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Research Article

Stability of Pexiderized Quadratic Functional Equation in Random 2-Normed Spaces

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The aim of this paper is to investigate the stability of Hyers-Ulam-Rassias type theorems by considering the pexiderized quadratic functional equation in the setting of random 2-normed spaces (RTNS), while the concept of random 2-normed space has been recently studied by Goleţ (2005).

1. Introduction and Preliminaries

In 1940, Ulam [1] proposed the famous "Ulam stability problem," which was solved by Hyers [2], in 1941, for additive mappings. In 1950, Aoki [3] solved this Ulam problem for weaker additive mappings; for some historical comments regarding the work of Aoki we refer to [4]. In 1978, Rassias [5] generalized the theorem of Hyers for linear mappings in which the Cauchy difference is allowed to be unbounded by replacing ϵ with a function depending on x and y in the Hyers theorem. The generalization of Hyers theorem was also presented by Rassias [6–9] in 1982–1989. Some important Ulam stability problems on Cauchy equation on semigroups, approximately additive mappings, and Jensen equation have been investigated by Gajda [10], Găvruta [11], and Jung [12], respectively. Until now, the stability problems for different types of functional equations in various spaces have been extensively studied, for instance, by Mirmostafaee and Moslehian [13, 14], Rassias [15], Chang et al. [16, 17], Xu et al. [18], Jun and Kim [19], Mursaleen et al. [20–22], and many others. Also very interesting results on additive, quadratic, and cubic functional equations have been achieved by Mohiuddine et al. [23–29]. This paper is inspired from the work of Alotaibi and Mohiuddine [30] in which they solved stability problem for cubic functional equation in random 2-normed spaces.

The pexiderized quadratic functional equation is of the form $f(x+y)+f(x-y)=2g(x)+2h(y)$. For $f=g=h$, it is called the quadratic functional equation.

The terminology and notations used below are standard as in [31–33].

A function $f:\mathbb{R}\to\mathbb{R}_0^+$ is called a distribution function if it is nondecreasing and is left continuous with $\inf_{t\in\mathbb{R}}f(t)=0$ and $\sup_{t\in\mathbb{R}}f(t)=1$. By D^+, we denote the set of all distribution functions such that $f(0)=0$.

If $a\in\mathbb{R}_0^+$, then $H_a\in D^+$, where

$$H_a(t)=\begin{cases}0 & \text{if } t\le a;\\ 1 & \text{if } t>a.\end{cases}\quad(1)$$

It is obvious that $H_0\ge f$ for all $f\in D^+$.

A *t-norm* is a continuous mapping $*:[0,1]\times[0,1]\to[0,1]$ such that $([0,1],*)$ is abelian monoid with unit one and $c*d\ge a*b$ if $c\ge a$ and $d\ge b$ for all $a,b,c,d\in[0,1]$. A triangle function τ is a binary operation on D^+ which is commutative and associative and $\tau(f,H_0)=f$ for every $f\in D^+$.

Gähler [34] presented the following notion of 2-normed space.

Let X be a linear space of a dimension d $(2\le d<\infty)$. A function $\|\cdot,\cdot\|:X\times X\to\mathbb{R}$ is called 2-normed on X if it satisfied the following conditions: for every $x,y\in X$, (i) $\|x,y\|=0$ if and only if x and y are linearly dependent; (ii) $\|x,y\|=\|y,x\|$; (iii) $\|\alpha x,y\|=|\alpha|\|x,y\|$ for every $\alpha\in\mathbb{R}$; and (iv) $\|x+y,z\|\le\|x,z\|+\|y,z\|$ for every $x,y,z\in X$. In this case, $(X,\|\cdot,\cdot\|)$ is called a 2-*norm space*.

Goleţ [35] defined and studied the notion of random 2-normed space with the help of 2-norm of Gähler [34]. Recently, the notion of statistical convergence and lacunary statistical convergence have been studied by Mursaleen [36] and Mohiuddine and Aiyub [37], respectively, in random 2-normed spaces.

Let X be a linear space of a dimension greater than one and let τ be a triangle function. A function $\mathcal{F} : X \times X \to D^{+}$ is called a *probabilistic* 2*-norm* on X if it satisfies the following conditions:

(i) $\mathcal{F}_{x,y}(t) = H_0(t)$ $(\forall x, y \in X)$ if x and y are linearly dependent,

(ii) $\mathcal{F}_{x,y}(t) \neq H_0(t)$ if x and y are linearly independent,

(iii) $\mathcal{F}_{x,y}(t) = \mathcal{F}_{y,x}(t)$,

(iv) $\mathcal{F}_{\alpha x,y}(t) = \mathcal{F}_{x,y}(t/|\alpha|)$ for all $t > 0$, $\alpha \neq 0$ and $x, y \in X$,

(v) $\mathcal{F}_{x+y,z} \geq \tau(\mathcal{F}_{x,z}, \mathcal{F}_{y,z})$ whenever $x, y, z \in X$,

where $\mathcal{F}_{x,y}(t)$ denotes the value of $\mathcal{F}_{x,y}$ at $t \in \mathbb{R}$ and the triple $(X, \mathcal{F}, \tau)$ is called a *probabilistic* 2*-normed space*. If we replaced (v) by

(v′) $\mathcal{F}_{x+y,z}(t_1+t_2) \geq \mathcal{F}_{x,z}(t_1) * \mathcal{F}_{y,z}(t_2)$, for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}_0^{+}$,

then triple $(X, \mathcal{F}, *)$ is called a *random* 2*-normed space* (RTNS).

Example A. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space with $\|x, z\| = \|x_1 z_2 - x_2 z_1\|$, $x = (x_1, x_2)$, $z = (z_1, z_2)$, and $a * b = ab$ for $a, b \in [0, 1]$. For all $x \in X$, $t > 0$, and nonzero $z \in X$, consider

$$\mathcal{F}_{x,z}(t) = \begin{cases} \dfrac{t}{t + \|x, z\|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \tag{2}$$

Then $(X, \mathcal{F}, *)$ is a RTNS.

We remark that every 2-normed space $(X, \|\cdot,\cdot\|)$ can be made RTNS by considering $\mathcal{F}_{x,y}(t) = H_0(t - \|x, y\|)$, for every $x, y \in X$, $t > 0$, and $a * b = \min\{a, b\}$, where $a, b \in [0, 1]$.

The notions of convergence and Cauchy sequences have been recently studied by Alotaibi and Mohiuddine [30] in the setting of RTNS.

Let $(X, \mathcal{F}, *)$ be a RTNS. Then, a sequence $x = (x_k)$ is said to be

(i) *convergent* in $(X, \mathcal{F}, *)$ ($\mathcal{F}$*-convergent*) to L if for every $\epsilon > 0$ and $\theta \in (0, 1)$ there exists $k_0 \in \mathbb{N}$ such that $\mathcal{F}_{x_k - L,z}(\epsilon) > 1 - \theta$ whenever $k \geq k_0$ and nonzero $z \in X$. In this case we write $\mathcal{F}\text{-}\lim_{k\to\infty} x_k = L$;

(ii) *Cauchy sequence* in $(X, \mathcal{F}, *)$ ($\mathcal{F}$*-Cauchy*) if for every $\epsilon > 0$, $\theta > 0$, and nonzero $z \in X$ there exists a number $N = N(\epsilon, z)$ such that $\lim \mathcal{F}_{x_k - x_l,z}(\epsilon) > 1 - \theta$ for all $k, l \geq N$. We say that RTNS is *complete* if every $\mathcal{F}$-Cauchy sequence is $\mathcal{F}$-convergent. A complete RTNS is called random 2-Banach space.

2. Main Results

Throughout the paper, by Y, $(Z, \mathcal{F}', *)$, and $(Y, \mathcal{F}, *)$, we denote linear space, random 2-normed space, and random 2-Banach space, respectively. Firstly, we prove the stability of the pexiderized quadratic functional equation in RTNS for an odd case.

Let φ be a function from $X \times X$ to Z. A mapping $f : X \to Y$ is said to be φ-approximately pexiderized quadratic function if there exist mappings $g, h : X \to Y$ such that

$$\mathcal{F}_{f(x+y)+f(x-y)-2g(x)-2h(y),z}(t) \geq \mathcal{F}'_{\varphi(x,y),z}(t), \tag{3}$$

for all $x, y \in X$, $t > 0$, and nonzero $z \in X$.

Theorem 1. *Suppose that f, g and h are odd functions from X to Y satisfying (3). If for some real number α with* $0 < |\alpha| < 2$

$$\varphi(2x, 2y) = \alpha\varphi(x, y), \tag{4}$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to X$ such that

$$\begin{gathered} \mathcal{F}_{f(x)-T(x),z}(t) \geq \mathcal{F}''_{x,z}\left(\frac{2 - |\alpha|}{4} t\right), \\ \mathcal{F}_{g(x)+h(x)-T(x),z}(t) \geq \mathcal{F}''_{x,z}\left(\frac{6 - 3|\alpha|}{14 - |\alpha|} t\right), \end{gathered} \tag{5}$$

where

$$\mathcal{F}''_{x,z}(t) = \mathcal{F}'_{\varphi(x,x),z}\left(\frac{t}{3}\right) * \mathcal{F}'_{\varphi(x,0),z}\left(\frac{t}{3}\right) * \mathcal{F}'_{\varphi(0,x),z}\left(\frac{t}{3}\right), \tag{6}$$

for all $x \in X$, $t > 0$, and nonzero $z \in X$.

Proof. Replacing x by y and y by x in (3), we obtain

$$\mathcal{F}_{f(x+y)-f(x-y)-2g(y)-2h(x),z}(t) \geq \mathcal{F}'_{\varphi(y,x),z}(t) \tag{7}$$

for all $x, y \in X$, $t > 0$, and nonzero $z \in X$. It follows from (3) and (7) that

$$\mathcal{F}_{f(x+y)-g(x)-h(y)-g(y)-h(x),z}(t) \geq \mathcal{F}'_{\varphi(x,y),z}(t) * \mathcal{F}'_{\varphi(y,x),z}(t). \tag{8}$$

Substituting $y = 0$ in (8), we get

$$\mathcal{F}_{f(x)-g(x)-h(x),z}(t) \geq \mathcal{F}'_{\varphi(x,0),z}(t) * \mathcal{F}'_{\varphi(0,x),z}(t). \tag{9}$$

From (8) and (9), we conclude that

$$\begin{aligned} &\mathcal{F}_{f(x+y)-f(x)-f(y),z}(3t) \\ &\quad \geq \mathcal{F}'_{\varphi(x,y),z}(t) * \mathcal{F}'_{\varphi(y,x),z}(t) * \mathcal{F}'_{\varphi(x,0),z}(t) \\ &\qquad * \mathcal{F}'_{\varphi(0,x),z}(t) * \mathcal{F}'_{\varphi(y,0),z}(t) * \mathcal{F}'_{\varphi(0,y),z}(t), \end{aligned} \tag{10}$$

for every $x, y \in X$ $t > 0$ and nonzero $z \in X$. Then, by our assumption,

$$\mathcal{F}''_{2^n x,z}(t) = \mathcal{F}''_{x,z}\left(\frac{t}{\alpha^n}\right). \tag{11}$$

Taking $x = y$ in (10), for all $x \in X$, $t > 0$, and nonzero $z \in X$, we get

$$\mathscr{F}_{f(2x)-2f(x),z}(t) \geq \mathscr{F}''_{x,z}(t). \quad (12)$$

Putting $x = 2^n x$ in (12), we have

$$\begin{aligned} &\mathscr{F}_{f(2^{n+1}x)/2^{n+1}-f(2^n x)/2^n,z}(t) \\ &\quad = \mathscr{F}_{f(2^{n+1}x)-f(2^n x),z}(2^n t) \\ &\quad \geq \mathscr{F}''_{2^n x,z}(2^n t) \geq \mathscr{F}''_{x,z}\left(\left(\frac{2}{\alpha}\right)^n t\right). \end{aligned} \quad (13)$$

Thus,

$$\mathscr{F}_{f(2^{n+1}x)/2^{n+1}-f(2^n x)/2^n,z}\left(\left(\frac{\alpha}{2}\right)^n t\right) \geq \mathscr{F}''_{x,z}(t). \quad (14)$$

Therefore, for each $n > m \geq 0$,

$$\begin{aligned} &\mathscr{F}_{f(2^n x)/2^n-f(2^m x)/2^m,z}\left(\sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^{k-1} t\right) \\ &\quad = \mathscr{F}_{\sum_{k=m+1}^{n}(f(2^k x)/2^k - f(2^{k-1}x),z/(2k-1))}\left(\sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^{k-1} t\right) \\ &\quad \geq \prod_{k=m+1}^{n} \mathscr{F}_{f(2^k x)/2^k - f(2^{k-1}x)/(2k-1),z}\left(\left(\frac{\alpha}{2}\right)^{k-1} t\right) \\ &\quad \geq \mathscr{F}''_{x,z}(t), \end{aligned} \quad (15)$$

where $\prod_{j=1}^{n} a_j = a_1 * a_2 * \cdots * a_n$. Let $\epsilon > 0$ and $t_0 > 0$ be given. With the help of the definition of RTNS, we have $\mathscr{F}''_{x,z}(t) = 1$ and, therefore, we can find some $t_1 > t_0$ such that $\mathscr{F}''_{x,z}(t_1) > 1 - \epsilon$. The convergence of the series $\sum_{n=1}^{\infty}(\alpha/2)^n t_1$ gives some $n_0 \in \mathbb{N}$ such that for each $n > m \geq n_0$, $\sum_{k=m+1}^{n}(\alpha/2)^{k-1} t_1 < t_0$. Therefore,

$$\begin{aligned} &\mathscr{F}_{f(2^n x)/2^n-f(2^m x)/2^m,z}(t_0) \\ &\quad \geq \mathscr{F}_{f(2^n x)/2^n-f(2^m x)/2^m,z}\left(\sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^{k-1} t_1\right) \\ &\quad \geq \mathscr{F}''_{x,z}(t_1) > 1 - \epsilon. \end{aligned} \quad (16)$$

It follows that $(f(2^n x)/2^n)$ is a Cauchy sequence in $(Y, \mathscr{F}, *)$. Since $(Y, \mathscr{F}, *)$ is complete RTNS, this sequence converges to some point in Y; that is, $T(x) \in Y$. Therefore, a mapping T from X to Y is defined by $T(x) = \mathscr{F}\text{-}\lim_{n\to\infty}(f(2^n x)/2^n)$. Fix $x, y \in X$ and $t > 0$. From (10), we get that

$$\begin{aligned} &\mathscr{F}_{f(2^n(x+y))/2^n-f(2^n x)/2^n-f(2^n y)/2^n,z}\left(\frac{t}{4}\right) \\ &\quad = \mathscr{F}_{f(2^n(x+y))-f(2^n x)-f(2^n y),z}\left(\frac{2^n t}{4}\right) \\ &\quad \geq \mathscr{F}'_{\varphi(x,y),z}\left(\frac{2^n t}{12\alpha^n}\right) * \mathscr{F}'_{\varphi(y,x),z}\left(\frac{2^n t}{12\alpha^n}\right) \\ &\qquad * \mathscr{F}'_{\varphi(x,0),z}\left(\frac{2^n t}{12\alpha^n}\right) * \mathscr{F}'_{\varphi(0,x),z}\left(\frac{2^n t}{12\alpha^n}\right) \\ &\qquad * \mathscr{F}'_{\varphi(y,0),z}\left(\frac{2^n t}{12\alpha^n}\right) * \mathscr{F}'_{\varphi(0,y),z}\left(\frac{2^n t}{12\alpha^n}\right) \end{aligned} \quad (17)$$

for all n. Moreover,

$$\begin{aligned} &\mathscr{F}_{T(x+y)-T(x)-T(y),z}(t) \\ &\quad \geq \mathscr{F}_{T(x+y)-f(2^n(x+y))/2^n,z}\left(\frac{t}{4}\right) * \mathscr{F}_{T(x)-f(2^n x)/2^n,z}\left(\frac{t}{4}\right) \\ &\qquad * \mathscr{F}_{T(y)-f(2^n y)/2^n,z}\left(\frac{t}{4}\right) \\ &\qquad * \mathscr{F}_{f(2^n(x+y))/2^n-f(2^n x)/2^n-f(2^n y)/2^n,z}\left(\frac{t}{4}\right) \end{aligned} \quad (18)$$

for all n. From (17) and (18), we obtain

$$\mathscr{F}_{T(x+y)-T(x)-T(y),z}(t) = 1. \quad (19)$$

Thus, $T(x+y) = T(x) + T(y)$. Now by taking (15) with $m = 0$, we get

$$\begin{aligned} &\mathscr{F}_{T(x)-f(x),z}(t) \\ &\quad \geq \mathscr{F}_{T(x)-f(2^n x)/2^n,z}\left(\frac{t}{2}\right) * \mathscr{F}_{f(2^n x)/2^n-f(x),z}\left(\frac{t}{2}\right) \\ &\quad \geq \mathscr{F}_{T(x)-f(2^n x)/2^n,z}\left(\frac{t}{2}\right) * \mathscr{F}''_{x,z}\left(\frac{t}{2\sum_{k=1}^{n}(\alpha/2)^{k-1}}\right) \\ &\quad \geq \mathscr{F}''_{x,z}\left(\frac{t}{2\sum_{k=1}^{\infty}(\alpha/2)^{k-1}}\right) = \mathscr{F}''_{x,z}\left(\frac{2-\alpha}{4}t\right). \end{aligned} \quad (20)$$

It follows from (9) and (20) that

$$\begin{aligned} &\mathscr{F}_{g(x)+h(x)-T(x),z}\left(\frac{14-\alpha}{12}t\right) \\ &\quad \geq \mathscr{F}_{f(x)-T(x),z}(t) * \mathscr{F}_{g(x)+h(x)-f(x),z}\left(\frac{2-\alpha}{12}t\right) \\ &\quad \geq \mathscr{F}''_{x,z}\left(\frac{2-\alpha}{4}t\right) * \mathscr{F}'_{\varphi(x,0),z}\left(\frac{2-\alpha}{12}t\right) \\ &\qquad * \mathscr{F}'_{\varphi(0,x),z}\left(\frac{2-\alpha}{12}t\right) \\ &\quad \geq \mathscr{F}''_{x,z}\left(\frac{2-\alpha}{4}t\right). \end{aligned} \quad (21)$$

Thus, we obtained (5). Now we will prove the uniqueness of T. For this, we assume that T' is another additive mapping from X into Y, which satisfies the required inequality. Since, for each $n \in \mathbb{N}$, $T(2^n x) = 2^n T(x)$ and $T'(2^n x) = 2^n T'(x)$, then

$$\begin{aligned}
&\mathcal{F}_{T(x)-T'(x),z}(t) \\
&\quad = \mathcal{F}_{T(2^n x)-T'(2^n x),z}(2^n t) \\
&\quad \geq \mathcal{F}_{T'(2^n x)-f(2^n x),z}\left(\frac{2^n t}{2}\right) * \mathcal{F}_{f(2^n x)-T(2^n x),z}\left(\frac{2^n t}{2}\right) \\
&\quad \geq \mathcal{F}''_{2^n x,z}\left(\frac{(2-|\alpha|)\,2^n t}{8}\right) = \mathcal{F}''_{x,z}\left(\frac{(2/\alpha)^n (2-|\alpha|)\,t}{8}\right).
\end{aligned} \tag{22}$$

We obtain with the help of the definition of RTNS that

$$\mathcal{F}''_{x,z}\left(\frac{(2/\alpha)^n (2-|\alpha|)\,t}{8}\right) = 1. \tag{23}$$

Therefore, $\mathcal{F}_{T(x)-T'(x),z}(t) = 1$, for all $x \in X$, $t > 0$, and nonzero $z \in X$. Hence, $T(x) = T'(x)$ for all $x \in X$. □

Now, we are going to prove the stability of the pexiderized quadratic functional equation in RTNS for an even case.

Theorem 2. *If (4) holds for $0 < |\alpha| < 4$, let f, g, and h be three even functions from X to Y such that $f(0) = g(0) = h(0) = 0$ and satisfies (3). Then there is a unique quadratic mapping $C : X \to Y$ such that, for every $x \in X$, $t > 0$, and nonzero $z \in X$,*

$$\begin{aligned}
&\mathcal{F}_{C(x)-f(x),z}(t) \geq \mathcal{F}''_{x,z}\left(\frac{(4-|\alpha|)}{16}t\right), \\
&\mathcal{F}_{C(x)-g(x),z}(t) \geq \mathcal{F}''_{x,z}\left(\frac{(12-3|\alpha|)}{52-|\alpha|}t\right), \\
&\mathcal{F}_{C(x)-h(x),z}(t) \geq \mathcal{F}''_{x,z}\left(\frac{(12-3|\alpha|)}{52-|\alpha|}t\right),
\end{aligned} \tag{24}$$

where $\mathcal{F}''_{x,z}(t)$ is defined by (6).

Proof. Substitute x by y and y by x in (3). Then, for all $x, y \in X$, $t > 0$, and nonzero $z \in X$, we obtain

$$\mathcal{F}_{f(x+y)+f(x-y)-2g(y)-2h(x),z}(t) \geq \mathcal{F}'_{\varphi(y,x),z}(t). \tag{25}$$

Again substituting $y = x$ in (3), we get

$$\mathcal{F}_{f(2x)-2g(x)-2h(x),z}(t) \geq \mathcal{F}'_{\varphi(x,x),z}(t). \tag{26}$$

Putting $x = 0$ in (3), we get

$$\mathcal{F}_{2f(y)-2h(y),z}(t) \geq \mathcal{F}'_{\varphi(0,y),z}(t). \tag{27}$$

For $y = 0$, (3) becomes

$$\mathcal{F}_{2f(x)-2g(x),z}(t) \geq \mathcal{F}'_{\varphi(x,0),z}(t). \tag{28}$$

It follows from (25), (27), and (28) that

$$\begin{aligned}
&\mathcal{F}_{f(x+y)-f(x-y)-2f(x)-2f(y),z}(t) \\
&\quad \geq \mathcal{F}'_{\varphi(x,y),z}\left(\frac{t}{3}\right) * \mathcal{F}'_{\varphi(x,0),z}\left(\frac{t}{3}\right) * \mathcal{F}'_{\varphi(0,y),z}\left(\frac{t}{3}\right).
\end{aligned} \tag{29}$$

By substituting $y = x$ in (29), we get

$$\mathcal{F}_{2f(x)-4f(x),z}(t) \geq \mathcal{F}''_{x,z}(t). \tag{30}$$

From (4), we obtain

$$\mathcal{F}''_{2^n x,z}(t) = \mathcal{F}''_{x,z}\left(\frac{t}{\alpha^n}\right), \tag{31}$$

for every $x \in X$, nonzero $z \in X$ and for each $n \geq 0$. It follows from (30) and (31) that

$$\mathcal{F}_{f(2^{n+1}x)-4f(2^n x),z}(t) \geq \mathcal{F}''_{x,z}\left(\frac{t}{\alpha^n}\right). \tag{32}$$

From (32), we obtain

$$\begin{aligned}
&\mathcal{F}_{f(2^{n+1}x)/4^{n+1}-f(2^n x)/4^n,z}(t) \\
&\quad = \mathcal{F}_{f(2^{n+1}x)-4f(2^n x),z}\left(4^{n+1}t\right) \geq \mathcal{F}''_{x,z}\left(\frac{4^{n+1}t}{\alpha^n}\right)
\end{aligned} \tag{33}$$

or, equivalently,

$$\mathcal{F}_{f(2^{n+1}x)/4^{n+1}-f(2^n x)/4^n,z}\left(\frac{\alpha^n t}{4^{n+1}}\right) \geq \mathcal{F}''_{x,z}(t). \tag{34}$$

Therefore, for all $x \in X$, $t > 0$, and nonzero $z \in X$ and for each $n > m \geq 0$

$$\begin{aligned}
&\mathcal{F}_{f(2^n x)/4^n - f(2^m x)/4^m,z}\left(\sum_{k=m+1}^{n} \frac{\alpha^{k-1}t}{4^k}\right) \\
&\quad = \mathcal{F}_{\sum_{k=m+1}^{n}(f(2^k x)/4^k - f(2^{k-1}x)/4^{k-1}),z}\left(\sum_{k=m+1}^{n} \frac{\alpha^{k-1}t}{4^k}\right) \\
&\quad \geq \prod_{k=m+1}^{n} \mathcal{F}_{f(2^k x)/4^k - f(2^{k-1}x)/4^{k-1},z}\left(\frac{\alpha^{k-1}t}{4^k}\right) \geq \mathcal{F}''_{x,z}(t),
\end{aligned} \tag{35}$$

where $\prod$ is the same as Theorem 1. Given $\epsilon > 0$ and $t_0 > 0$, since $\mathcal{F}''_{x,z}(t) = 1$, there is some $t_1 > t_0$ such that $\mathcal{F}''_{x,z}(t_1) > 1-\epsilon$. By the convergence of $\sum_{k=1}^{\infty}(\alpha^{k-1}/4^k)t_1$, we can find some n_0 such that $\sum_{k=m+1}^{n}(\alpha^{k-1}/4^k)t_1 < t_0$ for each $n > m \geq n_0$. This gives that

$$\begin{aligned}
&\mathcal{F}_{f(2^n x)/4^n - f(2^m x)/4^m,z}(t_0) \\
&\quad \geq \mathcal{F}_{f(2^n x)/4^n - f(2^m x)/4^m,z}\left(\sum_{k=m+1}^{n} \frac{\alpha^{k-1}}{4^k} t_1\right) \\
&\quad \geq \mathcal{F}''_{x,z}(t_0) > 1-\epsilon.
\end{aligned} \tag{36}$$

We see that $(f(2^n x)/4^n)$ is a Cauchy sequence in $(Y, \mathscr{F}, *)$ and so it is convergent to some point $C(x) \in Y$. Therefore, a mapping C from X to Y is defined by $C(x) = \mathscr{F}\text{-}\lim_{n\to\infty}(f(2^n x)/4^n)$. Fix $x, y \in X$ and $t > 0$. Thus, (29) gives that

$$\begin{aligned}
&\mathscr{F}_{f(2^n(x+y))/4^n+f(2^n(x-y))/4^n-2(f(2^n x)/4^n)-2(f(2^n y)/4^n),z}\left(\frac{t}{5}\right)\\
&\quad= \mathscr{F}_{f(2^n(x+y))+f(2^n(x-y))-2f(2^n x)-2f(2^n y),z}\left(\frac{4^n t}{5}\right)\\
&\quad\geq \mathscr{F}'_{\varphi(x,y),z}\left(\frac{4^n t}{15\alpha^n}\right) * \mathscr{F}'_{\varphi(x,0),z}\left(\frac{4^n t}{15\alpha^n}\right)\\
&\qquad * \mathscr{F}'_{\varphi(0,y),z}\left(\frac{4^n t}{15\alpha^n}\right),
\end{aligned} \tag{37}$$

for all n. Furthermore,

$$\begin{aligned}
&\mathscr{F}_{C(x+y)+C(x-y)-2C(x)-2C(y),z}(t)\\
&\quad\geq \mathscr{F}_{C(x+y)-f(2^n(x+y))/4^n,z}\left(\frac{t}{5}\right)\\
&\qquad * \mathscr{F}_{C(x-y)-f(2^n(x-y))/4^n,z}\left(\frac{t}{5}\right)\\
&\qquad * \mathscr{F}_{2C(x)-2(f(2^n x)/4^n),z}\left(\frac{t}{5}\right) * \mathscr{F}_{2C(y)-2(f(2^n y)/4^n),z}\left(\frac{t}{5}\right)\\
&\qquad * \mathscr{F}_{f(2^n(x+y))/4^n+f(2^n(x-y))/4^n-2(f(2^n x)/4^n)-2(f(2^n y)/4^n),z}\left(\frac{t}{5}\right).
\end{aligned} \tag{38}$$

Equations (37) and (38) give that

$$\mathscr{F}_{C(x+y)+C(x-y)-2C(x)-2C(y),z}(t) = 1, \tag{39}$$

for all $x, y \in X$, $t > 0$, and nonzero $z \in X$. Thus, $C(x+y) + C(x-y) = 2C(x) + 2C(y)$. Using (35) with $m = 0$, we get

$$\begin{aligned}
&\mathscr{F}_{C(x)-f(x),z}(t)\\
&\quad\geq \mathscr{F}_{C(x)-f(2^n x)/4^n,z}\left(\frac{t}{2}\right) * \mathscr{F}_{f(2^n x)/4^n-f(x),z}\left(\frac{t}{2}\right)\\
&\quad\geq \mathscr{F}_{C(x)-f(2^n x)/4^n,z}\left(\frac{t}{2}\right) * \mathscr{F}''_{x,z}\left(\frac{4t}{2\sum_{k=1}^{n}(\alpha/4)^{k-1}}\right)\\
&\quad\geq \mathscr{F}''_{x,z}\left(\frac{4t}{2\sum_{k=0}^{\infty}(\alpha/4)^{k}}\right) = \mathscr{F}''_{x,z}\left(\frac{4-\alpha}{16}t\right)
\end{aligned} \tag{40}$$

for sufficiently large n. From (28) and (40), we conclude that

$$\begin{aligned}
&\mathscr{F}_{C(x)-g(x),z}\left(\frac{52-\alpha}{48}t\right)\\
&\quad\geq \mathscr{F}_{C(x)-f(x),z}(t) * \mathscr{F}_{f(x)-g(x),z}\left(\frac{4-\alpha}{48}t\right)\\
&\quad\geq \mathscr{F}''_{x,z}\left(\frac{4-\alpha}{16}t\right) * \mathscr{F}'_{\varphi(x,0),z}\left(\frac{4-\alpha}{48}t\right)\\
&\quad\geq \mathscr{F}''_{x,z}\left(\frac{4-\alpha}{16}t\right).
\end{aligned} \tag{41}$$

Thus,

$$\mathscr{F}_{C(x)-g(x),z}(t) \geq \mathscr{F}''_{x,z}\left(\frac{12-3\alpha}{52-\alpha}t\right). \tag{42}$$

Similarly, one can show that the above inequality also holds for h. We obtain the uniqueness assertion of this theorem by proceeding the same lines as in Theorem 1. □

Theorem 3. *Suppose that (4) holds with $0 < |\alpha| < 4$. If a map $f : X \to Y$ satisfies*

$$\mathscr{F}_{f(x+y)+f(x-y)-2f(x)-2f(y),z}(t) \geq \mathscr{F}'_{\varphi(x,y),z}(t), \tag{43}$$

for all $x, y \in X$, $t > 0$, and nonzero $z \in X$ with $f(0) = 0$. Then, there are unique mappings $T, C : X \to Y$ such that T is additive, C is quadratic, and

$$\mathscr{F}_{f(x)-T(x)-C(x),z}(t) \geq M_{x,z}\left(\left\{\left(\frac{2-\alpha}{8}\right) * \left(\frac{4-\alpha}{32}\right)\right\} t\right), \tag{44}$$

for all $x \in X$, $t > 0$, and nonzero $z \in X$, where

$$\begin{aligned}
M_{x,z}(t) = {}& \mathscr{F}'_{\varphi(x,x),z}\left(\frac{t}{3}\right) * \mathscr{F}'_{\varphi(-x,-x),z}\left(\frac{t}{3}\right)\\
&* \mathscr{F}'_{\varphi(x,0),z}\left(\frac{t}{3}\right) * \mathscr{F}'_{\varphi(0,x),z}\left(\frac{t}{3}\right)\\
&* \mathscr{F}'_{\varphi(-x,0),z}\left(\frac{t}{3}\right) * \mathscr{F}'_{\varphi(0,-x),z}\left(\frac{t}{3}\right).
\end{aligned} \tag{45}$$

Proof. Passing to the odd part f° and even part f^{e} of f, we deduce from (43) that

$$\begin{aligned}
&\mathscr{F}_{f^{\circ}(x+y)+f^{\circ}(x-y)-2f^{\circ}(x)-2f^{\circ}(y),z}(t)\\
&\quad\geq \mathscr{F}'_{\varphi(x,y),z}(t) * \mathscr{F}'_{\varphi(-x,-y),z}(t).
\end{aligned} \tag{46}$$

On the other hand,

$$\begin{aligned}
&\mathscr{F}_{f^{e}(x+y)+f^{e}(x-y)-2f^{e}(x)-2f^{e}(y),z}(t)\\
&\quad\geq \mathscr{F}'_{\varphi(x,y),z}(t) * \mathscr{F}'_{\varphi(-x,-y),z}(t).
\end{aligned} \tag{47}$$

With the help of the proofs of Theorems 1 and 2, we obtain unique additive and quadratic mappings T and C, respectively, satisfying

$$\begin{aligned}
&\mathscr{F}_{f^{\circ}(x)-T(x),z}(t) \geq M_{x,z}\left(\frac{2-|\alpha|}{4}t\right),\\
&\mathscr{F}_{f^{e}(x)-C(x),z}(t) \geq M_{x,z}\left(\frac{4-|\alpha|}{16}t\right).
\end{aligned} \tag{48}$$

Therefore,

$$\begin{aligned}
&\mathcal{F}_{f(x)-T(x)-C(x),z}(t)\\
&\geq \mathcal{F}_{f^{\circ}-T(x),z}\left(\frac{t}{2}\right) * \mathcal{F}_{f^{e}-C(x),z}\left(\frac{t}{2}\right)\\
&\geq M_{x,z}\left(\frac{2-|\alpha|}{8}t\right) * M_{x,z}\left(\frac{4-|\alpha|}{32}t\right)\\
&= M_{x,z}\left(\left\{\left(\frac{2-\alpha}{8}\right) * \left(\frac{4-\alpha}{32}\right)\right\} t\right),
\end{aligned} \tag{49}$$

for all $x \in X$, $t > 0$, and nonzero $z \in X$. □

Remark 4. Let $(X, \langle \cdot, \cdot \mid \cdot \rangle)$ be a 2-inner product space. We can define a 2-norm on $X \times X$ by $\|x_1, x_2\| = \sqrt{\langle x_1, x_1 \mid x_2 \rangle}$ for all $x_1, x_2 \in X$. In this case, parallelogram law is given by

$$\|x_1 + x_2, x_3\|^2 + \|x_1 - x_2, x_3\|^2 = 2\|x_1, x_3\|^2 + 2\|x_2, x_3\|^2, \tag{50}$$

for all $x_1, x_2, x_3 \in X$ (for more details of 2-inner product space we refer to [38]).

Now we give the following illustrative example.

Example 5. Let $(X, \langle \cdot, \cdot \mid \cdot \rangle)$ be a 2-inner product space. Let Y be a 2-normed space such that $\|x, z\| = \|x_1 z_2 - x_2 z_1\|$, where $x = (x_1, x_2)$ and $z = (z_1, z_2)$. Suppose that $a * b = ab$ for all $a, b \in [0, 1]$. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are two random 2-norms on Y and $\mathbb{R}$, respectively, which are given by Example A. Suppose that the random 2-norm $\mathcal{F}$ makes Y into an random 2-Banach space. Fixing $x_{\circ}, y_{\circ}, z_{\circ} \in Y$ and $a \in X$, we define

$$\begin{aligned}
f(x) &= \langle x, a \mid s_1 \rangle x_{\circ} + \|x, s_1\|^2 y_{\circ} + \sqrt{\|x, s_1\|} z_{\circ},\\
g(x) &= \langle x, a \mid s_1 \rangle x_{\circ} + \|x, s_1\|^2 y_{\circ},\\
h(x) &= \|x, s_1\|^2 y_{\circ} + \sqrt{\|x, s_1\|} z_{\circ},\\
\varphi(x, s_2) &= \left(\sqrt{\|x + s_2, s_1\|} + \sqrt{\|x - s_2, s_1\|} - 2\sqrt{\|s_2, s_1\|}\right) z_{\circ},
\end{aligned} \tag{51}$$

for each $x, s_1, s_2 \in X$. Using parallelogram law, one can easily verify that

$$\begin{aligned}
&f(x + s_2) + f(x - s_2) - 2g(x) - 2h(s_2)\\
&= \left(\sqrt{\|x + s_2, s_1\|} + \sqrt{\|x - s_2, s_1\|} - 2\sqrt{\|s_2, s_1\|}\right) z_{\circ},
\end{aligned} \tag{52}$$

for all $x, s_1, s_2 \in X$. Therefore,

$$\mathcal{F}_{f(x+s_2)+f(x-s_2)-2g(x)-2h(s_2),z}(t) = \mathcal{F}'_{\varphi(x,s_2),z}(t), \tag{53}$$

for each $x, s_2 \in X$, $t \in \mathbb{R}$, and nonzero $z \in X$. Moreover, $\varphi(2x, 2s_2) = \sqrt{2}\varphi(x, s_2)$ for each $x, s_2 \in X$. We can see that the conditions of Theorems 1 and 2 for f, g, h and $|\alpha| = \sqrt{2} < 2$ are satisfied. It follows that odd and even parts of f can be approximated by linear and quadratic functions, respectively. In fact f°, the odd part of f and $f^{\circ}(x) = \langle x, a \mid s_1 \rangle x_{\circ}$, is linear. The even part of f is f^{e}, and $f^{e}(x) = \|x, s_1\|^2 y_{\circ} + \sqrt{\|x, s_1\|} z_{\circ}$ contains a quadratic $C(x) = \|x, s_1\|^2 y_{\circ}$. Also

$$\begin{aligned}
\mathcal{F}_{f^{e}(x)-C(x),z}(t) &= \mathcal{F}'_{\sqrt{\|x,s_1\|}|z_{\circ}|,z}(t)\\
&\geq \mathcal{F}''_{x,z}\left(\frac{4-\sqrt{2}}{16}t\right).
\end{aligned} \tag{54}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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