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Research Article

Exact Solutions for the Wick-Type Stochastic Schamel-Korteweg-de Vries Equation

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We consider the Wick-type stochastic Schamel-Korteweg-de Vries equation with variable coefficients in this paper. With the aid of symbolic computation and Hermite transformation, by employing the $(G'/G, 1/G)$-expansion method, we derive the new exact travelling wave solutions, which include hyperbolic and trigonometric solutions for the considered equations.

1. Introduction

The partial differential equations arise in many physical fields like the condense matter physics, fluid mechanics, plasma physics, optics, and so on, which exhibit a rich variety of nonlinear phenomena. It is known that to find exact solutions of the partial differential equations is always one of the central themes in mathematics and physics.

In [1], Lee and Sakthivel used the exp-function method to obtain some exact travelling waves solutions for the following Schamel-Korteweg-de Vries equation [2–4].

$$u_t + \left(\alpha u^{1/2} + \beta u\right) u_x + \delta u_{xxx} = 0, \tag{1}$$

where $u = u(t, x)$ denotes the unknown function of the space variable x and time t and the parameters α, β, and δ are constants which refer to the activation trapping, the convection, and the dispersion coefficients, respectively. Equation (1) arises in number of scientific models, such as plasma physics and optical fibre. This equation describes the nonlinear interaction of ion-acoustic waves when electron trapping is present and also it governs the electrostatic potential for a certain electron distribution in velocity space. In addition, a generalized KdV equation is a special case of (1) which has been studied in a variety of mathematical physics contexts. Equation (1) incorporates the well-known KdV equation when $\alpha = 0$ and the Schamel equation when $\beta = 0$ [5].

In [6], Kangalgil used the extended (G'/G)-expansion method to obtain some hyperbolic function solutions and trigonometric functions with free parameters of (1). In [7], by using the sine-cosine method and the extended tanh method, Yang and Tang obtained the soliton-like solutions, the kink solutions, and the plural solutions of (1).

When the inhomogeneities of media and nonuniformity of boundaries are taken into account in various real physical situations, the variable coefficient partial differential equations often can provide more powerful and realistic models than their constant coefficient counterparts in describing a large variety of real phenomena. Recently, the importance of taking random effects into account in modeling, analyzing, simulating, and predicting complex phenomena has been widely recognized in geophysical and climate dynamics, materials science, chemistry, biology, and other fields [8, 9]. Stochastic partial differential equations are appropriate mathematical models for complex systems under random influence in fluids.

The Wick-type stochastic Schamel-Korteweg-de Vries equation with variable coefficients is one of the most important stochastic partial differential equations and it has many applications. In [10], Holden et al. gave white noise functional approach to research stochastic partial differential equations in Wick versions. In [11], Li et al. introduced a new direct method called the $(G'/G, 1/G)$-expansion method to look for travelling wave solutions of nonlinear evolution equations. The determination of exact solutions to the variable coefficients partial differential equations is a complicated problem that challenges researchers greatly. We will use their theory and the $(G'/G, 1/G)$-expansion method to give exact solutions of the following Wick-type stochastic Schamel-Korteweg-de Vries equation with variable coefficients:

$$u_t + \left[\alpha(t) \diamondsuit u^{1/2} + \beta(t) \diamondsuit u\right] \diamondsuit u_x + \delta(t) \diamondsuit u_{xxx} = 0, \quad (2)$$

where $\diamondsuit$ is the Wick production on the Hida distribution space $(S(R^d))^*$, namely, $(S(R^d))^*$ is the white noise functional space which is defined in [12], and $\alpha(t), \beta(t)$, and $\delta(t)$ are white noise functionals.

This paper is organized as follows. In Section 2, we briefly describe some basic concepts on Wick-type and main steps of finding solutions. In Section 3, we describe the $(G'/G, 1/G)$-expansion method. In Section 4, we use white noise analysis and Hermite transformation to obtain a number of Wick versions of hyperbolic, trigonometric, and rational solutions. The conclusions are given in the final section.

2. Some Basic Concepts and Main Steps

Assume that $\mathcal{S}(\mathbb{R}^d)$ and $(\mathcal{S}(\mathbb{R}^d))^*$ are the Hida test function space and the Hida disturbance space on $\mathbb{R}^d$, respectively. And let $h_n(x)$ be the d-order Hermite polynomials. Put

$$\xi_n(x) = e^{-(1/2)x^2} \frac{h_n\left(\sqrt{2}x\right)}{(\pi(n-1)!)^{1/2}}, \quad n \geq 1. \quad (3)$$

We have that the collection $\{\xi_n\}_{n\geq 1}$ constitutes an orthogonal basis of $L^2(\mathbb{R})$.

If we denote $\alpha = (\alpha_1, \dots, \alpha_d)$ as d-dimensional multi-indices with $\alpha_1, \dots, \alpha_d \in \mathbb{N}$, we have that the family of tensor products $\xi_{(\alpha_1,\dots,\alpha_d)} = \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_d}$ $(\alpha \in \mathbb{N}^d)$ forms an orthogonal basis for $L^2(\mathbb{R}^d)$. Let $\alpha^{(i)} = (\alpha_1^{(i)}, \dots, \alpha_d^{(i)})$ be the ith multi-index number in some fixed ordering of all d-dimensional multi-indices $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$. We assume that this ordering has the following property:

$$\begin{gathered} i < j \Longrightarrow \\ \alpha_1^{(i)} + \cdots + \alpha_d^{(i)} \leq \alpha_1^{(j)} + \cdots + \alpha_d^{(j)}. \end{gathered} \quad (4)$$

We define

$$\eta_i = \xi_{\alpha^{(i)}} = \xi_{\alpha_1^{(i)}} \otimes \cdots \otimes \xi_{\alpha_d^{(i)}}, \quad (i \geq 1), \quad (5)$$

and denote multi-indices as elements of the space $(\mathbb{N}_0^{\mathbb{N}})_c$ of all sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ with elements $\alpha_i \in \mathbb{N}_0$ and with compact support, that is, with only finitely many $\alpha_i \neq 0$. We denote $\mathcal{J} \equiv (\mathbb{N}_0^{\mathbb{N}})_c$.

Fixing $n \in \mathbb{N}$, let $(\mathcal{S})_1^n$ consist of those

$$x = \sum_\alpha c_\alpha H_\alpha(w) \in \bigoplus_{k=1}^n L^2(\mu), \quad (6)$$

where $c_\alpha \in \mathbb{R}^n$, $\|x\|_{1,k}^2 \equiv \sum_\alpha c_\alpha^2 (\alpha!)^2 (2\mathbb{N})^{k\alpha} < \infty$, $\forall k \in \mathbb{N}$ with $c_\alpha^2 = |c_\alpha|^2 = \sum_{k=1}^n (c_\alpha^{(k)})^2$, $c_\alpha = (c_\alpha^{(1)}, \dots, c_\alpha^{(n)}) \in \mathbb{R}^n$, $\alpha! = \prod_{k=1}^\infty \alpha_k!$, and $(2\mathbb{N})^\alpha = \prod_j (2j)^{\alpha_j}$, where μ is the white noise measure on $(\mathcal{S}^*(\mathbb{R}), \mathcal{B}(\mathcal{S}^*(\mathbb{R})))$,

$$\begin{gathered} H_\alpha(w) = \prod_{i=1}^\infty H_{\alpha_i}(\langle w, \eta_i \rangle), \\ w \in \left(\mathcal{S}\left(\mathbb{R}^d\right)\right)^*, \ \alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{J}. \end{gathered} \quad (7)$$

The space $(\mathcal{S})_{-1}^n$ consists of all formal expansions

$$X = \sum_\alpha b_\alpha H_\alpha, \quad (8)$$

where $b_\alpha \in \mathbb{R}^n$, $\|X\|_{-1,-q} \equiv \sum_\alpha b_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$, $\forall q \in \mathbb{N}$. The family of seminorms $\|x\|_{1,k}$, $k \in \mathbb{N}$ gives rise to a topology on $(\mathcal{S})_1^n$, and we can regard $(\mathcal{S})_{-1}^n$ as the dual of $(\mathcal{S})_1^n$ by the action

$$\langle X, x \rangle = \sum_\alpha (b_\alpha, c_\alpha) \alpha!, \quad (9)$$

where (b_α, c_α) is the usual inner product in $\mathbb{R}^n$.

For $X = \sum_\alpha a_\alpha H_\alpha$, $Y = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})_{-1}^n$ with $a_\alpha, b_\alpha \in \mathbb{R}^n$,

$$X \diamondsuit Y = \sum_{\alpha,\beta} \left(a_\alpha, b_\beta\right) H_{\alpha+\beta} \quad (10)$$

is called the Wick product of X and Y.

We can prove that the spaces $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)^*$, $(\mathcal{S})_1$, and $(\mathcal{S})_{-1}$ are closed under Wick products.

For $X = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})_{-1}^n$ with $a_\alpha \in \mathbb{R}^n$, the Hermite transform of X is denoted as follows:

$$\begin{gathered} \mathcal{H}(X) = \widetilde{X}(z) = \sum_\alpha a_\alpha z^\alpha \in \mathbb{C}^n, \\ (\text{when convergent}), \end{gathered} \quad (11)$$

where $z = (z_1, z_2, \dots) \in \mathbb{C}^n$ (the set of all sequences of complex numbers), $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \cdots$ when $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{J}$.

For $X, Y \in (\mathcal{S})_{-1}^n$, we can find

$$\widetilde{X \diamondsuit Y}(z) = \widetilde{X}(z) \widetilde{Y}(z), \quad (12)$$

for all z such that $\widetilde{X}(z)$ and $\widetilde{Y}(z)$ exist. The product on the right-hand side of the above formula is the complex bilinear product between two elements of $\mathbb{C}^{\mathbb{N}}$ defined by

$$\left(z_1^1, \dots, z_n^1\right) \cdot \left(z_1^2, \dots, z_n^2\right) = \sum_{k=1}^n z_k^1 z_k^2, \quad z_k^i \in \mathbb{C}. \quad (13)$$

Let $X = \sum_\alpha a_\alpha H_\alpha, \in (\mathcal{S})_{-1}^n$. Then the vector $c_0 = \widetilde{X}(0) \in \mathbb{R}^n$ is called the generalized expectation of X and is denoted by $E(X)$. Suppose that $f : V \to \mathbb{C}^m$ is an analytic function, where V is a neighborhood of $E(X)$. Assume that the Taylor series of f around $E(X)$ has coefficients in $\mathbb{R}^n$, and we can find the Wick version $f^{\diamondsuit}(X) = \mathcal{H}^{-1}(f \circ \widetilde{X}) \in (S)_{-1}^m$.

We define the Wick exponential of $X \in (\mathcal{S})_{-1}^{n}$ as follows:

$$\exp^{\diamond}\{X\} = \sum_{n=0}^{\infty} \frac{X^{\diamond n}}{n!}. \tag{14}$$

We can find that the Wick exponential has the same algebraic properties as the usual exponential with the use of the Hermite transformation. For example, $\exp^{\diamond}\{X + Y\} = \exp^{\diamond}\{X\} \diamond \exp^{\diamond}\{Y\}$.

Suppose that modeling considerations lead us to consider an SPDE expressed formally as $A(t, x, \partial_t, \nabla_x, U, \omega) = 0$, where A is some given function and $U = U(t, x, \omega)$ is unknown (generalized) stochastic process, where the operators

$$\begin{gathered} \partial_t = \frac{\partial}{\partial t}, \\ \nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d. \end{gathered} \tag{15}$$

Firstly, we interpret all products as Wick products and all functions as their Wick versions indicate this as

$$A^{\diamond}(t, x, \partial_t, \nabla_x, U, \omega) = 0. \tag{16}$$

Secondly, we take the Hermite transformation of (16). This turns Wick products into ordinary products and the equation takes the following form:

$$\widetilde{A}\left(t, x, \partial_t, \nabla_x, \widetilde{U}, z_1, z_2, \dots\right) = 0, \tag{17}$$

where $\widetilde{U} = \mathscr{H}(U)$ is the Hermite transformation of U and $z_1, z_2, \dots$ are complex number. Suppose we find a solution $u = u(t, x, z)$ of the equation $\widetilde{A}(t, x, \partial_t, \nabla_x, u, z) = 0$ for each $z = (z_1, z_2, \dots) \in \mathbb{K}_q(r)$, where

$$\begin{aligned} &\mathbb{K}_q(r) \\ &= \left\{ z = (z_1, z_2, \dots) \in \mathbb{C}^{\mathbb{N}}, \ \sum_{\alpha \neq 0} \left|z^{\alpha}\right|^2 (2\mathbb{N})^{q\alpha} < r^2 \right\}. \end{aligned} \tag{18}$$

Then, under certain conditions, we can take the inverse Hermite transformation $U = \mathscr{H}^{-1} u \in (S)_{-1}$ and obtain a solution U of the original Wick equation (16). We have the following theorem, which was proved by Holden et al. [10].

Theorem 1. *Suppose $u(t, x, z)$ is a solution (in the usual strong, pointwise sense) of (17) for (t, x) in some bounded open set $G \subset \mathbb{R} \times \mathbb{R}^d$, and for all $z \in \mathbb{K}_q(r)$, for some q, r. Moreover, suppose that $u(t, x, z)$ and all its partial derivatives, which are involved in (17), are bounded for $(t, x, z) \in G \times \mathbb{K}_q(r)$, continuous with respect to $(t, x) \in G$ for all $z \in \mathbb{K}_q(r)$, and analytic with respect to $z \in \mathbb{K}_q(r)$, for all $(t, x) \in G$.*

Then there exists $U(t, x) \in (\mathcal{S})_{-1}$ such that $u(t, x, z) = (\widetilde{U}(t, x))(z)$ for all $(t, x, z) \in G \times \mathbb{K}_q(r)$, and $U(t, x)$ solves (in the strong sense in $(\mathcal{S})_{-1}$) (16) in $(\mathcal{S})_{-1}$.

3. Description of the $(G'/G, 1/G)$-Expansion Method

In this section, we describe the main steps of the $(G'/G, 1/G)$-expansion method for finding travelling wave solutions of nonlinear evolution equations. As preparations, consider the second order linear ordinary differential equation

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{19}$$

and we let

$$\begin{gathered} \phi = \frac{G'}{G}, \\ \psi = \frac{1}{G}, \end{gathered} \tag{20}$$

for simplicity here and after. Using (19) and (20) yields

$$\begin{gathered} \phi' = -\phi^2 + \mu\psi - \lambda, \\ \psi' = -\phi\psi. \end{gathered} \tag{21}$$

From the three cases of general solutions of the linear ordinary differential equation (19), we have the following.

Case 1. When $\lambda < 0$, the general solution of the linear ordinary differential equation (19) is

$$G(\xi) = A_1 \sinh \sqrt{-\lambda}\xi + A_2 \cosh \sqrt{-\lambda}\xi + \frac{\mu}{\lambda}, \tag{22}$$

and we have

$$\psi^2 = \frac{-\lambda}{\lambda^2 \sigma_1 + \mu^2} \left(\phi^2 - 2\mu\psi + \lambda\right), \tag{23}$$

where A_1 and A_2 are two arbitrary constants and $\sigma_1 = A_1^2 - A_2^2$.

Case 2. When $\lambda > 0$, the general solution of the linear ordinary differential equation (19) is

$$G(\xi) = A_1 \sin \sqrt{\lambda}\xi + A_2 \cos \sqrt{\lambda}\xi + \frac{\mu}{\lambda}, \tag{24}$$

and we have

$$\psi^2 = \frac{\lambda}{\lambda^2 \sigma_2 - \mu^2} \left(\phi^2 - 2\mu\psi + \lambda\right), \tag{25}$$

where A_1 and A_2 are two arbitrary constants and $\sigma_2 = A_1^2 + A_2^2$.

Case 3. When $\lambda = 0$, the general solution of the linear ordinary differential equation (19) is

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{26}$$

and we have

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2} \left(\phi^2 - 2\mu\psi\right), \tag{27}$$

where A_1 and A_2 are two arbitrary constants.

Now we consider a nonlinear evolution equation, say in two independent variables x and t,

$$P\left(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots\right) = 0. \tag{28}$$

In general, the left-hand side of (28) is a polynomial in u and its various partial derivatives. The main steps of the $(G'/G, 1/G)$-expansion method are as follows.

Step 1. By coordinates transformation $\xi = x - Vt$ and with $u(x,t) = u(\xi)$, (28) can be reduced to an ordinary differential equation on $u(\xi)$ with

$$P\left(u, -Vu', u', V^2u'', -Vu'', u'', \ldots\right) = 0. \tag{29}$$

Step 2. Suppose that the solution of the ordinary differential equation (29) can be expressed by a polynomial in ϕ and ψ as

$$u(\xi) = \sum_{i=0}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi, \tag{30}$$

where $G = G(\xi)$ satisfies the second-order linear ordinary differential equation (19), a_i $(i = 0, \ldots, N)$, b_i $(i = 1, \ldots, N)$, V, λ, and μ are constants to be determined later, and $a_N^2 + b_N^2 \neq 0$.

Step 3. Determine the positive integer N in (30) by using the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the ordinary differential equation (29). More precisely, we define the degree of $u(\xi)$ as $D[u(\xi)] = N$, which gives rise to the degree of other expressions as follows:

$$\begin{gathered} D\left[\frac{d^q u}{d\xi^q}\right] = N + q, \\ D\left[u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right] = Np + s(q + N). \end{gathered} \tag{31}$$

Therefore, we can get the value of N in (30). In some nonlinear equations the balance number N is not a positive integer. In this case, we make the following transformations [13]:

(a) When $N = q/p$, where q/p is a fraction in the lowest terms, we let

$$u(\xi) = v^{q/p}(\xi), \tag{32}$$

then substituting (32) into (29) to get a new equation in the new function $v(\xi)$ with a positive integer balance number.

(b) When N is a negative number, we let

$$u(\xi) = v^N(\xi), \tag{33}$$

then substituting (33) into (29) to get a new equation in the new function $v(\xi)$ with a positive integer balance number.

Step 4. Substituting (30) into (29), using (21) and (23) (here Case 1 is taken as an example), the left-hand side of (29) can be converted into a polynomial in ϕ and ψ, in which the degree of ψ is not larger than one. Equating each coefficient of the polynomial to zero yields a system of algebraic equations about a_i $(i = 0, \ldots, N)$, b_i $(i = 1, \ldots, N)$, V, λ, μ, A_1 and A_2. Then we solve the algebraic equations with the aid of Maple or Mathematica. Substituting the values of a_i $(i = 0, \ldots, N)$, b_i $(i = 1, \ldots, N)$, V, λ, μ, A_1, and A_2 obtained into (30), one can obtain the travelling wave solutions expressed by the hyperbolic functions of (29).

Step 5. Similar to Steps 3 and 4, substituting (30) into (29), using (21) and (25) (or (21) and (27)), we obtain the travelling wave solutions of (29) expressed by trigonometric functions (or expressed by rational functions).

4. Exact Solutions of Stochastic Schamel-Korteweg-de Vries Equation

In this section, we will give exact solutions of (2).

Taking the Hermite transformation of (2), we can get the equation

$$\begin{aligned} &\widetilde{u}_t(t,x,z) + \left[\widetilde{\alpha}(t,z)\,\widetilde{u}^{1/2}(t,x,z) + \widetilde{\beta}(t,z)\,\widetilde{u}(t,x,z)\right] \\ &\quad \cdot \widetilde{u}_x(t,x,z) + \widetilde{\delta}(t,z)\,\widetilde{u}_{xxx}(t,x,z) = 0, \end{aligned} \tag{34}$$

where $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ is a parameter.

For the sake of simplicity, we denote $u(t,x,z) = \widetilde{u}(t,x,z)$, $\alpha(t,z) = \widetilde{\alpha}(t,z)$, $\beta(t,z) = \widetilde{\beta}(t,z)$, and $\delta(t,z) = \widetilde{\delta}(t,z)$. In the following, we apply the $(G'/G, 1/G)$-expansion method to construct the travelling wave solutions of (34). In order to obtain exact solutions of (34), we consider the transformation

$$u(t,x,z) = u(\xi), \quad \xi = p(t,z)\,x + q(t,z), \tag{35}$$

where $p(t,z)$ and $q(t,z)$ will be determined later. Substituting (35) into (34), we have

$$\begin{aligned} &u_t(t,x,z) + \left[\alpha(t,z)\,u^{1/2}(t,x,z) + \beta(t,z)\,u(t,x,z)\right] \\ &\quad \cdot u_x(t,x,z) + \delta(t,z)\,u_{xxx}(t,x,z) = u'(\xi) \\ &\quad \cdot \left[p_t(t,z)\,x + q_t(t,z)\right] \\ &\quad + \left[\alpha(t,z)\,u^{1/2}(\xi) + \beta(t,z)\,u(\xi)\right]u'(\xi)\,p(t,z) \\ &\quad + \delta(t,z)\,u'''(\xi)\,p^3(t,z) = 0. \end{aligned} \tag{36}$$

This implies

$$u'(\xi)\,p_t(t,z)\,x = 0, \tag{37}$$

$$\begin{aligned} &u'(\xi)\,q_t(t,z) \\ &\quad + \left[\alpha(t,z)\,u^{1/2}(\xi) + \beta(t,z)\,u(\xi)\right]u'(\xi)\,p(t,z) \\ &\quad + \delta(t,z)\,u'''(\xi)\,p^3(t,z) = 0. \end{aligned} \tag{38}$$

Equation (37) means $p_t(t,z) = 0$. These imply that $p(t,z)$ is a constant; denote it by p. Suppose $q_t(t,z) = c\alpha(t,z)$, $\beta(t,z) = \beta\alpha(t,z)$ and $\delta(t,z) = \delta\alpha(t,z)$, where c, β, and δ are real constants. From (38), we obtain

$$q(t,z) = c\int_0^t \alpha(s,z)\,ds. \tag{39}$$

Then by (38) and (39), we get the following ordinary differential equations with constant coefficients:

$$cu'(\xi) + \left[pu^{1/2}(\xi) + p\beta u(\xi)\right]u'(\xi) + p^3\delta u'''(\xi) = 0. \tag{40}$$

Integrating (40) once, and considering the constants of integration as zero, we can find

$$cu(\xi) + \frac{2}{3}pu^{3/2}(\xi) + \frac{1}{2}p\beta u^2(\xi) + p^3\delta u''(\xi) = 0. \tag{41}$$

When we consider the transformations

$$\begin{aligned} u(\xi) &= v^2(\xi), \\ u'(\xi) &= 2v(\xi)v'(\xi), \\ u''(\xi) &= 2\left\{\left[v'(\xi)\right]^2 + v(\xi)v''(\xi)\right\}, \end{aligned} \tag{42}$$

where $\xi = px + q(t,z)$, we can rewrite (41) as

$$\begin{aligned} &cv^2(\xi) + \frac{2}{3}pv^3(\xi) + \frac{1}{2}p\beta v^4(\xi) + 2p^3\delta\left[v'(\xi)\right]^2 \\ &\quad + 2p^3\delta v(\xi)v''(\xi) = 0. \end{aligned} \tag{43}$$

For simplicity, (43) can be written as

$$v^2(\xi) + \vartheta v^3(\xi) + \iota v^4(\xi) + \kappa\left[v'(\xi)\right]^2 + \kappa v(\xi)v''(\xi) = 0, \tag{44}$$

where $\vartheta = 2p/3c$, $\iota = p\beta/2c$, $\kappa = 2p^3\delta/c$, and $c \neq 0$.

By balancing between $v(\xi)v''(\xi)$ and $v^4(\xi)$ in (44), we get

$$\begin{aligned} N + N + 2 = 4N &\Longrightarrow \\ N &= 1. \end{aligned} \tag{45}$$

Consequently, we assume

$$v(\xi) = a_1\phi(\xi) + b_1\psi(\xi), \tag{46}$$

where a_1 and b_1 are constants to be determined later satisfying $a_1^2 + b_1^2 \neq 0$.

By employing the $(G'/G, 1/G)$-expansion method with the aid of symbolic computation, we derive new exact travelling wave solutions, which include hyperbolic, trigonometric, and rational solutions for (34); there are three cases to be discussed as follows.

Case 1 (hyperbolic function solution). If $\lambda < 0$, substituting (46) into (44) and using (21) and (23), the left-hand side of (44) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero, we yield a system of algebraic equations in a_1, b_1, μ, and λ as follows:

$$\begin{aligned}
&\phi^4: a_1^4\iota + a_1^2\kappa - \frac{\lambda\left(6a_1^2b_1^2\iota + 3\kappa b_1^2\right)}{\sigma_1\lambda^2+\mu^2} + \frac{b_1^4\lambda^2\iota}{(\sigma_1\lambda^2+\mu^2)^2} = 0, \\
&\phi^3: a_1^3\vartheta + \frac{\lambda\left(6\kappa\mu a_1b_1 - 3\vartheta a_1b_1^2\right)}{\sigma_1\lambda^2+\mu^2} - \frac{8\iota\mu\lambda^2a_1b_1^3}{(\sigma_1\lambda^2+\mu^2)^2} = 0, \\
&\phi^3\psi: 2\kappa a_1b_1 + \iota a_1^3b_1 - \frac{\iota\lambda a_1b_1^3}{\sigma_1\lambda^2+\mu^2} = 0, \\
&\phi^2: a_1^2 - 4\lambda\kappa a_1^2 - \frac{\lambda}{\sigma_1\lambda^2+\mu^2}\left[b_1^2 + \lambda\left(6\iota a_1^2b_1^2 + 4\kappa b_1^2\right) + \kappa\mu^2a_1^2\right] \\
&\quad + \frac{2\iota\lambda^3b_1^4}{(\sigma_1\lambda^2+\mu^2)^2} + \frac{2\mu\lambda^2\left(\kappa\mu b_1^2 - 2\vartheta b_1^3\right)}{(\sigma_1\lambda^2+\mu^2)^2} - \frac{4\iota\mu^2\lambda^3b_1^4}{(\sigma_1\lambda^2+\mu^2)^3} = 0, \\
&\phi^2\psi: 3\vartheta a_1^2b_1 - 5\mu\kappa a_1^2 + \frac{\lambda\left(5\mu\kappa b_1^2 - \vartheta b_1^3\right)}{\sigma_1\lambda^2+\mu^2} + \frac{2\lambda\mu\left(6\iota a_1^2b_1^2 + \kappa b_1^2\right)}{\sigma_1\lambda^2+\mu^2} - \frac{16\kappa\mu\lambda^2b_1^4}{(\sigma_1\lambda^2+\mu^2)^2} = 0, \\
&\phi: \frac{6\mu\kappa a_1b_1 - 3\vartheta a_1b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{8\iota\mu a_1\lambda b_1^3}{(\sigma_1\lambda^2+\mu^2)^2} = 0, \\
&\phi\psi: 2a_1b_1 + 5\lambda\kappa a_1b_1 + \frac{2\lambda\mu\left(3\vartheta a_1b_1^2 - 6\kappa\mu a_1b_1\right)}{\sigma_1\lambda^2+\mu^2} - \frac{4\iota\lambda^2a_1b_1^3}{\sigma_1\lambda^2+\mu^2} + \frac{16\iota\mu^2\lambda^2a_1b_1^3}{(\sigma_1\lambda^2+\mu^2)^2} = 0, \\
&\psi: 2\mu\kappa a_1^2 - \frac{\lambda\left(3\mu\kappa b_1^2 - \vartheta b_1^3\right)}{\sigma_1\lambda^2+\mu^2} - \frac{2\mu\left(b_1^2 + 2\kappa\mu^2a_1^2\right)}{\sigma_1\lambda^2+\mu^2} \\
&\quad - \frac{4\mu^2\lambda\left(\vartheta b_1^3 - \kappa\mu b_1^2\right)}{(\sigma_1\lambda^2+\mu^2)^2} - \frac{8\iota\mu^3\lambda^2b_1^4}{(\sigma_1\lambda^2+\mu^2)^3} + \frac{4\iota\mu\lambda^2b_1^4}{(\sigma_1\lambda^2+\mu^2)^2} = 0, \\
&\psi^0: a_1^2\kappa - \frac{b_1^2 - a_1^2\mu^2\kappa - b_1^2\lambda\kappa}{\sigma_1\lambda^2+\mu^2} + \frac{2\lambda\mu\left(b_1^2\mu\kappa - \vartheta b_1^3\right)}{(\sigma_1\lambda^2+\mu^2)^2} + \frac{\iota\lambda^2b_1^4}{(\sigma_1\lambda^2+\mu^2)^2} - \frac{4\iota\mu^2\lambda^2b_1^4}{(\sigma_1\lambda^2+\mu^2)^3} = 0.
\end{aligned} \tag{47}$$

Solving the above algebraic equations, we get the following solutions: when $3\kappa\sigma_1/(\iota\lambda\kappa^2(7\iota-48\kappa)^2-3\iota\kappa\vartheta^2)>0$, then

$$\begin{aligned} a_1 &= 0, \\ b_1 &= \pm\kappa\lambda(7\iota-48\kappa)\sqrt{\frac{3\kappa\sigma_1}{\iota\lambda\kappa^2(7\iota-48\kappa)^2-3\iota\kappa\vartheta^2}}, \\ \lambda &= \frac{30\iota\vartheta^2+12\iota\vartheta^2(7\iota-48\kappa)+(7\iota-48\kappa)^2}{2\kappa(7\iota-48\kappa)^2}, \\ \mu &= \pm\vartheta\iota\lambda\sqrt{\frac{3\kappa\sigma_1}{\iota\lambda\kappa^2(7\iota-48\kappa)^2-3\iota\kappa\vartheta^2}}. \end{aligned} \tag{48}$$

From these solutions, we obtain the hyperbolic function solutions of (34).

Theorem 2. *Suppose that* $(30\iota\vartheta^2+12\iota\vartheta^2(7\iota-48\kappa)+(7\iota-48\kappa)^2)/2\kappa(7\iota-48\kappa)^2<0$ *and* $3\kappa\sigma_1/(\iota\lambda\kappa^2(7\iota-48\kappa)^2-3\iota\kappa\vartheta^2)>0$. *Then (34) admits an exact solution*

$$u(t,x,z)=\frac{3\kappa^3\lambda^2\sigma_1(7\iota-48\kappa)^2}{\left[\iota\lambda\kappa^2(7\iota-48\kappa)^2-3\iota\kappa\vartheta^2\right]d^2(t,x,z)}, \tag{49}$$

where

$$\begin{aligned} d(t,x,z) &= A_1 \\ &\cdot \sinh\left[\sqrt{-\frac{30\iota\vartheta^2+12\iota\vartheta^2(7\iota-48\kappa)+(7\iota-48\kappa)^2}{2\kappa(7\iota-48\kappa)^2}}\xi\right] \\ &+ A_2 \\ &\cdot \cosh\left[\sqrt{-\frac{30\iota\vartheta^2+12\iota\vartheta^2(7\iota-48\kappa)+(7\iota-48\kappa)^2}{2\kappa(7\iota-48\kappa)^2}}\xi\right] \\ &\pm \vartheta\iota\sqrt{\frac{3\kappa\sigma_1}{\iota\lambda\kappa^2(7\iota-48\kappa)^2-3\iota\kappa\vartheta^2}}, \end{aligned} \tag{50}$$

$\xi = px + c\int_0^t \alpha(s,z)ds$, $\vartheta = 2p/3c$, $\iota = p\beta/2c$, $\kappa = 2p^3\delta/c$, $c \neq 0$, $\sigma_1 = A_1^2 - A_2^2$, *and* A_1 *and* A_2 *are two arbitrary constants.*

Case 2 (trigonometric function solution). If $\lambda > 0$, substituting (46) into (44) and using (21) and (25), the left-hand side of (44) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero, we yield a system of algebraic equations in a_1, b_1, μ, and λ as follows:

$$\begin{aligned}
\phi^4 &: 3\kappa a_1^2+\kappa a_1^4+\frac{\lambda\left(6\iota a_1^2b_1^2+3\kappa b_1^2\right)}{\sigma_2\lambda^2-\mu^2} + \frac{\lambda\iota b_1^4}{\left(\sigma_2\lambda^2-\mu^2\right)^2}=0, \\
\phi^3 &: a_1^3\vartheta+\frac{\lambda\left(3\vartheta a_1b_1^2-6\kappa\mu a_1b_1\right)}{\sigma_2\lambda^2-\mu^2}-\frac{8\iota\mu\lambda^2b_1^3}{\left(\sigma_2\lambda^2-\mu^2\right)^2}=0, \\
\phi^3\psi &: 2\kappa a_1b_1+\iota a_1^3b_1+\frac{\iota\lambda a_1b_1^3}{\sigma_2\lambda^2-\mu^2}=0, \\
\phi^2 &: a_1^2+4\kappa\lambda a_1^2 + \frac{\lambda}{\sigma_2\lambda^2-\mu^2}\left[b_1^2+\lambda\left(4\kappa b_1^2+6\iota a_1^2b_1^2\right)+\kappa a_1^2\mu^2\right] \\
&\quad + \frac{2\iota\lambda^3b_1^4}{\left(\sigma_2\lambda^2-\mu^2\right)^2}+\frac{2\mu\lambda^2\left(\kappa\mu b_1^2-\vartheta b_1^3\right)}{\left(\sigma_2\lambda^2-\mu^2\right)^2} + \frac{4\iota\mu^2\lambda^3b_1^4}{\left(\sigma_2\lambda^2-\mu^2\right)^3}=0, \\
\phi^2\psi &: -5\kappa\mu a_1^2+3\vartheta a_1^2b_1+\frac{\lambda\left(\vartheta b_1^3-3\kappa\mu b_1^2\right)}{\sigma_2\lambda^2-\mu^2} - \frac{2\lambda\mu\left(6\iota a_1^2b_1^2+2\kappa b_1^2\right)}{\sigma_2\lambda^2-\mu^2}-\frac{4\iota\mu\lambda^2b_1^4}{\left(\sigma_2\lambda^2-\mu^2\right)^2}=0, \\
\phi &: \frac{3\vartheta a_1b_1^2-6\kappa a_1b_1}{\sigma_2\lambda^2-\mu^2}+\frac{8\iota\mu\lambda a_1b_1^3}{\left(\sigma_2\lambda^2-\mu^2\right)^2}=0, \\
\phi\psi &: 2a_1b_1+5\kappa\lambda a_1b_1+\frac{2\mu\lambda\left(6\mu\kappa a_1b_1-3\vartheta a_1b_1^2\right)}{\sigma_2\lambda^2-\mu^2} + \frac{4\iota\lambda^2a_1b_1^3}{\sigma_2\lambda^2-\mu^2}+\frac{16\iota\mu^2\lambda^2a_1b_1^3}{\left(\sigma_2\lambda^2-\mu^2\right)^2}=0, \\
\psi &: 2\kappa\mu a_1^2-\frac{\lambda\left(\vartheta b_1^3-3\kappa\mu b_1^2\right)}{\sigma_2\lambda^2-\mu^2}+\frac{2\mu\left(b_1^2+\kappa\mu^2a_1^2\right)}{\sigma_2\lambda^2-\mu^2} - \frac{4\lambda\mu^2\left(\vartheta b_1^3-\kappa\mu b_1^2\right)}{\left(\sigma_2\lambda^2-\mu^2\right)^2}+\frac{8\iota\mu^3\lambda^2b_1^4}{\left(\sigma_2\lambda^2-\mu^2\right)^3}=0, \\
\psi^0 &: \kappa a_1^2+\frac{b_1^2+\kappa\mu^2a_1^2+\kappa\lambda b_1^2}{\sigma_2\lambda^2-\mu^2}+\frac{2\mu\lambda\left(\kappa\mu b_1^2-2\vartheta b_1^3\right)}{\left(\sigma_2\lambda^2-\mu^2\right)^2} + \frac{\iota\lambda^2b_1^4}{\left(\sigma_2\lambda^2-\mu^2\right)^2}+\frac{4\iota\mu^2\lambda^2b_1^4}{\left(\sigma_2\lambda^2-\mu^2\right)^3}=0.
\end{aligned} \tag{51}$$

Solving the above algebraic equations, we get the following solutions: when $3\sigma_2/(12\vartheta^2-100\iota)>0$, then

$$\begin{aligned} a_1 &= 0, \\ b_1 &= \pm 5\sqrt{\frac{3\sigma_2}{12\vartheta^2-100\iota}}, \\ \lambda &= \frac{1}{2\kappa}, \\ \mu &= \mp\frac{\vartheta}{\kappa}\sqrt{\frac{3\sigma_2}{12\vartheta^2-100\iota}}. \end{aligned} \tag{52}$$

From these solutions, we obtain the hyperbolic function solutions of (34).

Theorem 3. *Suppose that* $3\sigma_2/(12\vartheta^2 - 100\iota) > 0$ *and* $\kappa > 0$. *Then (34) admits an exact solution*

$$u(t,x,z) = \frac{75\sigma_2}{(12\vartheta^2 - 100\iota)\, e^2(t,x,z)}, \tag{53}$$

where

$$e(t,x,z) = A_1 \sin\frac{\xi}{\sqrt{2\kappa}} + A_2 \cos\frac{\xi}{\sqrt{2\kappa}} \mp 2\vartheta\sqrt{\frac{3\sigma_2}{12\vartheta^2 - 100\iota}}, \tag{54}$$

$\xi = px + c\int_0^t \alpha(s,z)ds$, $\vartheta = 2p/3c$, $\iota = p\beta/2c$, $\kappa = 2p^3\delta/c$, $c \neq 0$, $\sigma_2 = A_1^2 + A_2^2$, *and* A_1 *and* A_2 *are two arbitrary constants.*

Case 3 (rational function solutions). If $\lambda = 0$, substituting (46) into (44) and using (21) and (27), the left-hand side of (44) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero, we yield a system of algebraic equations in a_1, b_1, μ, and λ as follows:

$$\begin{aligned}
&\phi^4: 3\kappa a_1^2 + \iota a_1^4 + \frac{3b_1^2(\kappa + 2\iota a_1^2)}{A_1^2 - 2\mu A_2} + \frac{\iota b_1^4}{(A_1^2 - 2\mu A_2)^2} = 0,\\
&\phi^3: \vartheta a_1^3 + \frac{3\vartheta a_1 b_1^2 - 6\kappa\mu a_1 b_1}{A_1^2 - 2\mu A_2} - \frac{8\iota\mu a_1 b_1^3}{(A_1^2 - 2\mu A_2)^2} = 0,\\
&\phi^3\psi: 3\kappa a_1 b_1 + 2\iota a_1^3 b_1 + \frac{2\iota a_1 b_1^3}{A_1^2 - 2\mu A_2} = 0,\\
&\phi^2: a_1^2 + 4\kappa\lambda a_1^2 + \frac{b_1^2 + \kappa\mu^2 a_1^2}{A_1^2 - 2\mu A_2} + \frac{\lambda\kappa b_1^2}{A_1^2 - 2\mu A_2} + \frac{2\mu(\kappa\mu b_1^2 - \vartheta b_1^3)}{(A_1^2 - 2\mu A_2)^2} + \frac{4\iota\mu^2 b_1^4}{(A_1^2 - 2\mu A_2)^3} = 0,\\
&\phi^2\psi: 3\vartheta a_1^2 b_1 - 5\kappa\mu a_1^2 + \frac{\vartheta b_1^3 - 12\iota\mu a_1^2 b_1^2 - 7\kappa\mu b_1^2}{A_1^2 - 2\mu A_2} - \frac{4\iota\mu b_1^4}{(A_1^2 - 2\mu A_2)^2} = 0,\\
&\phi\psi: 2a_1 b_1 + 5\kappa\lambda a_1 b_1 + \frac{2\mu(6\kappa\mu a_1 b_1 - 3\vartheta a_1 b_1^2)}{A_1^2 - 2\mu A_2} + \frac{16\iota\mu^2 a_1 b_1^3}{(A_1^2 - 2\mu A_2)^2} = 0,\\
&\psi: \mu\kappa\lambda a_1^2 + \frac{\mu(b_1^2 + \mu^2 a_1^2 + \kappa\lambda b_1^2)}{A_1^2 - 2\mu A_2} + \frac{2\mu^2(\kappa\mu b_1^2 - \vartheta b_1^3)}{(A_1^2 - 2\mu A_2)^2} + \frac{4\iota\mu^3 b_1^4}{(A_1^2 - 2\mu A_2)^3} = 0,\\
&\psi^0: \kappa a_1^2 \lambda^2 = 0.
\end{aligned} \tag{55}$$

Solving the above algebraic equations, we get the following solutions: when $3\kappa(A_1^2 - 2\mu A_2)/\iota < 0$, then

$$\begin{aligned}
a_1 &= 0,\\
b_1 &= \pm\sqrt{-\frac{3\kappa(A_1^2 - 2\mu A_2)}{\iota}},\\
\mu &= \mp\frac{\vartheta}{5\kappa}\sqrt{-\frac{3\kappa(A_1^2 - 2\mu A_2)}{\iota}},\\
\iota &= 0.
\end{aligned} \tag{56}$$

From the above expression of b_1, μ, and ι, we can get a contradiction. In Case 3, (34) does not admit an exact solution.

In order to get exact solutions of (2), we give the following conditions: $(\star)$ suppose $\alpha(t)$, $\beta(t)$, and $\delta(t)$ satisfy the conditions that there exist a bounded open set $G \subset \mathbb{R} \times \mathbb{R}^d$, $q > 0$, and $r > 0$ such that $u_t(t,x,z)$, $u^{1/2}(t,x,z)u_x(t,x,z)$, $u(t,x,z)u_x(t,x,z)$, and $u_{xxx}(t,x,z)$ are uniformly bounded for all $(t,x,z) \in G \times \mathbb{K}_q(r)$, continuous with respect to $(t,x) \in G$ for all $z \in \mathbb{K}_q(r)$ and analytic with respect to $z \in \mathbb{K}_q(r)$, for all $(t,x) \in G$.

Under condition $(\star)$ Theorem 1 implies that there exists $u(t,x) \in (\mathcal{S})_{-1}$ such that

$$u(t,x,z) = \mathcal{H}u(t,x), \tag{57}$$

which solves (34) for all $(t,x,z) \in G \times \mathbb{K}_q(r)$, $u(t,x)$ which solves (2). From (57), we can find that $u(t,x)$ is the inverse Hermite transformation of $u(t,x,z)$.

Hence, by Theorems 2 and 3, we yield the solutions of (2) as follows.

Theorem 4. *Suppose that* $(30\iota\vartheta^2 + 12\iota\vartheta^2(7\iota - 48\kappa) + (7\iota - 48\kappa)^2)/2\kappa(7\iota - 48\kappa)^2 < 0$ *and* $3\kappa\sigma_1/(\iota\lambda\kappa^2(7\iota - 48\kappa)^2 - 3\iota\kappa\vartheta^2) > 0$. *Then (2) admits an exact solution*

$$u(t,x) = \frac{3\kappa^3\lambda^2\sigma_1(7\iota - 48\kappa)^2}{\left[\iota\lambda\kappa^2(7\iota - 48\kappa)^2 - 3\iota\kappa\vartheta^2\right][h(t,x)]^{\diamond 2}}, \tag{58}$$

where

$$h(t,x) = A_1 \cdot \sinh^{\diamond}\left\{\sqrt{-\frac{30\iota\vartheta^2 + 12\iota\vartheta^2(7\iota - 48\kappa) + (7\iota - 48\kappa)^2}{2\kappa(7\iota - 48\kappa)^2}}\left[px + c\int_0^t \alpha(s)\,ds\right]\right\} + A_2$$

$$\cdot \cosh^{\diamond}\left\{\sqrt{-\frac{30\iota\vartheta^2+12\iota\vartheta^2(7\iota-48\kappa)+(7\iota-48\kappa)^2}{2\kappa(7\iota-48\kappa)^2}}\left[px + c\int_0^t \alpha(s)\,ds\right]\right\} \pm \vartheta\iota\sqrt{\frac{3\kappa\sigma_1}{\iota\lambda\kappa^2(7\iota-48\kappa)^2-3\iota\kappa\vartheta^2}}, \tag{59}$$

$\vartheta = 2p/3c$, $\iota = p\beta/2c$, $\kappa = 2p^3\delta/c$, $c \neq 0$, $\sigma_1 = A_1^2 - A_2^2$, *A_1 and A_2 are two arbitrary constants.*

Theorem 5. *Suppose that $3\sigma_2/(12\vartheta^2 - 100\iota) > 0$ and $\kappa > 0$. Then (2) admits an exact solution*

$$u(t,x) = \frac{75\sigma_2}{(12\vartheta^2 - 100\iota)[i(t,x)]^{\diamond 2}}, \tag{60}$$

where

$$\begin{aligned} i(t,x) &= A_1\sin^{\diamond}\left\{\frac{1}{\sqrt{2\kappa}}\left[px + c\int_0^t \alpha(s)\,ds\right]\right\} \\ &\quad + A_2\cos^{\diamond}\left\{\frac{1}{\sqrt{2\kappa}}\left[px + c\int_0^t \alpha(s)\,ds\right]\right\} \\ &\quad \mp 2\vartheta\sqrt{\frac{3\sigma_2}{12\vartheta^2-100\iota}}, \end{aligned} \tag{61}$$

$\vartheta = 2p/3c$, $\iota = p\beta/2c$, $\kappa = 2p^3\delta/c$, $c \neq 0$, $\sigma_2 = A_1^2 + A_2^2$, *and A_1 and A_2 are two arbitrary constants.*

Since Wick versions of functions are usually difficult to evaluate, we will give some non-Wick versions of solutions of (2) in special cases.

Let $\alpha(t) = f(t) + aW(t)$, where a is a constant. $f(t)$ is an integral or bounded measurable function on $\mathbb{R}_+$. $W(t)$ is Wiener white noise: that is, $W(t) = \dot{B}_t$, B_t is a Brown motion. We have the Hermite transforms: $\alpha(t,z) = f(t) + a\widetilde{W}(t,z)$, where $\widetilde{W}(t,z) = \sum_{k=1}^{\infty}\int_0^t \eta_k(s)ds\, z_k$, $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ is parameter, and $\eta_k(s)$ is defined in the second section.

Using $\exp^{\diamond}\{X\} = \exp\{X\}$ for nonrandom X, $\exp^{\diamond}\{B_t\} = \exp\{B_t - (1/2)t^2\}$ and the definitions of $\sin(\xi)$, $\cos(\xi)$, $\sinh(\xi)$, and $\cosh(\xi)$, we have

$$\begin{aligned} \sin^{\diamond}B_t &= \frac{1}{2}\left[\exp^{\diamond}(iB_t) - \exp^{\diamond}(-iB_t)\right] \\ &= \sin\left(B_t - \frac{1}{2}t^2\right), \\ \cos^{\diamond}B_t &= \frac{1}{2}\left[\exp^{\diamond}(iB_t) + \exp^{\diamond}(-iB_t)\right] \\ &= \cos\left(B_t - \frac{1}{2}t^2\right), \\ \sinh^{\diamond}B_t &= \frac{1}{2}\left[\exp^{\diamond}(B_t) - \exp^{\diamond}(-B_t)\right] \\ &= \sinh\left(B_t - \frac{1}{2}t^2\right), \\ \cosh^{\diamond}B_t &= \frac{1}{2}\left[\exp^{\diamond}(B_t) + \exp^{\diamond}(-B_t)\right] \\ &= \cosh\left(B_t - \frac{1}{2}t^2\right). \end{aligned} \tag{62}$$

Hence, by Theorems 4 and 5, we yield the solutions of (2) in special cases as follows.

Theorem 6. *Suppose that $(30\iota\vartheta^2 + 12\iota\vartheta^2(7\iota - 48\kappa) + (7\iota - 48\kappa)^2)/2\kappa(7\iota-48\kappa)^2 < 0$ and $3\kappa\sigma_1/(\iota\lambda\kappa^2(7\iota-48\kappa)^2 - 3\iota\kappa\vartheta^2) > 0$. Then (2) admits an exact solution*

$$u(t,x) = \frac{3\kappa^3\lambda^2\sigma_1(7\iota-48\kappa)^2}{\left[\iota\lambda\kappa^2(7\iota-48\kappa)^2 - 3\iota\kappa\vartheta^2\right][m(t,x)]^2}, \tag{63}$$

where

$$\begin{aligned} m(t,x) &= A_1 \\ &\quad \cdot \sinh\left\{\sqrt{-\frac{30\iota\vartheta^2+12\iota\vartheta^2(7\iota-48\kappa)+(7\iota-48\kappa)^2}{2\kappa(7\iota-48\kappa)^2}}\left[px + c\int_0^t f(s)\,ds + ca\left(B_t - \frac{1}{2}t^2\right)\right]\right\} + A_2 \\ &\quad \cdot \cosh\left\{\sqrt{-\frac{30\iota\vartheta^2+12\iota\vartheta^2(7\iota-48\kappa)+(7\iota-48\kappa)^2}{2\kappa(7\iota-48\kappa)^2}}\left[px + c\int_0^t f(s)\,ds + ca\left(B_t - \frac{1}{2}t^2\right)\right]\right\} \\ &\quad \pm \vartheta\iota\sqrt{\frac{3\kappa\sigma_1}{\iota\lambda\kappa^2(7\iota-48\kappa)^2-3\iota\kappa\vartheta^2}}, \end{aligned} \tag{64}$$

$\vartheta = 2p/3c$, $\iota = p\beta/2c$, $\kappa = 2p^3\delta/c$, $c \neq 0$, $\sigma_1 = A_1^2 - A_2^2$, *and A_1 and A_2 are two arbitrary constants.*

Theorem 7. *Suppose that $3\sigma_2/(12\vartheta^2 - 100\iota) > 0$ and $\kappa > 0$. Then (2) admits an exact solution*

$$u(t,x) = \frac{75\sigma_2}{(12\vartheta^2 - 100\iota)[n(t,x)]^2}, \tag{65}$$

where

$$\begin{aligned} n(t,x) &= A_1\sin\left\{\frac{1}{\sqrt{2\kappa}}\left[px + c\int_0^t f(s)\,ds + ca\left(B_t - \frac{1}{2}t^2\right)\right]\right\} + A_2\cos\left\{\frac{1}{\sqrt{2\kappa}}\left[px + c\int_0^t f(s)\,ds + ca\left(B_t - \frac{1}{2}t^2\right)\right]\right\} \\ &\quad \mp 2\vartheta\sqrt{\frac{3\sigma_2}{12\vartheta^2-100\iota}}, \end{aligned} \tag{66}$$

$\vartheta = 2p/3c$, $\iota = p\beta/2c$, $\kappa = 2p^3\delta/c$, $c \neq 0$, $\sigma_2 = A_1^2 + A_2^2$, *and A_1 and A_2 are two arbitrary constants.*

5. Conclusion

On the earlier works given in the reference list, the authors considered the exact solutions of the Schamel-Korteweg-de Vries equation. But in this paper, we use the $(G'/G, 1/G)$-expansion method to study the Wick-type stochastic Schamel-Korteweg-de Vries equation. We derive some new exact travelling wave solutions, which include hyperbolic and trigonometric solutions for the considered equations. The obtained solutions with free parameters may be important to expose most complex physical phenomena or to find new phenomena. It is shown in this paper that the $(G'/G, 1/G)$-expansion method, with the help of symbolic computation like Maple or Mathematica, is direct, concise, and elementary.

Compared to other methods, like the exp-function method [1], the extended (G'/G)-expansion method [6], the sine-cosine method, and the extended tanh method [7], the $(G'/G, 1/G)$-expansion method is effective and powerful in finding exact solutions of many other nonlinear evolution equations in mathematical physics, applied mathematics, and engineering. The equations are very difficult to be solved by traditional methods.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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