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# Some Remarks on Second Order Linear Difference Equations*

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**We obtain some further results for comparison theorems and oscillation criteria of second order linear difference equations.**

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## 1. INTRODUCTION

Oscillation and comparison theorems for the linear difference equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \quad n = 1, 2, \ldots, \quad (1.1)$$

has been investigated intensively [1–5].

Equation (1.1) is equivalent to the self adjoint equation

$$\Delta(c_{n-1}\Delta x_{n-1}) + a_n x_n = 0, \quad n = 1, 2, \ldots, \quad (1.2)$$

where $a_n = c_n + c_{n-1} - b_n$.

A nontrivial solution $\{x_n\}$ of Eq. (1.1) is said to be oscillatory, if the terms $x_n$ of the solution are neither eventually all positive nor eventually all negative. Otherwise, the solution is called non-oscillatory. It is well known that if one nontrivial solution of (1.1) is oscillatory, then all solutions are oscillatory, and so we can say that (1.1) is oscillatory.

In Section 2, we want to show some further results on the comparison theorem and oscillation criteria for (1.1), which improve some known results. In Section 3, we consider the forced oscillation.

## 2. COMPARISON THEOREMS AND OSCILLATION

We assume that $a_n > 0$ and $b_n > 0$ for all large $n$. Let $\{x_n\}$ be an eventually positive solution of (1.1),

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say $x_n > 0$ for $n \geq N$. Taking Riccati type transformation

$$s_n = (b_{n+1}x_{n+1})/(c_n x_n), \quad n \geq N, \tag{2.1}$$

(1.1) becomes

$$q_n s_n + 1/s_{n-1} = 1, \quad \text{for } n \geq N+1, \tag{2.2}$$

where $q_n = c_n^2/(b_n b_{n+1})$.

It is known [1] that (1.1) is nonoscillatory if and only if (2.2) has an eventually positive solution.

We consider (1.1) and (2.2) together with

$$C_n y_{n+1} + C_{n-1} y_{n-1} = B_n y_n, \quad n = 1, 2, \ldots, \tag{2.3}$$

and

$$Q_n S_n + 1/S_{n-1} = 1, \tag{2.4}$$

where $Q_n = C_n^2/(B_n B_{n+1})$.

THEOREM 2.1 *Suppose that* $Q_n Q_{n+1} \geq q_n q_{n+1}$ *and* $Q_n + Q_{n+1} \geq q_n + q_{n+1}$ *for all large n. If* (2.3) *is nonoscillatory, so is Eq.* (1.1).

*Proof* To prove that (1.1) has a positive solution, it is sufficient to prove that (2.2) has a positive solution $\{s_n\}$ for $n \geq N$. Since $q_n + q_{n+1} \leq Q_n + Q_{n+1}$ for all large $n$, then there exists a positive integer $n_1 > N$ such that $Q_{n_1+1} \geq q_{n_1+1}$. From (2.4), $S_n > 1$ for $n \geq n_1$. Choose $s_{n_1} \geq S_{n_1} > 1$ and define $s_{n_1+1}$ by (2.2). In view of (2.2) and (2.4), we have

$$\begin{aligned} q_{n_1+1} s_{n_1+1} &= 1 - 1/s_{n_1} \\ &= Q_{n_1+1} S_{n_1+1} + 1/S_{n_1} \\ &\quad - 1/s_{n_1} \geq Q_{n_1+1} S_{n_1+1}. \end{aligned}$$

Hence

$$s_{n_1+1} \geq \frac{Q_{n_1+1}}{q_{n_1+1}} S_{n_1+1} > 0$$

and $s_{n_1} s_{n_1+1} \geq S_{n_1} S_{n_1+1}$. By induction, we can prove that (2.2) has a positive solution $\{s_n\}$, $n \geq n_1$, which implies that (1.1) has a nonoscillatory solution. The proof is complete.

*Remark 2.1* Theorem 2.1 improves Theorem 6.8.4 in [1].

We write (1.1) in the form

$$x_{n+1} - \frac{b_n}{c_n} x_n + \frac{c_{n-1}}{c_n} x_{n-1} = 0$$

and let $y_n = \left[\prod_{i=N}^{n-1} (c_i/b_i)\right] x_n$. Then (1.1) becomes

$$y_{n+1} - y_n + q_n y_{n-1} = 0. \tag{2.5}$$

The oscillation of (1.1) and (2.5) is equivalent. By known results [1, Theorems 6.20.3 and 6.20.4] or [2], if

$$\liminf_{n\to\infty} q_n > \tfrac{1}{4}, \tag{2.6}$$

then (1.1) is oscillatory and if

$$\limsup_{n\to\infty} q_n < \tfrac{1}{4}, \tag{2.7}$$

then (1.1) is nonoscillatory. In particular, the equation

$$y_{n+1} - y_n + \tfrac{1}{4} y_{n-1} = 0 \tag{2.8}$$

is nonoscillatory.

Combining the above results and Theorem 2.1, we obtain the following corollaries.

COROLLARY 2.1 *If* $q_n + q_{n+1} \leq 1/2$ *for all large n. Then* (1.1) *is nonoscillatory.*

In fact, let $Q_n \equiv 1/4$, Corollary 2.1 follows from Theorem 2.1.

*Remark 2.2* Corollary 2.1 improves Theorem 6.5.5 in [1].

COROLLARY 2.2 *If* $q_n q_{n+1} \geq 1/16 + \epsilon_0$, *for some* $\epsilon_0 > 0$ *and all large n, then* (1.1) *is oscillatory.*

*Proof* Let $\epsilon_1$ be a positive number such that $1/(4-\epsilon_1) \leq \sqrt{1/16 + \epsilon_0}$ and $Q_n = 1/(4-\epsilon_1)$ for all large $n$. Then

$$q_n q_{n+1} \geq \frac{1}{16} + \epsilon_0 \geq \frac{1}{(4-\epsilon_1)^2} = Q_n Q_{n+1}$$

and

$$q_n + q_{n+1} \geq 2\sqrt{q_n q_{n+1}} \geq \frac{2}{4-\epsilon_1} = Q_n + Q_{n+1}$$

for all large $n$. Since $Q_n \equiv 1/(4-\epsilon_1)$ implies that (2.3) is oscillatory. By Theorem 2.1, (1.1) is oscillatory also.

*Remark 2.3* Corollary 2.2 improves Theorem 6.5.3 in [1].

*Example 2.1* Consider the difference equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = x_n, \tag{2.9}$$

where

$$c_{n-1} = \begin{cases} \sqrt{14.1/15}, & n\text{: even}, \\ \sqrt{1/15}, & n\text{: odd}. \end{cases}$$

Then

$$q_n = c_{n-1}^2 = \begin{cases} 14.1/15, & n\text{: even}, \\ 1/15, & n\text{: odd}. \end{cases}$$

Hence $q_n q_{n+1} = 14.1/(15)^2 > 1/16$. By Corollary 2.2, every solution of (2.9) is oscillatory.

Oscillation criteria in [1] are not valid for (2.9).

Define two sequences $\{R_n\}$ and $\{r_n\}$ as follows:

$$\begin{aligned} R_n = {} & q_n + q_{n-1} + q_n q_{n+1} + q_{n-1} q_{n-2} + q_n q_{n+1}^2 \\ & + q_n q_{n+1} q_{n+2} + q_{n-1} q_{n-2}^2 + q_n q_{n+1}^2 q_{n+2} \\ & + q_{n-2}^2 q_{n-1} q_{n-3} + q_{n-1} q_{n-2} q_{n-3}, \quad n \geq 4, \end{aligned} \tag{2.10}$$

and

$$r_n = q_n + q_{n-1} + q_n q_{n+1} + q_{n-1} q_{n-2}, \quad n \geq 3. \tag{2.11}$$

THEOREM 2.2 *Assume that there exists an increasing sequence* $\{n_k\}$ such that $R_{n_k} \geq 1$. *Then* (1.1) *is oscillatory.*

*Proof* Suppose to the contrary, let (1.1) be non-oscillatory. Then (2.2) has a positive solution $\{s_n\}$ defined for $n \geq N$. From (2.2), by the iterating substitution, we have

$$\begin{aligned} 1 &= q_n q_{n+1} s_n s_{n+1} + q_n + q_{n+1} + 1/(s_{n-1} s_{n-2}) \\ &= q_n q_{n+1} s_n s_{n+1} (q_{n+2} s_{n+2} + 1/s_{n+1}) \\ &\quad \times (q_{n+1} s_{n+1} + 1/s_n) + q_n + q_{n-1} \\ &\quad + (q_{n-1} s_{n-1} + 1/s_{n-2})(q_{n-2} s_{n-2} + 1/s_{n-3}) \\ &\quad /(s_{n-1} s_{n-2}) \\ &= q_n + q_{n-1} + q_n q_{n+1} + q_{n-1} q_{n-2} \\ &\quad + q_n q_{n+1}^2 s_n s_{n+1} + q_n q_{n+1} q_{n+2} s_{n+1} s_{n+2} \\ &\quad + q_n q_{n+1}^2 q_{n+2} s_n s_{n+1}^2 s_{n+2} + q_{n-2}/(s_{n-1} s_{n-2}) \\ &\quad + q_{n-1}/(s_{n-2} s_{n-3}) + 1/(s_{n-1} s_{n-2}^2 s_{n-3}) \\ &> q_n + q_{n-1} + q_n q_{n+1} + q_{n-1} q_{n-2} + q_n q_{n+1}^2 \\ &\quad + q_n q_{n+1} q_{n+2} + q_n q_{n+1}^2 q_{n+2} + q_{n-1} q_{n-2}^2 \\ &\quad + q_{n-1} q_{n-2} q_{n-3} + q_{n-1} q_{n-2}^2 q_{n-3} \\ &= R_n \end{aligned}$$

which contradicts the assumption. The proof is complete.

COROLLARY 2.3 *If* $\limsup_{n\to\infty} R_n > 1$, *then* (1.1) *is oscillatory.*

It is easy to see that $\limsup_{n\to\infty} r_n > 1$, then $\limsup_{n\to\infty} R_n > 1$.

Example 2.1 satisfies conditions of Corollary 2.3.

*Remark 2.3* Corollary 2.3 improves Corollary 6.5.11 in [1].

## 3. FORCED OSCILLATION

We consider the forced equation

$$\Delta^2 x_n + p_n x_{n+1} = f_n, \quad n = 0, 1, \ldots, \tag{3.1}$$

and the homogeneous equation

$$\Delta^2 x_n + p_n x_{n+1} = 0. \tag{3.2}$$

LEMMA 3.1 *Let* $\{\phi_n\}$ *be a solution of* (3.2) *and* $\{x_n\}$ *be a solution of* (3.1). *Let* $x_n = \phi_n y_n$, *then* $\{y_n\}$ *satisfies*

$$\Delta(\phi_n \phi_{n+1} \Delta y_n) = \phi_{n+1} f_n. \tag{3.3}$$

*Proof* Clearly,

$$\phi_n \Delta x_n = \phi_n \Delta \phi_n y_n + \phi_n \phi_{n+1} \Delta y_n.$$

Hence

$$\begin{aligned}
&\Delta(\phi_n \phi_{n+1} \Delta y_n) \\
&\quad = \Delta(\phi_n \Delta x_n) - \Delta(\phi_n \Delta \phi_n y_n) \\
&\quad = \phi_{n+1} \Delta^2 x_n + \Delta \phi_n \Delta x_n - \phi_{n+1} \Delta \phi_{n+1} \Delta y_n \\
&\qquad - \Delta(\phi_n \Delta \phi_n) y_n \\
&\quad = \phi_{n+1}(f_n - p_n \phi_{n+1} y_{n+1}) \\
&\qquad + \Delta \phi_n (\Delta \phi_n y_n + \phi_{n+1} \Delta y_n) \\
&\qquad - \phi_{n+1} \Delta \phi_{n+1} \Delta y_n - (\phi_{n+1} \Delta^2 \phi_n + (\Delta \phi_n)^2) y_n \\
&\quad = \phi_{n+1} f_n - \phi_{n+1}^2 y_{n+1} p_n \\
&\qquad + \phi_{n+1} \Delta y_n (\Delta \phi_n - \Delta \phi_{n+1}) - \phi_{n+1} \Delta^2 \phi_n y_n \\
&\quad = \phi_{n+1} f_n - \phi_{n+1}^2 y_{n+1} p_n \\
&\qquad - \Delta^2 \phi_n \phi_{n+1} (y_{n+1} - y_n) - \phi_{n+1} \Delta^2 \phi_n y_n \\
&\quad = \phi_{n+1} f_n - y_{n+1} \phi_{n+1} (p_n \phi_{n+1} + \Delta^2 \phi_n) \\
&\quad = \phi_{n+1} f_n.
\end{aligned}$$

The proof is complete.

THEOREM 3.1 *Let* $\{\phi_n\}$ *be a positive solution of* (3.2). *Assume that there exists a positive integer* $N$ *such that*

(i)

$$\liminf_{n\to\infty} \sum_{i=N}^{n} \phi_{i+1} f_i = -\infty$$

and

$$\limsup_{n\to\infty} \sum_{i=N}^{n} \phi_{i+1} f_i = \infty,$$

(ii)

$$\sum_{i=N}^{\infty} \frac{1}{\phi_i \phi_{i+1}} = \infty,$$

(iii)

$$\liminf_{n\to\infty} \sum_{i=N}^{n} \frac{1}{\phi_i \phi_{i+1}} \sum_{j=N}^{i-1} \phi_{j+1} f_j = -\infty,$$

$$\limsup_{n\to\infty} \sum_{i=N}^{n} \frac{1}{\phi_i \phi_{i+1}} \sum_{j=N}^{i-1} \phi_{j+1} f_j = \infty.$$

*Then every solution of* (3.1) *is oscillatory.*

*Proof* Suppose to the contrary, let $\{x_n\}$ be a positive solution of (3.1) and $x_n = \phi_n y_n$. By Lemma 3.1, $y_n$ satisfies (3.3).

Summing (3.3) from $N$ to $n-1$, we obtain

$$\phi_n \phi_{n+1} \Delta y_n - \phi_N \phi_{N+1} \Delta y_N = \sum_{i=N}^{n-1} \phi_{i+1} f_i. \tag{3.4}$$

Condition (i) implies that

$$\liminf_{n\to\infty} \phi_n \phi_{n+1} \Delta y_n = -\infty.$$

Let $N_1$ be a large integer that $\phi_{N_1} \phi_{N_1+1} \Delta y_{N_1} < -M$, $M > 0$. From (3.4), we obtain

$$\begin{aligned}
\Delta y_n &= \frac{\phi_{N_1} \phi_{N_1+1} \Delta y_{N_1}}{\phi_n \phi_{n+1}} + \frac{1}{\phi_n \phi_{n+1}} \sum_{i=N_1}^{n-1} \phi_{i+1} f_i \\
&< -\frac{M}{\phi_n \phi_{n+1}} + \frac{1}{\phi_n \phi_{n+1}} \sum_{i=N_1}^{n-1} \phi_{i+1} f_i.
\end{aligned} \tag{3.5}$$

Summing (3.5) from $N_1$ to $n-1$, we obtain

$$\begin{aligned}
y_n - y_{N_1} \leq &- M \sum_{i=N_1}^{n-1} \frac{1}{\phi_i \phi_{i+1}} \\
&+ \sum_{i=N_1}^{n-1} \frac{1}{\phi_i \phi_{i+1}} \sum_{j=N_1}^{i-1} \phi_{j+1} f_j.
\end{aligned} \tag{3.6}$$

Condition (iii) and (3.6) imply that there exists a sequence $\{n_i\}$ such that $y_{n_i} < 0$ for all large $i$, which is a contradiction.

We can prove this theorem in a similar manner for negative solutions of (3.1).

From (3.6), we obtain the following result.

THEOREM 3.2 *Let* $\{\phi_n\}$ *be a positive solution of* (3.2) *with* $\sum_{i=N}^{\infty} 1/(\phi_i \phi_{i+1}) < \infty$. *Assume that* (iii) *of Theorem* 3.1 *holds. Then every solution of* (3.1) *is oscillatory.*

*Example 3.1* Consider

$$\begin{aligned}
&\Delta^2 x_n + \frac{2}{(n+1)^2 (n+3)} x_{n+1} \\
&\quad = (-1)^n \frac{(2n-1)(n+2)}{n+1}, \quad n = 1, 2, \ldots
\end{aligned} \tag{3.7}$$

It is easy to see that the equation

$$\Delta^2 x_n + \frac{2}{(n+1)^2(n+3)} x_{n+1} = 0 \tag{3.8}$$

has a solution $\{\phi_n = n/(n+1)\}$, which satisfies (ii). On the other hand,

$$\sum_{i=N}^{n} \phi_{i+1} f_i = \sum_{i=N}^{n} (-1)^i (2i-1) = (-1)^n n + c, \tag{3.9}$$

where $c$ is a constant. Then (3.9) implies that (i) is satisfied. Also, (iii) is satisfied. By Theorem 3.1, every solution of (3.7) is oscillatory.

*Remark 3.1* Theorems 3.1 and 3.2 treat the oscillation of (3.1), which is caused by the forced term.

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