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OPTIMALITY AND EXISTENCE FOR LIPSCHITZ EQUATIONS

JOHNNY HENDERSON

Department of Algebra, Combinatorics, & Analysis
Auburn University
Auburn, Alabama 36849

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ABSTRACT: Solutions of certain boundary value problems are shown to exist for the nth order differential equation $y^{(n)} = f(t,y,y',\ldots,y^{(n-1)})$, where f is continuous on a slab $(a,b) \times \mathbb{R}^n$ and f satisfies a Lipschitz condition on the slab. Optimal length subintervals of (a,b) are determined, in terms of the Lipschitz coefficients, on which there exist unique solutions.

KEY WORDS AND PHRASES: Ordinary differential equation, boundary value problem, Lipschitz condition, optimal length interval, uniqueness imples existence.
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1. INTRODUCTION.

We will be concerned with the existence of solutions of boundary value problems for the nth order differential equation

$$y^{(n)} = f(t,y,y',\ldots,y^{(n-1)}), \tag{1.1}$$

where f is continuous on a slab $(a,b) \times \mathbb{R}^n$ and satisfies a Lipschitz condition,

$$\left|f(t,y_1,\ldots,y_n)-f(t,z_1,\ldots,z_n)\right| \le \sum_{i=1}^{n} k_i\left|y_i - z_i\right|, \tag{1.2}$$

on the slab.

A number of papers have appeared in which optimal length subintervals of (a,b) are determined, in terms of the Lipschitz coefficients k_i, $1 \le i \le n$, on which solutions of certain boundary value problems for (1.1) are unique; see, for example [1-15]. Of motivational importance in this work are the papers by Jackson [10-11] in which he applied methods from control theory in establishing optimal length subintervals, in terms of the Lipschitz coefficients, on which solutions of conjugate boundary value problems and right focal point boundary value problems for (1.1) are unique. It then follows from uniqueness implies existence results due to Hartman [16-17] and Klaasen [18] in the conjugate case and Henderson [19] in the right focal point case, that unique solutions exist on the optimal intervals given in [10-11].

In [7-8], we adapted Jackson's control theory arguments, in conjunction with uniqueness implies existence results, and determined optimal length subintervals of (a,b) on which there exist unique solutions of several classes of boundary value problems for third and fourth order ordinary differential equations satisfying Lipschitz conditions. In a recent work [9], we followed the pattern of [7-8, 10-11], by applying the Pontryagin Maximum Principle to a linearization of (1.1), and determined optimal length subintervals of (a,b), in terms of k_i, on which solutions are unique for boundary value problems for (1.1) satisfying

$$y^{(i)}(t_1') = y_{i+1}, \qquad 0 \le i \le n-h+k-1,$$
$$y^{(i)}(t_i) = y_{n-h+(i+1)}, \qquad k \le i \le h-1, \tag{1.3}$$

where $a < t_1' < t_k \le \cdots \le t_{h-1} < b$, $0 \le k < h \le n$, and $y_i \varepsilon\ \mathbb{R}$, $1 \le i \le n$.

In this work, we now address the problem of existence of solutions of (1.1), (1.3) on the optimal intervals for uniqueness given in [9]. We state in Section 2 some of the results concerning optimality and uniqueness obtained in [9] which are pertinent to the arguments here. Then in Section 3, we are able to prove that on subintervals of length less than the optimal length given in Section 2 and for certain values of k and h, solutions of (1.1), (1.3) exist. For this restricted set of k and h, the existence result is in some sense analogous to the uniqueness implies existence results in [16-19].

2. OPTIMALITY AND UNIQUENESS.

In this section, we state a Theorem and a Corollary from [9, Thm. 3 & the Cor.], in which optimal length subintervals of (a,b) in terms of the Lipschitz coefficients k_i, $1 \le i \le n$, are determined on which solutions of (1.1), (1.3) are unique.

THEOREM 1. Let $0 \le k < h \le n$ be given and let $\gamma = \min\{\gamma_\ell \mid k \le \ell < h\}$, where γ_ℓ is the smallest positive number such that there exists a solution $x(t)$ of the boundary value problem

$$x^{(n)} = (-1)^{h-\ell}[k_1 x + \sum_{i=1}^{n} k_i |x^{(i-1)}|],$$
$$x^{(i)}(0) = 0,\ 0 \le i \le n-h+\ell-1,$$
$$x^{(i)}(\gamma_\ell) = 0,\ \ell \le i \le h-1,$$

with $x(t) > 0$ on $(0,\gamma_\ell)$, or $\gamma_\ell = +\infty$ if no such solution exists. For any $k \le \ell < h$, if $y(t)$ and $z(t)$ are distinct solutions of (1.1) such that

$$y^{(i)}(t_1') = z^{(i)}(t_1'),\ 0 \le i \le n-h+\ell-1,$$
$$y^{(i)}(t_i) = z^{(i)}(t_i),\ \ell \le i \le h-1,$$

$a < t_1' < t_\ell \le \cdots \le t_{k-1} < b$, and if $t_{h-1} - t_1' < \gamma$, it follows that $y(t) \equiv z(t)$ on (a,b), and this is best possible for the class of all differential equations satisfying the Lipschitz condition (1.2).

REMARK. Jackson [11] has proved Theorem 1 for the case when $h = n$ and $k = 0$. Now, on subintervals of length less than the constant γ in Theorem 1, it follows from Rolle's Theorem that solutions of a number of other boundary value problems for (1.1) are unique. For example, we can state the following.

COROLLARY 2. Let γ be as in Theorem 1. For any $k \leq \ell < h$ and $h - \ell \leq j \leq h$, if $y(t)$ and $z(t)$ are solutions of (1.1) such that

$$y^{(i)}(t_1') = z^{(i)}(t_1'),\ 0 \leq i \leq n-h+\ell-1,$$

$$y^{(i+j-h)}(t_i) = y^{(i+j-h)}(t_i),\ \ell \leq i \leq h-1,$$

$a < t_1' < t_\ell \leq \dots \leq t_{h-1} < b$, and if $t_{h-1} - t_1' < \gamma$, it follows that $y(t) \equiv z(t)$ on (a,b), and this is best possible.

3. EXISTENCE OF SOLUTIONS.

Analogous to uniqueness implies existence results proved by Hartman [16-17], Klaasen [18], and Henderson [19], we give a proof in this section that, on the optimal subintervals for uniqueness determined in Section 2 and for a restricted set of values of h and k, the boundary value problems (1.1), (1.3) have unique solutions. For the proof, we use somewhat standard shooting methods. We will give the proof only for two-point problems, with the proof for multipoint problems being similar.

THEOREM 3. Let $[n/2] \leq h \leq n$ be given, ($[\cdot]$ denotes the greatest integer function). Let $k = 0$ and let $\gamma = \min\{\gamma_\ell \mid 0 \leq \ell < h\}$ be as defined in Theorem 1. Then the boundary value problem

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}),$$

$$y^{(i)}(t_1) = y_{i+1},\ 0 \leq i \leq n-h+\ell-1,$$

$$y^{(i)}(t_2) = y_{n-h+(i+1)},\ \ell \leq i \leq h-1,$$

where $a < t_1 < t_2 < b$, $0 \leq \ell < h$, has a unique solution for any assignment of $y_i \in \mathbf{R}$, $1 \leq i \leq n$, provided $t_2 - t_1 < \gamma$. Furthermore, this result is best possible for the class of all differential equations which satisfy the Lipschitz condition (1.2).

PROOF. Let $a < t_1 < t_2 < b$, with $t_2 - t_1 < \gamma$, and $y_i \in \mathbf{R}$, $1 \leq i \leq n$, be given. We prove the existence of solutions for a much larger family of boundary value problems than those in the statement of the theorem. In fact, we prove the existence of solutions of the two-point problems which belong to the class of problems in Corollary 2. For induction purposes, we arrange these problems in a lower triangular array,

$$\begin{array}{llll} (1,1) & & & \\ (2,1) & (2,2) & & \\ \vdots & \vdots & \ddots & \\ (h,1) & (h,2) & \cdots & (h,h) \end{array}$$

where the boundary value problem for (1) associated with the (μ,ν)-position,

$1 \le \nu \le \mu \le h$, satisfies

$$y^{(i)}(t_1) = y_{i+1},\ 0 \le i \le n-\nu-1,$$

$$y^{(i)}(t_2) = y_{n-\mu+(i+1)},\ \mu-\nu \le i \le \mu-1.$$

Under this arrangement, the boundary value problems for (1.1) along the principal diagonal (μ,μ), $1 \le \mu \le h$, are conjugate type problems, whereas the boundary value problems in the statement of this theorem are associated with the entries along the bottom row (h,ν), $1 \le \nu \le h$.

By Corollary 2, solutions of all the problems in this array are unique on subintervals of length less than γ. Moreover, by the constraints on h and k, it follows that solutions of all conjugate type boundary value problems for (1.1) are unique. Then, it follows from the uniqueness implies existence result of Hartman [16-17] and Klaasen [18] that the conjugate boundary value problems, and in particular those associated with the entries on the main diagonal, have unique solutions. (This is the reason for the constraints on h and k.) For existence of solutions of the remaining problems associated with the array, we will use the shooting method coupled with an induction along the subdiagonals on the array.

In that direction, choose any boundary value problem for (1) associated with the first subdiagonal $(\mu,\mu-1)$, where $2 \le \mu \le h$; that is, we are concerned with solutions of (1) satisfying

$$y^{(i)}(t_1) = y_{i+1},\ 0 \le i \le n-\mu,$$

$$y^{(i)}(t_2) = y_{n-\mu+(i+1)},\ 1 \le i \le \mu-1.$$

In applying the shooting method, let $z(t)$ be the solution of (1) satisfying conditions associated with the (μ,μ)-position,

$$z^{(i)}(t_1) = y_{i+1},\ 0 \le i \le n-\mu-1,$$

$$z(t_2) = 0,$$

$$z^{(i)}(t_2) = y_{n-\mu+(i+1)},\ 1 \le i \le \mu-1,$$

and define $S = \{y^{(n-\mu)}(t_1) \mid y(t)$ is a solution of (1) satisfying $y^{(i)}(t_1) = z^{(i)}(t_1)$, $0 \le i \le n-\mu-1$, and $y^{(i)}(t_2) = z^{(i)}(t_2)$, $1 \le i \le \mu-1\}$. $S \ne \emptyset$ since $z^{(n-\mu)}(t_1) \in S$, and since solutions of the problems corresponding to $(\mu,\mu-1)$-position are unique, it follows from a standard application of the Brouwer Invariance of Domain Theorem that S is open, (see [20-21] for a typical argument).

We claim that S is also a closed subset of $\mathbb{R}$. Assuming the claim to be false, it follows that there is a limit point $r_0 \in \bar{S} \setminus S$. Hence, there exists a strictly monotone sequence $\{r_j\} \subset S$ of numbers converging to r_0. We may assume without loss of generality that $r_j \uparrow r_0$. For each $j \ge 1$, let $y_j(t)$ denote the solution of (1) given by the definition of S satisfying,

$$y_j^{(i)}(t_1) = z^{(i)}(t_1),\ 0 \le i \le n-\mu-1,$$
$$y_j^{(n-\mu)}(t_1) = r_j,$$
$$y_j^{(i)}(t_2) = z^{(i)}(t_2),\ 1 \le i \le \mu-1.$$

From Corollary 2, it follows that, for each $j \ge 1$, $y_j(t) < y_{j+1}(t)$ on $(t_1,t_2]$. Furthermore, since f satisfies the Lipschitz condition (2), it follows that a compactness condition on sequences of solutions of (1) is satisfied, (see [10]); from this compactness condition and the fact that $r_0 \notin S$, we have that $\{y_j(t)\}$ is not uniformly bounded on each compact subinterval of (a,b), and in particular, is not uniformly bounded above on each compact subinterval of $[t_1,t_2]$.

Now let $u(t)$ be the solution of the problem for (1) associated with the $(\mu-1,\mu-1)$-position,

$$u^{(i)}(t_1) = y_{i+1},\ 0 \le i \le n-\mu-1,$$
$$u^{(n-\mu)}(t_1) = r_0,$$
$$u(t_2) = 0,$$
$$u^{(i)}(t_2) = y_{n-\mu+(i+1)},\ 1 \le i \le \mu-2.$$

It follows that, for some $\delta > 0$, $y_1^{(i)}(t) < u^{(i)}(t)$ on $(t_1,\ t_1+\delta)$, $0 \le i \le n-\mu$, and either (i) $(-1)^{i+1}y_1{}^{(i)}(t) < (-1)^{i+1}u^{(i)}(t)$ on $(t_1-\delta,\ t_1)$, $0 \le i \le n-\mu$, when $n-\mu$ is odd, or (ii) $(-1)^i y_1{}^{(i)}(t) < (-1)^i u^{(i)}(t)$ on $(t_1-\delta,\ t_1)$, $0 \le i \le n-\mu$, when $n-\mu$ is even. We will assume that $n-\mu$ is odd and also that $t_2-(t_1-\delta) < \gamma$.

It follows that there exists a subsequence $\{y_{j_k}(t)\}$ such that, for each $k \ge 1$, $y_{j_k}{}^{(n-\mu)}(t)$ intersects $u^{(n-\mu)}(t)$ at a point $\rho_k \in (t_1,\ t_1+\delta)$ and $y_{j_k}{}^{(n-\mu)}(t)$ intersects $u^{(n-\mu)}(t)$ or $y_1{}^{(n-\mu)}(t)$ at a point $\sigma_k \in (t_1-\delta,\ t_1)$ and $\sigma_k \uparrow t_1$ and $\rho_k \downarrow t_1$. By choosing successive subsequences and relabeling, we may assume that $t_1-\delta < \sigma_k < t_1 < \rho_k < t_1+\delta$ are the first points where these intersections occur. Now, if there is an infinite subsequence, which we relabel as $\{y_{j_k}(t)\}$, such that $y_{j_k}{}^{(n-\mu)}(\sigma_k) = y_1{}^{(n-\mu)}(\sigma_k)$, we have that, for each k, $(-1)^{i+1}y_1{}^{(i)}(t)$ $< (-1)^{i+1}y_{j_k}{}^{(i)}(t) < (-1)^{i+1}u^{(i)}(t)$ on $(\sigma_k,\ t_1)$, $0 \le i \le n-\mu$. In this case $\lim_{k\to\infty} y_{j_k}{}^{(i)}(\sigma_k) = y_1{}^{(i)}(t_1)$, $0 < i < n-\mu$. But, it is also the case that $y_{j_k}^{(i)}(t_2) = y_1{}^{(i)}(t_2)$, $1 < i < \mu-1$, and so from the continuous dependence of solutions on boundary conditions of problems associated with the $(\mu,\mu-1)$-position, it follows that $\{y_{j_k}^{(i)}(t)\}$ converges uniformly to $y_1^{(i)}(t)$ on compact subintervals of (a,b), $0 \leq i \leq n-1$. This is impossible, since $y_{j_k}^{(n-\mu)}(t_1) = r_{j_k} + r_0 > r_1 = y_1^{(n-\mu)}(t_1)$.

In the case that there is an infinite subsequence, which we relabel again as $\{y_{j_k}(t)\}$, such that $y_{j_k}^{(n-\mu)}(\sigma_k) = u^{(n-\mu)}(\sigma_k)$ and $y_{j_k}^{(n-\mu)}(\rho_k) = u^{(n-\mu)}(\rho_k)$, it follows that, for each k, $(-1)^{i+1}y_1^{(i)}(t) < (-1)^{i+1}y_{j_k}^{(i)}(t) < (-1)^{i+1}u^{(i)}(t)$ on (σ_k,t_1) and $y_1^{(i)}(t) < y_{j_k}^{(i)}(t) < u^{(i)}(t)$ on (t_1,ρ_k), $0 \leq i \leq n-\mu$, and $y_{j_k}^{(n-\mu+1)}(\tau_k) = u^{(n-\mu+1)}(\tau_k)$, some $\tau_k \in (\sigma_k,\rho_k)$. It follows that $\lim_{k\to\infty} y_{j_k}^{(i)}(\sigma_k) = u^{(i)}(t_1)$, $0 \leq i \leq n-\mu$, and that $\lim_{k\to\infty} y_{j_k}^{(n-\mu+1)}(\tau_k) = u^{(n-\mu+1)}(t_1)$; it is also the case that $y_{j_k}^{(i)}(t_2) = u^{(i)}(t_2)$, $1 \leq i \leq \mu-2$. From uniqueness of solutions of boundary value problems for (1) corresponding to the $(\mu-1, \mu-2)$-position coupled with an argument similar to the one used in the proof of the first theorem of [13, Thm. 1] and the fact that $t_2 - (t_1 - \delta) < \gamma$, solutions of this latter type of problem for (1) are unique and thus depend continuously upon boundary conditions; it follows that $\{y_{j_k}^{(i)}(t)\}$ converges uniformly to $u^{(i)}(t)$ on compact subintervals of (a,b), $0 \leq i \leq n-1$. In particular, $u^{(\mu-1)}(t_2) = y_{j_k}^{(\mu-1)}(t_2)$, for all k, and it then follows that $u^{(n-\mu)}(t_1) = r_0 \in S$; again, a contradiction.

Thus, S is also closed and hence $S \equiv \mathbf{R}$. Choosing $y_{n-\mu} \in S$, the corresponding solution of (1) satisfies the boundary value problem corresponding to the $(\mu,\mu-1)$-position. Hence, boundary value problems for (1) associated with the first subdiagonal, $(\mu,\mu-1)$, $2 \leq \mu \leq h$, have unique solutions.

For the induction, assume now that $2 < m+1 \leq h$ and that, for each $1 \leq s \leq m$, the boundary value problems for (1) associated with the subdiagonals $(\mu,\mu-(s-1))$, $s \leq \mu \leq h$, have unique solutions.

For $s = m+1$, we now argue that boundary value problems for (1) corresponding to the subdiagonal $(\mu,\mu-m)$, where $m+1 \leq \mu \leq h$, have unique solutions. Choosing any such $(\mu,\mu-m)$, we are concerned with solutions of (1) satisfying

$$y^{(i)}(t_1) = y_{i+1},\ 0 < i < n-\mu+m-1,$$

$$y^{(i)}(t_2) = y_{n-\mu+(i+1)},\ m < i < \mu-1.$$

For the shooting scheme here, let $z(t)$ be the solution of (1) corresponding to the

$(\mu,\mu-(m-1))$-position satisfying

$$z^{(i)}(t_1) = y_{i+1},\quad 0 \le i \le n-\mu+m-2,$$

$$z^{(m-1)}(t_2) = 0,$$

$$z^{(i)}(t_2) = y_{n-\mu+(i+1)},\quad m \le i \le \mu-1.$$

In this case, define $S_1 = \{y^{(n-\mu+m-1)}(t_1) \mid y(t)$ is a solution of (1) satisfying $y^{(i)}(t_1) = z^{(i)}(t_1)$, $0 \le i \le n-\mu+m-2$, and $y^{(i)}(t_2) = z^{(i)}(t_2)$, $m \le i \le \mu-1\}$. In a manner analogous to above, it can be argued that S_1 is a nonempty subset of $\mathbf{R}$ which is both open and closed, so that $S_1 \equiv \mathbf{R}$. Choosing $y_{n-\mu+m} \in S_1$, the corresponding solution of (1) is the desired solution. Hence, boundary value problems for (1) associated with the diagonal $(\mu,\mu-m)$, $m+1 \le \mu \le h$, have unique solutions.

Therefore, by induction, boundary value problems for (1) associated with each entry in the triangular array have unique solutions, and the conclusion of the theorem holds from the cases corresponding to the bottom row (h,ν), $1 \le \nu \le h$.

Using shooting methods and an induction similar to above, one can prove existence of solutions of multipoint boundary value problems (1), (3) with the same constraints on h and k.

THEOREM 4. Let $[n/2] \le h \le n$ be given. Let $k = 0$ and let $\gamma = \min\{\gamma_\ell \mid 0 \le \ell < h\}$ be as defined in Theorem 1. Then the boundary value problem

$$y^{(n)} = f(t,y,y',\ldots,y^{(n-1)}),$$

$$y^{(i)}(t_1') = y_{i+1},\quad 0 \le i \le n-h+\ell-1,$$

$$y^{(i)}(t_i) = y_{n-h+(i+1)},\quad \ell \le i \le h-1,$$

where $a < t_1' < t_\ell \le \cdots \le t_{h-1} < b$, $0 \le \ell < h$, has a unique solution for any assignment of $y_i \in \mathbf{R}$, $1 \le i \le n$, provided $t_{h-1} - t_1' < \gamma$. This result is best possible for the class of all differential equations which satisfy the Lipschitz condition (1.2).

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