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Existence of *p*-adic quasi Gibbs measure for countable state Potts model on the Cayley tree

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Abstract

In the present article, we provide a new construction of measure, called *p*-adic quasi Gibbs measure, for countable state of *p*-adic Potts model on the Cayley tree. Such a construction depends on a parameter $\mathfrak{p}$ and wights. In particular case, i.e., if $\mathfrak{p} = \exp_p$, the defined measure coincides with *p*-adic Gibbs measure. In this article, under some condition on weights we establish the existence of *p*-adic quasi Gibbs measures associated with the model. Note that this condition does not depend on values of the prime *p*. An analogues fact is not valid when the number of spins is finite.
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1. Introduction

Interest in the physics of non-Archimedean quantum models [1-5] is based on the idea that the structure of space-time for very short distances might conveniently be described in terms of non-Archimedean numbers. One of the ways to describe this violation of the Archimedean axiom, is the using *p*-adic analysis. Numerous applications of this analysis to mathematical physics have been proposed in [4,6-8]. It is known [8] that a number of *p*-adic models in physics cannot be described using ordinary Kolmogorov's probability theory. New probability models–*p*-adic probability models were investigated in [9-11]. In [12-14], the theory of stochastic processes with values in *p*-adic and more general non-Archimedean fields having probability distributions with non-Archimedean values has been developed. This gives a possibility to develop the theory of statistical mechanics in the context of the *p*-adic theory, since it lies on the base of the theory of probability and stochastic processes. The central problems of that theory [15] is the study of infinite-volume Gibbs measures corresponding to a given Hamiltonian, which also includes a description of the set of Gibbs measures. In most cases such analysis depend on a specific properties of Hamiltonian, and complete description is often a difficult problem. This problem, in particular, relates to a phase transitions problem.

In [16,17], a notion of ultrametric Markovianity, which describes independence of contributions to random field from different ultrametric balls has been introduced, and showed that Gaussian random fields on general ultrametric spaces (which were related with hierarchical trees), which were defined as a solution of pseudodifferential stochastic

equation, satisfy the Markovianity. Some applications of the results to replica matrices, related to general ultrametric spaces have been investigated in [18] (see also [19]).

In [20,21], we have developed of p-adic probability theory approaches to study $q + 1$-state nearest-neighbor p-adic Potts model on Cayley tree (see [22]). We constructed infinite volume p-adic Gibbs measures for the mentioned model, and moreover, established the existence of a phase transition (here the phase transition means the existence of two distinct p-adic Gibbs measures for the given model). Further, in [23] we have introduced a new kind of p-adic measures for the mentioned model, called *p-adic quasi Gibbs measure*. For such a model, we investigated a phase transition phenomena from the associated dynamical system point of view. Namely, we established that if q is not divisible by p, then there occurs the quasi phase transition. Note that such kind of measures present more natural concrete examples of p-adic Markov processes (see [12], for definitions). In [24-26], we investigated a countable state p-adic Potts model on the Cayley tree, [a] and provided a construction of p-adic Gibbs measures which depends on weights λ. Moreover, the uniqueness of such measures under certain conditions to the weight was proved.

In this article, we continue our investigations on countable state p-adic Potts model. Namely, we are going to provide a new construction of measure, called p-adic quasi Gibbs measure, for the mentioned model on the Cayley tree. Such a construction depends on a parameter $\mathfrak{p}$ and wights $\{\lambda(i)\}_{i \in \mathbb{N}}$. In particular case, i.e., if $\mathfrak{p} = \exp_p$, the defined measure coincides with p-adic Gibbs measure (see [24,25]). In this article, under some condition on weights we establish the existence of p-adic quasi Gibbs measures associated with the model. Note that this condition does not depend on values of the prime p. An analogues fact is not valid when the number of spins is finite. Moreover, our result extends the previous proved ones in [25,26]. Note that in comparison to a real case, in a p-adic setting, à priori the existence of such kind of measures for the model is not known, since there is not much information on topological properties of the set of all p-adic measures defined even on compact spaces. However, in the real case, there is the so called the Dobrushin's Theorem [27,28] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians. Note that when states are finite, then the corresponding p-adic Potts models on the same trees have been studied in [23].

2. Preliminaries

Fix a prime number p, which throughout the article will be a fixed >3, and let $\mathbb{Q}_p$ denotes the field of p-adic filed, formed by completing $\mathbb{Q}$ with respect to the unique absolute value satisfying $|p| = 1/p$. The absolute value $|\cdot|$, is non-Archimedean, meaning that it satisfies the ultrametric triangle inequality $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.

Given $a \in \mathbb{Q}_p$ and $r > 0$ put

$$B(a, r) = \left\{ x \in \mathbb{Q}_p : |x - a|_p < r \right\}, \quad S(a, r) = \left\{ x \in \mathbb{Q}_p : |x - a|_p = r \right\}.$$

The *p-adic exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Lemma 2.1. [24]*If* $|a_i|_p \leq 1$, $|b_i|_p \leq 1$, $i = 1,..., n$, *then*

$$\left| \prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i \right|_p \leq \max_{i \leq i \leq n}\{|a_i - b_i|_p\}$$

Note the basics of p-adic analysis, p-adic mathematical physics are explained in [5,29].

Let $(X, \mathscr{B})$ be a measurable space, where $\mathscr{B}$ is an algebra of subsets X. A function μ: $\mathscr{B} \rightarrow \mathbb{Q}_p$ is said to be a *p-adic measure* if for any $A_1,..., A_n \subset \mathscr{B}$ such that $A_i \cap A_j = \emptyset$ ($i \neq j$) the equality holds

$$\mu \left(\bigcup_{j=1}^{n} A_j \right) = \sum_{j=1}^{n} \mu \left(A_j \right).$$

A p-adic measure is called a *probability measure* if $\mu(X) = 1$. For more detail information about p-adic measures we refer to [9,10,30].

Let $\Gamma^k = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (whose each vertex has exactly $k + 1$ edges, except for the root x^0, which has k edges). Here V is the set of vertices and L is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = < x, y >$ if there exists an edge connecting them. A collection of the pairs $< x, x_1 >,..., < x_{d-1}, y >$ is called a *path* from the point x to the point y. The distance $d(x, y)$, $x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y. Let us set

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \bigcup_{m=1}^{n} W_m, \quad L_n = \{l = < x, y > \in L | x, y \in V_n\}.$$

The set of *direct successors* of x is defined by

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n. \tag{2.1}$$

Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k + 1$.

Now we are going to introduce a coordinate structure in Γ^k. Every vertex x (except for x_0) of Γ^k has coordinates $(i_1,..., i_n)$, here $i_m \in \{1,..., k\}$, $1 \leq m \leq n$ and for the vertex x_0 we put. $\emptyset$. Namely, the symbol $\emptyset$ constitutes level 0 and the sites $(i_1,..., i_n)$ form level n of the lattice. In this notation for $x \in \Gamma^k$, $x = (i_1,..., i_n)$ we have

$$S(x) = \{(x, i) : 1 \leq i \leq k\},$$

here (x, i) means that $(i_1,..., i_n, i)$.

Let us define on Γ^k a binary operation $o: \Gamma^k \times \Gamma^k \to \Gamma^k$ as follows: for any two elements $x = (i_1,..., i_n)$ and $y = (j_1,..., j_m)$ put

$$x \circ y = (i_1, \ldots, i_n) \circ (j_1, \ldots, j_m) = (i_1, \ldots, i_n, j_1, \ldots, j_m) \tag{2.2}$$

and

$$x \circ x^0 = x^0 \circ x = (i_1, \ldots, i_n) \circ (0) = (i_1, \ldots, i_n). \tag{2.3}$$

By means of the defined operation Γ^k becomes a noncommutative semigroup with a unit.

Using this semigroup structure one defines translations $\tau_g : \Gamma^k \to \Gamma^k$, $g \in \Gamma^k$ by

$$\tau_g(x) = g \circ x. \tag{2.4}$$

It is clear that $\tau_{(0)} = id$.

Similarly, by means of τ_g one can define translations $\tilde{\tau}_g : L \to L$ of L. Namely,

$$\tilde{\tau}_g(<x, \, y \, >) = < \tau_g(x), \, \tau_g(y) \, > .$$

Let $G \subset \Gamma^k$ be a sub-semigroup of Γ^k and $h : \Gamma^k \to \mathbb{R}$ be a function defined on Γ^k. We say that h is *G-periodic* if $h(\tau_g(x)) = h(x)$ for all $g \in G$ and $x \in \Gamma^2$. Any Γ^k-periodic function is called *translation invariant*. Put

$$G_k = \left\{ x \in \Gamma^k : d\left(x, x^0\right) \equiv 0 \, (\mathrm{mod}\, k) \right\}, \quad k \geq 2 \tag{2.5}$$

One can check that G_k is a sub-semigroup with a unit.

3. The *p*-adic Potts model

We consider the *p*-adic Potts model where spin takes values in the set $\Phi = \{0, 1, 2, \cdots, \}$ and is assigned to the vertices of the tree $\Gamma^k = (V, L)$. A configuration σ on V is then defined as a function $x \in V \to \sigma(x) \in \Phi$; in a similar manner one defines configurations σ_n and $\sigma^{(n)}$ on V_n and W_n, respectively. The set of all configurations on V (resp. V_n, W_n) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{w_n} = \Phi^{W_n}$). One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Using this, for given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\sigma^{(n)} \in \Omega_{W_n}$ we define their concatenations by

$$\sigma_{n-1} \vee \sigma^{(n)} = \{\{\sigma_n(x), \, x \in V_{n-1}\}, \, \{\sigma^{(n)}(y), \, y \in W_n\}\}.$$

It is clear that $\sigma_{n-1} \vee \sigma^{(n)} \in \Omega_{V_n}$.

The Hamiltonian $H_n : \Omega_{V_n} \to \mathbb{Q}_p$ of the *p*-adic countable state Potts model has the form

$$H_n(\sigma) = \sum_{<x, y> \in L_n} N_{xy} \delta_{\sigma(x), \sigma(y)}, \quad n \in \mathbb{N}, \tag{3.1}$$

here $\sigma \in \Omega_n$, $N_{xy} \in \mathbb{N}$ $(x, y \in V)$ and δ is the Kronecker symbol.

We are going to construct *p*-adic quasi Gibbs measures for the model.

Let us consider a function $\mathbf{h} : x \in V \to \mathbf{h}_x = \{h_{i,x}\}_{i \in \Phi} \in \mathbb{Q}_p^\Phi$ of $x \in V \setminus \{x^{(0)}\}$. Here $\mathbb{Q}_p^\Phi = \{\{x_i\}_{i \in \Phi} : x_i \in \mathbb{Q}_p\}$.

Fix a sequence $\lambda \in \mathbb{Q}_p^\Phi$ such that

$$|\lambda(n)|_p \to 0 \text{ as } n \to \infty. \tag{3.2}$$

Such kind of sequences are called *weights*.

Given $n = 1, 2, \ldots$ define a *p*-adic probability measure $\mu_{\mathbf{h}}^{(n)}$ on Ω_{V_n} by

$$\mu_{\mathbf{h}}^{(n)}(\sigma_n) = \frac{1}{Z_n^{(\mathbf{h})}} \mathfrak{p}^{H_n(\sigma_n)} \prod_{x \in W_n} h_{\sigma_n(x), x} \prod_{x \in V_n} \lambda(\sigma_n(x)) \tag{3.3}$$

where $\mathfrak{p}$ is any p-adic number such that $|\mathfrak{p}|_p \leq 1$. In particular, it could be $\mathfrak{p} = p$. Here, as before, $\sigma_n : x \in V_n \to \sigma_n(x) \in \Phi$ and $Z_n^{(\mathbf{h})}$ is the corresponding partition function:

$$Z_n^{(\mathbf{h})} = \sum_{\sigma \in \Omega_{V_n}} \mathfrak{p}^{H_n(\sigma)} \prod_{x \in W_n} h_{\sigma(x),x} \prod_{x \in V_n} \lambda(\sigma(x)). \tag{3.4}$$

One of the central results of the theory of probability concerns a construction of an infinite volume distribution with given finite-dimensional distributions, which is called *Kolmogorov's Theorem* [31]. Therefore, in this article we are interested in the same question but in a p-adic context. More exactly, we want to define a p-adic probability measure μ on Ω which is compatible with defined ones $\mu_{\mathbf{h}}^{(n)}$, i.e.,

$$\mu(\sigma \in \Omega : \sigma|_{V_n} = \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n), \quad \text{for all } \sigma_n \in \Omega_{V_n}, \ n \in \mathbb{N}. \tag{3.5}$$

In general, à priori the existence such a kind of measure μ is not known, since there is not much information on topological properties, such as compactness, of the set of all p-adic measures defined even on compact spaces.[b] Note that certain properties of the set of p-adic measures has been studied in [32,33], but those properties are not enough to prove the existence of the limiting measure. Therefore, at a moment, we can only use the p-adic Kolmogorov extension Theorem (see [12,34]) which based on so called *compatibility condition* for the measures $\mu_{\mathbf{h}}^{(n)}$, $n \geq 1$, i.e.,

$$\sum_{\omega \in \Omega_{W_n}} \mu_{\mathbf{h}}^{(n)}(\sigma_{n-1} \vee \omega) = \mu_{\mathbf{h}}^{(n-1)}(\sigma_{n-1}), \tag{3.6}$$

for any $\sigma_{n-1} \in \Omega_{V_{n-1}}$. This condition according to the theorem implies the existence of a unique p-adic measure μ defined on Ω with a required condition (3.5). Note that more general theory of p-adic measures has been developed in [35].

So, if for some function $\mathbf{h}$ the measures $\mu_{\mathbf{h}}^{(n)}$ satisfy the compatibility condition, then there is a unique p-adic probability measure, which we denote by $\mu_{\mathbf{h}}$, since it depends on $\mathbf{h}$. Such a measure $\mu_{\mathbf{h}}$ is said to be *a p-adic quasi Gibbs measure* corresponding to the p-adic Potts model. By $Q\mathcal{G}(H)$ we denote the set of all p-adic quasi Gibbs measures associated with functions $\mathbf{h} = \{\mathbf{h}_x \in V\}$. If there are at least two distinct p-adic quasi Gibbs measures $\mu, \nu \in Q\mathcal{G}(H)$ such that μ is bounded and ν is unbounded, then we say that *a phase transition* occurs. By another words, one can find two different functions $\mathbf{s}$ and $\mathbf{h}$ defined on $\mathbb{N}$ such that there exist the corresponding measures $\mu_{\mathbf{s}}$ and $\mu_{\mathbf{h}}$, for which one is bounded, another one is unbounded. If there are two different functions $\mathbf{s}$ and $\mathbf{h}$ defined on $\mathbb{N}$ such that there exist the corresponding measures $\mu_{\mathbf{s}}$, $\mu_{\mathbf{h}}$, and they are bounded, then we say there is a *quasi phase transition*.

Remark 3.1. Note that in [24] we considered the following sequence of p-adic measures defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma_n) = \frac{1}{\tilde{Z}_n^{(\mathbf{h})}} \exp_p \left\{ H_n(\sigma_n) + \sum_{x \in W_n} h_{\sigma_n(x),x} \right\} \prod_{x \in V_n} \lambda(\sigma_n(x)) \tag{3.7}$$

here as usual $\tilde{Z}_n^{(h)}$ is the corresponding normalizing factor. A limiting p-adic measures generated by (3.7) was called *p-adic Gibbs measure*. Such kind of measures have been studied in [24].

Now let us find for what kind of functions $\mathbf{h} = \{\mathbf{h}_x : x \in V\}$ the measures defined by (3.3) would satisfy the compatibility condition (3.6). The following statement describes conditions on $\mathbf{h}$ guaranteeing the compatibility condition for the measures $\mu_{\mathbf{h}}^{(n)}$.

Theorem 3.1. *The measures* $\mu_{\mathbf{h}}^{(n)}$, $n = 1, 2, \ldots$ *satisfy the compatibility condition* (3.6) *if and only if for any* $x \in V \setminus \{x^{(0)}\}$ *the following equation holds:*

$$\hat{h}_{i,x} = \frac{\lambda(i)}{\lambda(0)} \prod_{y \in S(x)} F_i\left(\hat{h}_y; \theta_{xy}\right), \quad i \in \mathbb{N} \tag{3.8}$$

here and below $\theta_{xy} = \mathfrak{p}^{N_{xy}}$, *a vector* $\hat{\mathbf{h}} = \{\hat{h}_i\}_{i \in \mathbb{N}} \in \mathbb{Q}_p^{\mathbb{N}}$ *is defined by a vector* $\mathbf{h} = \{h_i\}_{i \in \Phi}$ *as follows*

$$\hat{h}_i = \frac{h_i \lambda(i)}{h_0 \lambda(0)}, \quad i \in \mathbb{N} \tag{3.9}$$

and a mapping $F : \mathbb{Q}_p^{\mathbb{N}} \to \mathbb{Q}_p^{\mathbb{N}}$ *is* $F(\mathbf{x}; \theta) = \{F_i(\mathbf{x}; \theta)\}_{i \in \mathbb{N}}$ *with*

$$F_i(\mathbf{x}; \theta) = \frac{(\theta - 1)x_i + \sum\limits_{j=1}^{\infty} x_j + 1}{\sum\limits_{j=1}^{\infty} x_j + \theta}, \quad \mathbf{x} = \{x_i\}_{i \in \mathbb{N}}, \quad i \in \mathbb{N}. \tag{3.10}$$

The proof consists of checking the condition (3.6) for the measures (3.3) (cp. [24,36]).

Remark 3.2. Note that thanks to non-Archimedeanity of the norm $|\cdot|_p$ the series $\sum\limits_{k=1}^{\infty} x_k$ converges iff the sequence $\{x_n\}$ converges to 0 (see [29]). Therefore, in what follows we should assume that $\sum\limits_{k=1}^{\infty} \hat{h}_{k,x}$ converges, otherwise the Equation (3.8) has no sense.

Remark 3.3. In what follows, without loss of generality, we may assume that $h_0 = 1$ and $\lambda(0) = 1$. Otherwise, in (3.3) we multiply and divide the expression on the right hand side by $\prod_{x \in W_n} h_{0,x} \prod_{x \in V_n} \lambda(0)$, and after replacing h_i by h_i/h_0 and $\lambda(k)$ by $\lambda(k)/\lambda(0)$, respectively, we get the desired equality.

Observation 3.1. Here we are going to underline a connection between q-state Potts model with the defined one. First recall that q-state Potts model is defined by the same Hamiltonian (3.1), but with the state space $\Phi_q = \{0, 1, \ldots, q - 1\}$. Similarly, one can define p-adic quasi Gibbs measures for the q-state Potts model, here instead of the weight $\{\lambda(i)\}$ we will take a collection $\{\lambda(0), \lambda(1), \ldots, \lambda(q - 1)\} \subset \mathbb{Q}_p$.

Now consider countable Potts model with a weight $\{\lambda(i)\}$ such that

$$\lambda(k) = 0 \text{ for all } k \geq q, \, q > 1. \tag{3.11}$$

In this case the corresponding p-adic quasi Gibbs measures will coincide with those of q-state Potts model. Indeed, let

$$\Omega^c = \left\{ \sigma \in \Omega : \exists j \in \mathbb{Z}_+ : \sigma(j) \geq q \right\}$$
$$\Omega^{(q)} = \left\{ \sigma \in \Omega : \sigma(j) \leq q - 1 \ \forall j \in \mathbb{Z}_+ \right\}$$

It is clear that $\Omega^{(q)} = \Phi_q^{\mathbb{Z}_+}$. Let μ be a p-adic quasi Gibbs measure of the countable Potts model with the given weight corresponding to a solution $\mathbf{h}_n = \{h_{i,\,n}\}_{i \in \Phi}$ of (3.8). From the definition (3.3) we see that the restriction of μ to Ω^c is zero, i.e., $\mu \lceil \Omega^c = 0$. Moreover, from (3.8) and (3.11) we conclude that $h_{i,\,n} = 0$ for all $i \geq q$. This means that vectors $\mathbf{h}_n^{(q)} = \{h_{i,n}\}_{i \in \Phi_q}$ will be a solution of (3.8) corresponding to the q-state Potts model. Therefore, the restriction of μ to $\Omega^{(q)}$ coincides with p-adic quasi Gibbs measure of q-state Potts model with a weight $\{\lambda(0), \lambda(1),..., \lambda(q-1)\}$ corresponding to a solution of $\mathbf{h}_n^{(q)}$.

Hence, we conclude that under the condition (3.11) all p-adic quasi Gibbs measures corresponding to countable Potts model are described by those measures of q-state Potts model.

4. Existence of p-adic quasi Gibbs measure

In this section, we are going to provide a condition for the existence of the p-adic quasi Gibbs measure.

Taking into account Remark 3.2, we consider the following space

$$c_0 = \{\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{Q}_p : |x_n|_p \to 0 \text{ as } n \to \infty\}$$

with a norm $\|x\| = \max_n |x_n|_p$. Define

$$\mathbf{B}_r = \{\{x_n\}_{n \in \mathbb{N}} \in c_0 : \|x\| \leq r\}, \ r > 0.$$

It is clear that $\mathbf{B}_r$ is a closed set.

Lemma 4.1. *Let $|\theta|_p \geq 1$. Then*

$$\left| F_i(\mathbf{x}, \theta) - F_i(\mathbf{y}, \theta) \right|_p \leq \|\mathbf{x} - \mathbf{y}\| \quad \text{for every } \mathbf{x}, \mathbf{y} \in \mathbf{B}_\delta. \tag{4.1}$$

Proof. Let $\mathbf{x} = (x_i)$, $\mathbf{y} = (y_i) \in \mathbf{B}_\delta$, then it is clear that $|x_i|_p \leq \delta$ and $|y_i|_p \leq \delta$ for all $i \in \mathbb{N}$.

Therefore, the strong triangle inequality implies that

$$\left| \sum_{j=1}^{\infty} x_j + \theta \right|_p = |\theta|_p, \quad \left| \sum_{j=1}^{\infty} y_j + \theta \right|_p = |\theta|_p. \tag{4.2}$$

From (3.10) one gets

$$
\begin{aligned}
\left| F_i(\mathbf{x}, \theta) - F_i(\mathbf{y}, \theta) \right|_p &= \frac{|\theta - 1|_p}{|\theta|_p^2} \left| \left(\sum_{j=1}^{\infty} x_j + \theta \right)(x_i - y_i) + (1 - x_i) \sum_{j=1}^{\infty} (x_j - y_j) \right|_p \\
&\leq \frac{1}{|\theta|_p} \max \left\{ |\theta|_p |x_i - y_i|_p, \ |1 - x_i|_p \left| \sum_{j=1}^{\infty} (x_j - y_j) \right|_p \right\} \\
&\leq \max_i |x_i - y_i|_p \\
&= \|\mathbf{x} - \mathbf{y}\|.
\end{aligned}
$$

This completes the proof. □

Before going to Equation (3.8), let us first enumerate $S(x)$ for any $x \in V$ as follows $S(x) = \{x_1,..., x_k\}$, here as before $S(x)$ is the set of direct successors of x (see (2.1)). Using this enumeration one can rewrite (3.8) by

$$\hat{h}_{i,x} = \lambda(i) \prod_{m=1}^{k} F_i\left(\hat{\mathbf{h}}_{xm}, \theta_i\right), \quad i \in \mathbb{N}, \text{ for every } x \in V \setminus \{x^{(0)}\}. \tag{4.3}$$

Now we can formulate the main result.

Theorem 4.2. *Assume that* $N_{xy} \leq 0$ *for all* $< x, y > \in L$, *i.e.*, $|\theta_{xy}|_p \geq 1$ *and for* λ *the condition*

$$|\lambda(i)|_p \leq \delta \qquad \forall_i \in \mathbb{N}, \tag{4.4}$$

is satisfied. Then any two solution of (4.3) *belonging to* $\mathbf{B}_\delta$ *coincides with each other.*

Proof. Assume that $\hat{\mathbf{h}} = \{\mathbf{h}_x, x \in V \setminus \{x^0\}\}$, $\hat{\mathbf{s}} = \{\hat{\mathbf{s}}_x, x \in V \setminus \{x^0\}\}$ be two solutions of (4.3). Now we are going to show that they coincide with each other. Indeed, let $x \in V \setminus \{x^{(0)}\}$ be an arbitrary vertex. Using Lemma 2.1 from (4.3) one can find

$$\left|\hat{h}_{i,x} - \hat{s}_{i,x}\right|_p \leq |\lambda(i)|_p \max_{1 \leq m \leq k} \left\{ \left|F_i(\hat{\mathbf{h}}_{xm}))_i - F_i\left(\hat{\mathbf{s}}_{x_m}\right)\right|_p \right\},$$

which with (4.1) and (4.4) implies that

$$\left\|\hat{\mathbf{h}}_x - \hat{\mathbf{s}}_x\right\| \leq \delta \max_{1 \leq m \leq k} \left\{ \left\|\hat{\mathbf{h}}_{x_m} - \hat{\mathbf{s}}_{x_m}\right\| \right\} \tag{4.5}$$

Now take an arbitrary $\varepsilon > 0$. Let $n_0 \in \mathbb{N}$ such that $1/p^{n_0} < \varepsilon$. Iterating (4.5) n_0 times one gets $\|\hat{\mathbf{h}}_x - \hat{\mathbf{s}}_x\| < \varepsilon$.

Hence, from the arbitrariness of ε we obtain $\hat{\mathbf{h}}_x = \hat{\mathbf{s}}_x$ for every $x \in V \setminus \{x^{(0)}\}$. This completes the proof. □

The provided theorem states, if Equation (4.3) has solution belonging to $\mathbf{B}_\delta$, then it is unique. But, in general, we do not whether the equation has a solution or not. Below, we provide a sufficient condition for the existence of solution of (4.3).

Remark 4.1. The proved Theorem extends the main results of [25,26] to more general kind of measures, since there it was taken $\mathfrak{p} = \exp_p$.

4.1. Homogeneous case

In this subsection, we shall assume that $N_{xy} := N$ for all $< x, y > \in Ł$. Namely, we want to consider homogeneous case.

Recall that a function $\mathbf{h} = \{\mathbf{h}_x\}_{x \in V \setminus \{x^0\}}$ is called *translation-invariant* if $\mathbf{h}_{\tau_x(y)} = \mathbf{h}_y$ for all $x, y \in V \setminus \{x^0\}$. Let us restrict ourselves to the description of translation-invariant solutions of (3.8), namely $\mathbf{h}_x = \mathbf{h}(= (h_0, h_1,...,))$ for all $x \in V$.

Define the following mapping

$$(\mathcal{F}_\theta(\mathbf{x}))_i = \lambda(i)(F_i(\mathbf{x}, \theta))^k, \quad i \in \mathbb{N}, \tag{4.6}$$

where $\mathbf{x} = \{x_n\} \in c_0$. Here, $\theta = \mathfrak{p}^N$. Due to Remark 3.2, the mapping $\mathscr{F}_\theta$ is well defined.

Theorem 4.3. *Let $N \leq 0$, i.e., $|\theta|_p \geq 1$. Assume that for λ the condition (4.4) is satisfied. Then $\mathscr{F}_\theta(\mathbf{B}_\delta) \subset \mathbf{B}_\delta$, and*

$$\left\| \mathcal{F}_\theta(\mathbf{x}) - \mathcal{F}_\theta(\mathbf{y}) \right\| \leq \delta \left\| \mathbf{x} - \mathbf{y} \right\| \quad \text{for every } \mathbf{x}, \mathbf{y} \in \mathbf{B}_\delta. \tag{4.7}$$

Proof. Let $\mathbf{x} = (x_i)$, $\mathbf{y} = (y_i) \in \mathbf{B}_\delta$. Then from (3.10) with (4.2) we have

$$|F_i(\mathbf{x}, \theta)|_p = \frac{\left| \theta x_i + \sum_{j=1, j\neq i}^{\infty} x_j + 1 \right|_p}{|\theta|} \leq \frac{\max\{|\theta|_p |x_i|_p, 1\}}{|\theta|_p} \leq 1 \tag{4.8}$$

this with (4.6) yields that $\mathscr{F}_\theta(\mathbf{B}_\delta) \subset \mathbf{B}_\delta$.

Now using (4.1), (4.6), and (4.8) we obtain

$$\left\| \mathcal{F}_\theta(\mathbf{x}) - \mathcal{F}_\theta(\mathbf{y}) \right\| = \max_i |\lambda(i)|_p \max_i \left| (F_i(\mathbf{x}, \theta))^k - (F_i(\mathbf{y}, \theta))^k \right|_p$$

$$\leq \delta \left| F_i(\mathbf{x}, \theta) - F_i(\mathbf{y}, \theta) \right|_p \max_i \left| \sum_{\ell=0}^{k-1} F_i^{k-\ell}(\mathbf{x}, \theta) F_i^\ell(\mathbf{y}, \theta) \right|_p$$

$$\leq \delta \left\| \mathbf{x} - \mathbf{y} \right\|.$$

This completes the proof. □

Remark 4.2. We should stress that if $N > 0$ then the similar methods are no longer applicable for F_θ, therefore, it needs other kind of techniques. So, such a case will be considered elsewhere.

Now thanks to Lemma 4.3 we can apply the fixed point theorem to $\mathscr{F}_\theta$, which implies the existence of unique fixed point $\mathbf{x}_0 \in \mathbf{B}_\delta$. This, according to Theorem 4.2, means that there exists a unique solution of (3.8). Hence, due to Theorem 3.1, such a solution defines the p-adic quasi Gibbs measure μ_0.

Remark 4.3. Let us emphasize the following notes:

(a) Note that in [23] we have proved for the $q+1$-state Potts model the p-adic quasi Gibbs measure is unique if q and p are relatively prime. Therefore, the proved Theorem 4.2 shows the difference between finite and countable state Potts models. Moreover, in the real case such kind of result is unknown (see [36,37]).

(b) We also should stress that the condition (4.4) is important. If we replace δ with 1, then Theorem 4.2 may not be valid. Namely, in that case it may occur a quasi phase transition. Indeed, if $\lambda(1) = \lambda(2) = 1$ and $\lambda(k) = 0$ for every $k \geq 3$, then clearly (4.4) is not satisfied. On the other hand, our model reduces to 3-state Potts model. For such a model in [23] the existence of the quasi phase transition has been proved at $p = 2$.

4.2. Periodic case

In this section, we consider when N_{xy} is G_m-periodic. This means that $N_{\tau_g(x)\tau_g(y)} = N_{xy}$ for all $g \in G_m$ and $x, y \in \Gamma^m$. Therefore, let us denote

$$\theta_i = \mathfrak{p}^{N_{xy}}, \quad \text{if } d(x, x^0) \equiv i \pmod{m}, \ i = 0, \ldots, m-1.$$

We want to find G_m-periodic solution of (3.8). Recall that a function $\mathbf{h} = \{\mathbf{h}_x\}_{x \in V_\infty \setminus \{x^0\}}$ is called G_m-*periodic* if $\mathbf{h}_{\tau g(x)} = \mathbf{h}_x$ for all $g \in G_m$, $x \in V \setminus \{x^0\}$. Note that if $\mathbf{h} = \{\mathbf{h}_x\}$ is a G_k-periodic, then it can be defined by a k-collection of vectors $(\mathbf{h}_0, ..., \mathbf{h}_{k-1})$, where $\mathbf{h}_i = \{h_j^{(i)}\}_{i \in \mathbb{N}}$, i.e., $\mathbf{h}_x = \mathbf{h}_i$, if $d(x, x^0) \equiv i \pmod{m}$, $i = 0, ..., k - 1$.

Then Equation (3.8) is reduced to the following one

$$
\begin{cases}
h_j^{(i)} = \lambda_j \left(F_j \left(\mathbf{h}^{(i+1)}, \theta_{i+1} \right) \right)^k, \quad i = 0, \ldots, m - 2 \\
h_j^{(m-1)} = \lambda_j \left(F_j \left(\mathbf{h}^{(0)}, \theta_0 \right) \right)^k,
\end{cases} \quad j \in \mathbb{N} \tag{4.9}
$$

Hence, define the following mapping

$$
(\mathcal{H}(\mathbf{x})) = \mathcal{F}_{\theta_{m-1}} \left(\mathcal{F}_{\theta_{m-2}} \left(\cdot \cdot \cdot \left(\mathcal{F}_{\theta_0}(\mathbf{x}) \right) \cdot \cdot \cdot \right) \right) \tag{4.10}
$$

where $\mathbf{x} = \{x_n\} \in c_0$.

It is clear that the fixed point of (4.10) defines a solution of (4.9)

Theorem 4.4. *Let* $|\theta_i|_p \geq 1$, $i = 0, ..., m - 1$. *Assume that for* λ *the condition* (4.4) *is satisfied. Then* $\mathcal{H}(\mathbf{B}_\delta) \subset \mathbf{B}_\delta$, *and*

$$
\|\mathcal{H}(\mathbf{x}) - \mathcal{H}(\mathbf{y})\| \leq \delta^m \|\mathbf{x} - \mathbf{y}\| \quad \text{for every } \mathbf{x}, \mathbf{y} \in \mathbf{B}_\delta. \tag{4.11}
$$

The proof immediately follows from Theorem 4.3.

Consequently, Theorem 4.4 allows us to apply the fixed point theorem to $\mathcal{H}$, which implies the existence of unique fixed point $\mathbf{x}_0 \in \mathbf{B}_\delta$. This, according to Theorem 4.2, means that there exists a unique solution of (3.8). Hence, due to Theorem 3.1, such a solution defines the p-adic quasi Gibbs measure μ_m.

5. Conclusion

In the present study, we have provided a new construction of measure, called p-adic quasi Gibbs measure, for countable state p-adic Potts model on the Cayley tree. Note that the construction depends on a parameter $\mathfrak{p}$ and wights $\{\lambda(i)\}_{i \in \mathbb{N}}$. In particular case, i.e., if $\mathfrak{p} = \exp_p$, the defined measure coincides with p-adic Gibbs measure (see [24,25]). In this article, under some condition on weights we proved the existence of p-adic quasi Gibbs measures associated with the model. Note that this condition does not depend on values of the prime p. Moreover, our result extends the previous proved ones in [25,26]. An analogues fact is not valid when the number of spins is finite. We should stress that when states are finite and $\rho = p$, then the corresponding p-adic quasi Gibbs measures have been investigated in [23].

Endnotes

[a]The classical (real value) counterparts of such models were considered in [15,36]. [b]In the real case, when the state space is compact, then the existence follows from the compactness of the set of all probability measures (i.e., Prohorov's Theorem). When the state space is non-compact, then there is a Dobrushin's Theorem [27,28] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians.

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