Arab J Math (2012) 1:329–339
DOI 10.1007/s40065-012-0027-0

RESEARCH ARTICLE

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# Coupled coincidence points for mixed monotone operators in partially ordered metric spaces

Received: 17 November 2011 / Accepted: 28 January 2012 / Published online: 11 April 2012


**Abstract** In this paper, we give and prove some coupled coincidence point theorems for mappings $F : X \times X \to X$ and $g : X \to X$ in partially ordered metric space $X$, where $F$ has the mixed $g$-monotone property. Our results improve and generalize the results of Bhaskar and Lakshmikantham (Nonlinear Anal TMA 65:1379–1393, 2006), Luong and Thuan (Bull Math Anal Appl 2(4):16–24, 2010), Harjani et al. (Nonlinear Anal 74:1749–1760, 2011) and Choudhury et al. (Ann Univ Ferrara 57:1–16, 2011). We also give some examples to illustrate our results.

**Mathematics Subject Classification (2010)** 47H10 · 54H25

**الملخص**

في هذه الورقة، نعطي ونثبت بعض مبرهنات نقطة الصدفة المضاعفة للراسمين $F: X \times X \to X$ وَ $g: X \to X$ في فضاءات مترية ذات ترتيب جزئي $X$، حيث تمتلك $F$ خاصية الـ g-رتابة المختلطة. نتائجنا تطور وتعمّم نتائج هاسكار ولكشميكنتام (Nonlinear Anal TMA 65:1379–1393, 2006)، ولونغ وثوان (Bull Math Anal Appl 2(4):16–24, 2010)، وهرجاني وآخرون (Nonlinear Anal 74:1749–1760, 2011) و شاوذري وآخرون (Ann Univ Ferrara 57:1–16, 2011). كما أننا نعطي بعض الأمثلة لتوضيح نتائجنا.

## 1 Introduction and preliminaries

In 2006, Bhaskar and Lakshmikantham [7] introduced the notions of mixed monotone mapping and coupled fixed point and proved some coupled fixed point theorems for the mixed monotone mappings and also discussed the existence and uniqueness of solution for a periodic boundary value problem. These concepts are defined as follows.

**Definition 1.1** [7] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping $F$ is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in $x$ and is monotone non-increasing in $y$, i.e., for any $x, y \in X$,

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$$x_1, x_2 \in X,\ x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X,\ y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

**Definition 1.2** [7] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$x = F(x, y) \quad \text{and} \quad y = F(y, x)$$

The following are the main results in [7].

**Theorem 1.3** [7] *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on $X$. Assume that there exists $k \in [0, 1)$ with*

$$d\left(F(x, y), F(u, v)\right) \leq \frac{k}{2}\left[d(x, u) + d(y, v)\right] \tag{1.1}$$

*for all $x \succeq u$ and $y \preceq v$. If there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \textit{and} \quad y_0 \succeq F(y_0, x_0)$$

*then there exist $x, y \in X$ such that*

$$x = F(x, y) \quad \textit{and} \quad y = F(y, x).$$

**Theorem 1.4** [7] *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Assume that $X$ has the following properties:*

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n$,*
(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all $n$.*

*Let $F : X \times X \to X$ be a mapping having the mixed monotone property on $X$. Assume that there exists $k \in [0, 1)$ with*

$$d\left(F(x, y), F(u, v)\right) \leq \frac{k}{2}\left[d(x, u) + d(y, v)\right]$$

*for all $x \succeq u$ and $y \preceq v$. If there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \textit{and} \quad y_0 \succeq F(y_0, x_0)$$

*then there exist $x, y \in X$ such that*

$$x = F(x, y) \quad \textit{and} \quad y = F(y, x).$$

Afterward, the theory of coupled fixed point in partially ordered metric spaces has developed rapidly (see [1,4,9,11–16,22] and references therein). Luong and Thuan [15] proved the following result.

**Theorem 1.5** [15] *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on $X$ such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \textit{and} \quad y_0 \succeq F(y_0, x_0)$$

*Suppose there exist non-negative real numbers $\alpha, \beta$ and $L$ with $\alpha + \beta < 1$ such that*

$$\begin{aligned} d\left(F(x, y), F(u, v)\right) &\leq \alpha d(x, u) + \beta d(y, v) \\ &\quad + L \min\left\{\begin{array}{l} d\left(F(x, y), u\right), d\left(F(u, v), x\right), \\ d\left(F(x, y), x\right), d\left(F(u, v), u\right) \end{array}\right\} \end{aligned} \tag{1.2}$$

*for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either*

(a) *$F$ is continuous or*
(b) *$X$ has the following properties:*
   (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$, for all $n$,*
   (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$, for all $n$.*

*then there exist $x, y \in X$ such that*

$$x = F(x, y) \quad and \quad y = F(y, x),$$

*i.e., $F$ has a coupled fixed point in $X$.*

Harjani et al. [9] proved some generalizations of the main results in [7] and discussed the existence and uniqueness of the solution of non-linear integral equations.

**Theorem 1.6** [9] *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on $X$ such that*

$$\varphi\,(d\,(F\,(x, y), F\,(u, v))) \leq \varphi\,(\max\{d\,(x, u), d\,(y, v)\}) - \phi\,(\max\{d\,(x, u), d\,(y, v)\}) \tag{1.3}$$

*for all $x \succeq u$ and $y \preceq v$, where $\varphi, \phi$ are altering distance functions. Suppose either*

(a) *$F$ is continuous or*
(b) *$X$ has the following properties:*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $gx_n \preceq gx$ for all $n$,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $gy \preceq gy_n$ for all $n$.*
  *If there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F\,(x_0, y_0) \quad and \quad y_0 \succeq F\,(y_0, x_0)$$

*then there exist $x, y \in X$ such that*

$$x = F(x, y) \quad and \quad y = F(y, x).$$

On the other hand, Lakshmikantham and Ciric [13] established coupled coincidence and coupled fixed point theorems for two mappings $F : X \times X \to X$ and $g : X \to X$, where $F$ has the mixed $g$-monotone property and the functions $F$ and $g$ commute, as an extension of the fixed point results in [7].

Later, Choudhury and Kundu [5] introduced the concept of compatibility and proved the result established in [13] under a different set of conditions. Precisely, they established their result by assuming that $F$ and $g$ are compatible mappings and the function $g$ is monotone increasing.

**Definition 1.7** [13] Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$ and $g : X \to X$ be two mappings. We say $F$ has the mixed g-monotone property if $F(x, y)$ is g- non-decreasing in its first argument and is g- non-increasing in its second argument, i.e., for any $x, y \in X$,

$$x_1, x_2 \in X, \quad gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \preceq gy_2 \Rightarrow F(x, y_1) \succeq F(x, y_2)$$

**Definition 1.8** [13] An element $(x, y) \in X \times X$ is called a coupled coincident point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$gx = F(x, y) \quad \text{and} \quad gy = F(y, x)$$

**Definition 1.9** [5] The mappings $F$ and $g$ where $F : X \times X \to X$, $g : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0$$

where $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ for all $x, y \in X$ are satisfied.

Using the concept of compatibility, Choudhury et al. [4] proved a generalization of Theorem 1.6.

**Theorem 1.10** [4] *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $\phi : [0, \infty) \to [0, \infty)$ be continuous and $\phi(t) = 0$ if and only if $t = 0$, and $\psi$ be an altering distance function. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $g$-monotone property and*

$$\psi\left(d\left(F(x, y), F(u, v)\right)\right) \leq \psi\left(\max\{d(gx, gu), d(gy, gv)\}\right) - \phi\left(\max\{d(gx, gu), d(gy, gv)\}\right) \tag{1.4}$$

*for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Let $F(X \times X) \subseteq g(X)$, $g$ be continuous and let $F$ and $g$ be compatible mappings. Suppose also that*

(a) *$F$ is continuous or*
(b) *$X$ has the following properties:*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $gx_n \preceq gx$ for all $n$,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $gy \preceq gy_n$ for all $n$.*

*If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$, i.e., $F$ and $g$ have a coupled coincidence point in $X$.*

Denote $\Phi$ the set of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying:

(i) $\varphi$ is continuous,
(ii) $\varphi(t) < t$ for all $t > 0$ and $\varphi(t) = 0$ if and only if $t = 0$.

Then from the results of Jachymski [10] , the condition (1.4) is equivalent to

$$d\left(F(x, y), F(u, v)\right) \leq \varphi\left(\max\{d(gx, gu), d(gy, gv)\}\right) \tag{1.5}$$

where $\varphi \in \Phi$.

In Sect. 2, we will prove some coupled coincident point theorems which are generalizations of the results of Bhaskar and Lakshmikantham [7], Luong and Thuan [15], Choudhury et al. [4], and Harjani et al. [9]. More precisely, we will prove some coupled coincidence point theorems for mappings $F : X \times X \to X$ and $g : X \to X$ satisfying condition

$$\begin{aligned} d\left(F(x, y), F(u, v)\right) \leq\ & \varphi\left(\max\{d(gx, gu), d(gy, gv)\}\right) \\ & + L \min\left\{\begin{array}{l} d(F(x, y), gu), d(F(u, v), gx), \\ d(F(x, y), gx), d(F(u, v), gu) \end{array}\right\} \end{aligned} \tag{1.6}$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$, where $\varphi \in \Phi$ and $L \geq 0$.

## 2 Coupled coincidence point theorems

**Theorem 2.1** *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $g$-monotone property and satisfies (1.6). Let $F(X \times X) \subseteq g(X)$, $F, g$ are continuous and let $F$ and $g$ be compatible mappings. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then $F$ and $g$ have a coupled coincidence point in $X$.*

*Proof* Let $x_0, y_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n) \quad \text{for all } n \geq 0 \tag{2.1}$$

Using the mathematical induction and the mixed $g$-monotone property of $F$, we can show that

$$gx_n \preceq gx_{n+1} \quad \text{and} \quad gy_n \succeq gy_{n+1} \quad \text{for all } n \geq 0 \tag{2.2}$$

Since $gx_n \succeq gx_{n-1}$ and $gy_n \preceq gy_{n-1}$, from (1.6) and (2.1), we have

$$\begin{aligned} d\left(F(x_n, y_n), F(x_{n-1}, y_{n-1})\right) \leq\ & \varphi\left(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}\right) \\ & + L \min\left\{\begin{array}{l} d(F(x_n, y_n), gx_{n-1}), d(F(x_{n-1}, y_{n-1}), gx_n), \\ d(F(x_n, y_n), gx_n), d(F(x_{n-1}, y_{n-1}), gx_{n-1}) \end{array}\right\} \end{aligned}$$

or

$$d(gx_{n+1}, gx_n) \leq \varphi\left(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}\right) \tag{2.3}$$

Similarly, since $gy_{n-1} \succeq gy_n$ and $gx_{n-1} \preceq gx_n$, we have

$$d\left(F\left(y_{n-1}, x_{n-1}\right), F\left(y_n, x_n\right)\right) \leq \varphi \max\{d\left(gy_{n-1}, gy_n\right), d\left(gx_{n-1}, gx_n\right)\} + L \min\left\{\begin{array}{l} d\left(F\left(y_{n-1}, x_{n-1}\right), gy_n\right), d\left(F\left(y_n, x_n\right), gy_{n-1}\right), \\ d\left(F\left(y_{n-1}, x_{n-1}\right), gy_{n-1}\right), d\left(F\left(y_n, x_n\right), gy_n\right) \end{array}\right\}$$

or

$$d(gy_n, gy_{n+1}) \leq \varphi\left(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}\right) \tag{2.4}$$

From (2.3) and (2.4), we get

$$\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} \leq \varphi\left(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}\right) \tag{2.5}$$

Since $\varphi(t) \leq t$ for all $t \geq 0$, from (2.5), we have

$$\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} \leq \max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}$$

Set $d_n = \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}$, then $\{d_n\}$ is a non-increasing sequence of positive real numbers. Thus, there is $d \geq 0$ such that

$$\lim_{n\to\infty} d_n = d$$

Suppose that $d > 0$, letting $n \to \infty$ in two sides of (2.5) and using the properties of $\varphi$, we have

$$d = \lim_{n\to\infty} d(gy_n, gy_{n+1}) \leq \lim_{n\to\infty} \varphi\left(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}\right) = \varphi(d) < d$$

which is a contradiction. Hence $d = 0$, i.e.,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} = 0 \tag{2.6}$$

We shall show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{gx_n\}$ or $\{gy_n\}$ is not a Cauchy sequence. This means that there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}, \{gx_{m(k)}\}$ of $\{gx_n\}$ and $\{gy_{n(k)}\}, \{gy_{m(k)}\}$ of $\{gy_n\}$ with $n(k) > m(k) \geq k$ such that

$$\max\left\{d\left(gx_{n(k)}, gx_{m(k)}\right), d\left(gy_{n(k)}, gy_{m(k)}\right)\right\} \geq \varepsilon \tag{2.7}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies (2.7). Then,

$$\max\left\{d\left(gx_{n(k)-1}, gx_{m(k)}\right), d\left(gy_{n(k)-1}, gy_{m(k)}\right)\right\} < \varepsilon \tag{2.8}$$

Using the triangle inequality and (2.8), we have

$$d\left(gx_{n(k)}, gx_{m(k)}\right) \leq d\left(gx_{n(k)}, gx_{n(k)-1}\right) + d\left(gx_{n(k)-1}, gx_{m(k)}\right) < d\left(gx_{n(k)}, gx_{n(k)-1}\right) + \varepsilon \tag{2.9}$$

and

$$d\left(gy_{n(k)}, gy_{m(k)}\right) \leq d\left(gy_{n(k)}, gy_{n(k)-1}\right) + d\left(gy_{n(k)-1}, gy_{m(k)}\right) < d\left(gy_{n(k)}, gy_{n(k)-1}\right) + \varepsilon \tag{2.10}$$

From (2.7), (2.9) and (2.10), we have

$$\begin{aligned} \varepsilon &\leq \max\left\{d\left(gx_{n(k)}, gx_{m(k)}\right), d\left(gy_{n(k)}, gy_{m(k)}\right)\right\} \\ &< \max\left\{d\left(gx_{n(k)}, gx_{n(k)-1}\right), d\left(gy_{n(k)}, gy_{n(k)-1}\right)\right\} + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in the inequalities above and using (2.6), we get

$$\lim_{k\to\infty} \max\left\{d\left(gx_{n(k)}, gx_{m(k)}\right), d\left(gy_{n(k)}, gy_{m(k)}\right)\right\} = \varepsilon. \tag{2.11}$$

By the triangle inequality

$$d\left(gx_{n(k)}, gx_{m(k)}\right) \leq d\left(gx_{n(k)}, gx_{n(k)-1}\right) + d\left(gx_{n(k)-1}, gx_{m(k)-1}\right) + d\left(gx_{m(k)-1}, gx_{m(k)}\right)$$

and

$$d\left(gy_{n(k)}, gy_{m(k)}\right) \leq d\left(gy_{n(k)}, gy_{n(k)-1}\right) + d\left(gy_{n(k)-1}, gy_{m(k)-1}\right) + d\left(gy_{m(k-1}, gy_{m(k)}\right).$$

From the last two inequalities and (2.7), we have

$$\begin{aligned} \varepsilon &\leq \max\left\{d\left(gx_{n(k)}, gx_{m(k)}\right), d\left(gy_{n(k)}, gy_{m(k)}\right)\right\} \\ &\leq \max\left\{d\left(gx_{n(k)}, gx_{n(k)-1}\right), d\left(gy_{n(k)}, gy_{n(k)-1}\right)\right\} \\ &\quad + \max\left\{d\left(gx_{m(k)-1}, gx_{m(k)}\right), d\left(gy_{m(k)-1}, gy_{m(k)}\right)\right\} \\ &\quad + \max\left\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right\}. \end{aligned} \tag{2.12}$$

Again, by the triangle inequality,

$$\begin{aligned} d\left(gx_{n(k)-1}, gx_{m(k)-1}\right) &\leq d\left(gx_{n(k)-1}, gx_{m(k)}\right) + d\left(gx_{m(k)}, gx_{m(k)-1}\right) \\ &< d\left(gx_{m(k)}, gx_{m(k)-1}\right) + \varepsilon \end{aligned}$$

and

$$\begin{aligned} d\left(gy_{n(k)-1}, gy_{m(k)-1}\right) &\leq d\left(gy_{n(k)-1}, gy_{m(k)}\right) + d\left(gy_{m(k)}, gy_{m(k)-1}\right) \\ &< d\left(gy_{m(k)}, gy_{m(k)-1}\right) + \varepsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} &\max\left\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right\} \\ &\quad < \max\left\{d\left(gx_{m(k)}, gx_{m(k)-1}\right), d\left(gy_{m(k)}, gy_{m(k)-1}\right)\right\} + \varepsilon. \end{aligned} \tag{2.13}$$

Taking $k \to \infty$ in (2.12) and (2.13) and using (2.6), (2.11), we have

$$\lim_{k\to\infty} \max\left\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right\} = \varepsilon. \tag{2.14}$$

Since $n(k) > m(k)$, $gx_{n(k)-1} \succeq gx_{m(k)-1}$ and $gy_{n(k)-1} \preceq gy_{m(k)-1}$. Then from (1.6) and (2.1), we have

$$\begin{aligned} d(gx_{n(k)}, gx_{m(k)}) &= d(F(x_{n(k)-1}, y_{n(k)-1}), F(x_{m(k)-1}, y_{m(k)-1})) \\ &\leq \varphi\left(\max\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\}\right) \\ &\quad + L \min\left\{\begin{array}{l} d(F(x_{n(k)-1}, y_{n(k)-1}), gx_{m(k)-1}), d(F(x_{m(k)-1}, y_{m(k)-1}), gx_{n(k)-1}), \\ d(F(x_{n(k)-1}, y_{n(k)-1}), gx_{n(k)-1}), d(F(x_{m(k)-1}, y_{m(k)-1}), gx_{m(k)-1}) \end{array}\right\} \\ &\leq \varphi\left(\max\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\}\right) \\ &\quad + L \min\{d\left(gx_{n(k)}, gx_{n(k)-1}\right), d\left(gx_{m(k)}, gx_{m(k)-1}\right)\} \end{aligned} \tag{2.15}$$

Similarly,

$$\begin{aligned} d(gy_{m(k)}, gy_{n(k)}) \leq \varphi&\left(\max\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\}\right) \\ &+ L \min\{d(gy_{m(k)}, gy_{m(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\} \end{aligned} \tag{2.16}$$

From (2.15) and (2.16), we have

$$\begin{aligned} &\max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} \\ &\quad \leq \varphi\left(\max\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\}\right) \\ &\qquad + L \min\{d\left(gx_{n(k)}, gx_{n(k)-1}\right), d\left(gx_{m(k)}, gx_{m(k)-1}\right)\} \\ &\qquad + L \min\{d(gy_{m(k)}, gy_{m(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\} \end{aligned}$$

Letting $n \to \infty$ in the above inequality and using (2.6), (2.11), (2.14) and the properties of $\varphi$, we have

$$\varepsilon \leq \varphi(\varepsilon) + 2L \min\{0, 0\} < \varepsilon$$

which is a contradiction. This means that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Since $X$ is complete, there are $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x \quad \text{and} \quad \lim_{n\to\infty} gy_n = y \tag{2.17}$$

Thus,

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y \tag{2.18}$$

By (2.18) and the compatibility of $F$ and $g$, we have

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0 \tag{2.19}$$

and

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0 \tag{2.20}$$

Taking the limit as $n \to \infty$ in the following inequality

$$d(gx, F(gx_n, gy_n)) \leq d(gx, gF(x_n, y_n)) + d(gF(x_n, y_n), F(gx_n, gy_n))$$

and using (2.17), (2.19) and the continuity of $F$, $g$, we get $d(gx, F(x, y)) \leq 0$. This implies $gx = F(x, y)$. Similarly, we can show that $gy = F(y, x)$.
The proof is complete. □

**Theorem 2.2** *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $g$-monotone property and satisfies (1.6). Let $F(X \times X) \subseteq g(X)$, $g$ be continuous and let $F$ and $g$ be compatible mappings. Suppose $X$ has the following properties:*

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $gx_n \preceq gx$ for all $n$,*
(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $gy \preceq gy_n$ for all $n$.*

*If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then $F$ and $g$ have a coupled coincidence point in $X$.*

*Proof* Construct two sequences $\{x_n\}$ and $\{y_n\}$ as in Theorem 2.1. As in the proof of Theorem 2.1, $\{gx_n\}$ is non-decreasing sequence and $gx_n \to x$ and $\{gy_n\}$ is non-increasing sequence and $gy_n \to y$, by the assumption, we have $ggx_n \succeq gx$ and $ggy_n \preceq gy$ for all $n$.

From (2.18), (2.19) and (2.20), we have

$$\lim_{n\to\infty} F(gx_n, gy_n) = \lim_{n\to\infty} gF(x_n, y_n) = \lim_{n\to\infty} ggx_n = gx \tag{2.21}$$

and

$$\lim_{n\to\infty} F(gy_n, gx_n) = \lim_{n\to\infty} gF(y_n, x_n) = \lim_{n\to\infty} ggy_n = gy \tag{2.22}$$

Since $ggx_n \succeq gx$ and $ggy_n \preceq gy$, we have

$$\begin{aligned} d(F(x, y), gx) &\leq d(F(x, y), F(gx_n, gy_n)) + d(F(gx_n, gy_n), gx) \\ &\leq d(F(gx_n, gy_n), gx) + \varphi\left(\max\{d(gx, ggx_n), d(gy, ggy_n)\}\right) \\ &\quad + L \min\left\{\begin{array}{l} d(F(x, y), ggx_n), d(F(gx_n, gy_n), gx), \\ d(F(x, y), gx), d(F(gx_n, gy_n), ggx_n) \end{array}\right\} \end{aligned}$$

Taking $n \to \infty$ in the above inequality and using (2.21),(2.22) and the properties of $\varphi$, we have

$$d(F(x, y), gx) \leq \varphi\left(\max\{0, 0\}\right) + L \min d(F(x, y), gx), 0 = 0$$

Hence $F(x, y) = gx$. Similarly, one can show that $F(y, x) = gy$.
The proof is complete. □

**Theorem 2.3** *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $g$-monotone property and satisfies (1.6). Let $F(X \times X) \subseteq g(X)$, $g$ be continuous and let $F$ and $g$ be compatible mappings. Suppose $g$ is monotone and $X$ has the following properties:*

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n$,*
(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all $n$.*

*If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then $F$ and $g$ have a coupled coincidence point in $X$.*

*Proof* Construct two sequences $\{x_n\}$ and $\{y_n\}$ as in Theorem 2.1. Since $g$ is monotone, we may assume that $g$ is decreasing. Since $\{gx_n\}$ is non-decreasing sequence and $gx_n \to x$ and as $\{gy_n\}$ is non-increasing sequence and $gy_n \to y$ , we have $gx_n \succeq x$ and $gy_n \preceq y$ for all $n$. Since $g$ is decreasing, $ggx_n \preceq gx$ and $ggy_n \succeq gy$ for all $n$. Since $ggy_n \succeq gy$ and $ggx_n \preceq gx$ , we have

$$\begin{aligned} d(gy, F(y,x)) &\le d(gy, F(gy_n, gx_n)) + d(F(gy_n, gx_n), F(y,x)) + \\ &\le d(gy, F(gy_n, gx_n)) + \varphi\left(\max\{d(ggy_n, gy), d(ggx_n, gx)\}\right) \\ &\quad + L \min \left\{ \begin{array}{l} d(F(gy_n, gx_n), gy), d(F(y,x), ggy_n), \\ d(F(gy_n, gx_n), ggy_n), d(F(y,x), gy) \end{array} \right\} \end{aligned}$$

Taking $n \to \infty$ in the previous inequality and using (2.21), (2.22) and the properties of $\varphi$, we have

$$d(gy, F(y,x)) \le \varphi\left(\max\{0,0\}\right) + L \min d(F(y,x), gy), 0 = 0$$

Hence $F(y, x) = gy$. Similarly, one can show that $F(x, y) = gx$. □

In the following theorem, we replace the continuity of $g$, the compatibility of $F$ and $g$ and the completeness of $X$ by assuming that $g(X)$ is a complete subspace of $X$.

**Theorem 2.4** *Let $(X, \preceq, d)$ be a partially ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $g$-monotone property and satisfies (1.6). Let $F(X \times X) \subseteq g(X)$, $g(X)$ is a complete subspace of $X$. Suppose also that $X$ has the following properties:*

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n$,*
(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all $n$.*

*If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then $F$ and $g$ have a coupled coincidence point in $X$.*

*Proof* We construct two sequences $\{x_n\}$ and $\{y_n\}$ as in Theorem 2.1. As in the proof of Theorem 2.1, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Since $g(X)$ is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = gx \quad \text{and} \quad \lim_{n\to\infty} gy_n = gy \tag{2.23}$$

Since $\{gx_n\}$ is non-decreasing, $gx_n \to gx$ and $\{gy_n\}$ is non-increasing, $gy_n \to gy$, by the assumption, we have $gx_n \preceq gx$ and $gy_n \succeq gy$ for all $n$. Since $gy_n \succeq gy$ and $gx_n \preceq gx$, we have

$$\begin{aligned} d(gy, F(y,x)) &\le d(gy, gy_{n+1}) + d(gy_{n+1}, F(y,x)) \\ &= d(gy, gy_{n+1}) + d(F(y_n, x_n), F(y,x)) \\ &\le d(gy, gy_{n+1}) + \varphi\left(\max\{d(gy_n, gy), d(gx_n, gx)\}\right) \\ &\quad + L \min \left\{ \begin{array}{l} d\,(F(y_n, x_n), gy), d\,(F(y,x), gy_n), \\ d\,(F(y_n, x_n), gy_n), d\,(F(y,x), gy) \end{array} \right\} \\ &= d(gy, gy_{n+1}) + \varphi\left(\max\{d(gy_n, gy), d(gx_n, gx)\}\right) \\ &\quad + L \min \left\{ \begin{array}{l} d\,(gy_{n+1}, gy), d\,(F(y,x), gy_n), \\ d\,(gy_{n+1}, gy_n), d\,(F(y,x), gy) \end{array} \right\} \end{aligned}$$

On taking $n \to \infty$ and using (2.23), we obtain

$$d(gy, F(y,x)) \le \varphi(0) + L \min\{0, d(F(y,x), gy)\} = 0$$

This means $gy = F(y, x)$. Similarly, it can be shown that $gx = F(x, y)$.
Thus, $F$ and $g$ have a coupled coincidence point in $X$. □

In Theorems 2.1 and 2.2 (or 2.3 or 2.4), letting $gx = x$ for all $x \in X$, we get

**Corollary 2.5** *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on $X$ such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0)$$

*Suppose there exist a real number $L \geq 0$ and $\varphi \in \Phi$ such that*

$$d\,(F\,(x, y), F\,(u, v)) \leq \varphi\,(\max\{d\,(x, u), d\,(y, v)\}) + L \min \left\{ \begin{array}{l} d\,(F\,(x, y), u), d\,(F\,(u, v), x), \\ d\,(F\,(x, y), x), d\,(F\,(u, v), u) \end{array} \right\} \tag{2.24}$$

*for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either*

(a) *$F$ is continuous or*
(b) *$X$ has the following properties:*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$, for all $n$,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$, for all $n$.*

*then there exist $x, y \in X$ such that*

$$x = F(x, y) \quad \text{and} \quad y = F(y, x),$$

*i.e., $F$ has a coupled fixed point in $X$.*

*Remark 2.6* (1) In Theorems 2.1 and 2.2, letting $L = 0$, we get the results of Choudhury et al. [4].
(2) For all $x, y, u, v \in X, \alpha, \beta \geq 0, \alpha + \beta < 1$, we have

$$\alpha d(x, u) + \beta d(y, v) \leq (\alpha + \beta) \max\{d(x, u), d(y, v)\} = \varphi\,(\max\{d(x, u), d(y, v)\}),$$

where $\varphi(t) = (\alpha + \beta)\,(t)$, for all $t \geq 0$, is in $\Phi$. Therefore, Theorem 1.10 is a consequence of Corollary 2.5

## 3 Examples

*Example 3.1* Let $X = [1, 3)$ with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$. We consider the following order relation on $X$

$$x, y \in X\; x \preceq y \;\Leftrightarrow\; x = y \text{ or } (x, y) \in \{(1, 1), (1, 2), (2, 2)\}.$$

Let $F : X \times X \to X$ be given by

$$F\,(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 2 & \text{otherwise} \end{cases}$$

and $g : X \to X$ be defined by

$$gx = \begin{cases} 1 & \text{if } 1 \leq x \leq 3/2 \\ 2 & \text{if } 3/2 < x \leq 2 \\ 3 - x & \text{if } 2 < x < 5/2 \\ 3/2 & \text{if } 5/2 \leq x < 3 \end{cases}$$

and let $\varphi : [0, \infty) \to [0, \infty)$ be defined by $\varphi(t) = t/2$ for all $t \geq 0$. Then, all the conditions of Theorem 2.4 are satisfied. Applying Theorem 2.4, we conclude that $F$ and $g$ have a coupled coincidence point and it is seen that $(1, 1)$ is a coupled coincidence point of $F$ and $g$.
However,

(i) $X$ is not complete, $F$, $g$ are not continuous.
(ii) $F$ and $g$ are not compatible. Indeed, let $\{x_n\}$, $\{y_n\}$ in $X$ with

$$x_n = y_n = 2 + \frac{1}{n+1}$$

For all $n$, we have

$$F(x_n, y_n) = F(y_n, x_n) = F\left(2 + \frac{1}{n+1}, 2 + \frac{1}{n+1}\right) = 2,$$

$$gx_n = gy_n = g\left(2 + \frac{1}{n+1}\right) = 2 - \frac{1}{n+1} \to 2 \text{ as } n \to \infty$$

but

$$d\left(F(gx_n, gy_n), gF(x_n, y_n)\right) = d\left(F\left(2 - \frac{1}{n+1}, 2 - \frac{1}{n+1}\right) gF\left(2 + \frac{1}{n+1}, 2 + \frac{1}{n+1}\right)\right) = 2 \nrightarrow 0 \text{ as } n \to \infty$$

*Example 3.2* Let $X = [1, \infty)$ with the usual metric $d(x, y) = |x - y|$, $\forall x, y \in X$ and the usual ordering. Let $F : X \times X \to X$ be given by

$$F(x, y) = \frac{1 + \sqrt{3}}{2} \quad \text{for all } x, y \in X$$

and $g : X \to X$ be given by

$$gx = 1 + \frac{1}{2x} \quad \text{for all } x \in X$$

It is easy to see that all the conditions of Theorems 2.1 and 2.3 are satisfied. Applying these Theorems, we conclude that $F$ and $g$ have a coupled coincidence point. However, since $g(X) = (1, 2]$ is not a complete subspace of $X$, we cannot apply Theorem 2.4 to this example. Moreover, if $\{x_n\}$ is a non-decreasing sequence in $X$, $x_n \to x$ then $gx_n \geq gx$. Therefore, we cannot apply Theorem 2.2 to this example.



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