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On refined Hardy-Knopp type inequalities in Orlicz spaces and some related results

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Abstract

In this paper, we construct a new integral operator $T_k^{(r)}$ which generalizes the classical Hardy-Knopp type integral operator A_k by considering the power mean of the non-negative measurable functions. We state and prove a new refined Hardy-Knopp type inequality related to the weighted Lebesgue spaces. As a special case of our results, the refinements of multidimensional Hardy-Knopp type inequalities are obtained. Finally, we also apply a similar idea to prove some new norm inequalities in Orlicz spaces in which the properties of N-functions and superquadratic functions are involved.

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Keywords: Hardy-Knopp type inequalities; Orlicz spaces; norm inequalities; convex functions; superquadratic functions

1 Introduction

It is well known that the classical Hardy inequality in [1] reads

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \tag{1.1}$$

for any $f \in L^p(\mathbb{R}_+)$ with $1 < p < \infty$; the constant $(\frac{p}{p-1})^p$ is the best possible. After that, the inequality has been tremendously studied and applied in an almost unbelievable way. By replacing f with $f^{\frac{1}{p}}$ and letting $p \to \infty$ in (1.1), we obtain the limiting case which is referred to as Knopp's inequality:

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)\,dt\right) dx \le e\int_0^\infty f(x)\,dx, \tag{1.2}$$

for all positive functions $f \in L^1(\mathbb{R}_+)$. Another important classical Hardy-Hilbert inequality is closely associated with (1.1):

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\,dx\right)^p dy \le \left(\frac{\pi}{\sin(\frac{\pi}{p})}\right)^p \int_0^\infty f^p(x)\,dx. \tag{1.3}$$

As we know, since the above inequalities (1.1), (1.2), and (1.3) were established, they have been developed and generalized in different directions; see [1–4]. It should be particularly

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emphasized that the above inequalities are various special cases of the following Hardy-Knopp type inequality, which was pointed out by Oguntuase *et al.* in [5] and Kaijser *et al.* in [6]:

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \le \int_0^\infty \Phi\bigl(f(x)\bigr)\frac{dx}{x}, \tag{1.4}$$

where Φ is a convex function on $(0,\infty)$. Note that the above inequality (1.4) can be proved by using Jensen's inequality and Fubini's theorem, whose idea comes from those papers [7–9].

Recently, Krulić *et al.* [10] unified the all above results in an abstract way by introducing the Hardy-Knopp type integral operator A_k in the measure space. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, respectively. Suppose that $u : \Omega_1 \to \mathbb{R}$ and $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ are two non-negative measurable functions with

$$K(x) := \int_{\Omega_2} k(x,y)\,d\mu_2(y) < \infty, \quad x \in \Omega_1. \tag{1.5}$$

If f is a real-valued measurable function defined on Ω_2, the general Hardy-Knopp type operator A_k is defined by

$$A_k f(x) := \frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\,d\mu_2(y), \quad x \in \Omega_1. \tag{1.6}$$

Then we have the following modular Hardy type inequality in [10]: for $0 < p \le q < \infty$ and any measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $f(\Omega_2) \subseteq I$ we have

$$\left(\int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}\bigl(A_k f(x)\bigr)\,d\mu_1(x)\right)^{\frac{1}{q}} \le \left(\int_{\Omega_2} v(y)\Phi\bigl(f(y)\bigr)\,d\mu_2(y)\right)^{\frac{1}{p}}, \tag{1.7}$$

where Φ is a non-negative convex function defined on a convex set $I \subseteq \mathbb{R}$ and

$$v(y) = \left[\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(y)\right]^{\frac{p}{q}}, \quad y \in \Omega_2. \tag{1.8}$$

In addition, Čižmešija *et al.* [11] obtained a class of new sufficient conditions for a weighted modular inequality involving the above operator A_k, so that they refined the classical Godunova inequality. Adeleke *et al.* [3] generalized the classical Hardy-Knopp type inequality to the class of arbitrary non-negative functions bounded from below and above with a convex function multiplied by positive real constants.

Motivated by the idea from [3, 6, 10–12], in this paper we will establish a generalized Hardy-Knopp type inequality by introducing a new integral operator $T_k^{(r)}$ as follows: for a non-negative measurable function f defined on Ω_2 and a real number $r > 0$, let

$$T_k^{(r)} f(x) := \left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f^r(y)\,d\mu_2(y)\right)^{\frac{1}{r}}, \quad x \in \Omega_1. \tag{1.9}$$

Then we will attain a strengthened Hardy-Knopp type inequality which includes all the above results, so as to give a refined version with multidimensional form as corollary.

Moreover, some new norm inequalities in Orlicz spaces are established. The assertion that the Orlicz norm $\|A_k f\|_{\Phi(u)}$ is bounded by a constant K if the N-function Φ satisfies the Δ_2-condition is proved. Additionally, under the assumption that the composition of two N-functions $\Phi_1 \circ \Phi_2^{-1}$ is also an N-function, we prove a new norm inequality $\|A_k f\|_{\Phi_2(u)} \leq C\|f\|_{\Phi_1(u)}$ which may characterize the Hardy-Knopp operators in abstract spaces. Further, we obtain the upper bound of the operator norm $\|A_k\|_*$ which implies the continuity of the Hardy-Knopp operator between two different Orlicz spaces. This conclusion is also applied to some useful examples.

The paper is organized as follows. To make the proofs as self-contained as possible, some notations of Orlicz spaces and superquadratic functions are stated in Section 2 and we also present some preliminaries. In Section 3, we prove the generalized Hardy-Knopp type inequalities as regards the operator T_k^r and derive the corresponding conclusions in a multidimensional form. The norm inequalities in Orlicz spaces are formulated and discussed in Section 4.

2 Preliminaries

Throughout this paper, all measures are assumed to be positive and all functions are assumed to be measurable. For a real parameter $p > 1$, we denote its conjugate exponent by p' and $p' = \frac{p}{p-1}$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, by a weight function we mean a non-negative measurable function on the actual interval or more general set.

Before stating and proving the related norm inequality on the integral operator A_k in Orlicz spaces, let us first describe some properties of the Δ_2-condition and superquadratic functions involved later. We know that the seminal textbook by Krasnosel'skii *et al.* [13] contains all the fundamental properties about Orlicz spaces. More recently, the textbooks by Rao and Ren [14] or by Adams and Fournier [15] were concerned with very general situations including the possible pathologies of Young's functions and the concept of the Orlicz-Sobolev space. Following the notations in [13, 16], we use the class of 'N-functions' as defining functions Φ for Orlicz spaces. This class is not as wide as the class of Young's functions used in [17]. However, N-functions are simpler to deal with and are adequate for our purpose. First, we recall the concepts of an N-function and its complement (see [13, 14] for details).

Definition 2.1 A real-valued function $\Phi(x) = \int_0^x \phi(t)\,dt$ is called an N-function if ϕ is a real-valued function defined on $[0,\infty)$ and satisfies the following conditions:

(a) $\phi(0) = 0$, $\phi(t) > 0$ whenever $t > 0$, $\lim_{t\to\infty} \phi(t) = \infty$;
(b) $\phi(t)$ is non-decreasing;
(c) $\phi(t)$ is right continuous.

Definition 2.2 Given any ϕ with the assumptions (a)-(c) above, we let $\phi^{-1}(s) := \sup\{t > 0 : \phi(t) \leq s\}$ be a right continuous inverse function of ϕ. Denoting

$$\Psi(x) = \int_0^x \phi^{-1}(s)\,ds, \tag{2.1}$$

then $\Psi(x)$ is called the complementary function of $\Phi(x)$. Note that it is an N-function itself.

Definition 2.3 An N-function Φ is said to satisfy the Δ_2-condition (globally) if there is a positive constant C such that

$$\Phi(2t) \le C\Phi(t) \quad \text{for all } t \in \mathbb{R}_+. \tag{2.2}$$

Next, we recall the concept of the Orlicz space $L_{\Phi(u)}$; see [13] for details.

Definition 2.4 Let $u(x)$ be a weight function and $\Phi(x)$ be an N-function on a σ-finite measure space (Ω, Σ, μ). The Orlicz space $L_{\Phi(u)}$ consists of all non-negative measurable functions f (module equivalent almost everywhere) with

(a) the Luxemburg norm

$$\|f\|_{\Phi(u)} := \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{f(x)}{\lambda}\right) u(x)\, d\mu(x) \le 1\right\} < +\infty.$$

(b) The Orlicz norm

$$\|f\|'_{\Phi(u)} := \sup\left\{\int_\Omega f(x)g(x)u(x)\, d\mu(x) : \int_\Omega \Psi\big(g(x)\big)u(x)\, d\mu(x) \le 1\right\} < +\infty,$$

where Ψ is the complementary function of Φ.

Now, we give some basic properties of the Orlicz space $L_{\Phi(u)}$ (*cf.* [13]), which will be used to prove our main results.

Proposition 2.5 *Let Φ be an N-function with $\Phi(0) = 0$. Then*

(a) $\Phi(\alpha x) \le \alpha\Phi(x)$ *for* $0 \le \alpha \le 1$.
(b) $\alpha\Phi(x) \le \Phi(\alpha x)$ *for* $1 \le \alpha < \infty$.
(c) *For any measurable function* $f \ge 0$, $\|f\|_{\Phi(u)} \le 1$ *if and only if* $\int_\Omega \Phi(f(x))u(x)\, d\mu(x) \le 1$.

Proposition 2.6 *Let Φ be an N-function and Ψ be the complementary of Φ. Then we have*

(a) *L_Φ is a Banach space such that the Luxemburg and Orlicz norms are equivalent; indeed,*

$$\|f\|_{\Phi(u)} \le \|f\|'_{\Phi(u)} \le 2\|f\|_{\Phi(u)}. \tag{2.3}$$

(b) *Hölder's inequality*:

$$\int_\Omega f(x)g(x)u(x)\, d\mu(x) \le \|f\|_{\Phi(u)}\|g\|'_{\Psi(u)}. \tag{2.4}$$

(c) *If an N-function satisfies the Δ_2-condition, then there are constants α and β with $1 \le \beta \le \alpha < \infty$ such that $s^\beta\Phi(t) \le \Phi(st) \le s^\alpha\Phi(t)$ when $s \ge 1$ and $t \ge 0$, and $s^\alpha\Phi(t) \le \Phi(st) \le s^\beta\Phi(t)$ when $0 \le s \le 1$ and $t \ge 0$.*

In fact, the verification of propositions above can be found in pp.23-26 in [13] and pp.59-62 in [14]. Another main tool in the proofs is to use superquadratic functions and a generalization of Jensen's inequality given by Abramovich *et al.* in [18].

Definition 2.7 A function $f : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for each $x \ge 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$f(y) - f(x) - f\bigl(|y-x|\bigr) \ge C_x(y-x), \quad \forall y \ge 0. \tag{2.5}$$

Lemma 2.8 *A function $\phi : [0,\infty] \to \mathbb{R}$ is continuously differentiable and $\phi(0) \le 0$. If ϕ' is superadditive or $\frac{\phi'(x)}{x}$ is non-decreasing, then ϕ is superquadratic.*

Lemma 2.9 (Refinement of Jensen's inequality) *Let (Ω,μ) be a probability measure space. The inequality*

$$\phi\left(\int_\Omega f(s)\,d\mu(s)\right) \le \int_\Omega \phi\bigl(f(s)\bigr)\,d\mu(s) - \int_\Omega \phi\left(\left|f(s) - \int_\Omega f(s)\,d\mu(s)\right|\right) d\mu(s) \tag{2.6}$$

holds for all probability measures μ and all non-negative μ-integrable functions f if and only if ϕ is superquadratic.

For convenience later, we also recall the following convexity concepts and Jensen's inequalities in n dimensional variables; see [19] for details.

Definition 2.10 A function $\Phi : D \to \mathbb{R}$ for which D is a convex set of $\mathbb{R}^n$ is said to be convex on D if for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^n$ and $\lambda \in [0,1]$ we have

$$\Phi\bigl(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\bigr) \le \lambda\Phi(\mathbf{x}) + (1-\lambda)\Phi(\mathbf{y}). \tag{2.7}$$

Lemma 2.11 *Let $D \subseteq \mathbb{R}^n$ be convex and open, $\phi : D \to \mathbb{R}$ be twice differentiable. Then ϕ is convex on D if and only if its Hessian matrix $(\frac{\partial^2 f}{\partial x_i\,\partial y_j}(\mathbf{x}))_{n\times n}$ is positive semi-definite for all $\mathbf{x} \in D \subset \mathbb{R}^n$.*

Lemma 2.12 (n-variable Jensen's inequality) *Let $p(x)$ be a non-negative continuous function on $I = [a,b] \subseteq \mathbb{R}$ such that $\int_I p(t)\,dt > 0$. If $f_i : I \to [m_i, M_i]$ is a real-valued continuous function for each $i \in 1,2,\ldots,n$ on $[a,b]$ and Φ is convex on $\Delta_n = \prod_{i=1}^n [m_i, M_i] \subseteq \mathbb{R}^n$, then we have*

$$\Phi\left(\frac{\int_I f_1(t)p(t)\,dt}{\int_I p(t)\,dt}, \ldots, \frac{\int_I f_n(t)p(t)\,dt}{\int_I p(t)\,dt}\right) \le \frac{\int_I \Phi(f_1(t),\ldots,f_n(t))p(t)\,dt}{\int_I p(t)\,dt}. \tag{2.8}$$

Remark 2.13 The Orlicz spaces really extend the usual L_p spaces. In fact, the function $\Phi(x) = x^p$ entering the definition of L_p is replaced by a more general convex N-function $\Phi(x)$. The Propositions 2.5 and 2.6 are crucial for the proofs of the norm inequalities in Orlicz space. The concepts of a superquadratic function and Jensen's inequality in n variables are used to prove the generalized Hardy-Knopp type inequalities.

3 Generalizations for Hardy-Knopp type operators in weighted Lebesgue spaces

Our analysis starts with a powerful sufficient condition for a new inequality related to the operator $T_k^{(r)}$. As its conclusion, a new norm inequality in weighted Lebesgue space is obtained. Now, we point out the monotonicity of $T_k^{(r)}(x)$ on r when x is fixed.

Lemma 3.1 *Fix $x \in \Omega_1$ and define $M(r) = T_k^{(r)} f(x)$ for each $r > 0$, then the function $M : \mathbb{R}^+ \to [0, \infty)$ is non-decreasing.*

Proof (I) First the case of $0 < r < 1$. Let $p = \frac{1}{r}$. By Hölder's inequality we have

$$\begin{aligned} T_k^{(r)} f(x) &= \left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) f^{\frac{1}{p}}(y)\, d\mu_2(y) \right)^p \\ &\le \frac{1}{K^p(x)} \left(\int_{\Omega_2} k(x,y) f(y)\, d\mu_2(y) \right) \left(\int_{\Omega_2} k(x,y)\, d\mu_2(y) \right)^{\frac{p}{p'}} = T_k^{(1)} f(x). \end{aligned}$$

Therefore, for any $0 < s_1 < s_2 \le 1$ let $r = \frac{s_1}{s_2} < 1$. Then, by replacing f with f^{s_2} one deduces $M(s_1) \le M(s_2) \le M(1)$.

(II) Next the case of $r > 1$. Since

$$\begin{aligned} T_k^{(r)} f(x) &= \frac{1}{K(x)} \left(\int_{\Omega_2} k(x,y) f^r(y)\, d\mu_2(y) \right)^{\frac{1}{r}} \left(\int_{\Omega_2} k(x,y)\, d\mu_2(y) \right)^{\frac{1}{r'}} \\ &\ge \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y)\, d\mu_2(y) = T_k^{(1)} f(x), \end{aligned}$$

similar to the case above, one gets $M(1) \le M(r_1) \le M(r_2)$ for any $1 \le r_1 < r_2$. This completes the proof. □

Theorem 3.2 *For $1 < \beta \le q$, $0 < p \le \beta$, and $0 < r \le 1$, let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, u be a positive weight function on Ω_1, v be a positive weight function on Ω_2 and $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative measurable function. Suppose that $K : \Omega_1 \to \mathbb{R}$ is as in* (1.5) *so that the function $x \to u(x)(\frac{k(x,y)}{K(x)})^q$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Assume Φ is a non-negative increasing convex function on an interval $I \subseteq [0, \infty)$ and there is a positive measurable function $w : \Omega_2 \to \mathbb{R}$ such that*

$$C_w(\beta) = D_w(\beta) \sup_{y \in \Omega_2} w^{\frac{1}{\beta'}}(y) \left(\int_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)} \right)^q d\mu_1(x) \right)^{\frac{1}{q}} < \infty,$$

where

$$D_w(\beta) = \left(\int_{\Omega_2} v^{-\frac{\beta'}{p}}(y) w^{-1}(y)\, d\mu_2(y) \right)^{\frac{1}{\beta'}}.$$

Then the following inequality:

$$\left\| \Phi\left(T_k^{(r)} f\right) \right\|_{L_u^q(\Omega_1, \mu_1)} \le C_w(\beta) \left\| \Phi(f) v^{\frac{\beta}{p} - 1} \right\|_{L_v^\beta(\Omega_2, \mu_2)} \tag{3.1}$$

is valid for all measurable functions $f : \Omega_2 \to I \subseteq \mathbb{R}$ and $T_k^{(r)}$ is defined by (1.9).

Proof Denote $g(y) = v(y)\Phi^p(f(y))$, then $\Phi(f(y)) = g^{\frac{1}{p}}(y) v^{-\frac{1}{p}}(y)$. First, by Hölder's inequality, we have the following estimate:

$$\begin{aligned}
&\left(\int_{\Omega_2} k(x,y)\Phi\big(f(y)\big)\,d\mu_2(y)\right)^q \\
&= \left(\int_{\Omega_2} k(x,y)g^{\frac{1}{p}}(y)v^{-\frac{1}{p}}(y)w^{\frac{1}{\beta}-1}(y)w^{1-\frac{1}{\beta}}(y)\,d\mu_2(y)\right)^q \\
&\leq \left(\int_{\Omega_2} k^\beta(x,y)g^{\frac{\beta}{p}}(y)w^{\beta-1}(y)\,d\mu_2(y)\right)^{\frac{q}{\beta}}\left(\int_{\Omega_2} v^{-\frac{\beta'}{p}}(y)w^{-1}(y)\,d\mu_2(y)\right)^{\frac{q}{\beta'}} \\
&= D_w^q(\beta)\left(\int_{\Omega_2} k^\beta(x,y)g^{\frac{\beta}{p}}(y)w^{\beta-1}(y)\,d\mu_2(y)\right)^{\frac{q}{\beta}}. \qquad (3.2)
\end{aligned}$$

Notice that $T_k^{(r)}f(x) \in I$, $x \in \Omega_1$ and inequality (3.2). Applying Jensen's inequality, Minkowski's inequality as well as monotonicity of the convex function $\Phi(x)$ on I and $M(r)$ on $(0,\infty)$, by Lemma 3.1 for any measurable function $f:\Omega_2 \to I$ we get a series of inequalities:

$$\begin{aligned}
&\left(\int_{\Omega_1} u(x)\Phi^q\big(T_k^{(r)}f(x)\big)\,d\mu_1(x)\right)^{\frac{1}{q}} \\
&\leq \left(\int_{\Omega_1} u(x)\Phi^q\big(T_k^{(1)}f(x)\big)\,d\mu_1(x)\right)^{\frac{1}{q}} \\
&\leq \left[\int_{\Omega_1} \frac{u(x)}{K^q(x)}\left(\int_{\Omega_2} k(x,y)\Phi\big(f(y)\big)\,d\mu_2(y)\right)^q d\mu_1(x)\right]^{\frac{1}{q}} \\
&\leq D_w(\beta)\left[\int_{\Omega_1} \frac{u(x)}{K^q(x)}\left(\int_{\Omega_2} k^\beta(x,y)g^{\frac{\beta}{p}}(y)w^{\beta-1}(y)\,d\mu_2(y)\right)^{\frac{q}{\beta}} d\mu_1(x)\right]^{\frac{1}{q}} \\
&\leq D_w(\beta)\left[\int_{\Omega_2} g^{\frac{\beta}{p}}(y)w^{\frac{\beta}{\beta'}}(y)\left(\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^q d\mu_1(x)\right)^{\frac{\beta}{q}} d\mu_2(y)\right]^{\frac{1}{\beta}} \\
&\leq C_w(\beta)\left(\int_{\Omega_2} v^{\frac{\beta}{p}}(y)\Phi^\beta\big(f(y)\big)\,d\mu_2(y)\right)^{\frac{1}{\beta}}. \qquad (3.3)
\end{aligned}$$

Immediately, it yields (3.1) from (3.3). □

Remark 3.3 Suppose that the weight function v is defined by

$$v(y) = \left(\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^q d\mu_1(x)\right)^{\frac{1}{q}} < \infty \quad \text{for each } y \in \Omega_2$$

in Theorem 3.2. Then we can proceed to the inequalities (3.3) in the following way:

$$\begin{aligned}
&\left(\int_{\Omega_1} u(x)\Phi^q\big(T_k^{(r)}f(x)\big)\,d\mu_1(x)\right)^{\frac{1}{q}} \\
&\leq \left(\int_{\Omega_1} u(x)\Phi^q\big(T_k^{(1)}f(x)\big)\,d\mu_1(x)\right)^{\frac{1}{q}} \\
&\leq \left(\int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi\big(f(y)\big)\,d\mu_2(y)\right)^q d\mu_1(x)\right)^{\frac{1}{q}} \\
&= \left[\int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi\big(f(y)\big)\,d\mu_2(y)\right)^q d\mu_1(x)\right]^{\frac{1}{q}}
\end{aligned}$$

$$\leq \int_{\Omega_2} \Phi\bigl(f(y)\bigr)\biggl[\int_{\Omega_1} u(x)\biggl(\frac{k(x,y)}{K(x)}\biggr)^q d\mu_1(x)\biggr]^{\frac{1}{q}} d\mu_2(y)$$
$$\leq \int_{\Omega_2} v(y)\Phi\bigl(f(y)\bigr)\,d\mu_2(y), \tag{3.4}$$

where $q \geq 1$. Let $r = 1$ and replace q with $\frac{p}{q}$ in (3.4); we also get the modular Hardy type inequality (1.7).

Remark 3.4 If K is the best possible constant in (3.1), then

$$K \leq \inf_{1<\beta\leq q, w>0} C_w(\beta).$$

Theorem 3.2 is our main result in the first part of Section 3. Enlightened by the work of Lour [20], we will derive a series of examples based on it, including several averaging operators and integral transforms in the weighted Lebesgue spaces. Before stating their descriptions we need to give some notations.

First, for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$ we denote $\frac{\mathbf{y}}{\mathbf{x}} = (\frac{y_1}{x_1}, \ldots, \frac{y_n}{x_n})$, $\mathbf{x}^{\mathbf{y}} = {x_1}^{y_1} \cdots {x_n}^{y_n}$; in particular, $\mathbf{x}^{\mathbf{1}} = \prod_{i=1}^n x_i$. Additionally, let $S = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$ be the unit sphere in $\mathbb{R}^n$ with the standard Euclidean norm $|\mathbf{x}|$ of $\mathbf{x}$, and $E \subseteq \mathbb{R}^n$ be a spherical cone with $E = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = r\mathbf{b}, 0 < r < \infty, \mathbf{b} \in A\}$ for any measurable subset A of S. Suppose that $\Omega_1 = \Omega_2 = E$ in Theorem 3.2, $d\mu_1(\mathbf{x}) = d\mathbf{x}$, and $d\mu_2(\mathbf{y}) = d\mathbf{y}$. For all non-negative functions f on E, we list the following examples with the averaging integral operators.

Example 3.5 (Averaging operator of Laplace type) Consider the case that $k(\mathbf{x},\mathbf{y}) = |\mathbf{x}|^n e^{-|\mathbf{x}||\mathbf{y}|}$, $1 < \beta = p \leq q$, $r = 1$, and $w(\mathbf{y}) \equiv 1$. Then we have $K(\mathbf{x}) = \int_E k(\mathbf{x},\mathbf{y})\,d\mathbf{y} = |A|(n-1)!$, and consequently

$$Lf(\mathbf{x}) = A_k f(\mathbf{x}) = \frac{|\mathbf{x}|^n}{|A|(n-1)!}\int_E e^{-|\mathbf{x}||\mathbf{y}|} f(\mathbf{y})\,d\mathbf{y}$$

as an averaging operator of Laplace type. According to Theorem 3.2 it follows that

$$\bigl\|\Phi(Lf)\bigr\|_{L_u^q(\Omega_1,\mu_1)} \leq C\bigl\|\Phi(f)\bigr\|_{L_v^p(\Omega_2,\mu_2)},$$

for any non-negative measurable functions $f : E \to I$, where

$$C = \biggl(\int_{\Omega_2} v^{-\frac{p'}{p}}(\mathbf{y})\,d\mu_2(\mathbf{y})\biggr)^{\frac{1}{p'}} \sup_{y\in\Omega_2}\biggl(\int_{\Omega_1} u(\mathbf{x})\biggl(\frac{k(\mathbf{x},\mathbf{y})}{K(\mathbf{x})}\biggr)^q d\mu_1(\mathbf{x})\biggr)^{\frac{1}{q}} < \infty.$$

Example 3.6 (Averaging operator of Stieltjes type) Consider the case that $k(\mathbf{x},\mathbf{y}) = (|\mathbf{x}| + |\mathbf{y}|)^{-\rho}$ ($\rho > 0$), $1 < \beta = p \leq q$, $r = 1$, and $w(\mathbf{y}) \equiv 1$. Then we have $K(\mathbf{x}) = \int_E k(\mathbf{x},\mathbf{y})\,d\mathbf{y} = |A|B(\rho - n, n)|\mathbf{x}|^{-\rho+n}$ with a Beta function $B(\cdot,\cdot)$, and consequently

$$Sf(\mathbf{x}) = A_k f(\mathbf{x}) = \frac{|\mathbf{x}|^{\rho-n}}{|A|B(\rho-n,n)}\int_E \frac{f(\mathbf{y})}{(|\mathbf{x}| + |\mathbf{y}|)^\rho}\,d\mathbf{y}$$

as an averaging operator of Stieltjes type. By Theorem 3.2 we obtain

$$\bigl\|\Phi(Sf)\bigr\|_{L_u^q(\Omega_1,\mu_1)} \leq C\bigl\|\Phi(f)\bigr\|_{L_v^p(\Omega_2,\mu_2)}$$

with any non-negative measurable functions $f : E \to I$, where

$$C = \left(\int_{\Omega_2} v^{-\frac{p'}{p}}(\mathbf{y})\, d\mu_2(\mathbf{y}) \right)^{\frac{1}{p'}} \sup_{y \in \Omega_2} \left(\int_{\Omega_1} u(\mathbf{x}) \left(\frac{k(\mathbf{x},\mathbf{y})}{K(\mathbf{x})} \right)^q d\mu_1(\mathbf{x}) \right)^{\frac{1}{q}} < \infty.$$

Example 3.7 (Averaging operator of Lambert type) Finally, consider the case that $k(\mathbf{x},\mathbf{y}) = |\mathbf{y}|(e^{|\mathbf{x}||\mathbf{y}|} - 1)^{-1}$, $1 < \beta = p \le q$, $r = 1$, and $w(\mathbf{y}) \equiv 1$. Then we attain $K(\mathbf{x}) = \int_E k(\mathbf{x},\mathbf{y})\, d\mathbf{y} = |A| l_1 |\mathbf{x}|^{-n-1}$ with $l_r = \int_0^\infty t^{r+n-1}(e^t - 1)^{-r}\, dt$, $r > 0$, and consequently

$$Ef(\mathbf{x}) = A_k f(\mathbf{x}) = \frac{|\mathbf{x}|^{n+1}}{|A| l_1} \int_E \frac{|\mathbf{y}|}{e^{|\mathbf{x}||\mathbf{y}|} - 1} f(\mathbf{y})\, d\mathbf{y}$$

as an averaging operator of Stieltjes type. In terms of Theorem 3.2 we deduce

$$\left\| \Phi(Ef) \right\|_{L^q_u(\Omega_1,\mu_1)} \le C \left\| \Phi(f) \right\|_{L^p_v(\Omega_2,\mu_2)},$$

where $f : E \to I$ is a non-negative measurable function, and

$$C = \left(\int_{\Omega_2} v^{-\frac{p'}{p}}(\mathbf{y})\, d\mu_2(\mathbf{y}) \right)^{\frac{1}{p'}} \sup_{y \in \Omega_2} \left(\int_{\Omega_1} u(\mathbf{x}) \left(\frac{k(\mathbf{x},\mathbf{y})}{K(\mathbf{x})} \right)^q d\mu_1(\mathbf{x}) \right)^{\frac{1}{q}} < \infty.$$

Indeed, the above conclusions can be reformulated with particular convex functions such as power or exponential functions, especially with the N-function $\Phi = \int_0^x \phi(t)\, dt$. This leads to multidimensional analogs of corollaries and examples by way of the previous theorems.

Now, we are in the position to consider the superquadratic function Φ. On the basis of a refinement of Jensen's inequality (2.6), we can refine the inequality (3.4) above with respect to the operator A_k. Therefore, we get the following theorem as the second part of this section.

Theorem 3.8 *Let $t \in [1,\infty)$, $(\Omega_1, \Sigma_1, \mu_1)$, and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, u be a weight function on Ω_1, and $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative measurable function. Suppose that $K : \Omega_1 \to \mathbb{R}$ is as in* (1.5), *that the function $x \to u(x)(\frac{k(x,y)}{K(x)})^t$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, and that the weight function v is defined by*

$$v(y) = \left(\int_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)} \right)^t d\mu_1(x) \right)^{\frac{1}{t}} < \infty, \quad y \in \Omega_2.$$

If Φ is a non-negative superquadratic function on an interval $I \subseteq [0,\infty)$, then we have

$$\begin{aligned} &\int_{\Omega_1} u(x) \Phi^t\big(A_k f(x)\big)\, d\mu_1(x) \\ &\quad + t \int_{\Omega_1} \frac{u(x)}{K(x)} \Phi^{t-1}\big(A_k f(x)\big) \left(\int_{\Omega_2} k(x,y) \Phi\big(\big|f(y) - A_k f(x)\big|\big)\, d\mu_2(y) \right) d\mu_1(x) \\ &\le \left(\int_{\Omega_2} v(y) \Phi\big(f(y)\big)\, d\mu_2(y) \right)^t, \end{aligned} \tag{3.5}$$

for any non-negative measurable functions $f : \Omega_2 \to I \subseteq \mathbb{R}$, *where* $A_k f$ *defined on* Ω_1 *by* (1.6).

Proof According to Lemma 2.9 it yields

$$\begin{aligned}
&\Phi\big(A_k f(x)\big) + \frac{1}{K(x)} \int_{\Omega_2} k(x,y) \Phi\big(\big|f(y) - A_k f(x)\big|\big)\, d\mu_2(y) \\
&\quad \le \frac{1}{K(x)} \int_{\Omega_2} k(x,y) \Phi\big(f(y)\big)\, d\mu_2(y).
\end{aligned}$$

As a consequence of Bernoulli's inequality, we derive

$$\begin{aligned}
&\Phi^t\big(A_k f(x)\big) + t \frac{\Phi^{t-1}(A_k f(x))}{K(x)} \int_{\Omega_2} k(x,y) \Phi\big(\big|f(y) - A_k f(x)\big|\big)\, d\mu_2(y) \\
&\quad \le \left(\Phi\big(A_k f(x)\big) + \frac{1}{K(x)} \int_{\Omega_2} k(x,y) \Phi\big(\big|f(y) - A_k f(x)\big|\big)\, d\mu_2(y) \right)^t \\
&\quad \le \left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) \Phi\big(f(y)\big)\, d\mu_2(y) \right)^t. \qquad (3.6)
\end{aligned}$$

Multiplying (3.6) by $u(x)$ and integrating it over Ω_1, by Minkowski's inequality, it follows that

$$\begin{aligned}
&\int_{\Omega_1} u(x) \Phi^t\big(A_k f(x)\big)\, d\mu_1(x) \\
&\qquad + t \int_{\Omega_1} \frac{u(x)}{K(x)} \Phi^{t-1}\big(A_k f(x)\big) \left(\int_{\Omega_2} k(x,y) \Phi\big(\big|f(y) - A_k f(x)\big|\big)\, d\mu_2(y) \right) d\mu_1(x) \\
&\quad \le \int_{\Omega_1} u(x) \left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) \Phi\big(f(y)\big)\, d\mu_2(y) \right)^t d\mu_1(x) \\
&\quad = \left\{ \left[\int_{\Omega_1} u(x) \left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) \Phi\big(f(y)\big)\, d\mu_2(y) \right)^t d\mu_1(x) \right]^{\frac{1}{t}} \right\}^t \\
&\quad \le \left\{ \int_{\Omega_2} \Phi\big(f(y)\big) \left[\int_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)} \right)^t d\mu_1(x) \right]^{\frac{1}{t}} d\mu_2(y) \right\}^t \\
&\quad = \left(\int_{\Omega_2} v(y) \Phi\big(f(y)\big)\, d\mu_2(y) \right)^t. \qquad (3.7)
\end{aligned}$$

So, the inequality (3.5) follows from (3.7). □

Remark 3.9 Observe that for $t = 1$ the inequality (3.5) may result from Theorem 5.1 in [3]. Moreover, the above conclusions can be rewritten by a special convex functions such as a power function, an exponential function, and an N-function $\Phi = \int_0^x \phi(t)\, dt$ with a continuous function ϕ such that $\frac{\phi(t)}{t}$ is non-decreasing or $\phi(t)$ is superadditive on $[0, \infty)$, since the N-function Φ is a superquadratic function by Lemma 2.8.

Let $\Omega_1 = \Omega_2 = \mathbb{R}^n_+$, $d\mu_1(\mathbf{x}) = d\mathbf{x}$, $d\mu_2(\mathbf{y}) = d\mathbf{y}$, and the kernel k in (1.5) be as the form $k(\mathbf{x}, \mathbf{y}) = h(\frac{\mathbf{y}}{\mathbf{x}})$, where $h : \mathbb{R}^n_+ \to \mathbb{R}$ is a non-negative measurable function. If $u(\mathbf{x})$ and $v(\mathbf{y})$

are substituted by $\frac{u(\mathbf{x})}{\mathbf{x}^{\mathbf{1}}}$ and $\frac{w(\mathbf{y})}{\mathbf{y}^{\mathbf{1}}}$, recall that $\mathbf{x}^{\mathbf{1}} = \prod_1^n x_i$ above, and by Theorem 3.8 we have the following corollary.

Corollary 3.10 *Let $t \in [1,\infty)$, and u be a weight function on $\mathbb{R}_+^n$ such that $H(\mathbf{x}) = \mathbf{x}^{\mathbf{1}} \int_{\mathbb{R}_+^n} h(\mathbf{y})\,d\mathbf{y}$ satisfies $0 < H(\mathbf{x}) < \infty$ for all $\mathbf{x} \in \mathbb{R}_+^n$ and that the function $\mathbf{x} \to u(\mathbf{x})(\frac{\frac{\mathbf{y}}{\mathbf{x}}}{H(\mathbf{x})})^t$ is integrable on $\mathbb{R}_+^n$ for each fixed $\mathbf{y} \in \mathbb{R}_+^n$. The weight function w is defined by*

$$w(\mathbf{y}) = \mathbf{y}^{\mathbf{1}} \left(\int_{\mathbb{R}_+^n} u(\mathbf{x}) \left(\frac{h(\frac{\mathbf{y}}{\mathbf{x}})}{H(\mathbf{x})} \right)^t \frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{1}}} \right)^{\frac{1}{t}}.$$

If Φ is a non-negative increasing superquadratic function on an interval $I \subseteq [0,\infty)$, then we have the following inequality:

$$\begin{aligned} &\int_{\mathbb{R}_+^n} u(\mathbf{x}) \Phi^t \big(A_k f(\mathbf{x}) \big) \frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{1}}} \\ &\quad + t \int_{\Omega_1} \frac{u(\mathbf{x})}{H(\mathbf{x})\mathbf{x}^{\mathbf{1}}} \Phi^{t-1} \big(A_k f(\mathbf{x}) \big) \left(\int_{\Omega_2} h\left(\frac{\mathbf{y}}{\mathbf{x}} \right) \Phi \big(\big| f(\mathbf{y}) - A_k f(\mathbf{x}) \big| \big) \, d\mu_2(\mathbf{y}) \right) d\mu_1(\mathbf{x}) \\ &\quad \le \left(\int_{\mathbb{R}_+^n} w(\mathbf{y}) \Phi \big(f(\mathbf{y}) \big) \frac{d\mathbf{y}}{\mathbf{y}^{\mathbf{1}}} \right)^t \end{aligned} \tag{3.8}$$

with any non-negative measurable functions $f : \mathbb{R}_+^n \to \mathbb{R}$ with values in I and $A_k f$ as in (1.9).

In virtue of the above corollary, one can deduce a generalization of Godunova's inequality in [21]. The following result is based on Lemma 2.12 and its proof is similar to the proof of Theorem 3.2 above.

Theorem 3.11 *Suppose that $t \in [1,\infty)$, $I_1 = [a,b] \subseteq \mathbb{R}$, $I_2 = [c,d] \subseteq \mathbb{R}$, and $p(x)$ is as in Lemma* 2.12. *Let u be a weight function on I_1, $k : I_1 \times I_2 \to \mathbb{R}$ be a non-negative measurable function, $K(x) = \int_{I_2} k(x,y) d(y) > 0$, $x \in I_1$, the function $x \to u(x)(\frac{k(x,y)}{K(x)})^t$ be integrable on I_1 for each fixed $y \in I_2$, and the weight function v be defined by*

$$v(y) = \left(\int_{I_1} u(x) \left(\frac{k(x,y)}{K(x)} \right)^t dx \right)^{\frac{1}{t}} < \infty, \quad y \in I_2.$$

If Φ is a non-negative convex function on $\Delta_n = \prod_{i=1}^n [m_i, M_i] \subseteq \mathbb{R}^n$, then we have

$$\begin{aligned} &\int_{I_1} u(x) \Phi^t \left(\frac{\int_{I_2} f_1(y) k(x,y)\, dy}{\int_{I_2} k(x,y)\, dy}, \dots, \frac{\int_{I_2} f_n(y) k(x,y)\, dy}{\int_{I_2} k(x,y)\, dy} \right) dx \\ &\quad \le \left(\int_{I_2} v(y) \Phi \big(f_1(y), \dots, f_n(y) \big)\, dy \right)^t \end{aligned} \tag{3.9}$$

with any non-negative measurable functions $f_i : I_2 \to [m_i, M_i]$. Further, inequality (3.9) *holds in the reversed direction if Φ is a non-negative concave function and $t \in (0,1]$.*

Proof By using Lemma 2.12, Minkowski's inequality, and Fubini's theorem, we observe that

$$\begin{aligned}
&\int_{I_1} u(x)\Phi^t\left(\frac{\int_{I_2} f_1(y)k(x,y)\,dy}{\int_{I_2} k(x,y)\,dy},\ldots,\frac{\int_{I_2} f_n(y)k(x,y)\,dy}{\int_{I_2} k(x,y)\,dy}\right)dx \\
&\quad\le \int_{I_1} u(x)\left(\frac{1}{K(x)}\int_{I_2} k(x,y)\Phi\bigl(f_1(y),\ldots,f_n(y)\bigr)\,dy\right)^t dx \\
&\quad= \left\{\left[\int_{I_1} u(x)\left(\frac{1}{K(x)}\int_{I_2} k(x,y)\Phi\bigl(f_1(y),\ldots,f_n(y)\bigr)\,dy\right)^t dx\right]^{\frac{1}{t}}\right\}^t \\
&\quad\le \left\{\int_{I_2} \Phi\bigl(f_1(y),\ldots,f_n(y)\bigr)\left[\int_{I_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^t dx\right]^{\frac{1}{t}} dy\right\}^t \\
&\quad\le \left(\int_{I_2} v(y)\Phi\bigl(f_1(y),\ldots,f_n(y)\bigr)\,dy\right)^t.
\end{aligned} \tag{3.10}$$

Note that if Φ is a non-negative concave function and $t \in (0,1]$, it is completed by reversing the inequality sign in (3.10). □

In the process of proving Theorem 3.9, assume that $\Phi : H \to \mathbb{R}$ is a twice differentiable function on an open convex set H which contains the compact set $\Delta_n = \prod_{i=1}^n [m_i, M_i]$ such that its Hessian matrix $(\frac{\partial^2 f}{\partial x_i \partial y_j}(\mathbf{x}))_{n\times n}$ is positive semi-definite for all $\mathbf{x} \in \mathbb{R}^n$. Then according to Lemma 2.11 Φ is a convex function on H and inequality (3.9) holds for any non-negative measurable functions $f_i : I_2 \to [m_i, M_i]$. As a special case, the following corollary is derived.

Corollary 3.12 *Under the same conditions as Theorem* 3.11, *let* $\Phi(x_1, x_2, \ldots, x_n) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ *be a quadratic form in n independent variables, with associated symmetric matrix* $\mathbf{A}$ *which is positive semi-definite. Then we have the following inequality*:

$$\int_{I_1} u(x)\bigl(\mathbf{x}^T\mathbf{A}\mathbf{x}\bigr)^t dx \le \left(\int_{I_2} v(y)\mathbf{y}^T\mathbf{A}\mathbf{y}\,dy\right)^t, \tag{3.11}$$

for any non-negative measurable functions $f_i : I_2 \to \mathbb{R}$, *where*

$$\mathbf{x} = \left(\frac{\int_{I_2} f_1(y)k(x,y)\,dy}{\int_{I_2} k(x,y)\,dy},\ldots,\frac{\int_{I_2} f_n(y)k(x,y)\,dy}{\int_{I_2} k(x,y)\,dy}\right) \quad \text{and} \quad \mathbf{y} = \bigl(f_1(y),\ldots,f_n(y)\bigr).$$

Reversely, the inequality (3.11) *holds in the reversed direction if* $\mathbf{A}$ *is a negative semi-definite and* $t \in (0,1]$.

4 The norm inequalities in Orlicz spaces

In this section, by combining some basic properties of Orlicz spaces and the arguments of the preceding sections, we establish some new norm inequalities which may characterize the Hardy-Knopp type operators in abstract spaces.

Theorem 4.1 *Suppose that* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$, $u(x)$, $k(x,y)$, *and* $K(x)$ *are as in Theorem* 3.2. *Let the function* $x \to \frac{u(x)k(x,y)}{K(x)}$ *be an integrable function on* Ω_1 *for each fixed*

$y \in \Omega_2$, and the weight function ω be defined by

$$\omega(y) = \int_{\Omega_1} \frac{u(x)k(x,y)}{K(x)}\, d\mu_1(x) < \infty, \quad y \in \Omega_2.$$

If Φ is a N-function satisfying the Δ_2-condition, then there are constants α and β with $1 \le \beta \le \alpha < \infty$ such that the following norm inequality holds:

$$\|A_k f\|_{\Phi(u)} \le M_f^{\frac{1}{\alpha}}, \quad \text{if } M_f < 1; \quad \text{or} \quad \|A_k f\|_{\Phi(u)} \le M_f^{\frac{1}{\beta}}, \quad \text{if } M_f \ge 1;$$

for any non-negative measurable functions $f : \Omega_2 \to [0,\infty)$, where

$$M_f = \int_{\Omega_2} \omega(y)\Phi\bigl(f(y)\bigr)\, d\mu_2(y)$$

and $A_k f$ is defined on Ω_1 by (1.6).

Moreover, if $\phi(t)$ is a continuous function such that $\frac{\phi(t)}{t}$ is non-decreasing or $\phi(t)$ is superadditive on $[0,\infty)$, then the following refined normal inequality holds:

$$\|A_k f\|_{\Phi(u)} \le N_f^{\frac{1}{\alpha}}, \quad \text{if } N_f < 1; \quad \text{or} \quad \|A_k f\|_{\Phi(u)} \le N_f^{\frac{1}{\beta}}, \quad \text{if } N_f \ge 1;$$

for all non-negative measurable functions $f : \Omega_2 \to [0,\infty)$, where

$$N_f = \int_{\Omega_2} \omega(y)\Phi\bigl(f(y)\bigr)\, d\mu_2(y) - \int_{\Omega_1 \times \Omega_2} u(x)\frac{k(x,y)}{K(x)}\Phi\bigl(\bigl|f(y) - A_k f(x)\bigr|\bigr)\, d\mu_1(x) \times d\mu_2(y)$$

and $A_k f$ is defined on Ω_1 by (1.6).

Proof By Proposition 2.6, there are constants α and β with $1 \le \beta \le \alpha < \infty$ such that $s^\beta \Phi(t) \le \Phi(st) \le s^\alpha \Phi(t)$ when $s \ge 1$ and $t \ge 0$, and $s^\alpha \Phi(t) \le \Phi(st) \le s^\beta \Phi(t)$ when $0 \le s \le 1$ and $t \ge 0$.

Case I. If $\lambda \le 1$, let $s = \frac{1}{\lambda}$. Then it follows that

$$\begin{aligned}
\int_{\Omega_1} u(x)\Phi\left(\frac{A_k f(x)}{\lambda}\right) d\mu_1(x) &\le \left(\frac{1}{\lambda}\right)^\alpha \int_{\Omega_1} u(x)\Phi\bigl(A_k f(x)\bigr)\, d\mu_1(x) \\
&\le \left(\frac{1}{\lambda}\right)^\alpha \int_{\Omega_1} \left(\int_{\Omega_2} \frac{u(x)k(x,y)}{K(x)}\Phi\bigl(f(y)\bigr)\, d\mu_2(y)\right) d\mu_1(x) \\
&= \left(\frac{1}{\lambda}\right)^\alpha \int_{\Omega_2} \Phi\bigl(f(y)\bigr)\left[\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right) d\mu_1(x)\right] d\mu_2(y) \\
&= \left(\frac{1}{\lambda}\right)^\alpha \int_{\Omega_2} \Phi\bigl(f(y)\bigr)\omega(y)\, d\mu_2(y) = \left(\frac{1}{\lambda}\right)^\alpha M_f. \qquad (4.1)
\end{aligned}$$

Case II. If $\lambda > 1$, let $s = \frac{1}{\lambda}$. Similarly to (4.1), we can get

$$\int_{\Omega_1} u(x)\Phi\left(\frac{A_k f(x)}{\lambda}\right) d\mu_1(x) \le \left(\frac{1}{\lambda}\right)^\beta \int_{\Omega_2} \Phi\bigl(f(y)\bigr)\omega(y)\, d\mu_2(y) = \left(\frac{1}{\lambda}\right)^\beta M_f. \qquad (4.2)$$

First, we consider the case of $M_f < 1$, by letting $\lambda = 1$ in (4.1) then we have $\|A_k f\|_{\Phi(u)} \leq 1$. Hence, it is sufficient to consider the case that $\lambda \leq 1$. If $\lambda \geq M_f^{\frac{1}{\alpha}}$ then

$$\int_{\Omega_1} u(x)\Phi\left(\frac{A_k f(x)}{\lambda}\right) d\mu_1(x) \leq 1$$

due to inequality (4.1). Consequently, $\|A_k f\|_{\Phi(u)} \leq M_f^{\frac{1}{\alpha}} < 1$ by the definition of the Luxemburg norm. Now, we are in a position to consider another case of $M_f \geq 1$. If $\|A_k f\|_{\Phi(u)} \geq 1$, we have the norm inequality $\|A_k f\|_{\Phi(u)} \leq M_f^{\frac{1}{\beta}}$ due to inequality (4.2). Therefore, $\|A_k f\|_{\Phi(u)} \leq \max(1, M_f^{\frac{1}{\beta}}) = M_f^{\frac{1}{\beta}}$, which completes the first part of this theorem.

Finally, let $\phi(t)$ be a continuous function such that $\frac{\phi(t)}{t}$ is a non-decreasing or $\phi(t)$ is superadditive on $[0, \infty)$. Then $\Phi(x)$ is superquadratic by Lemma 2.8, and consequently we employ the refinement Jensen's inequality as follows (*cf.* Lemma 2.6):

$$\begin{aligned}
&\Phi\bigl(A_k f(x)\bigr) + \frac{1}{K(x)} \int_{\Omega_2} k(x,y)\Phi\bigl(\bigl|f(y) - A_k f(x)\bigr|\bigr)\, d\mu_2(y) \\
&\quad \leq \frac{1}{K(x)} \int_{\Omega_2} k(x,y)\Phi\bigl(f(y)\bigr)\, d\mu_2(y),
\end{aligned}$$

which completes the rest of the proof by way of repeating the above discussion (4.1) and (4.2). □

Theorem 4.2 *Let (Ω, Σ, μ) be a measure space with positive σ-finite measure, u be a weight function on Ω, and $k : \Omega \times \Omega \to \mathbb{R}$ be a non-negative measurable function. Let the weight function v be defined by*

$$v(x) = \int_{\Omega} u(t)\left(\frac{k(t,x)}{K(t)}\right) d\mu(t) < \infty, \quad x \in \Omega.$$

Suppose that Φ_1 and Φ_2 are N-functions, where Φ_2 satisfies the Δ_2-condition, so that $\Phi_1 \circ \Phi_2^{-1}$ is an N-function. The complementary function of $\Phi_1 \circ \Phi_2^{-1}$ is denoted by Ψ. If $\|\frac{v}{u}\|_{\Psi(u)} < \infty$, then there exists a constant C such that the following norm inequality:

$$\|A_k f\|_{\Phi_2(u)} \leq C\|f\|_{\Phi_1(u)}, \tag{4.3}$$

holds for any non-negative measurable function f. Moreover, if there exists a constant C such that the inequality

$$\|f\|_{\Phi_2(u)} \leq C\|f\|_{\Phi_1(v)} \tag{4.4}$$

holds for any non-negative function f, then we have $\|\frac{u}{v}\|_{\Psi(v)} < \infty$.

Proof Without loss of generality, to prove the first statement we may assume that $\|f\|_{\Phi_1(u)} = 1$, which implies $\|\Phi_2(f)\|_{\Phi_1 \circ \Phi_2^{-1}(u)} \leq 1$. By Hölder's inequality in Orlicz spaces

(2.6) it yields

$$\begin{aligned}\int_\Omega u(t)\Phi_2\bigl(A_k f(t)\bigr)\,d\mu(t) &\le \int_\Omega \frac{u(t)}{K(t)}\biggl(\int_\Omega k(t,x)\Phi_2\bigl(f(x)\bigr)\,d\mu(x)\biggr)\,d\mu(t)\\ &= \int_\Omega \Phi_2\bigl(f(x)\bigr)\biggl(\int_\Omega \frac{u(t)}{K(t)}k(t,x)\,d\mu(t)\biggr)\,d\mu(x)\\ &= \int_\Omega \Phi_2\bigl(f(x)\bigr)v(x)\,d\mu(x)\\ &\le 2\bigl\|\Phi_2(f)\bigr\|_{\Phi_1\circ\Phi_2^{-1}(u)}\biggl\|\frac{v}{u}\biggr\|_{\Psi(u)} \le 2\biggl\|\frac{v}{u}\biggr\|_{\Psi(u)}.\end{aligned}$$

Now we take $C = \max(1, 2\|\frac{v}{u}\|_{\Psi(u)})$, then one deduces $\int_\Omega u(t)\Phi_2(\frac{A_k f(t)}{C})\,d\mu(t) \le 1$. This proves (4.3).

Conversely, since the Luxemburg norm is dominated by the Orlicz norm itself, it suffices to show that

$$\biggl\|\frac{u}{v}\biggr\|_{\Psi(v)} \le \sup\biggl\{\int_\Omega u(x)f(x)\,d\mu(x) : \int_\Omega \Phi_1\circ\Phi_2^{-1}\bigl(f(x)\bigr)v(x)\,d\mu(x) \le 1\biggr\} < \infty.$$

Let $\int_\Omega \Phi_1\circ\Phi_2^{-1}(f(x))v(x)\,d\mu(x) \le 1$, then $\|\Phi_2^{-1}(f)\|_{\Phi_1(v)} \le 1$. By (4.4) we have

$$\bigl\|\Phi_2^{-1}(f)\bigr\|_{\Phi_2(u)} \le C.$$

According to the definition of the Luxemburg norm this shows that

$$\int_\Omega \Phi_2\biggl(\frac{\Phi_2^{-1}(f(x))}{C}\biggr)u(x)\,d\mu(x) \le 1.$$

Note that Φ_2 satisfies the Δ_2-condition, Proposition 2.5, and hence the inequality $\int_\Omega f(x)u(x)\,d\mu(x) \le C_1$ holds for some constant C_1. Then we have $\|\frac{u}{v}\|_{\Psi(v)} \le C_1 < \infty$. □

Corollary 4.3 *Suppose that* (Ω, Σ, μ), u, k, *and* v *are as in Theorem* 4.2. *Let* Φ_1 *and* Φ_2 *be* N*-functions such that* Φ_2 *satisfies the* Δ_2*-condition and* $\Phi_1\circ\Phi_2^{-1}$ *is an* N*-function. Denote by* Ψ *the complementary function of* $\Phi_1\circ\Phi_2^{-1}$. *If the inequality* $\|\frac{v}{u}\|_{\Psi(u)} < \infty$ *holds, then the linear operator* $A_k : L_{\Phi_1(u)} \to L_{\Phi_2(u)}$ *is continuous and we have the following estimate*:

$$\|A_k\|_* \le \max\biggl(1, 2\biggl\|\frac{v}{u}\biggr\|_{\Psi(u)}\biggr).$$

Here $\|\cdot\|_*$ *is the operator norm.*

Proof According to the proof of inequality (4.3), we conclude that $\frac{\|A_k f\|_{\Phi_2(u)}}{\|f\|_{\Phi_1(u)}} \le \max(1, 2\|\frac{v}{u}\|_{\Psi(u)})$ holds for any non-negative function $f(x)$. Then we have $\|A_k\|_* \le \max(1, 2\|\frac{v}{u}\|_{\Psi(u)})$ and hence A_k is continuous. □

Let $\Phi_1(x) = \frac{1}{p}x^p$ and $\Phi_2(x) = \frac{1}{q}x^q$ in Theorem 4.2, where $1 < q < p < \infty$. It is clear that Φ_1, Φ_2 are N-functions satisfying the Δ_2-condition, and $\Phi_1\circ\Phi_2^{-1} = \int_0^x q^{\frac{p}{q}-1}t^{\frac{p}{q}-1}\,dt$ is also an N-function. Furthermore, the complementary N-function of Φ is calculated by $\Psi(x) = \frac{p-q}{pq}x^{\frac{p}{p-q}}$. Then we have the following conclusion.

Corollary 4.4 *Let (Ω, Σ, μ) be a measure space with positive σ-finite measure, $u(x)$, $k(x,y)$, $v(x)$ be as in Theorem 4.2. Suppose that $\Phi_1(x) = \frac{1}{p}x^p$ and $\Phi_2(x) = \frac{1}{q}x^q$ where $1 < q < p < \infty$. Then there exists a constant C such that the norm inequality holds*:

$$\|A_k f\|_{\Phi_2(u)} \leq C\|f\|_{\Phi_1(u)},$$

for any non-negative function f and $\|\frac{v}{u}\|_{\Psi(u)} < \infty$ with $\Psi(x) = \frac{p-q}{pq}x^{\frac{p}{p-q}}$. Moreover, if there exists a constant C such that the following inequality is valid:

$$\|f\|_{\Phi_2(u)} \leq C\|f\|_{\Phi_1(v)},$$

for any non-negative function f, then $\|\frac{u}{v}\|_{\Psi(v)} < \infty$ holds.

Proposition 4.5 *Suppose that $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are σ-finite measure spaces and that T is a linear operator which maps any non-negative measurable functions on Ω_2 to some non-negative measurable functions on Ω_1. Let $\Phi(x)$ be an N-function, then*

$$\int_{\Omega_1} \Phi\bigl(Tf(x)\bigr)\,d\mu_1(x) \leq \int_{\Omega_2} \Phi\bigl(Cf(y)\bigr)\,d\mu_2(y),$$

if and only if

$$\|Tf\|_{\Phi(\epsilon)} \leq C\|f\|_{\Phi(\epsilon)}$$

holds for all $\epsilon > 0$ with C independent of ϵ (see Bloom's paper in [22]).

Corollary 4.6 *Assume that the assumptions in Proposition* 4.5 *are satisfied. Let $T_k^{(r)}$ be the linear operator defined in* (1.9) *and $\Phi(x)$ be an N-function, then*

$$\int_{\Omega_1} \Phi\bigl(T_k^{(r)}f(x)\bigr)\,d\mu_1(x) \leq \int_{\Omega_2} \Phi\bigl(Cf(y)\bigr)\,d\mu_2(y),$$

if and only if

$$\bigl\|T_k^{(r)}f\bigr\|_{\Phi(\epsilon)} \leq C\|f\|_{\Phi(\epsilon)}$$

holds for all $\epsilon > 0$ with C independent of ϵ.

It is clear that $\Phi(x) = \int_0^x \phi(t)\,dt$ in which $\phi(t) = e^t - 1$ is an N-function. Then, by applying Proposition 4.5 to the linear operator $T_k^{(r)}$ and replacing $f(x)$ by $\ln f(x)$, we obtain the following important example.

Example 4.7 Assume that the assumptions in Proposition 4.5 are satisfied and that $f(x)$ is a measurable function such that $f(x) \geq 1$ for all $x \in \Omega_2$. Then the following inequality:

$$\int_{\Omega_1} \exp\bigl\{T_k^{(r)} \ln f(x)\bigr\}\,d\mu_1(x) + I \leq \int_{\Omega_2} f^c(y)\,d\mu_2(y),$$

where $I = \int_{\Omega_2} (\ln f(y) + 1)\, d\mu_2(y) - \int_{\Omega_1} (T_k^{(r)} \ln f(x) + 1)\, d\mu_1(x)$ holds, if and only if

$$\left\| T_k^{(r)} f \right\|_{\Phi(\epsilon)} \le C\|f\|_{\Phi(\epsilon)} \quad \text{with } \Phi(x) = e^x - x - 1$$

holds for all $\epsilon > 0$ with C independent of ϵ.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally, and they all read and approved the final manuscript.

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