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Research Article

Affine Fullerene C_{60} in a GS-Quasigroup

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It will be shown that the affine fullerene C_{60}, which is defined as an affine image of buckminsterfullerene C_{60}, can be obtained only by means of the golden section. The concept of the affine fullerene C_{60} will be constructed in a general GS-quasigroup using the statements about the relationships between affine regular pentagons and affine regular hexagons. The geometrical interpretation of all discovered relations in a general GS-quasigroup will be given in the GS-quasigroup $\mathbb{C}((1/2)(1+\sqrt{5}))$.

1. Introduction

The fullerenes are closed carbon-cage molecules containing only pentagonal and hexagonal rings.

C_{60} is the first fullerene that was theoretically conceived and experimentally obtained. The geometrical structure of C_{60} is a truncated icosahedron with a carbon atom at the corners of each hexagon and a bond along each edge (Figure 1). The sixty-carbon cluster with the geometry of a truncated icosahedron is named buckminsterfullerene [1, 2].

The affine regular icosahedron is defined as an affine image of a regular icosahedron. Let the affine regular icosahedron be given with the pairs of opposite vertices a, a'; b, b'; c, c'; d, d'; e, e'; f, f' as in Figure 2. Let us divide each edge of this icosahedron into three equal parts and then omit two lateral parts. On each of the twenty faces of the icosahedron on the sides the three segments in their middle parts are left. Let us connect the adjacent ends of these segments, so that an affine regular hexagon is formed on each side of an icosahedron, and an affine regular pentagon appears in the neighborhood of each vertex of icosahedron (Figure 3).

The obtained polyhedron consists of twelve affine regular pentagons and twenty affine regular hexagons. It is an affine version of buckminsterfullerene C_{60} which will be called *affine fullerene* C_{60}. It is presented in Figure 3, from where it is obvious how the labels of vertices of that polyhedron are chosen, starting from the labels of vertices of the affine regular icosahedron.

We will prove that the complete affine fullerene C_{60} can be presented only by means of the golden section. The concept of a GS-quasigroup will be used in this consideration.

2. GS-Quasigroup

A quasigroup $(Q,\cdot)$ is said to be a golden section quasigroup or shortly a GS-quasigroup [3] if it satisfies the (mutually equivalent) identities

$$a(ab\cdot c)\cdot c = b, \tag{1}$$

$$a\cdot(a\cdot bc)c = b, \tag{2}$$

and the identity of idempotence

$$aa = a. \tag{3}$$

GS-quasigroups are medial quasigroups; that is, the identity

$$ab\cdot cd = ac\cdot bd \tag{4}$$

is valid [4].

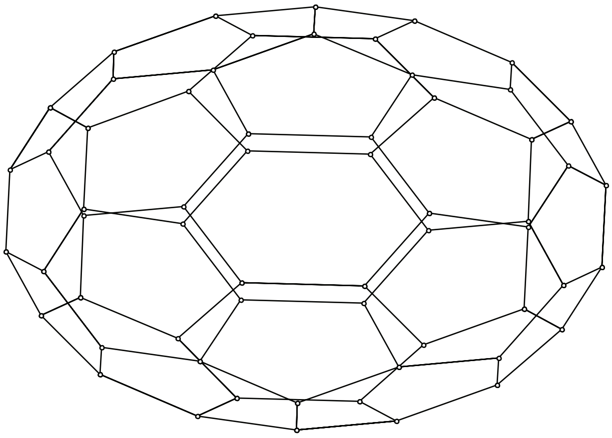

Figure 1

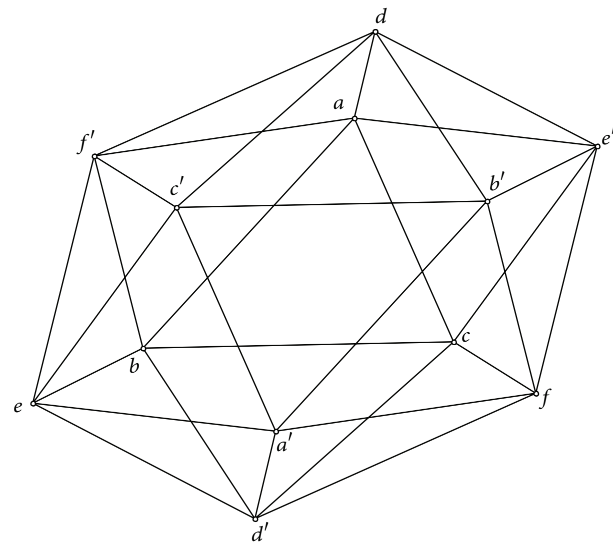

Figure 2

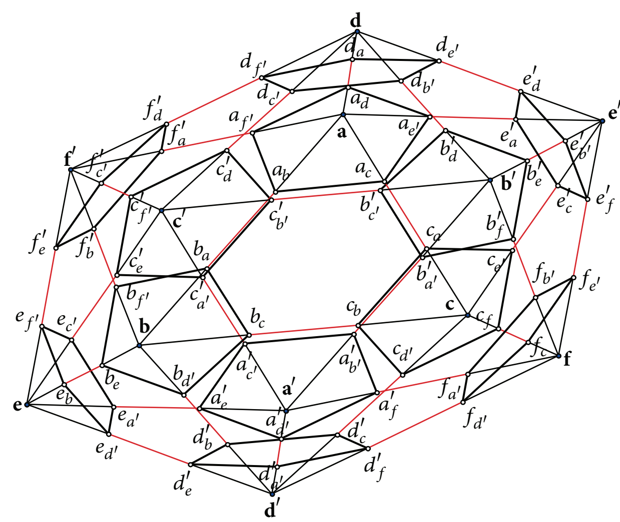

Figure 3

As a consequence of the identity of mediality, the considered GS-quasigroup $(Q, \cdot)$ satisfies the identities of elasticity and left and right distributivity; that is, we have these identities:

$$ab \cdot a = a \cdot ba, \tag{5}$$

$$a \cdot bc = ab \cdot ac, \tag{6}$$

$$ab \cdot c = ac \cdot bc. \tag{7}$$

Further, the identities

$$a\,(ab \cdot b) = b, \tag{8}$$

$$(b \cdot ba)\,a = b, \tag{9}$$

$$a\,(ab \cdot c) = b \cdot bc, \tag{10}$$

$$(c \cdot ba)\,a = cb \cdot b, \tag{11}$$

$$a\,(a \cdot bc) = b\,(b \cdot ac)\,, \tag{12}$$

$$(cb \cdot a)\,a = (ca \cdot b)\,b \tag{13}$$

and equivalencies

$$ab = c \Longleftrightarrow a = c \cdot cb, \tag{14}$$

$$ab = c \Longleftrightarrow b = ac \cdot c \tag{15}$$

also hold.

Let $\mathbb{C}$ be the set of points of the Euclidean plane. For any two different points a, b we define $ab = c$ if the point b divides the pair a, c in the ratio of the golden section. In [3], it is proved that $(\mathbb{C}, \cdot)$ is a GS-quasigroup. We will denote that quasigroup by $\mathbb{C}((1/2)(1 + \sqrt{5}))$ because we have $c = (1/2)(1+\sqrt{5})$ if $a = 0$ and $b = 1$. The figures in this quasigroup $\mathbb{C}((1/2)(1 + \sqrt{5}))$ can be used for illustration of "geometrical" concepts and relations in any GS-quasigroup.

3. Affine Regular Pentagons and Hexagons in GS-Quasigroups

From now on, let $(Q, \cdot)$ be any GS-quasigroup. The elements of the set Q are said to be *points*.

In each medial quasigroup, the concept of a parallelogram can be introduced by means of two auxiliary points. In [5], it is proved that the points a, b, c, d are the vertices of a *parallelogram* denoted by $\mathrm{Par}(a, b, c, d)$, if and only if there are two points p and q such that $pa = qb$ and $pd = qc$. It is also shown that if the statement $\mathrm{Par}(a, b, c, d)$ holds, then the equalities $pa = qb$ and $pd = qc$ are equivalent. In a general GS-quasigroup, the notation of a parallelogram can

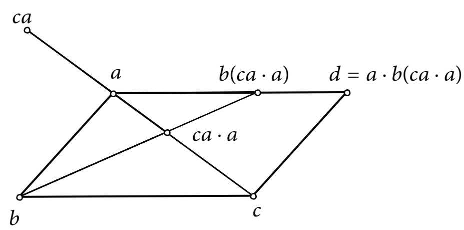

Figure 4

be characterized by the equivalency $\mathrm{Par}(a,b,c,d) \Leftrightarrow a \cdot b(ca \cdot a) = d$ (Figure 4).

In [3], some properties of the quaternary relation Par on the set Q are proved. We will mention only the property which will be used afterwards.

Lemma 1. *From* $Par(a,b,c,d)$ *and* $Par(c,d,e,f)$ *there follows* $Par(a,b,f,e)$.

We will say that b is the *midpoint* of the pair of points a, c and we write $M(a,b,c)$ if and only if $\mathrm{Par}(a,b,c,b)$ holds. The statement $M(a,b,c)$ holds if and only if $c = ba \cdot b$ [3].

The concept of the affine regular hexagon [6] in a GS-quasigroup is defined in the following way. We will say that $(a_1,a_2,a_3,a_4,a_5,a_6)$ is an *affine regular hexagon* with the vertices a_1,a_2,a_3,a_4,a_5,a_6 and the center o and we write $\mathrm{ARH}_o(a_1,a_2,a_3,a_4,a_5,a_6)$ if the statements $\mathrm{Par}(o,a_{i-1},a_i,a_{i+1})$ hold ($i = 1,2,3,4,5,6$), where indexes are taken modulo 6 (Figure 5). The following statement can be proved [6].

Lemma 2. *An affine regular hexagon is uniquely determined by any three consecutive vertices.*

The points o, a_1, a_2, a_3, a_4 determine the figure which will be denoted by the symbol $\mathrm{HARH}_o(a_1,a_2,a_3,a_4)$, "half" of the affine regular hexagon with the center o (Figure 5).

The following results [6] will be very useful.

Lemma 3. *Let* $n \in \mathbb{N}$, $n \geq 3$. *If the statements* $HARH_{c_{12}}(b_1,a_1,a_2,b_2)$, $HARH_{c_{23}}(b_2,a_2,a_3,b_3), \ldots,$ *and* $HARH_{c_{n-1,n}}(b_{n-1},a_{n-1},a_n,b_n)$ *are valid, then there exists a unique point* c_{n1} *so that the statement* $HARH_{c_{n1}}(b_n,a_n,a_1,b_1)$ *is valid too. (The case for* $n = 5$ *is illustrated in Figure 6.)*

Lemma 3 implies the following statement.

Lemma 4. *Let* $n \in \mathbb{N}$, $n \geq 3$. *If the statements* $ARH_{c_{12}}(b_1,a_1,a_2,b_2,d_{21},d_{12})$, $ARH_{c_{23}}(b_2,a_2,a_3,b_3,d_{32},d_{23})$, $\ldots, ARH_{c_{n-1,n}}(b_{n-1},a_{n-1},a_n,b_n,d_{n,n-1},d_{n-1,n})$ *are valid, then there exist unique points* c_{n1}, d_{n1}, d_{1n} *so that the statement* $ARH_{c_{n1}}(b_n,a_n,a_1,b_1,d_{1n},d_{n1})$ *is valid, too.*

The points a, b, c, d successively are said to be the vertices of the *golden section trapezoid* [7] denoted by $\mathrm{GST}(a,b,c,d)$ if the identity $a \cdot ab = d \cdot dc$ holds (Figure 7). It can be proved that the following equivalency $\mathrm{GST}(a,b,c,d) \Leftrightarrow c = a(db \cdot b)$ holds. The following statement is also valid.

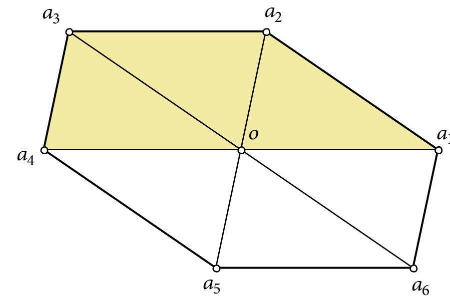

Figure 5

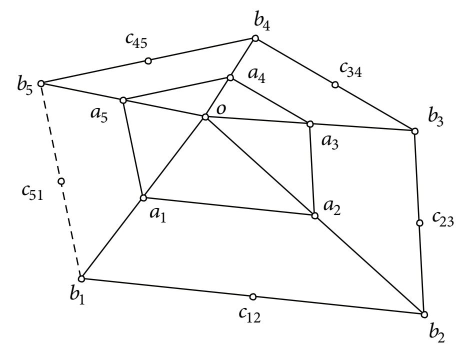

Figure 6

Lemma 5. *Any of the three statements* $GST(a,b,c,d)$, $GST(b,e,f,c)$, *and* $ae = df$ *is a consequence of the two remaining statements.*

In [8], it is proved that any two of the five statements

$$\begin{gathered} \mathrm{GST}(a,b,c,d), \quad \mathrm{GST}(b,c,d,e), \quad \mathrm{GST}(c,d,e,a), \\ \mathrm{GST}(d,e,a,b), \quad \mathrm{GST}(e,a,b,c) \end{gathered} \tag{16}$$

imply the remaining statements.

The points a,b,c,d,e successively are said to be the vertices of the *affine regular pentagon* [8] denoted by $\mathrm{ARP}(a,b,c,d,e)$ if any two (and then all five) of the five statements (16) are valid (Figure 7).

Lemma 6. *An affine regular pentagon is uniquely determined by any three of its vertices.*

Now, we are going to study the relationships between the previously defined geometrical concepts in a general GS-quasigroup.

Lemma 7. *If the statements* $Par(d_1,o,b_2,a_1)$, $Par(d_2,o,b_1,a_2)$ *are valid, then*

(i) *the statements* $d_1b_1 = d_2b_2$ *and* $GST(d_1,a_1,a_2,d_2)$ *are equivalent (Figure 8);*

(ii) *the statements* $d_1b_1 = d_2b_2$ *and* $GST(a_1,b_1,b_2,a_2)$ *are equivalent (Figure 8).*

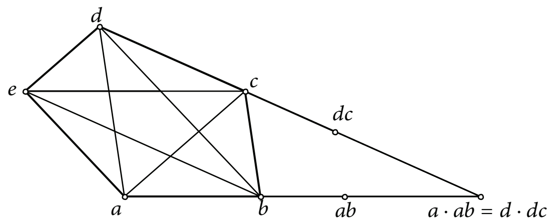

Figure 7

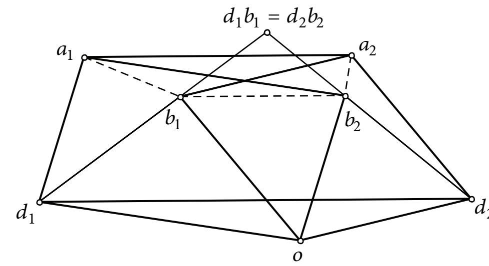

Figure 8

Proof. (i) We have the equalities

$$a_1 = d_1 \cdot o\left(b_2 d_1 \cdot d_1\right), \qquad a_2 = d_2 \cdot o\left(b_1 d_2 \cdot d_2\right), \tag{17}$$

and we have to prove the equivalency of the equalities

$$d_1 \cdot d_1 a_1 = d_2 \cdot d_2 a_2, \qquad d_1 b_1 = d_2 b_2. \tag{18}$$

However, we get

$$\begin{aligned} a_1 &= d_1 \cdot o\left(b_2 d_1 \cdot d_1\right) \overset{(6)}{=} d_1 o \cdot d_1\left(b_2 d_1 \cdot d_1\right) \\ &\overset{(5)}{=} d_1 o \cdot \left(d_1 \cdot b_2 d_1\right) d_1 \\ &\overset{(11)}{=} d_1 o \cdot \left(d_1 b_2 \cdot b_2\right) \overset{(4)}{=} \left(d_1 \cdot d_1 b_2\right) \cdot o b_2 \\ &\overset{(10)}{=} o\left(o d_1 \cdot b_2\right) \cdot o b_2 \\ &\overset{(6)}{=} o \cdot \left(o d_1 \cdot b_2\right) b_2 \overset{(13)}{=} o \cdot \left(o b_2 \cdot d_1\right) d_1 \end{aligned} \tag{19}$$

and thus we get

$$\begin{aligned} & d_1 \cdot d_1 a_1 \\ &\quad = d_1 \cdot d_1\left[o \cdot \left(o b_2 \cdot d_1\right) d_1\right] \\ &\quad \overset{(6)}{=} \left(d_1 \cdot d_1 o\right) \cdot d_1\left[d_1 \cdot \left(o b_2 \cdot d_1\right) d_1\right] \\ &\quad \overset{(5),(2)}{=} \left(d_1 \cdot d_1 o\right) \cdot o b_2 \overset{(4)}{=} d_1 o \cdot \left(d_1 o \cdot b_2\right) \end{aligned} \tag{20}$$

and analogously $d_2 \cdot d_2 a_2 = d_2 o \cdot (d_2 o \cdot b_1)$. Because of that, it is necessary to prove the equivalency of the equality $d_1 o \cdot (d_1 o \cdot b_2) = d_2 o \cdot (d_2 o \cdot b_1)$ and the equality $b_2 = (d_2 \cdot d_1 b_1) \cdot d_1 b_1$ which is, according to (15), equivalent to $d_1 b_1 = d_2 b_2$. As we get

$$\begin{aligned} & d_2 o \cdot \left(d_2 o \cdot b_1\right) \\ &\quad \overset{(8)}{=} d_2 o \cdot \left[d_2 o \cdot d_2\left(d_2 b_1 \cdot b_1\right)\right] \\ &\quad \overset{(6)}{=} d_2\left[o \cdot o\left(d_2 b_1 \cdot b_1\right)\right] \\ &\quad \overset{(2),(5)}{=} \left[d_1 \cdot d_1\left(d_2 d_1 \cdot d_1\right)\right]\left[o \cdot o\left(d_2 b_1 \cdot b_1\right)\right] \\ &\quad \overset{(4)}{=} d_1 o \cdot \left[d_1 o \cdot \left(d_2 d_1 \cdot d_2 b_1\right)\left(d_1 b_1\right)\right] \\ &\quad \overset{(6)}{=} d_1 o \cdot \left[d_1 o \cdot \left(d_2 \cdot d_1 b_1\right)\left(d_1 b_1\right)\right], \end{aligned} \tag{21}$$

these should be equivalent equalities:

$$\begin{aligned} & d_1 o \cdot \left(d_1 o \cdot b_2\right) = d_1 o \cdot \left[d_1 o \cdot \left(d_2 \cdot d_1 b_1\right)\left(d_1 b_1\right)\right], \\ & b_2 = \left(d_2 \cdot d_1 b_1\right) \cdot d_1 b_1, \end{aligned} \tag{22}$$

which is obvious.

(ii) Firstly, let us prove that from $\mathrm{GST}(a_1, b_1, b_2, a_2)$ there follows $d_1 b_1 = d_2 b_2$.

We have the equalities

$$\begin{aligned} d_1 &= a_1 \cdot b_2\left(o a_1 \cdot a_1\right), \\ d_2 &= a_2 \cdot b_1\left(o a_2 \cdot a_2\right), \\ b_2 &= a_1\left(a_2 b_1 \cdot b_1\right). \end{aligned} \tag{23}$$

Therefore we get

$$\begin{aligned} d_1 b_1 &= \left[a_1 \cdot b_2\left(o a_1 \cdot a_1\right)\right] b_1 \\ &= a_1\left[a_1\left(a_2 b_1 \cdot b_1\right) \cdot \left(o a_1 \cdot a_1\right)\right] \cdot b_1 \\ &\overset{(6)}{=} \left[a_1 \cdot a_1\left(a_2 b_1 \cdot b_1\right)\right]\left[a_1\left(o a_1 \cdot a_1\right)\right] \cdot b_1 \\ &\overset{(12)}{=} \left[\left(a_2 b_1\right)\left(a_2 b_1 \cdot a_1 b_1\right) \cdot a_1\left(o a_1 \cdot a_1\right)\right] b_1 \\ &\overset{(7)}{=} \left[\left(a_2 \cdot a_2 a_1\right) b_1 \cdot a_1\left(o a_1 \cdot a_1\right)\right] b_1 \\ &\overset{(4)}{=} \left[\left(a_2 \cdot a_2 a_1\right) a_1 \cdot b_1\left(o a_1 \cdot a_1\right)\right] b_1 \\ &\overset{(9)}{=} \left[a_2 \cdot b_1\left(o a_1 \cdot a_1\right)\right] b_1 \\ &\overset{(6)}{=} \left[a_2 b_1 \cdot a_2\left(o a_1 \cdot a_1\right)\right] b_1 \\ &\overset{(7)}{=} \left(a_2 b_1 \cdot b_1\right)\left[a_2\left(o a_1 \cdot a_1\right) \cdot b_1\right], \\ d_2 b_2 &= \left[a_2 \cdot b_1\left(o a_2 \cdot a_2\right)\right] \cdot a_1\left(a_2 b_1 \cdot b_1\right) \\ &\overset{(6)}{=} \left[a_2 b_1 \cdot a_2\left(o a_2 \cdot a_2\right)\right] \cdot a_1\left(a_2 b_1 \cdot b_1\right) \end{aligned}$$

$$\begin{aligned}
&\overset{(4)}{=} (a_2b_1 \cdot a_1)\,[a_2\,(oa_2 \cdot a_2) \cdot (a_2b_1 \cdot b_1)] \\
&\overset{(7)}{=} (a_2b_1 \cdot a_1)\,[a_2\,(a_2b_1 \cdot b_1) \cdot (oa_2 \cdot a_2)\,(a_2b_1 \cdot b_1)] \\
&\overset{(8)}{=} (a_2b_1 \cdot a_1)\,[b_1 \cdot (oa_2 \cdot a_2)\,(a_2b_1 \cdot b_1)] \\
&\overset{(4)}{=} (a_2b_1 \cdot b_1)\,[a_1 \cdot (oa_2 \cdot a_2)\,(a_2b_1 \cdot b_1)],
\end{aligned} \tag{24}$$

so it is necessary to prove the equality

$$a_2\,(oa_1 \cdot a_1) \cdot b_1 = a_1 \cdot (oa_2 \cdot a_2)\,(a_2b_1 \cdot b_1). \tag{25}$$

Really, we have

$$\begin{aligned}
&a_1 \cdot (oa_2 \cdot a_2)\,(a_2b_1 \cdot b_1) \\
&\overset{(9),(11)}{=} (a_1 \cdot a_1b_1)\,b_1 \cdot [(o \cdot a_2b_1)\,b_1 \cdot (a_2b_1 \cdot b_1)] \\
&\overset{(7)}{=} [(a_1 \cdot a_1b_1) \cdot (o \cdot a_2b_1)\,(a_2b_1)]\,b_1 \\
&\overset{(4)}{=} [a_1\,(o \cdot a_2b_1) \cdot (a_1b_1 \cdot a_2b_1)]\,b_1 \\
&\overset{(7)}{=} [a_1\,(o \cdot a_2b_1) \cdot (a_1a_2 \cdot b_1)]\,b_1 \\
&\overset{(4)}{=} [(a_1 \cdot a_1a_2) \cdot (o \cdot a_2b_1)\,b_1]\,b_1 \\
&\overset{(11)}{=} (a_1 \cdot a_1a_2)\,(oa_2 \cdot a_2) \cdot b_1 \\
&\overset{(10)}{=} [a_2\,(a_2a_1 \cdot a_2) \cdot (oa_2 \cdot a_2)]\,b_1 \\
&\overset{(5)}{=} [(a_2 \cdot a_2a_1)\,a_2 \cdot (oa_2 \cdot a_2)]\,b_1 \\
&\overset{(7)}{=} [(a_2 \cdot a_2a_1)\,(oa_2) \cdot a_2]\,b_1 \\
&\overset{(4)}{=} [(a_2o)\,(a_2a_1 \cdot a_2) \cdot a_2]\,b_1 \\
&\overset{(5)}{=} [(a_2o)\,(a_2 \cdot a_1a_2) \cdot a_2]\,b_1 \\
&\overset{(6)}{=} [a_2\,(o \cdot a_1a_2) \cdot a_2]\,b_1 \\
&\overset{(5)}{=} [a_2 \cdot (o \cdot a_1a_2)\,a_2]\,b_1 \\
&\overset{(11)}{=} a_2\,(oa_1 \cdot a_1) \cdot b_1.
\end{aligned} \tag{26}$$

Now, we are going to prove that $d_1b_1 = d_2b_2$ implies $\mathrm{GST}(a_1, b_1, b_2, a_2)$. According to (i), from the hypotheses of (ii), there follows the statement $\mathrm{GST}(d_1, a_1, a_2, d_2)$ and, from this statement and the equality $d_1b_1 = d_2b_2$, according to Lemma 5, there follows the statement $\mathrm{GST}(a_1, b_1, b_2, a_2)$. □

Lemma 8. *If the statements $HARH_o\,(c_1, b_1, b_2, c_2)$, $HARH_{d_1}\,(a_1, b_1, c_1, e_1)$, and $HARH_{d_2}(a_2, b_2, c_2, e_2)$ are valid, then the statement $GST(a_1, b_1, b_2, a_2)$ and equality $d_1b_1 = d_2b_2$ are equivalent (Figure 9).*

Proof. The assumptions of the lemma imply the statements $\mathrm{Par}(o, b_2, b_1, c_1)$, $\mathrm{Par}(o, b_1, b_2, c_2)$, $\mathrm{Par}(b_1, c_1, d_1, a_1)$, and

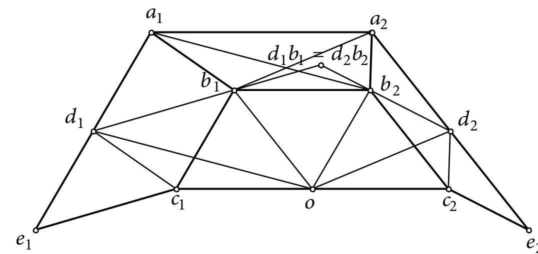

Figure 9

$\mathrm{Par}(b_2, c_2, d_2, a_2)$, and then, according to Lemma 1, parallelograms $\mathrm{Par}(o, b_2, a_1, d_1)$, $\mathrm{Par}(o, b_1, a_2, d_2)$ follow from the first and the third, and the second and the fourth parallelogram, respectively. Owing to these last statements, according to Lemma 7(ii), statements $\mathrm{GST}(a_1, b_1, b_2, a_2)$ and $d_1b_1 = d_2b_2$ are equivalent. □

Lemma 9. *With the assumption ARH $(a_b, b_a, b_c, c_b, c_a, a_c)$, the statement $f'a_b = e'a_c$ follows from the equalities $d'b_c = f'b_a$, $d'c_b = e'c_a$ (Figure 10).*

Proof. Supposing that a more precise statement $\mathrm{ARH}_o(a_b, b_a, b_c, c_b, c_a, a_c)$ is valid, so the statements $\mathrm{Par}(c_b, c_a, o, b_c)$ and $\mathrm{Par}(b_a, o, a_c, a_b)$ are valid. From the statements $\mathrm{Par}(c_b, c_a, o, b_c)$, $d'c_b = e'c_a$ there follows $d'b_c = e'o$, which together with $d'b_c = f'b_a$ gives the equality $f'b_a = e'o$, and this statement and the statement $\mathrm{Par}(b_a, o, a_c, a_b)$ imply the equality $f'a_b = e'a_c$. □

Theorem 10. *The statement $ARP(a_b, a_{f'}, a_d, a_{e'}, a_c)$ follows from the statements $ARH(a_b, b_a, b_c, c_b, c_a, a_c)$, $ARH(b_c, b_{d'}, d'_b, d'_c, c_{d'}, c_b)$, $ARP(b_c, b_{d'}, b_e, b_{f'}, b_a)$, $ARP(c_b, c_{d'}, c_f, c_{e'}, c_a)$, $ARH(b_a, b_{f'}, f'_b, f'_a, a_{f'}, a_b)$, and $ARH(c_a, c_{e'}, e'_c, e'_a, a_{e'}, a_c)$ (Figure 11).*

Proof. It is sufficient to prove that the statement $\mathrm{GST}(a_{f'}, a_b, a_c, a_{e'})$ follows from the statements $\mathrm{ARH}_o(a_b, b_a, b_c, c_b, c_a, a_c)$, $\mathrm{HARH}_{d'}(b_{d'}, b_c, c_b, c_{d'})$, $\mathrm{GST}(b_{d'}, b_c, b_a, b_{f'})$, $\mathrm{GST}(c_{d'}, c_b, c_a, c_{e'})$, $\mathrm{HARH}_{f'}(b_{f'}, b_a, a_b, a_{f'})$, and $\mathrm{HARH}_{e'}(c_{e'}, c_a, a_c, a_{e'})$. Firstly, according to Lemma 8, the equality $d'b_c = f'b_a$ follows from the statements $\mathrm{HARH}_o(c_b, b_c, b_a, a_b)$, $\mathrm{HARH}_{d'}(b_{d'}, b_c, c_b, c_{d'})$, $\mathrm{HARH}_{f'}(b_{f'}, b_a, a_b, a_{f'})$, and $\mathrm{GST}(b_{d'}, b_c, b_a, b_{f'})$. Owing to the same lemma, the equality $d'c_b = e'c_a$ follows from the statements $\mathrm{HARH}_o(b_c, c_b, c_a, a_c)$, $\mathrm{HARH}_{d'}(c_{d'}, c_b, b_c, b_{d'})$, $\mathrm{HARH}_{e'}(c_{e'}, c_a, a_c, a_{e'})$, and $\mathrm{GST}(c_{d'}, c_b, c_a, c_{e'})$.

Now, all requirements of Lemma 9 are satisfied and the equality $f'a_b = e'a_c$ follows accordingly.

Finally, according to Lemma 8, the statements $\mathrm{HARH}_o(b_a, a_b, a_c, c_a)$, $\mathrm{HARH}_{f'}(a_{f'}, a_b, b_a, b_{f'})$, and $\mathrm{HARH}_{e'}(a_{e'}, a_c, c_a, c_{e'})$ and equality $f'a_b = e'a_c$ imply the statement $\mathrm{GST}(a_{f'}, a_b, a_c, a'_e)$. □

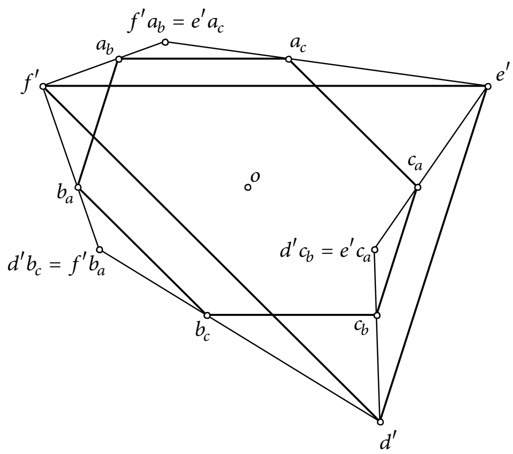

Figure 10

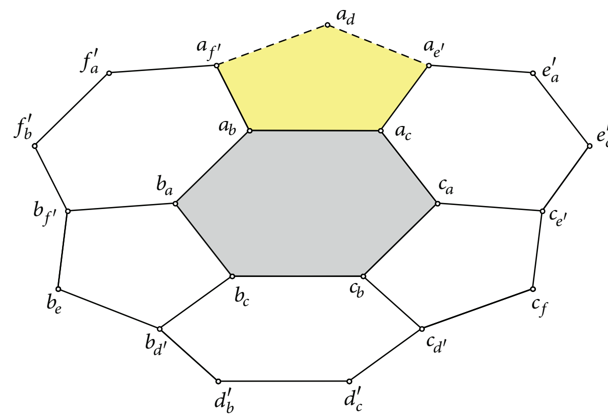

Figure 13

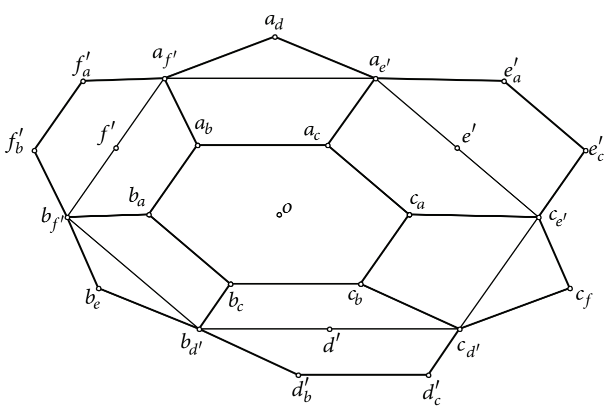

Figure 11

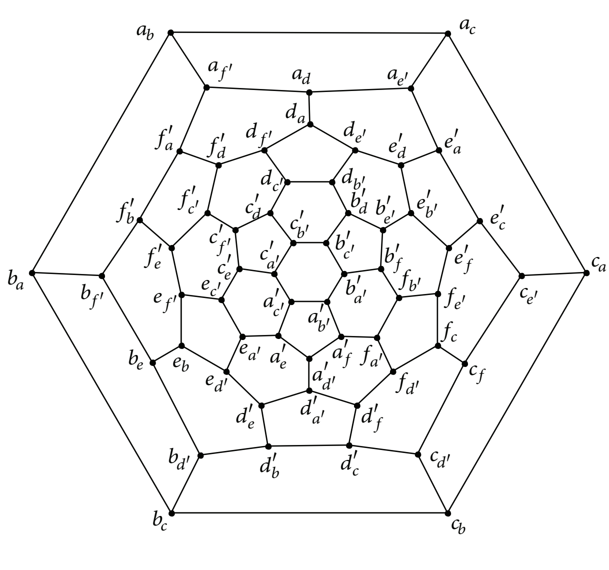

Figure 12

4. Construction of an Affine Fullerene C_{60} in a GS-Quasigroup

In this section, we are going to construct an affine fullerene C_{60} in a general GS-quasigroup by means of the previously discovered statements about affine regular pentagons and hexagons in a general GS-quasigroup.

Theorem 11. *An affine fullerene C_{60} can be constructed in each GS-quasigroup.*

Proof. For the sake of clarity, each step of the proof is precisely presented in figures in the GS-quasigroup $\mathbb{C}((1/2)(1+\sqrt{5}))$ and each sequence of the proof of the theorem can be followed on the Schlegel diagram (Figure 12).

Let us start with the four given points $b_a, b_c, c_b, b_{d'}$. The affine regular hexagons $\mathrm{ARH}(a_b, b_a, b_c, c_b, c_a, a_c)$ and $\mathrm{ARH}(b_c, b_{d'}, d'_b, d'_c, c_{d'}, c_b)$ can be constructed according to Lemma 2.

Owing to Lemma 6, affine regular pentagons $\mathrm{ARP}(b_c, b_{d'}, b_e, b_{f'}, b_a)$ and $\mathrm{ARP}(c_b, c'_d, c_f, c_{e'}, c_a)$ can be obtained. If we apply Lemma 2 again, we can get $\mathrm{ARH}(b_a, b_{f'}, f'_b, f'_a, c_{f'}, a_b)$ and $\mathrm{ARH}(c_a, c_{e'}, e'_c, e'_a, a_{e'}, a_c)$.

According to Theorem 10, the existence of these obtained affine regular hexagons and affine regular pentagons around an affine regular hexagon will result in the existence of the affine regular pentagon $\mathrm{ARP}(a_b, a_{f'}, a_d, a_{e'}, a_c)$ (Figure 13).

According to Lemma 2, the already obtained points $a_d, a_{f'}, f'_a$ uniquely determine $\mathrm{ARH}(d_a, a_d, a_{f'}, f'_a, f'_d, d_{f'})$ and then, because of Lemma 4, the statements $\mathrm{ARH}(d_a, a_d, a_{f'}, f'_a, f'_d, d_{f'})$, $\mathrm{ARH}(f'_a, a_{f'}, a_b, b_a, b_{f'}, f'_b)$, $\mathrm{ARH}(b_a, a_b, a_c, c_a, c_b, b_c)$, and $\mathrm{ARH}(c_a, a_c, a_{e'}, e'_a, e'_c, c_{e'})$ imply the statement $\mathrm{ARH}(e'_a, a_{e'}, a_d, d_a, d_{e'}, e'_d)$. In the same way, the statements $\mathrm{ARH}(e_b, b_e, b_{d'}, d'_b, d'_e, e_{d'})$, $\mathrm{ARH}(d'_b, b_{d'}, b_c, c_b, c_{d'}, d'_c)$, $\mathrm{ARH}(c_b, b_c, b_a, a_b, a_c, c_a)$, and $\mathrm{ARH}(a_b, b_a, b_{f'}, f'_b, f'_a, a_{f'})$ imply $\mathrm{ARH}(f'_b, b_{f'}, b_e, e_b, e_{f'}, f'_e)$ and, analogously, the statements $\mathrm{ARH}(f_c, c_f, c_{e'}, e'_c, e'_f, f_{e'})$, $\mathrm{ARH}(e'_c, c_{e'}, c_a, a_c, a_{e'}, e'_a)$, $\mathrm{ARH}(a_c, c_a, c_b, b_c, b_a, a_b)$, and $\mathrm{ARH}(b_c, c_b, c_{d'}, d'_c, d'_b, b_{d'})$ imply $\mathrm{ARH}(d'_c, c_{d'}, c_f, f_c, f_{d'}, d'_f)$.

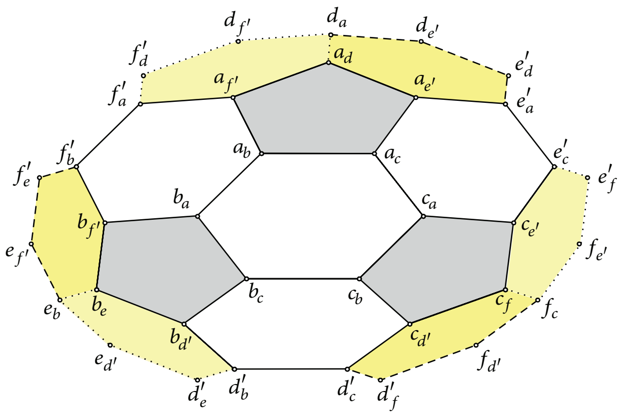

Figure 14

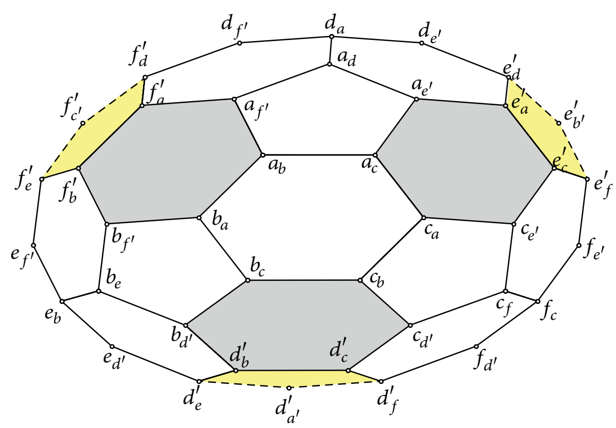

Figure 15

This consideration is presented in Figure 14.

These two precisely described procedures will also be used later. It will also be denoted which figure is used for the geometrical presentation of the obtained implications in the GS-quasigroup $\mathbb{C}((1/2)(1+\sqrt{5}))$.

Now, according to Theorem 10, the statements ARH $(d'_b, b_{d'}, b_c, c_b, c_{d'}, d'_c)$, ARH$(b_c, b_a, a_b, a_c, c_a, c_b)$, ARP$(b_c, b_a, b_{f'}, b_e, b_{d'})$, ARP$(c_b, c_a, c_{e'}, c_f, c_{d'})$, ARH$(b_{d'}, b_e, e_b, e_{d'}, d'_e, d'_b)$, and ARH$(c_{d'}, c_f, f_c, f_{d'}, d'_f, d'_c)$ imply ARP$(d'_b, d'_e, d'_{a'}, d'_f, d'_c)$. The statements ARP$(e'_c, e'_f, e'_{b'}, e'_d, e'_a)$ and ARP$(f'_a, f'_d, f'_{c'}, f'_e, f'_b)$ can be obtained similarly (Figure 15).

Thanks to Lemma 4, the statements ARH$(e_{d'}, d'_e, d'_b, b_{d'}, b_e, e_b)$, ARH$(b_{d'}, d'_b, d'_c, c_{d'}, c_b, b_c)$, ARH$(c_{d'}, d'_c, d'_f, f_{d'}, f_c, c_f)$, and ARH$(f_{d'}, d'_f, d'_{a'}, a'_{d'}, a'_f, f_{a'})$ imply ARH$(a'_{d'}, d'_{a'}, d'_e, e_{d'}, e_{a'}, a'_e)$. We can find the affine regular hexagons ARH$(b'_{e'}, e'_{b'}, e'_f, f_{e'}, f_{b'}, b'_f)$ and ARH$(c'_{f'}, f'_{c'}, f'_d, d_{f'}, d_{c'}, c'_d)$ in the same way (Figure 16).

If we apply Theorem 10, we will discover three new affine regular pentagons (Figure 17). The statement ARP$(d_{f'}, d_{c'}, d_{b'}, d_{e'}, d_a)$ follows from ARH$(d_{f'}, f'_d, f'_a, a_{f'}, a_d, d_a)$, ARH$(f'_a, f'_b, b_{f'}, b_a, a_b, a_{f'})$, ARP$(f'_a, f'_b, f'_e, f'_{c'}, f'_d)$, ARP$(a_{f'}, a_b, a_c, a_{e'}, a_d)$, ARH$(f'_d, f'_{c'}, c'_{f'}, c'_d, d_{c'}, d_{f'})$, and

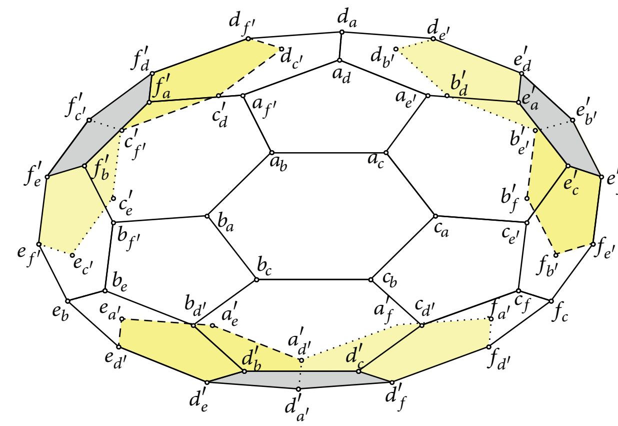

Figure 16

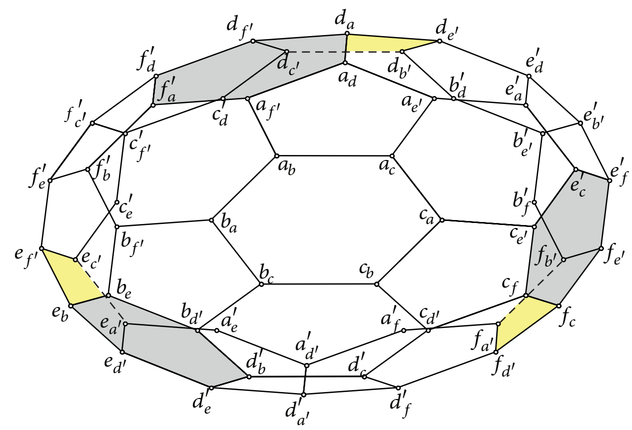

Figure 17

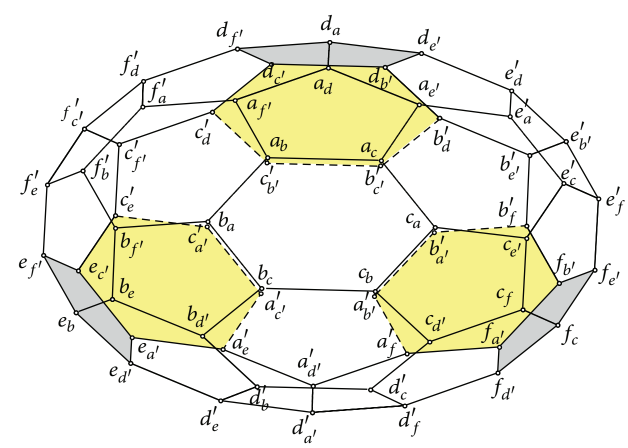

Figure 18

ARH$(a_d, a_{e'}, e'_a, e'_d, d_{e'}, d_a)$, and similarly we get ARP $(e_{d'}, e_{a'}, e_{c'}, e_{f'}, e_b)$ and ARP$(f_{e'}, f_{b'}, f_{a'}, f_{d'}, f_c)$.

Now, we are in a position to use Lemma 4 whose application gives the statements about three new affine regular hexagons (Figure 18). The statements ARH$(b'_d, d_{b'}, d_{e'}, e'_d, e'_{b'}, b'_{e'})$, ARH$(e'_d, d_{e'}, d_a, a_d, a_{e'}, e'_a)$, ARH $(a_d, d_a, d_{f'}, f'_d, f'_a, a_{f'})$, and ARH$(f'_d, d_{f'}, d_{c'}, c'_d, c'_{f'}, f'_{c'})$ imply ARH$(c'_d, d_{c'}, d_{b'}, b'_d, b'_{c'}, c'_{b'})$. Similarly, we can get

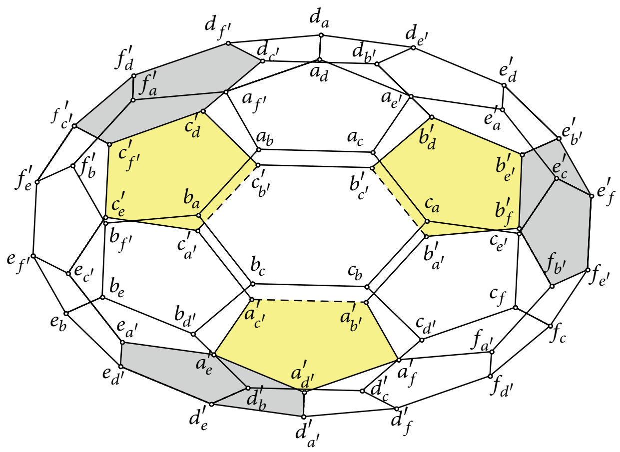

FIGURE 19

the affine regular hexagons $\mathrm{ARH}(a'_e, e_{a'}, e_{c'}, c'_e, c'_{a'}, a'_{c'})$ and $\mathrm{ARH}(b'_f, f_{b'}, f_{a'}, a'_f, a'_{b'}, b'_{a'})$.

If we apply Theorem 10, we will obtain the three new affine regular pentagons $\mathrm{ARP}(a'_e, a'_{c'}, a'_{b'}, a'_f, a'_{d'})$, $\mathrm{ARP}(b'_f, b'_{a'}, b'_{c'}, b'_d, b'_{e'})$, and $\mathrm{ARP}(c'_d, c'_{b'}, c'_{a'}, c'_e, c'_{f'})$ (Figure 19).

Finally, the application of Lemma 3 will allow us to close the complete structure. We have to prove that the points $b'_{a'}, a'_{b'}, a'_{c'}, c'_{a'}, c'_{b'}, b'_{c'}$ are the vertices of an affine regular hexagon.

By applying Lemma 3 we get that the statements $\mathrm{HARH}(c'_{a'}, a'_{c'}, a'_e, e_{a'})$, $\mathrm{HARH}(e_{a'}, a'_e, a'_{d'}, d'_{a'})$, $\mathrm{HARH}(d'_{a'}, a'_{d'}, a'_f, f_{a'})$, and $\mathrm{HARH}(f_{a'}, a'_f, a'_{b'}, b'_{a'})$ imply $\mathrm{HARH}(b'_{a'}, a'_{b'}, a'_{c'}, c'_{a'})$.

Analogously, we have that $\mathrm{HARH}(a'_{b'}, b'_{a'}, b'_f, f_{b'})$, $\mathrm{HARH}(f_{b'}, b'_f, b'_{e'}, e'_{b'})$, $\mathrm{HARH}(e'_{b'}, b'_{e'}, b'_d, d_{b'})$, and $\mathrm{HARH}(d_{b'}, b'_d, b'_{c'}, c'_{b'})$ imply $\mathrm{HARH}(c'_{b'}, b'_{c'}, b'_{a'}, a'_{b'})$, and $\mathrm{HARH}(b'_{c'}, c'_{b'}, c'_d, d_{c'})$, $\mathrm{HARH}(d_{c'}, c'_d, c'_{f'}, f'_{c'})$, $\mathrm{HARH}(f'_{c'}, c'_{f'}, c'_e, e_{c'})$, and $\mathrm{HARH}(e_{c'}, c'_e, c'_{a'}, a'_{c'})$ imply $\mathrm{HARH}(a'_{c'}, c'_{a'}, c'_{b'}, b'_{c'})$.

These obtained three halves of affine regular hexagons $\mathrm{HARH}(b'_{a'}, a'_{b'}, a'_{c'}, c'_{a'})$, $\mathrm{HARH}(c'_{b'}, b'_{c'}, b'_{a'}, a'_{b'})$, and $\mathrm{HARH}(a'_{c'}, c'_{a'}, c'_{b'}, b'_{c'})$ determine the affine regular hexagon $\mathrm{ARH}(b'_{a'}, a'_{b'}, a'_{c'}, c'_{a'}, c'_{b'}, b'_{c'})$ (Figure 19). This completes the proof of the theorem. □

Thus, we have proved that any affine fullerene C_{60} can be obtained only by applying the golden section.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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