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Research Article

A Chaotic System with an Infinite Number of Equilibrium Points: Dynamics, Horseshoe, and Synchronization

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Discovering systems with hidden attractors is a challenging topic which has received considerable interest of the scientific community recently. This work introduces a new chaotic system having hidden chaotic attractors with an infinite number of equilibrium points. We have studied dynamical properties of such special system via equilibrium analysis, bifurcation diagram, and maximal Lyapunov exponents. In order to confirm the system's chaotic behavior, the findings of topological horseshoes for the system are presented. In addition, the possibility of synchronization of two new chaotic systems with infinite equilibria is investigated by using adaptive control.

1. Introduction

Nonlinear systems with chaotic behavior have been exploited since the 1960s [1–4]. Their applications have been witnessed in numerous areas, for example, secure digital communication systems [5], multiple input multiple output radar [6], image encryption with random bit sequence [7], or optimization algorithms [8]. Although almost normal chaotic systems have a countable number of equilibrium points, few unusual systems with infinite number of equilibria have been investigated in the last five years [9]. Chaotic system with line equilibrium was reported in [9–11]. A new class of chaotic systems with circle and square equilibrium was discovered by using predefined general forms [12, 13]. In addition, hyperchaotic behavior was observed in a four-dimensional system with a curve of equilibria [14] or four-dimensional systems with a line of equilibria [15–17].

Remarkably, systems with an infinite number of equilibrium points are considered as systems with "hidden attractors" based on the view point of computation [18–21]. Hidden attractors cause unexpected effects for engineering systems [22–25]. However, the characteristics of hidden attractors are not well understood [26]. The community has raised some concerns about discovering hidden attractors in known systems [27, 28], finding new systems with hidden attractors [29, 30], studying synchronization schemes for systems with hidden attractors [31], or verifying chaotic dynamics in systems with hidden attractors with topological horseshoes [32, 33].

Motivated by special features of systems with hidden attractors, we introduce a new system with an open curve of equilibrium points in this work. In the next section, the model of the new system is described and its dynamics are discovered through different nonlinear tools. Chaotic dynamics of the proposed system are studied through topological horseshoes in Section 3. A possible synchronization of two new identical systems is discussed in Section 4. Finally, Section 5 concludes our work.

2. New System with an Infinite Number of Equilibrium Points and Its Properties

The new system proposed in the present work is a three-dimensional continuous system described as

$$\begin{aligned}\dot{x} &= -z,\\ \dot{y} &= xz^2,\\ \dot{z} &= x - y\tanh(y) + z\left(ay^2 - z^2\right),\end{aligned} \tag{1}$$

in which three state variables are x, y, and z. It is worth noting that there is only one positive parameter (a) in system (1).

It is straightforward to find the equilibrium points of the proposed system by setting the right hand side of (1) to equal zero, that is,

$$-z = 0, \tag{2}$$

$$xz^2 = 0, \tag{3}$$

$$x - y\tanh(y) + z\left(ay^2 - z^2\right) = 0. \tag{4}$$

Equation (2) reveals that $z = 0$. By substituting $z = 0$ into (3) and (4) we have

$$x - y\tanh(y) = 0. \tag{5}$$

In other words, system (1) has an infinite number of equilibrium points:

$$E = \left\{(x, y, z) \in R^3 \mid x = y^*\tanh(y^*),\ y = y^*,\ z = 0\right\}. \tag{6}$$

For the equilibrium E, the Jacobian matrix of system (1) is given by

$$\mathbf{J}_E = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & -\tanh(y^*) - y^*\left(1 - \tanh^2(y^*)\right) & a(y^*)^2 \end{bmatrix}. \tag{7}$$

On combining this result with $\det(\mathbf{J}_E - \lambda\mathbf{I}) = 0$, we obtain its characteristic equation

$$\lambda\left(\lambda^2 - a(y^*)^2\lambda + 1\right) = 0. \tag{8}$$

It is easy to verify that the characteristic equation (8) has one zero eigenvalue ($\lambda_1 = 0$) and two nonzero eigenvalues ($\lambda_{2,3}$) which depend on the sign of the discriminant:

$$\Delta = a^2(y^*)^4 - 4. \tag{9}$$

For $\Delta = 0$, we get positive eigenvalues $\lambda_{2,3} = a(y^*)^2/2$. Two nonzero eigenvalues are $\lambda_{2,3} = (a(y^*)^2 \pm \sqrt{\Delta})/2$ for the positive discriminant. When the discriminant (9) is negative, a pair of complex conjugate eigenvalues is $\lambda_{2,3} = (a(y^*)^2 \pm i\sqrt{\Delta})/2$. These eigenvalues state that the equilibrium point E is unstable for $a > 0$ and $y^* \neq 0$.

It is interesting that system (1) with uncountable equilibria is chaotic for $a = 2.9$ and the initial condition $(x(0), y(0), z(0)) = (0, 0.1, 0.2)$. Chaotic attractors of system (1) are presented in Figure 1. Its Lyapunov exponents and Kaplan–Yorke dimension are $L_1 = 0.0727$, $L_2 = 0$, $L_3 = -0.3122$, and $D_{KY} = 2.2329$, respectively. The well-known Wolf's method [34] has been applied to calculate the Lyapunov exponents in our work. The time of computation is 10,000. It is worth noted that, in general, in numerical experiments one cannot expect to get the same values of the finite-time local Lyapunov exponents and the Lyapunov dimension for different points [35–37]. Therefore, the maximum of the finite-time local Lyapunov dimensions on the grid of points has to be considered [35–37].

The value of parameter a has been changed to get detailed dynamics of system (1) with infinite equilibria. By decreasing the value of the parameter a from 3.4 to 2.8, the bifurcation diagram and maximal Lyapunov exponents (MLEs) of system (1) are shown in Figures 2 and 3, respectively. It is possible to observe a route from period-doubling limit cycles to chaos when decreasing the value of the parameter a. When $a > 3.048$, system (1) remains at periodical states, for example, periodical states for $a = 3.35$ are illustrated in Figure 4. System (1) can generate chaotic attractors for $a \leq 3.048$.

3. Horseshoe in the Chaotic System with Infinite Equilibria

Topological horseshoe is a different effective approach to investigate chaos in dynamical systems [38–44]. There is significant attention about seeking topological horseshoe in chaotic systems with hidden attractors [32, 33]. Therefore, in this section we will discover topological horseshoes in the proposed system with infinite equilibria (1).

In order to support the verification of chaos in system with infinite equilibria (1), the most important results of topological horseshoe [45–47] are reviewed briefly. We define X and D as a metric space and a compact subset while f is a map $f : D \to X$. We assume that there are m mutually disjoint compact subsets of D (i.e., $D_1, D_2, \ldots, D_m$) and the restriction of f to each D_i is continuous. A compact subset d of D satisfies $d_i = d \cap D_i$ for $1 \leq i \leq m$. In this case, d is a connection with respect to m mutually disjoint compact subsets of D. We denote F as a family of connections with respect to m mutually disjoint compact subsets of D. The family F is an f-connected family with respect to m mutually disjoint compact subsets of D when

$$\begin{aligned} d \in F &\Longrightarrow \\ f(d_i) &\in F. \end{aligned} \tag{10}$$

Horseshoe Lemma (see [48]). *If there is an f-connected family F with respect to m mutually disjoint compact subsets of D, then there is the presence of a compact invariant set $K \subset D$ and semiconjugate to m-shift dynamics is $f \mid K$.*

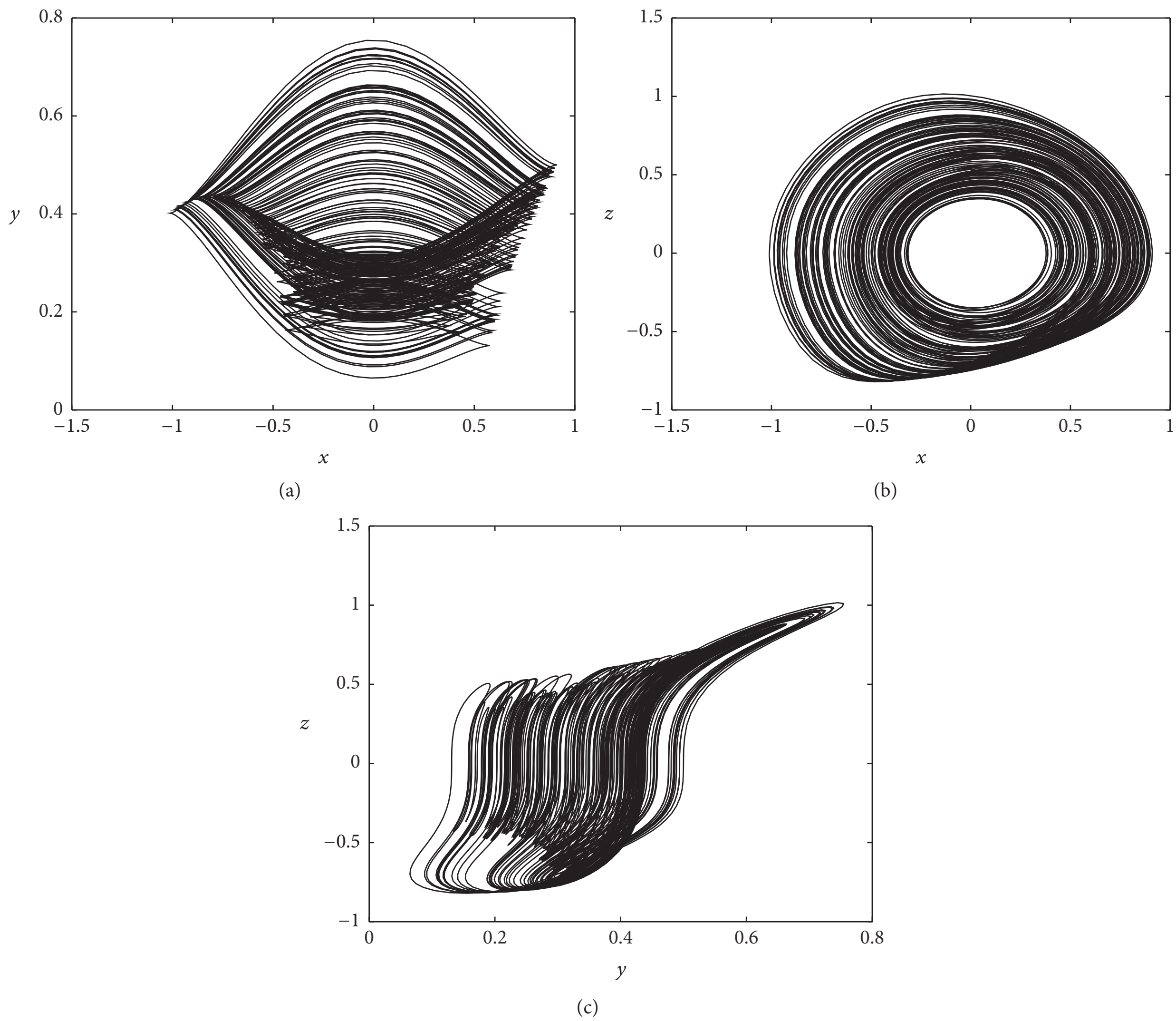

Figure 1: Chaotic attractor of the system with infinite equilibria (1) in (a) x-y plane, (b) x-z plane, and (c) y-z plane.

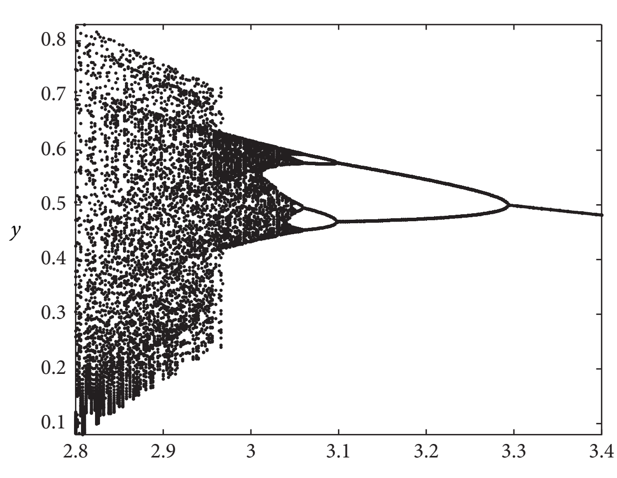

Figure 2: Bifurcation diagram of the system with infinite equilibria (1) when varying a.

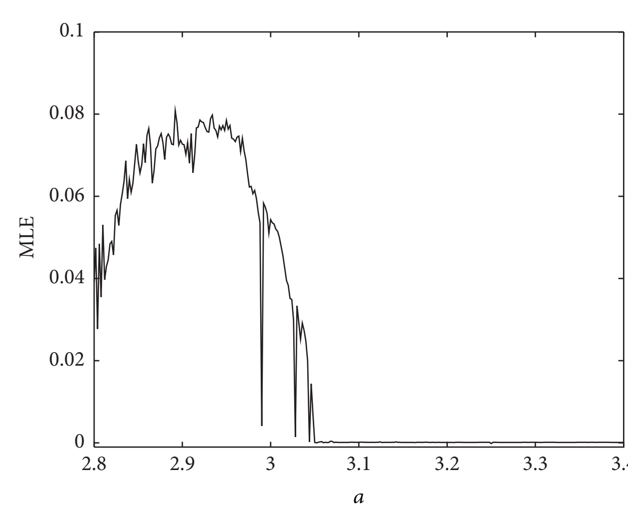

Figure 3: Maximal Lyapunov exponents of system (1) with respect to the bifurcation parameter a.

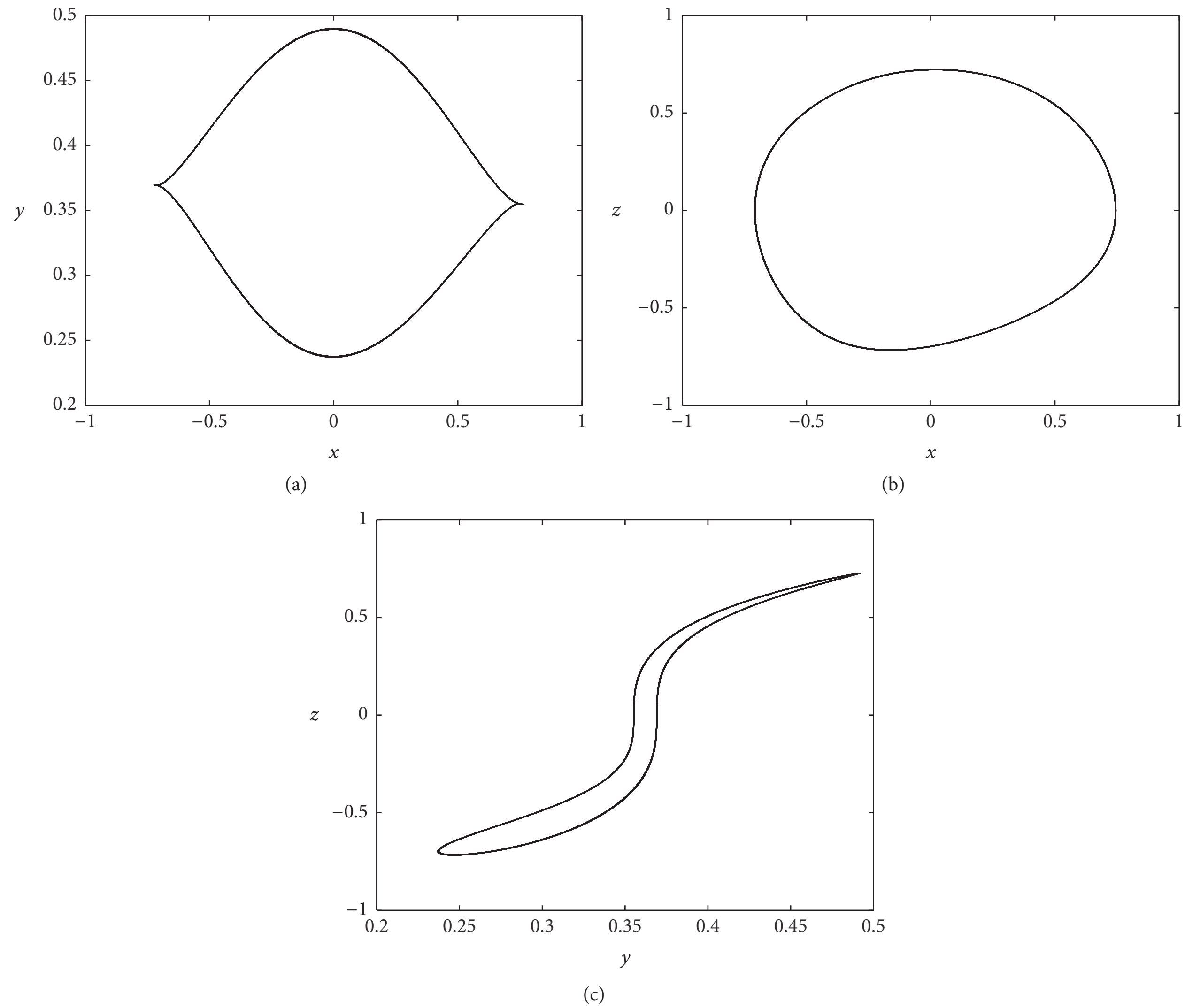

Figure 4: Limit cycle of the system with infinite equilibria (1) in (a) x-y plane, (b) x-z plane, and (c) y-z plane for $a = 3.35$.

In order to find the topology horseshoe, we select two polygon subsets D_1, D_2 in the Poincaré map Γ of the system with infinite equilibria (1):

$$\Gamma = \left\{ (x, y, z) \in R^3 \mid z = 0 \right\}. \tag{11}$$

The corresponding Poincaré map H is defined as

$$H : \Gamma \longrightarrow \Gamma. \tag{12}$$

Here $H(p)$ is the image of the initial p that returns back to Γ at the first time [48]. The same definition can be applied to the corresponding Poincaré map H^n. In this work, four vertices of the first polygon subset D_1 are selected as

$$\begin{gathered} (0.822470322, 0.438565370), \\ (0.823776275, 0.435278699), \\ (0.815679371, 0.429691358), \\ (0.813328658, 0.432978029), \end{gathered} \tag{13}$$

while four vertices of the second polygon subset D_2 are chosen as

$$\begin{gathered} (0.796873661, 0.424104017), \\ (0.798701994, 0.421146013), \\ (0.794261757, 0.418188009), \\ (0.792694615, 0.420160011). \end{gathered} \tag{14}$$

Two selected polygon subsets and their images are displayed in Figures 5 and 6. As shown in Figure 5, it is trivial to verify that $H^6(D_1)$ goes through both two polygon subsets D_1 and D_2. Similarly, $H^6(D_2)$ crosses two polygon subsets D_1 and D_2 as illustrated in Figure 6. According to the Horseshoe lemma, chaos of the system with infinite equilibria (1) is determined [45–47].

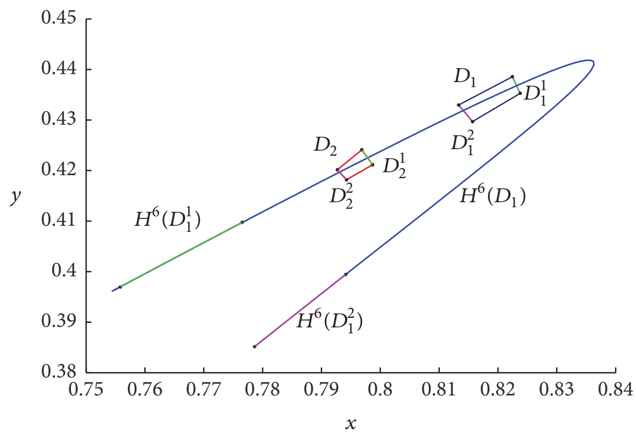

FIGURE 5: The quadrilateral subset D_1 and its image.

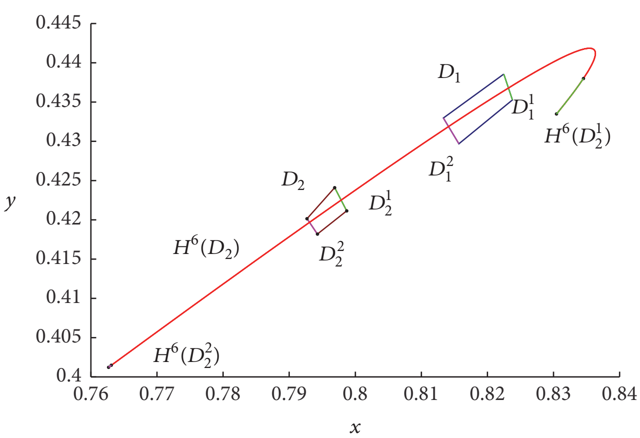

FIGURE 6: The quadrilateral subset D_2 and its image.

4. Synchronization of the Identical Systems with Infinite Equilibria

After the study of Pecora and Carroll about synchronization in chaotic systems [49], various synchronization techniques and related works were presented extensively [50–54]. Critically, the possibility of synchronization of two identical chaotic systems plays a vital role in practical applications [55–58]. In this section, we discover the synchronization of two new systems with infinite equilibria, called the master system and the slave system, by using an adaptive controller.

We consider the following master system with the unknown system parameter a:

$$\begin{aligned} \dot{x}_1 &= -z_1, \\ \dot{y}_1 &= x_1 z_1^2, \\ \dot{z}_1 &= x_1 - y_1 \tanh\left(y_1\right) + a y_1^2 z_1 - z_1^3. \end{aligned} \tag{15}$$

The slave system with adaptive control $\mathbf{u} = [u_x, u_y, u_z]^T$ is given as

$$\begin{aligned} \dot{x}_2 &= -z_2 + u_x, \\ \dot{y}_2 &= x_2 z_2^2 + u_y, \\ \dot{z}_2 &= x_2 - y_2 \tanh\left(y_2\right) + a y_2^2 z_2 - z_2^3 + u_z. \end{aligned} \tag{16}$$

The state errors between the slave system and the master system are calculated by

$$\begin{aligned} e_x &= x_2 - x_1, \\ e_y &= y_2 - y_1, \\ e_z &= z_2 - z_1. \end{aligned} \tag{17}$$

The parameter estimation error is defined as follows

$$e_a = a - \hat{a}, \tag{18}$$

in which $\hat{a}$ is the estimation of the unknown parameter a.

In order to synchronize the slave system and the master system, the adaptive control is constructed in the following form:

$$\begin{aligned} u_x &= e_z - k_x e_x, \\ u_y &= -x_2 z_2^2 + x_1 z_1^2 - k_y e_y, \\ u_z &= -e_x + y_2 \tanh\left(y_2\right) - y_1 \tanh\left(y_1\right) \\ &\quad - \hat{a}\left(y_2^2 z_2 - y_1^2 z_1\right) + z_2^3 - z_1^3 - k_z e_z, \end{aligned} \tag{19}$$

in which k_x, k_y, and k_z are three positive gain constants and the parameter update law is described by

$$\dot{\hat{a}} = e_z\left(y_2^2 z_2 - y_1^2 z_1\right). \tag{20}$$

By applying Lyapunov stability theory, we will prove that the master system (15) and the slave system (16) are synchronized when using the adaptive control (19).

In this work, the Lyapunov function is selected as

$$V\left(e_x, e_y, e_z, e_a\right) = \frac{1}{2}\left(e_x^2 + e_y^2 + e_z^2 + e_a^2\right). \tag{21}$$

Therefore, the differentiation of V is

$$\dot{V} = e_x \dot{e}_x + e_y \dot{e}_y + e_z \dot{e}_z + e_a \dot{e}_a. \tag{22}$$

From (17) and (18), we have

$$\begin{aligned} \dot{e}_x &= -k_x e_x, \\ \dot{e}_y &= -k_y e_y, \\ \dot{e}_z &= e_a\left(y_2^2 z_2 - y_1^2 z_1\right) - k_z e_z, \\ \dot{e}_a &= -\dot{\hat{a}}. \end{aligned} \tag{23}$$

By substituting (23) into (22), the differentiation of V can be expressed as

$$\dot{V} = -k_x e_x^2 - k_y e_y^2 - k_z e_z^2. \quad (24)$$

Because $\dot{V}$ is a negative semidefinite function, it is simply verified that $e_x \to 0$, $e_y \to 0$, and $e_z \to 0$ exponentially as $t \to \infty$ according to Barbalat's lemma [59]. In other words, we obtain the complete synchronization between the master system and the slave system.

We take an example to illustrate the calculation of the synchronization scheme. The parameter values of the master system and the slave system are fixed as

$$a = 2.9. \quad (25)$$

We assume that the initial states of the master system are

$$\begin{aligned} x_1(0) &= 0, \\ y_1(0) &= 0.1, \\ z_1(0) &= 0.2, \end{aligned} \quad (26)$$

while the initial states of the slave system are

$$\begin{aligned} x_2(0) &= -0.7, \\ y_2(0) &= 0.4, \\ z_2(0) &= -0.1. \end{aligned} \quad (27)$$

The positive gain constants are chosen as follows: $k_x = 4$, $k_y = 4$, and $k_z = 4$. We take the initial condition of the parameter estimate as

$$\hat{a}(0) = 3. \quad (28)$$

The time-history of the synchronization errors e_x, e_y, e_z is shown in Figure 7. It is straightforward to verify that Figure 7 depicts the synchronization of the master and slave systems.

5. Conclusions

A new chaotic system with a curve of equilibria has been introduced in this work. Interestingly, because of having an infinite number of equilibrium points, the system is a special system with hidden attractors, which is rarely reported in the literature. Basic dynamical characters of the system with infinite equilibria are investigated via phase portraits, equilibrium analysis, Kaplan–Yorke dimension, maximal Lyapunov exponents, and bifurcation diagram. Although it is great challenge for researchers to find a topological horseshoe in systems with hidden attractor, horseshoe in such new system with infinite equilibria has been discovered in our work. After studying the possibility of synchronization of two novel chaotic systems, we believe that potential applications of such a system should be considered further in future works.

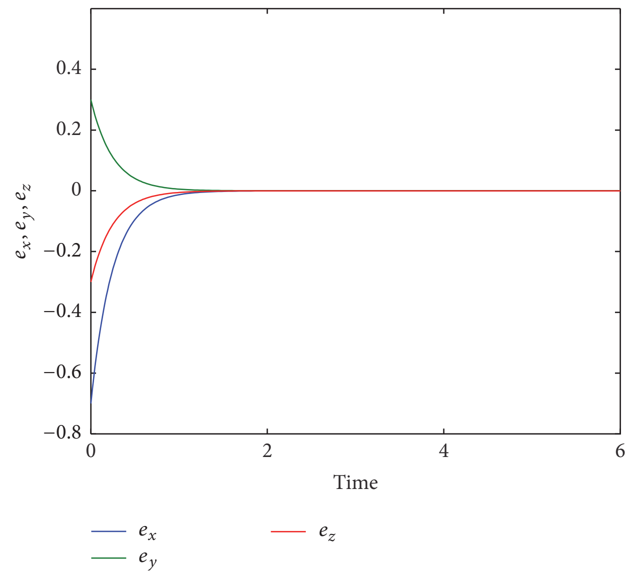

Figure 7: Time-history of the synchronization errors which indicates the synchronization between the master system and the slave system.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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