psi}_{j,k} \rangle \psi_{j,k}$ and $\sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle \widetilde{\psi}_{j,k}$ converge unconditionally in $L^p(\mathbb{R})$ so that they must converge to f. □

Similar results are found in the aforementioned papers in Section 1. They were obtained by the technique of CZOs. Perceivably, (mild) smoothness is necessary for wavelets, and thus they do not support the Haar wavelet (compactly supported).

Furthermore, (bi)orthonormal wavelet bases and tight frames are trivial cases being wavelet bi-frames. Compactly supported wavelets certainly satisfy condition $\mathcal{M}^1$ and thus Theorem 4.1 can be applied to *all* compactly supported wavelet bi-frames. We also note that Theorem 4.1 is significantly superior to the results [11, 20] based on decay-based approaches.

Theorem 4.1 ensures some typical wavelet frames are complete/unconditional bases for Lebesgue spaces. Of special interest are Haar wavelets, Daubechies' wavelets [10], biorthogonal wavelets [33], Lemarié-Meyer wavelets [34, 35], and spline wavelets of high order (orthogonal [36, 37], semi-orthogonal [26, 38, 39], biorthogonal [33]). Bownik has constructed a wavelet tight frame $\{\psi_{j,k}\}$ which is in the Schwartz class and $\hat{\psi} \in C^\infty$ with compact support [40], p.219. The last paper [41] we refer to was written by Lemvig; he has constructed pairs of dual band-limited wavelet frames which are band-limited and come with the desired time localization.

Lemma 4.2 plays a crucial role in deriving the Bernstein inequality. Let $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ be a pair of dual wavelet frames for $L^2(\mathbb{R})$. We consider the collection of all possible m-term expansions with elements from the primal wavelet frame $\{\psi_{j,k}^{(p)}\}$, $1 < p < \infty$:

$$\Upsilon_m := \left\{ \sum_{j,k\in\Gamma} a_{j,k} \psi_{j,k}^{(p)} : \{a_{j,k}\} \in l^\infty(\mathbb{Z} \times \mathbb{Z}), \operatorname{card}\Gamma \leq m, m \in \mathbb{N} \right\}. \tag{4.1}$$

Lemma 4.2 *Let $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ be a pair of dual wavelet frames for $L^2(\mathbb{R})$. Given $p \in (1,\infty)$, $\frac{1}{\tau} = \alpha + \frac{1}{p}$, $1 \le \tau < p$ and $\alpha > 0$, then, for any $f \in \Upsilon_m$, the following inequality holds*:

$$|f|^{\tau}_{\mathcal{B}^{\alpha}} \lesssim \sum_{j,k\in\Gamma} |a_{j,k}|^{\tau}. \tag{4.2}$$

In particular, if $\psi \in \mathcal{M}^{\tau}$ with $\tau \in (L_{\psi},1)$, (4.2) also holds.

Proof of Lemma 4.2 From Lemma 2.3 and given $x \in \mathbb{R}$, the series $\sum_{j,k\in\Gamma} a_{j,k}\psi^{(p)}_{j,k}$ is absolutely convergent. For $0 < \tau < 1$, $|\sum_{j,k\in\Gamma} a_{j,k}\psi^{(p)}_{j,k}|^{\tau} \le \sum_{j,k\in\Gamma} |a_{j,k}\psi^{(p)}_{j,k}|^{\tau}$.

For $\tau \ge 1$,

$$\left|\sum_{j,k\in\Gamma} a_{j,k}\psi^{(p)}_{j,k}\right|^{\tau} \le \Bigl(m \cdot \max_{j,k\in\Gamma}\bigl|a_{j,k}\psi^{(p)}_{j,k}\bigr|\Bigr)^{\tau} \le m^{\tau}\sum_{j,k\in\Gamma}\bigl|a_{j,k}\psi^{(p)}_{j,k}\bigr|^{\tau}.$$

From the definition of $|f|^{\tau}_{\mathcal{B}^{\alpha}}$ and the fact that the norm of operator $\triangle^r_h$ in $L^p(\mathbb{R})$, $1 \le p \le \infty$, does not exceed 2^r, we have, for $\tau \ge 1$,

$$\begin{aligned}
|f|^{\tau}_{\mathcal{B}^{\alpha}} &\lesssim \sum_{l\in\mathbb{Z}} 2^{\alpha\tau l} w^{\tau}_r\bigl(f,2^{-l}\bigr)_{\tau} \le \sum_{l\in\mathbb{Z}} 2^{\alpha\tau l}\Bigl(\sup_{|h|\le 2^{-l}}\bigl\|\triangle^r_h f\bigr\|_{\tau}\Bigr)^{\tau} \\
&\le 2^{r\tau}\sum_{l\in\mathbb{Z}} 2^{\alpha\tau l}\|f\|^{\tau}_{\tau} \lesssim \Biggl(\sum_{l\in\mathbb{N}} + \sum_{l\in\mathbb{Z},l\le 0}\Biggr) 2^{\alpha\tau l}\|f\|^{\tau}_{\tau} \\
&\le \Biggl(\sum_{l\in\mathbb{N}} + \sum_{l\in\mathbb{Z},l\le 0}\Biggr) m^{\tau} 2^{\alpha\tau l}\sum_{(j,k)\in\Gamma}\bigl\|a_{j,k}\psi^{(p)}_{j,k}\bigr\|^{\tau}_{\tau} \\
&= \Biggl(\sum_{l\in\mathbb{N}} + \sum_{l\in\mathbb{Z},l\le 0}\Biggr)\Biggl(\sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{N}}} + \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{Z},j\le 0}}\Biggr) m^{\tau} 2^{\alpha\tau l}\bigl\|a_{j,k}\psi^{(p)}_{j,k}\bigr\|^{\tau}_{\tau}.
\end{aligned}$$

Setting $u_1 := m^{\tau}2^{\alpha\tau l}(2^j x + k)$, we have

$$\begin{aligned}
&\sum_{l\in\mathbb{N}}\sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{Z},j\in\mathbb{N}}} m^{\tau}2^{\alpha\tau l}|a_{j,k}|^{\tau}\int_{\mathbb{R}} 2^{\frac{\tau j}{p}}|\psi|^{\tau}\bigl(2^j x + k\bigr)\,dx \\
&\quad\le \sum_{l\in\mathbb{N}}\sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{N}}} |a_{j,k}|^{\tau}\int_{\mathbb{R}} 2^{-\alpha\tau j}|\psi|^{\tau}\biggl(\frac{u_1}{m^{\tau}2^{\alpha\tau l}}\biggr)\,du_1 \\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{N}}} |a_{j,k}|^{\tau}\sum_{l\in\mathbb{N}}\sum_{s\in\mathbb{Z}}\int_s^{s+2^l} |\psi|^{\tau}\biggl(\frac{u_1}{m^{\tau}2^{\alpha\tau l}}\biggr)\,du_1 \\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{N}}} |a_{j,k}|^{\tau}\sum_{l\in\mathbb{N}}\sum_{s\in\mathbb{Z}}\sup_{u_1\in\frac{1}{m^{\tau}2^{\alpha\tau l}}[s,s+2^l]} |\psi|^{\tau}(u_1) \\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{N}}} |a_{j,k}|^{\tau}\Biggl[\sum_{l\in\mathbb{N}}\sum_{s\in\mathbb{Z}}\sup_{u_1\in 2^l[0,1]+s} |\psi|^{\tau}(u_1)\Biggr] \lesssim \sum_{(j,k)\in\Gamma} |a_{j,k}|^{\tau}.
\end{aligned}$$

Setting $u_2 := m^\tau 2^{\alpha\tau l}(2^{\frac{\tau j}{p}}x + 2^{-\tau\alpha j}k)$, we have

$$\begin{aligned}
&\sum_{l\in\mathbb{N}}\sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{Z},j\le 0}} m^\tau 2^{\alpha\tau l}|a_{j,k}|^\tau \int_{\mathbb{R}} 2^{\frac{\tau j}{p}}|\psi|^\tau\bigl(2^j x+k\bigr)\,dx\\
&\quad\le \sum_{l\in\mathbb{N}}\sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{Z},j\le 0}} |a_{j,k}|^\tau \int_{\mathbb{R}} |\psi|^\tau\left(\frac{2^{\tau\alpha(j-l)}}{m^\tau}u_2\right)du_2\\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{Z},j\le 0}} |a_{j,k}|^\tau \sum_{l\in\mathbb{N}}\sum_{s\in\mathbb{Z}} \int_s^{s+2^l} |\psi|^\tau\left(\frac{2^{\tau\alpha(j-l)}}{m^\tau}u_2\right)du_2\\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{Z},j\le 0}} |a_{j,k}|^\tau \sum_{l\in\mathbb{N}}\sum_{s\in\mathbb{Z}} \sup_{u_2\in\frac{2^{\tau\alpha(j-l)}}{m^\tau}[s,s+2^l]} |\psi|^\tau(u_2)\\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{Z},j\le 0}} |a_{j,k}|^\tau \Biggl[\sum_{l\in\mathbb{N}}\sum_{s\in\mathbb{Z}} \sup_{u_2\in 2^l[0,1]+s} |\psi|^\tau(u_2)\Biggr] \lesssim \sum_{(j,k)\in\Gamma}|a_{j,k}|^\tau.
\end{aligned}$$

Setting $u_3 := m^\tau(2^j x + k)$, we have

$$\begin{aligned}
&\sum_{l\in\mathbb{Z},l\le 0}\sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{Z},j\in\mathbb{N}}} m^\tau 2^{\alpha\tau l}|a_{j,k}|^\tau \int_{\mathbb{R}} 2^{\frac{\tau j}{p}}|\psi|^\tau\bigl(2^j x+k\bigr)\,dx\\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{N}}} |a_{j,k}|^\tau \sum_{l\in\mathbb{Z},l\le 0} 2^{\alpha\tau l}\int_{\mathbb{R}} 2^{-\alpha\tau j}|\psi|^\tau\left(\frac{u_3}{m^\tau}\right)du_3\\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{N}}} |a_{j,k}|^\tau \sum_{l\in\mathbb{Z},l\le 0} 2^{\alpha\tau l}\sum_{s\in\mathbb{Z}}\int_s^{s+1} |\psi|^\tau\left(\frac{u_3}{m^\tau}\right)du_3\\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{N}}} |a_{j,k}|^\tau \sum_{l\in\mathbb{Z},l\le 0} 2^{\alpha\tau l}\sum_{s\in\mathbb{Z}} \sup_{u_3\in\frac{1}{m^\tau}[s,s+1]} |\psi|^\tau(u_3)\\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{N}}} |a_{j,k}|^\tau \Biggl[\sum_{l\in\mathbb{Z},l\le 0} 2^{\alpha\tau l}\Biggr]\Biggl[\sum_{s\in\mathbb{Z}} \sup_{u_3\in[s,s+1]} |\psi|^\tau(u_3)\Biggr] \lesssim \sum_{(j,k)\in\Gamma}|a_{j,k}|^\tau.
\end{aligned}$$

Setting $u_4 := m^\tau(2^{\frac{\tau j}{p}}x + 2^{-\tau\alpha j}k)$, we have

$$\begin{aligned}
&\sum_{l\in\mathbb{Z},l\le 0}\sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{Z},j\le 0}} m^\tau 2^{\alpha\tau l}|a_{j,k}|^\tau \int_{\mathbb{R}} 2^{\frac{\tau j}{p}}|\psi|^\tau\bigl(2^j x+k\bigr)\,dx\\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{Z},j\le 0}} |a_{j,k}|^\tau \sum_{l\in\mathbb{Z},l\le 0} 2^{\alpha\tau l}\int_{\mathbb{R}} |\psi|^\tau\left(\frac{2^{\tau\alpha j}}{m^\tau}u_4\right)du_4\\
&\quad\le \sum_{\substack{(j,k)\in\Gamma\\ j\in\mathbb{Z},j\le 0}} |a_{j,k}|^\tau \sum_{l\in\mathbb{Z},l\le 0} 2^{\alpha\tau l}\sum_{s\in\mathbb{Z}}\int_s^{s+1} |\psi|^\tau\left(\frac{2^{\tau\alpha j}}{m^\tau}u_4\right)du_4
\end{aligned}$$

$$\leq \sum_{\substack{(j,k)\in\Gamma \\ j\in\mathbb{Z},j\leq 0}} |a_{j,k}|^\tau \sum_{l\in\mathbb{Z},l\leq 0} 2^{\alpha\tau l} \sum_{s\in\mathbb{Z}} \sup_{u_4\in\frac{2^{\tau\alpha j}}{m^\tau}[s,s+1]} |\psi|^\tau(u_4)$$

$$\leq \sum_{\substack{(j,k)\in\Gamma \\ j\in\mathbb{Z},j\leq 0}} |a_{j,k}|^\tau \Bigg[\sum_{l\in\mathbb{Z},l\leq 0} 2^{\alpha\tau l}\Bigg]\Bigg[\sum_{s\in\mathbb{Z}} \sup_{u_4\in[s,s+1]} |\psi|^\tau(u_4)\Bigg] \lesssim \sum_{(j,k)\in\Gamma} |a_{j,k}|^\tau.$$

Therefore, we conclude (4.2). By using the method described above, (4.2) also holds for $0 < \tau < 1$. □

The proof of Theorem 4.3 is based on the ideas expressed in [42], Theorem 3.1. However, our hypotheses are more flexible and clear. While DeVore's proof is only applicable to orthogonal wavelet bases, our result can be applied to a larger family.

Theorem 4.3 *Let $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ be a pair of dual wavelet frames for $L^2(\mathbb{R})$. Given $p \in (1,\infty)$, $\frac{1}{\tau} = \alpha + \frac{1}{p}$, $1 \leq \tau < p$ and $\alpha > 0$, then, for any $f \in \Upsilon_m$, the Bernstein inequality holds. That is,*

$$|f|_{B^\alpha} \lesssim m^\alpha \|f\|_p. \tag{4.3}$$

In particular, if $\psi \in \mathcal{M}^\tau$ with $\tau \in (L_\psi, 1)$, (4.3) also holds.

Proof of Theorem 4.3 We denote $I_{j,-k} := [\frac{-k}{2^j}, \frac{-k+1}{2^j})$ and $\chi_{I_{j,-k}}$ is the indicator function for $I_{j,-k}$. We order the intervals of Γ in order of non-decreasing size $I^1_{j_1,-k_1}, I^2_{j_2,-k_2}, \ldots, I^r_{j_r,-k_r}$ where $r \leq m$. We let $E_s := I^s_{j_s,-k_s} \setminus \bigcup\{I^t_{j_t,-k_t} : t < s\}$. Then the sets $\{E_s : s = 1,2,3,\ldots,r\}$ are disjoint. From Lemma 4.2, we have

$$\begin{aligned}
|f|^\tau_{B^\alpha} &\lesssim \sum_{(j,k)\in\Gamma} |a_{j,k}|^\tau \\
&= \int_{\mathbb{R}} \sum_{(j,k)\in\Gamma} \big(2^{j\alpha} 2^{\frac{j}{p}} |a_{j,k}| \chi_{I_{j,-k}}(x)\big)^\tau\, dx \\
&= \sum_{s=1}^r \int_{E_s} \Bigg[\sum_{(j,k)\in\Gamma} \big(2^{j\alpha} 2^{\frac{j}{p}} |a_{j,k}| \chi_{I_{j,-k}}(x)\big)^\tau\Bigg] dx \\
&\lesssim \sum_{s=1}^r \int_{E_s} \Bigg[\sum_{(j,k)\in\Gamma} \big(2^{j\alpha} \chi_{I_{j,-k}}(x)\big)^\tau\Bigg] \sup_{(j,k)\in\Gamma} \big[2^{\frac{j}{p}} |a_{j,k}| \chi_{I_{j,-k}}(x)\big]^\tau\, dx \\
&\lesssim \sum_{s=1}^r 2^{j_s\tau\alpha} \int_{E_s} \sup_{(j,k)\in\Gamma} \big[2^{\frac{j}{p}} |a_{j,k}| \chi_{I_{j,-k}}(x)\big]^\tau\, dx,
\end{aligned}$$

where $\sum_{(j,k)\in\Gamma} (2^{j\alpha} \chi_{I_{j,-k}}(x))^\tau \lesssim 2^{j_s\tau\alpha}$ for $x \in E_s$. Now,

$$\sup_{(j,k)\in\Gamma} 2^{\frac{j}{p}} |a_{j,k}| \chi_{I_{j,-k}}(x) \leq \Bigg[\sum_{(j,k)\in\Gamma} 2^j |a_{j,k}|^p \chi_{I_{j,-k}}(x)\Bigg]^{\frac{1}{p}} := f_0(x).$$

Since $|E_s| \le |I^s_{j_s,-k_s}|$, $1 = \tau\alpha + \frac{\tau}{p} = \frac{1}{\frac{1}{\tau\alpha}} + \frac{1}{\frac{p}{\tau}}$ and $\|f_0\|_p = \|\{a_{j,k}\}\|_{l^p} \sim \|f\|_p$ (see Theorem 3.3), we have

$$\begin{aligned}
|f|^\tau_{B^\alpha_\tau} &\lesssim \sum_{s=1}^r 2^{j_s\tau\alpha} \int_{E_s} |f_0|^\tau(x)\,dx \\
&\lesssim \sum_{s=1}^r 2^{j_s\tau\alpha} |E_s|^{1-\frac{\tau}{p}} \|f_0\|^\tau_{L^p(E_s)} \\
&\lesssim \sum_{s=1}^r \|f_0\|^\tau_{L^p(E_s)} = \sum_{s=1}^r \bigl(1 \cdot \|f_0\|^\tau_{L^p(E_s)}\bigr) \\
&\le r^{\tau\alpha} \left[\sum_{s=1}^r \|f_0\|^p_{L^p(E_s)}\right]^{\frac{\tau}{p}} \le m^{\tau\alpha} \|f_0\|^\tau_p \\
&\lesssim m^{\tau\alpha} \|f\|^\tau_p.
\end{aligned}$$ □

Special attention is given to the following.

Corollary 4.4 *Let $\mathcal{F}_\psi$ and $\mathcal{F}_{\tilde\psi}$ be a pair of dual wavelet frames with compactly supported for $L^2(\mathbb{R})$. Given $p \in (1,\infty)$, $\frac{1}{\tau} = \alpha + \frac{1}{p}$, $0 < \tau < p$ and $\alpha > 0$, then, for any $f \in \Upsilon_m$, the Bernstein inequality holds. That is,*

$$|f|_{B^\alpha} \lesssim m^\alpha \|f\|_p.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]PhD Program in Mechanical and Aeronautical Engineering, Feng-Chia University, Wenhwa Road, Taichung, 40724, Taiwan. [2]Department of Applied Mathematics, Feng-Chia University, Wenhwa Road, Taichung, 40724, Taiwan. [3]Department of Mechanical Engineering, Nan-Kai University of Technology, Zhongzheng Road, Nantou, 54243, Taiwan.

Acknowledgements
The many suggestions and detailed corrections of anonymous referees are gratefully acknowledged.

Received: 5 May 2016 Accepted: 28 September 2016 Published online: 06 October 2016

References
1. DeVore, RA, Lorentz, GG: Constructive Approximation. Springer, Berlin (1993)
2. Jia, R-Q: A Bernstein type inequality associated with wavelet decomposition. Constr. Approx. **9**, 299-318 (1993)
3. Borup, L, Gribonval, R, Nielsen, M: Tight wavelet frames in Lebesgue and Sobolev spaces. J. Funct. Spaces **2**, 227-252 (2004)
4. Borup, L, Gribonval, R, Nielsen, M: Bi-framelet systems with few vanishing moments characterize Besov spaces. Appl. Comput. Harmon. Anal. **17**, 3-28 (2004)
5. Ehler, M: Nonlinear approximation schemes associated with nonseparable wavelet bi-frames. J. Approx. Theory **161**, 292-313 (2009)
6. Ahmad, K, Kumar, R, Debnath, L: Existence of unconditional wavelet packet bases for the spaces $L^p(\mathbb{R})$ and $\mathcal{H}^1(\mathbb{R})$. Taiwan. J. Math. **10**, 851-863 (2006)
7. Bui, H-Q, Laugesen, RS: Wavelet frame bijectivity on Lebesgue and Hardy spaces. J. Fourier Anal. Appl. **19**, 376-409 (2013)
8. Chui, CK, Shi, X: On L^p-boundedness of affine frame operators. Indag. Math. **4**, 431-438 (1993)
9. Chui, CK, Shi, X: Affine operators and frames of multivariate wavelets. In: Dikshit, HP, Micchelli, CA (eds.) Advances in Computational Mathematics, pp. 139-156. World Scientific, Singapore (1994)
10. Daubechies, I: Ten Lectures on Wavelets. SIAM, Philadelphia (1992)
11. Gripenberg, G: Wavelet bases in $L^p(\mathbb{R})$. Stud. Math. **2**, 175-187 (1993)
12. Hernández, E, Weiss, G: A First Course on Wavelets. CRC Press, Boca Raton (1996)

13. Kumar, D: Wavelets convergence and unconditional bases for $L^p(\mathbb{R}^n)$ and $\mathcal{H}^1(\mathbb{R}^n)$. Chin. J. Math. **2014**, Article ID 984280 (2014)
14. Kumar, D: Convergence and characterization of wavelets associated with dilation matrix in variable L^p-spaces. J. Math. **2013**, Article ID 254694 (2013)
15. Kumar, D, Seikh, NA: Pseudo frames and non-orthogonal projections on to subspaces. Panam. Math. J. **22**(3), 45-55 (2012)
16. Kumar, D: Convergence of a class of non-orthogonal wavelet expansions in L^p, $1 \leq p \leq \infty$. Panam. Math. J. **19**(4), 61-70 (2009)
17. Li, K, Sun, W: Characterization of function spaces using wavelets. Numer. Funct. Anal. Optim. **34**, 1115-1128 (2013)
18. Meyer, Y: Wavelets and Operators. Cambridge University Press, Cambridge (1995)
19. Meyer, Y, Coifman, R: Wavelets: Calderón-Zygmund and Multilinear Operators. Cambridge University Press, Cambridge (2000)
20. Wojtaszczyk, P: Wavelets as unconditional bases in $L^p(\mathbb{R})$. J. Fourier Anal. Appl. **5**, 73-85 (1999)
21. Heil, C: A Basis Theory Primer: Expanded Edition. Springer, Berlin (2010)
22. Christensen, O: An Introduction to Frames and Riesz Bases. Birkhäuser Boston, Boston (2003)
23. Chui, CK, He, W, Stöckler, J: Compactly supported tight and sibling frames with maximum vanishing moments. Appl. Comput. Harmon. Anal. **13**, 224-262 (2002)
24. Daubechies, I, Han, B, Ron, A, Shen, Z: Framelets: MRA-based constructions of wavelet frames. Appl. Comput. Harmon. Anal. **14**, 1-46 (2003)
25. Stein, EM: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
26. Gun, C-Y, Wang, K-C, Yang, C-I, Chang, K-F: Pre-wavelet bases in Lebesgue spaces. J. Inequal. Appl. **2015**, 286 (2015)
27. Bownik, M, Lemvig, J: The canonical and alternate duals of a wavelet frame. Appl. Comput. Harmon. Anal. **23**, 263-272 (2007)
28. Tao, T: L^p non-invertibility of frame operators. http://www2.math.ucla.edu/tao/preprints/Expository/inverse.tex
29. Canuto, C, Tabacco, A: Multilevel decompositions of functional spaces. J. Fourier Anal. Appl. **3**, 715-742 (1997)
30. Härdle, W, Kerkyacharian, G, Picard, D, Tsybakov, A: Wavelets, Approximation, and Statistical Applications. Springer, Berlin (2012)
31. Wojtaszczyk, P: A Mathematical Introduction to Wavelets. Cambridge University Press, Cambridge (1997)
32. Yang, L: Unconditional basic sequence in $L^p(\mu)$ and its l^p-stability. Proc. Am. Math. Soc. **127**, 455-464 (1999)
33. Cohen, A, Daubechies, I, Feauveau, J-C: Biorthogonal bases of compactly supported wavelets. Commun. Pure Appl. Math. **45**, 485-560 (1992)
34. Battle, G: A block spin construction of ondelettes. Part I: Lemarié functions. Commun. Math. Phys. **110**, 601-615 (1987)
35. Lemarié, PG: Ondelettes à localisation exponentielles. J. Math. Pures Appl. **67**(3), 227-236 (1988)
36. Mallat, SG: A theory for multiresolution signal decomposition: the wavelet representation. IEEE Trans. Pattern Anal. Mach. Intell. **11**, 674-693 (1989)
37. Mallat, SG: Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$. Trans. Am. Math. Soc. **315**, 69-87 (1989)
38. Chui, CK, Wang, J: On compactly supported spline wavelets and a duality principle. Trans. Am. Math. Soc. **330**, 903-915 (1992)
39. Unser, M, Aldroubi, A, Eden, M: A family of polynomial spline wavelet transforms. Signal Process. **30**, 141-162 (1993)
40. Bownik, M: A characterization of affine dual frames in $L^2(\mathbb{R}^n)$. Appl. Comput. Harmon. Anal. **8**, 203-221 (2000)
41. Lemvig, J: Constructing pairs of dual bandlimited framelets with desired time localization. Adv. Comput. Math. **30**, 231-247 (2008)
42. DeVore, RA, Jawerth, B, Popov, V: Compression of wavelet decompositions. Am. J. Math. **114**, 737-785 (1992)