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Monotone variational inequalities, generalized equilibrium problems and fixed point methods

Shenghua Wang*

*Correspondence:
hdwangsh@yeah.net
School of Mathematics and Physics,
North China Electric Power
University, Baoding, 071003, China

Abstract

In this paper, we study monotone variational inequalities and generalized equilibrium problems. Weak convergence theorems are established based on a fixed point method in the framework of Hilbert spaces.

Keywords: fixed point; generalized equilibrium problem; monotone operator; nonexpansive mapping

1 Introduction and preliminaries

It is well known that many nonlinear problems can be reduced to the search for solutions of monotone variational inequalities. Fixed point methods are often used for finding and approximating such solutions. In this paper, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a nonempty closed and convex subset of H . Let $S : C \rightarrow C$ be a mapping. In this paper, we use $F(S)$ to stand for the set of fixed points. Recall that S is said to be nonexpansive iff $\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C$. S is said to be κ -strictly pseudocontractive iff there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \|x - y - Sx + Sy\|^2, \quad \forall x, y \in C.$$

The class of strict pseudocontractions was introduced by Browder and Petryshyn [1]. It is clear that the class of κ -strict pseudocontractions includes the class of nonexpansive mappings as a special case.

Let $A : C \rightarrow H$ be a mapping. Recall that A is said to be monotone iff $\langle Ax - Ay, x - y \rangle \geq 0, \forall x, y \in C$. A is said to be α -inverse strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is clear that α -inverse strongly monotone is monotone and Lipschitz continuous.

Recall that the classical variational inequality, denoted by $VI(C, A)$, is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \quad (1.1)$$

One can see that variational inequality (1.1) is equivalent to a fixed point problem. The element $x \in C$ is a solution of variational inequality (1.1) iff $x \in C$ satisfies the fixed point equation $x = \text{Proj}_C(x - \lambda Ax)$, where $\lambda > 0$ is a constant. This alternative equivalent formulation plays a significant role in the studies of variational inequalities and related optimization problems. If A is α -inverse-strongly monotone and $\lambda \in (0, 2\alpha]$, then the mapping $\text{Proj}_C(I - \lambda A)$ is nonexpansive; see [2] and the references therein.

A set-valued mapping $T : H \rightarrow 2^H$ is said to be monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle > 0$. A monotone mapping $T : H \rightarrow 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. The class of monotone operators is one of the most important classes of operators. Within the past several decades, many authors have been devoted to the studies on the existence and convergence of different schemes for zero points for maximal monotone operators; see [3–15] and the references therein.

Let \mathbb{R} denote the set of real numbers, and let F be a bifunction of $C \times C$ into \mathbb{R} . Recall that the following generalized equilibrium problem is to find an x such that

$$F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \quad (1.2)$$

In this paper, the solution set of the equilibrium problem is denoted by $GEP(F, A)$, i.e.,

$$GEP(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x \rangle \geq 0, \forall y \in C\}.$$

To study equilibrium problem (1.2), we may assume that F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

If $F \equiv 0$, the generalized equilibrium problem is reduced to variational inequality (1.1). If $A \equiv 0$, the generalized equilibrium problem is reduced to the following equilibrium problem: find an x such that

$$F(x, y) \geq 0, \quad \forall y \in C. \quad (1.3)$$

In this paper, the solution set of the equilibrium problem is denoted by $EP(F)$, i.e.,

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}.$$

Equilibrium problems, which were introduced by Blum and Oettli [16], have intensively been studied. It has been shown that equilibrium problems cover fixed point problems,

variational inequality problems, inclusion problems, saddle point problems, complementarity problem, minimization problem, and Nash equilibrium problem. Equilibrium problem has emerged as an effective and powerful tool for studying a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization. Recently, many authors studied the numerical solution of equilibriums (1.2) and (1.3) based on iterative methods. Convergence theorems are established in a different framework of spaces; see [17–24] and the references therein.

In this paper, an iterative algorithm is investigated for monotone variational inequalities and generalized equilibrium problems. Weak convergence of the algorithm is obtained in the framework of Hilbert spaces.

Recall that a space is said to satisfy Opial's condition [25] if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, where \rightharpoonup denotes the weak convergence, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Indeed, the above inequality is equivalent to the following:

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|.$$

Recall that a function is lower semi-continuous at some point x_0 if $\liminf_{x \rightarrow x_0} f(x) \geq f(x_0)$. It is known that the norm is lower semi-continuous. In order to prove our main results, we also need the following lemmas.

Lemma 1.1 [16, 17] *Let C be a nonempty closed convex subset of H , and let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $r > 0$ and $x \in H$. Then the following hold:

- (a) T_r is single-valued;
- (b) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

- (c) $F(T_r) = EP(F)$;
- (d) $EP(F)$ is closed and convex.

Lemma 1.2 [4] *Let A be a monotone mapping of C into H and $N_C v$ be the normal cone to C at $v \in C$, i.e.,*

$$N_C v = \{ w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C \}$$

and define a mapping T on C by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Av, u - v \rangle \geq 0$ for all $u \in C$.

Lemma 1.3 [26] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n \rightarrow \infty} a_n$ exists.

Lemma 1.4 [27] *Let $\{a_n\}_{n=1}^{\infty}$ be real numbers in $[0, 1]$ such that $\sum_{n=1}^{\infty} a_n = 1$. Then we have the following:*

$$\left\| \sum_{i=1}^{\infty} a_i x_i \right\|^2 \leq \sum_{i=1}^{\infty} a_i \|x_i\|^2,$$

for any given bounded sequence $\{x_n\}_{n=1}^{\infty}$ in H .

2 Main results

Theorem 2.1 *Let C be a nonempty closed and convex subset of H , and let $A : C \rightarrow H$ be an L -Lipschitz continuous and monotone mapping. Let F_m be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4), and let $B_m : C \rightarrow C$ be a κ_m -inverse strongly monotone mapping for each $m \geq 1$. Assume that $\mathcal{F} := \bigcap_{m=1}^{\infty} \text{GEP}(F_m, B_m) \cap \text{VI}(C, A)$ is not empty. Let $\{\alpha_n\}$, $\{\delta_{n,m}\}$ be real number sequences in $[0, 1]$. Let $\{\lambda_n\}$ and $\{r_{n,m}\}$ be positive real number sequences. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in C, \\ z_{n,m} \in C \text{ such that } F_m(z_{n,m}, z) + \langle B_m x_n, z - z_{n,m} \rangle + \frac{1}{r_{n,m}} \langle z - z_{n,m}, z_{n,m} - x_n \rangle \geq 0, \\ y_n = \text{Proj}_C(\sum_{m=1}^{\infty} \delta_{n,m} z_{n,m} - \lambda_n A \sum_{m=1}^{\infty} \delta_{n,m} z_{n,m}), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \text{Proj}_C(\sum_{m=1}^{\infty} \delta_{n,m} z_{n,m} - \lambda_n A y_n), \quad n \geq 1. \end{cases}$$

Assume that $\{\alpha_n\}$, $\{\delta_{n,m}\}$, $\{\lambda_n\}$, and $\{r_{n,m}\}$ satisfy the following restrictions:

- (a) $0 \leq \alpha_n \leq a' < 1$;
- (b) $\sum_{m=1}^{\infty} \delta_{n,m} = 1$, and $0 < \delta \leq \delta_{n,m} \leq 1$;
- (c) $\liminf_{n \rightarrow \infty} r_{n,m} > 0$, and $\lambda \leq \lambda_n \leq \lambda'$, where $\lambda, \lambda' \in (0, 1/L)$.

Then the sequence $\{x_n\}$ weakly converges to some point $\bar{x} \in \mathcal{F}$.

Proof We first show that the sequence $\{x_n\}$ is bounded. Set $u_n = \text{Proj}_C(v_n - \lambda_n A y_n)$, where $v_n = \sum_{m=1}^{\infty} \delta_{n,m} z_{n,m}$. Letting $p \in \mathcal{F}$, we see that

$$\begin{aligned} \|u_n - p\|^2 &\leq \|v_n - p\|^2 - \|v_n - u_n\|^2 + 2\lambda_n \langle A y_n, p - u_n \rangle \\ &= \|v_n - p\|^2 - \|v_n - u_n\|^2 + 2\lambda_n (\langle A y_n - A p, p - y_n \rangle + \langle A p, p - y_n \rangle) \end{aligned}$$

$$\begin{aligned}
 &+ \langle Ay_n, y_n - u_n \rangle \\
 &\leq \|v_n - p\|^2 - \|v_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\langle v_n - \lambda_n Ay_n - y_n, u_n - y_n \rangle.
 \end{aligned}$$

Since A is L -Lipschitz continuous and $y_n = \text{Proj}_C(v_n - \lambda_n Av_n)$, we find that

$$\begin{aligned}
 &\langle v_n - \lambda_n Ay_n - y_n, u_n - y_n \rangle \\
 &= \langle v_n - \lambda_n Av_n - y_n, u_n - y_n \rangle + \langle \lambda_n Av_n - \lambda_n Ay_n, u_n - y_n \rangle \\
 &\leq \lambda_n L \|v_n - y_n\| \|u_n - y_n\|.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \|u_n - p\|^2 &\leq \|v_n - p\|^2 - \|v_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\lambda_n L \|v_n - y_n\| \|u_n - y_n\| \\
 &\leq \|v_n - p\|^2 + (\lambda_n^2 L^2 - 1) \|v_n - y_n\|^2.
 \end{aligned} \tag{2.1}$$

On the other hand, we obtain from the restriction (c) that

$$\begin{aligned}
 \|v_n - p\|^2 &\leq \sum_{m=1}^{\infty} \delta_{n,m} \|z_{n,m} - p\|^2 \\
 &\leq \sum_{m=1}^{\infty} \delta_{n,m} \|T_{r_{n,m}}(I - r_{n,m} B_m)x_n - p\|^2 \\
 &\leq \|x_n - p\|^2.
 \end{aligned} \tag{2.2}$$

Substituting (2.2) into (2.1), we obtain that $\|u_n - p\|^2 \leq \|x_n - p\|^2 + (\lambda_n^2 L^2 - 1) \|v_n - y_n\|^2$. This in turn implies from the restriction (c) that

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|u_n - p\|^2 \\
 &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) (\|x_n - p\|^2 + (\lambda_n^2 L^2 - 1) \|v_n - y_n\|^2) \\
 &\leq \|x_n - p\|^2 + (1 - \alpha_n) (\lambda_n^2 L^2 - 1) \|v_n - y_n\|^2 \\
 &\leq \|x_n - p\|^2.
 \end{aligned} \tag{2.3}$$

It follows from Lemma 1.3 that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This in turn shows that $\{x_n\}$ is bounded. From (2.3), we also have that

$$(1 - \alpha_n) (1 - \lambda_n^2 L^2) \|v_n - y_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

This implies from the restrictions (a) and (c) that

$$\lim_{n \rightarrow \infty} \|v_n - y_n\| = 0. \tag{2.4}$$

Since $\|y_n - u_n\| \leq \lambda L \|v_n - y_n\|$, we find from (2.4) $\lim_{n \rightarrow \infty} \|y_n - u_n\| = 0$. Hence, we have

$$\lim_{n \rightarrow \infty} \|v_n - u_n\| = 0. \tag{2.5}$$

Notice that

$$\begin{aligned} \|z_{n,m} - p\|^2 &= \|T_{r_{n,m}}(I - r_{n,m}B_m)x_n - T_{r_{n,m}}(I - r_{n,m}B_m)p\|^2 \\ &\leq \langle (I - r_{n,m}B_m)x_n - (I - r_{n,m}B_m)p, z_{n,m} - p \rangle \\ &= \frac{1}{2} (\|(I - r_{n,m}B_m)x_n - (I - r_{n,m}B_m)p\|^2 + \|z_{n,m} - p\|^2 \\ &\quad - \|(I - r_{n,m}B_m)x_n - (I - r_{n,m}B_m)p - (z_{n,m} - p)\|^2) \\ &\leq \frac{1}{2} (\|x_n - p\|^2 + \|z_{n,m} - p\|^2 - \|(x_n - z_{n,m}) - r_{n,m}(B_mx_n - B_m p)\|^2) \\ &= \frac{1}{2} (\|x_n - p\|^2 + \|z_{n,m} - p\|^2 - \|x_n - z_{n,m}\|^2 \\ &\quad + 2r_{n,m} \langle x_n - z_{n,m}, B_mx_n - B_m p \rangle - r_{n,m}^2 \|B_mx_n - B_m p\|^2). \end{aligned}$$

This implies that

$$\|z_{n,m} - p\|^2 \leq \|x_n - p\|^2 - \|x_n - z_{n,m}\|^2 + 2r_{n,m} \|x_n - z_{n,m}\| \|B_mx_n - B_m p\|.$$

It follows from Lemma 1.4 that

$$\begin{aligned} \|v_n - p\|^2 &\leq \sum_{m=1}^{\infty} \delta_{n,m} \|z_{n,m} - p\|^2 \\ &\leq \sum_{m=1}^{\infty} \delta_{n,m} (\|x_n - p\|^2 - \|z_{n,m} - x_n\|^2 + 2r_{n,m} \|x_n - z_{n,m}\| \|B_mx_n - B_m p\|) \\ &= \|x_n - p\|^2 - \sum_{m=1}^{\infty} \delta_{n,m} \|z_{n,m} - x_n\|^2 + 2 \sum_{m=1}^{\infty} r_{n,m} \|x_n - z_{n,m}\| \|B_mx_n - B_m p\|. \end{aligned}$$

By use of (2.3), we find that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|u_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|v_n - p\|^2 \\ &\leq \|x_n - p\|^2 - (1 - \alpha_n) \sum_{m=1}^{\infty} \delta_{n,m} \|z_{n,m} - x_n\|^2 \\ &\quad + 2(1 - \alpha_n) \sum_{m=1}^{\infty} r_{n,m} \|x_n - z_{n,m}\| \|B_mx_n - B_m p\|. \end{aligned}$$

This implies that

$$\begin{aligned} (1 - \alpha_n) \sum_{m=1}^{\infty} \delta_{n,m} \|z_{n,m} - x_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad + 2(1 - \alpha_n) \sum_{m=1}^{\infty} r_{n,m} \|x_n - z_{n,m}\| \|B_mx_n - B_m p\|. \end{aligned}$$

Since $T_{r_{n,m}}$ is firmly nonexpansive, we find from the convexity of $\|\cdot\|^2$ that $\lim_{n \rightarrow \infty} \|B_m x_n - B_m p\| = 0$. It follows that

$$\lim_{n \rightarrow \infty} \|z_{n,m} - x_n\| = 0.$$

Since $\{x_n\}$ is bounded, we may assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ . It follows that $\{z_{n_i,m}\}$ converges weakly to ξ for each $m \geq 1$. Next, we show that $\xi \in GEP(F_m, B_m)$. Since $z_{n,m} = T_{r_{n,m}}(I - r_{n,m}B_m)x_n$, we have

$$F_m(z_{n,m}, z) + \langle B_m x_n, z - z_{n,m} \rangle + \frac{1}{r_{n,m}} \langle z - z_{n,m}, z_{n,m} - x_n \rangle \geq 0, \quad \forall z \in C.$$

From assumption (A2), we see that

$$\langle B_m x_n, z - z_{n,m} \rangle + \frac{1}{r_{n,m}} \langle z - z_{n,m}, z_{n,m} - x_n \rangle \geq F_m(z, z_{n,m}), \quad \forall z \in C.$$

Replacing n by n_i , we arrive at

$$\langle B_m x_{n_i}, z - z_{n_i,m} \rangle + \left\langle z - z_{n_i,m}, \frac{z_{n_i,m} - x_{n_i}}{r_{n_i,m}} \right\rangle \geq F_m(z, z_{n_i,m}), \quad \forall z \in C.$$

For t_m with $0 < t_m \leq 1$ and $\rho_m \in C$, let $\rho_m = t_m \rho + (1 - t_m)\xi$. Since $\rho \in C$ and $\xi \in C$, we have $\rho_m \in C$. It follows that

$$\begin{aligned} \langle \rho_m - z_{n_i,m}, B_m \rho_m \rangle &\geq \langle \rho_m - z_{n_i,m}, B_m \rho_m \rangle - \langle B_m x_{n_i}, \rho_m - z_{n_i,m} \rangle \\ &\quad - \left\langle \rho_m - z_{n_i,m}, \frac{z_{n_i,m} - x_{n_i}}{r_{n_i,m}} \right\rangle + F_m(\rho_m, z_{n_i,m}) \\ &= \langle \rho_m - z_{n_i,m}, B_m \rho_m - B_m z_{n_i,m} \rangle + \langle \rho_m - z_{n_i,m}, B_m z_{n_i,m} - B_m x_{n_i} \rangle \\ &\quad - \left\langle \rho_m - z_{n_i,m}, \frac{z_{n_i,m} - x_{n_i}}{r_{n_i,m}} \right\rangle + F_m(\rho_m, z_{n_i,m}). \end{aligned} \tag{2.6}$$

It follows from (A4) and (2.6) that

$$\langle \rho_m - \xi, B_m \rho_m \rangle \geq F_m(\rho_m, \xi) \quad \text{as } i \rightarrow \infty.$$

From (A1) and (A4), we see

$$\begin{aligned} 0 &= F_m(\rho_m, \rho_m) \leq t_m F_m(\rho_m, \rho) + (1 - t_m) F_m(\rho_m, \xi) \\ &\leq t_m F_m(\rho_m, \rho) + (1 - t_m) \langle \rho_m - \xi, B_m \rho_m \rangle \\ &= t_m F_m(\rho_m, \rho) + (1 - t_m) t_m \langle \rho - \xi, B_m \rho_m \rangle, \end{aligned}$$

which yields that

$$F_m(\rho_m, \rho) + (1 - t_m) \langle \rho - \xi, B_m \rho_m \rangle \geq 0.$$

Letting $t \rightarrow 0$ in the above inequality, we arrive at

$$F_m(\xi, \rho) + \langle \rho - \xi, B_m \xi \rangle \geq 0.$$

This completes the proof that $\xi \in \bigcap_{m=1}^{\infty} GEP(F_m, B_m)$. Next, we show that $\xi \in VI(C, A)$. In fact, let T be the maximal monotone mapping defined by

$$Tx = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

For any given $(x, y) \in G(T)$, we have $y - Ax \in N_C x$. So, we have $\langle x - m, y - Ax \rangle \geq 0$ for all $m \in C$. On the other hand, we have $u_n = \text{Proj}_C(v_n - \lambda_n A y_n)$. We obtain that $\langle v_n - \lambda_n A y_n - u_n, u_n - x \rangle \geq 0$ and hence $\langle x - u_n, \frac{u_n - v_n}{\lambda_n} + A y_n \rangle \geq 0$. In view of the monotonicity of A , we see that

$$\begin{aligned} \langle x - u_{n_i}, y \rangle &\geq \langle x - u_{n_i}, Ax \rangle \\ &\geq \langle x - u_{n_i}, Ax \rangle - \left\langle x - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\lambda_{n_i}} + A y_{n_i} \right\rangle \\ &= \langle x - u_{n_i}, Ax - A u_{n_i} \rangle + \langle x - u_{n_i}, A u_{n_i} - A y_{n_i} \rangle - \left\langle x - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\geq \langle x - u_{n_i}, A u_{n_i} - A y_{n_i} \rangle - \left\langle x - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\lambda_{n_i}} \right\rangle. \end{aligned} \tag{2.7}$$

On the other hand, we see that $\|v_n - x_n\| \leq \sum_{m=1}^{\infty} \delta_{n,m} \|z_{n,m} - x_n\|$. It follows that

$$\lim_{n \rightarrow \infty} \|v_n - x_n\| = 0. \tag{2.8}$$

Notice that

$$\|u_n - x_n\| \leq \|u_n - v_n\| + \|v_n - x_n\|.$$

Combining (2.5) with (2.8), one finds

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0.$$

This in turn implies that $u_{n_i} \rightharpoonup \xi$. It follows that $\langle x - \xi, y \rangle \geq 0$. Notice that T is maximal monotone and hence $0 \in T\xi$. This shows from Lemma 1.2 that $\xi \in VI(C, A)$. This completes the proof that $\xi \in \mathcal{F}$.

Finally, we show that the whole sequence $\{x_n\}$ weakly converges to ξ . Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ converging weakly to ξ' , where $\xi' \neq \xi$. In the same way, we can show that $\xi' \in \mathcal{F}$. Since the space H enjoys Opial's condition [25], we, therefore, obtain that

$$\begin{aligned} d &= \liminf_{i \rightarrow \infty} \|x_{n_i} - \xi\| < \liminf_{i \rightarrow \infty} \|x_{n_i} - \xi'\| \\ &= \liminf_{j \rightarrow \infty} \|x_j - \xi'\| < \liminf_{j \rightarrow \infty} \|x_j - \xi\| = d. \end{aligned}$$

This is a contradiction. Hence $\xi = \xi'$. This completes the proof. \square

If $B_m \equiv 0$, we have the following result on equilibrium problem (1.3).

Corollary 2.2 *Let C be a nonempty closed and convex subset of H , and let $A : C \rightarrow H$ be an L -Lipschitz continuous and monotone mapping. Let F_m be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4) for each $m \geq 1$. Assume that $\mathcal{F} := \bigcap_{m=1}^{\infty} EP(F_m) \cap VI(C, A)$ is not empty. Let $\{\alpha_n\}$, $\{\delta_{n,m}\}$ be real number sequences in $[0, 1]$. Let $\{\lambda_n\}$ and $\{r_{n,m}\}$ be positive real number sequences. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in C, \\ z_{n,m} \in C \text{ such that } F_m(z_{n,m}, z) + \frac{1}{r_{n,m}} \langle z - z_{n,m}, z_{n,m} - x_n \rangle \geq 0, \\ y_n = \text{Proj}_C(\sum_{m=1}^{\infty} \delta_{n,m} z_{n,m} - \lambda_n A \sum_{m=1}^{\infty} \delta_{n,m} z_{n,m}), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \text{Proj}_C(\sum_{m=1}^{\infty} \delta_{n,m} z_{n,m} - \lambda_n A y_n), \quad n \geq 1. \end{cases}$$

Assume that $\{\alpha_n\}$, $\{\delta_{n,m}\}$, $\{\lambda_n\}$, and $\{r_{n,m}\}$ satisfy the following restrictions:

- (a) $0 < a \leq \alpha_n \leq b < 1$;
- (b) $\sum_{m=1}^{\infty} \delta_{n,m} = 1$, and $0 < c \leq \delta_{n,m} \leq 1$;
- (c) $\liminf_{n \rightarrow \infty} r_{n,m} > 0$, and $d \leq \lambda_n \leq e$, where $d, e \in (0, 1/L)$.

Then the sequence $\{x_n\}$ weakly converges to some point $\bar{x} \in \mathcal{F}$.

Corollary 2.3 *Let C be a nonempty closed and convex subset of H , and let $A : H \rightarrow H$ be an L -Lipschitz continuous and monotone mapping. Assume that $A^{-1}(0)$ is not empty. Let $\{\alpha_n\}$ be a real number sequence in $[0, 1]$, and let $\{\lambda_n\}$ be a positive real number sequence. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(x_n - \lambda_n A(x_n - \lambda_n A x_n)), \quad n \geq 1.$$

Assume that $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the following restrictions:

- (a) $0 < a \leq \alpha_n \leq b < 1$;
- (b) $c \leq \lambda_n \leq d$, where $c, d \in (0, 1/L)$.

Then the sequence $\{x_n\}$ weakly converges to some point $\bar{x} \in A^{-1}(0)$.

Proof Put $F_m(x, y) \equiv 0$ for all $x, y \in C$, and $r_{n,m} \equiv 1$. Notice that $A^{-1}(0) = VI(H, A)$, and $P_H = I$, we easily find from Corollary 2.2 the desired conclusion. \square

Letting $F_m = 0$, $B_m = 0$, and $r_{n,m} = 1$ in Γ , we have the extragradient-type algorithm (see Table 1).

Corollary 2.4 *Let C be a nonempty closed and convex subset of H , and let $A : C \rightarrow H$ be an L -Lipschitz continuous and monotone mapping. Assume that $VI(C, A)$ is not empty.*

Table 1 The framework of the EA

EA:	Extragradient-type algorithm (for variational inequality (1.1))
Step 0:	Choose $x_1 \in C$, $\lambda_1 \in [\lambda, \lambda']$, $\alpha_1 \in [0, 1]$. Set $n := 1$.
Step 1:	Given $x_n \in C$. Choose $\lambda_n \in [\lambda, \lambda']$, $\alpha_n \in [0, 1]$ and compute $x_{n+1} \in C$ as $y_n = \text{Proj}_C(x_n - \lambda_n A x_n),$ $x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \text{Proj}_C(x_n - \lambda_n A y_n).$
	Update $n := n + 1$ and go to Step 1.

Table 2 The framework of the MA

MA:	Mann-type algorithm (for equilibrium problem (1.3))
Step 0:	Choose $x_1 \in C, r_1 \in (0, +\infty), \alpha_1 \in [0, 1]$. Set $n := 1$.
Step 1:	Given $x_n \in C$. Choose $r_n \in (0, +\infty), \alpha_n \in [0, 1]$ and compute $x_{n+1} \in C$ as $F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0,$ $x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n.$
	Update $n := n + 1$ and go to Step 1.

Let $\{\alpha_n\}$ be a real number sequence in $[0, 1]$. Let $\{\lambda_n\}$ and $\{r_{n,m}\}$ be positive real number sequences. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in C, \\ y_n = \text{Proj}_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \text{Proj}_C(x_n - \lambda_n Ay_n), \quad n \geq 1. \end{cases}$$

Assume that $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the following restrictions:

- (a) $0 \leq \alpha_n \leq a' < 1$;
- (b) $\lambda \leq \lambda_n \leq \lambda',$ where $\lambda, \lambda' \in (0, 1/L)$.

Then the sequence $\{x_n\}$ weakly converges to some point $\bar{x} \in VI(C, A)$.

Letting $F_m = F, B_m = B$ and $A = 0$ in Γ , we have the Mann-type algorithm (see Table 2).

Corollary 2.5 Let C be a nonempty closed and convex subset of H . Let F be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4) such that $EP(F)$ is not empty. Let $\{\alpha_n\}$ be a real number sequence in $[0, 1]$. Let $\{r_n\}$ be a positive real number sequence. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in C, \\ z_n \in C \text{ such that } F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n, \quad n \geq 1. \end{cases}$$

Assume that $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions:

- (a) $0 < a \leq \alpha_n \leq a' < 1$;
- (b) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then the sequence $\{x_n\}$ weakly converges to some point $\bar{x} \in EP(F)$.

3 Conclusion

Generalized equilibrium problems have emerged as an effective and powerful tool for studying a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, and network. The problems include variational inequality problems, saddle point problems, complementarity problems, Nash equilibrium problem in noncooperative games, and others as special cases. In this paper, we study monotone variational inequalities and generalized equilibrium problems based on a fixed point method. The extragradient algorithm is valid for a family of infinite bifunctions. We analyze the convergence of the algorithm and establish a weak convergence theorem for solutions under mild restrictions imposed on the control sequences.

Competing interests

The author declares that he has no competing interests.

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