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Research Article

New Operational Matrix of Integrations and Coupled System of Fredholm Integral Equations

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We study Legendre polynomials and develop new operational matrix of integration. Based on the operational matrix, we develop a new method to solve a coupled system of Fredholm integral equations of the form $U(x) + \lambda_{11}\int_0^1 K_{11}(x,t)U(t)dt + \lambda_{12}\int_0^1 K_{12}(x,t)V(t)dt = f(x)$, $V(x) + \lambda_{21}\int_0^1 K_{21}(x,t)U(t)dt + \lambda_{22}\int_0^1 K_{22}(x,t)V(t)dt = g(x)$, where λ_{11}, λ_{12}, λ_{21}, and λ_{22} are real constants and $f, g \in C([0,1])$. The method reduces the coupled system to a system of easily solvable algebraic equations without discretizing the original system. As an application, we provide examples and numerical simulations demonstrating that the results obtained using the new technique match very well with the exact solutions of the problems. To show the efficiency of the method, we compare our results with some of the results already studied with other available methods in the literature.

1. Introduction

Fredholm integral equations are frequently encountered in many physical processes such as dynamic stiffness of rigid rectangular foundations [1], soil mechanics and rock mechanics [2], diffraction of waves by randomly rough surface in two dimensions [3], thermoelasticity [4], and scattering problem [5], to name a few. For systems of such equations, various techniques such as extrapolation method, Galerkin discretization, collocation methods, and quadrature, iterative, spline, orthogonal polynomial, and multiple grid methods have been proposed to determine desired solutions (see, e.g., [6–9] and the references quoted there). These methods include approximate analytical and numerical approaches.

Recently, approximate solutions to system of integral equations have attracted the attention of many authors and they obtained solutions using various available techniques in the literature. For example, system of integral equations has been studied with wavelets techniques in [10, 11], with Adomian decomposition method in [12, 13], with Tau method in [14], with chebesheve polynomial and block pulse function in [15, 16], and with Taylor expansion and some modified methods based on taylor series expansion in [17–25].

In this paper, we use shifted Legendre polynomials and develop a new operational matrix of integration. Based on the operational matrix of integration, we develop a simple method to find solutions of the coupled system of Fredholm integral equations. The method reduces the coupled system to a system of easily solvable algebraic equations without discretizing the original system of equations. Besides simplicity, the method yields accurate results even for small value of M resulting in the reduction of the system to small system of algebraic equations. It is verified by examples and their numerical simulations demonstrating that the results obtained using the new technique match very well with the exact solutions of the problems. To show the efficiency of the method over some of the well-known techniques, we compare our results with some of the results already studied with other available methods such as Taylor series approximation method [19] and block pulse method [16]. We find that the new techniques provide highly accurate solutions as compared to Taylor series approximation method and block pulse method.

2. Main Results: New Operational Matrix of Integrations

The Legendre polynomials defined on $[-1,1]$ are given by the following recurrence relation:

$$Ł_{i+1}(z)=\frac{2i+1}{i+1}zŁ_i(z)-\frac{i}{i+1}Ł_{i-1}(z),\quad i=1,2,\ldots,\quad \text{where } Ł_0(z)=0,\quad Ł_1(z)=z. \tag{1}$$

The transformation $x=(z+1)/2$ transforms the interval $[-1,1]$ to $[0,1]$ and the polynomials transformed to the so called shifted Legendre polynomials given as [26] follows:

$$P_i(x)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{x^k}{(k!)^2},\quad i=0,1,2,3,\ldots, \tag{2}$$

where $P_i(0)=(-1)^i$, $P_i(1)=1$. The orthogonality condition is

$$\int_0^1 P_i(x)P_j(x)\,dx=\begin{cases}\dfrac{1}{2i+1}, & \text{if } i=j;\\ 0, & \text{if } i\neq j.\end{cases} \tag{3}$$

Consequently, any $f(x)\in C[0,1]$ can be approximated by shifted Legendre polynomial as follows:

$$f(x)\approx\sum_{a=0}^{m}c_aP_a(x),\quad \text{where } c_a=\langle f(x),P_a(x)\rangle=(2a+1)\int_0^1 f(x)P_a(x)\,dx. \tag{4}$$

In vector notation, we write

$$f(x)=K_M^T\widehat{P}_M, \tag{5}$$

where $M=m+1$, K is the coefficient vector, and $\widehat{P}$ is M terms vector function. In case of function of two variables, that is, $f\in C([0,1]\times[0,1])$, we write

$$f(x,t)\approx\sum_{i=0}^{m}\sum_{j=0}^{m}c_{ij}P_i(x)P_j(t),\quad \text{where } c_{ij}=(2i+1)(2j+1)\iint_0^1 f(x,t)P_i(x)P_j(t)\,dx\,dt. \tag{6}$$

The orthogonality condition of $P_i(x)P_j(t)$ is found to be

$$\iint_0^1 P_i(x)P_j(t)P_a(x)P_b(t)\,dx\,dt=\begin{cases}\dfrac{1}{(2i+1)(2j+1)}, & \text{if } a=i,\ b=j;\\ 0, & \text{otherwise.}\end{cases} \tag{7}$$

In vector notation, (6) can be written as

$$f(x,t)\approx\left(\widehat{P}_M(x)\right)^TC_{(M\times M)}\widehat{P}_M(t), \tag{8}$$

where $\widehat{P}_M(x)$ and $\widehat{P}_M(t)$ are column vectors containing Legendre polynomial and C is the coefficient matrix whose entries are obtained by using (6).

2.1. Error Analysis. For sufficiently smooth function $f(x,y)$ on $[0,1]\times[0,1]$, the error of the approximation is given by

$$\left\|f(x,y)-P_n(x,y)\right\|_2\le\left(C_1+C_2+C_3\frac{1}{M^{M+1}}\right)\frac{1}{M^{M+1}}, \tag{9}$$

where

$$C_1=\frac{1}{4}\max_{(x,y)\in[0,1]\times[0,1]}\left|\frac{\partial^{M+1}}{\partial x^{M+1}}f(x,y)\right|,$$
$$C_2=\frac{1}{4}\max_{(x,y)\in[0,1]\times[0,1]}\left|\frac{\partial^{M+1}}{\partial y^{M+1}}f(x,y)\right|, \tag{10}$$
$$C_3=\frac{1}{16}\max_{(x,y)\in[0,1]\times[0,1]}\left|\frac{\partial^{2M+2}}{\partial x^{M+1}\partial y^{M+1}}f(x,y)\right|.$$

We refer the reader to [27] for the proof of the above result.

Lemma 1. *Let $f(x,t)\in C([0,1]\times[0,1])$ and $g(t)\in C([0,1])$; then*

$$\int_0^1 f(x,t)g(t)\,dt\approx K_MG_{M\times M}\widehat{P}(x), \tag{11}$$

where K_M is the Legendre coefficient vector of $g(t)$ and the matrix $G=[q_{ji}]$, where $q_{ji}=(1/(2j+1))c_{ij}$.

Proof. In view of (5) and (6), we have

$$f(x,t)\approx\sum_{i=0}^{m}\sum_{j=0}^{m}c_{ij}P_i(x)P_j(t),\quad \text{where } c_{ij}=(2j+1)(2i+1)\iint_0^1 f(x,t)P_i(x)P_j(t)\,dx\,dt,$$
$$g(t)\approx\sum_{a=0}^{m}d_aP_a(t),\quad \text{where } d_a=(2a+1)\int_0^1 g(t)P_a(t)\,dt. \tag{12}$$

Using (12), we obtain

$$\int_0^1 f(x,t)g(t)\,dt\approx\int_0^1\left(\sum_{i=0}^{m}\sum_{j=0}^{m}c_{ij}P_i(x)P_j(t)\right)\left(\sum_{a=0}^{m}d_aP_a(t)\right)dt, \tag{13}$$

which implies that

$$\int_0^1 f(x,t)g(t)\,dt\approx\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{a=0}^{m}d_ac_{ij}P_i(x)\int_0^1 P_j(t)P_a(t)\,dt. \tag{14}$$

Using the orthogonality relation, we get

$$\int_0^1 f(x,t)\,g(t)\,dt \approx \sum_{i=0}^{m}\sum_{j=0}^{m} d_j c_{ij} P_i(x)\left(\frac{1}{2j+1}\right) = \sum_{j=0}^{m}\sum_{i=0}^{m} d_j q_{ji} P_i(x), \tag{15}$$

where $q_{ji} = (1/(2j+1))c_{ij}$. In matrix form, we have

$$\int_0^1 f(x,t)\,g(t)\,dt \approx K_M G_{M\times M}\widehat{P}(x). \tag{16}$$

□

3. System of Fredholm Integral Equations

Consider the following coupled system of Fredholm integral equations:

$$\begin{aligned} U(x) + \lambda_{11}\int_0^1 K_{11}(x,t)\,U(t)\,dt + \lambda_{12}\int_0^1 K_{12}(x,t)\,V(t)\,dt &= f(x), \\ V(x) + \lambda_{21}\int_0^1 K_{21}(x,t)\,U(t)\,dt + \lambda_{22}\int_0^1 K_{22}(x,t)\,V(t)\,dt &= g(x), \end{aligned} \tag{17}$$

where λ_{11}, λ_{12}, λ_{21}, and λ_{22} are real constants, $f, g \in C([0,1])$, $K_{11}, K_{12}, K_{21}, K_{22} \in C([0,1]\times[0,1])$, and $U(x)$, $V(x)$ are unknown functions to be determined. Approximating $U(x)$ and $V(x)$ in terms of Legendre polynomials, we obtain

$$U(x) \approx H_M^T \widehat{P}(x), \qquad V(x) \approx N_M^T \widehat{P}(x). \tag{18}$$

Using Lemma 1, we have the following approximations:

$$\begin{aligned} \int_0^1 K_{11}(x,t)\,U(t)\,dt &\approx H_M^T G_{11}\widehat{P}(x), \\ \int_0^1 K_{12}(x,t)\,V(t)\,dt &\approx N_M^T G_{12}\widehat{P}(x), \\ \int_0^1 K_{21}(x,t)\,U(t)\,dt &\approx H_M^T G_{21}\widehat{P}(x), \\ \int_0^1 K_{22}(x,t)\,V(t)\,dt &\approx N_M^T G_{22}\widehat{P}(x). \end{aligned} \tag{19}$$

Using (18) and (19) in the coupled system (17), we obtain the following system of algebraic equations

$$\begin{aligned} H_M^T\widehat{P}(x) + \lambda_{11}H_M^T G_{11}\widehat{P}(x) + \lambda_{12}N_M^T G_{12}\widehat{P}(x) &= F_1\widehat{P}(x), \\ N_M^T\widehat{P}(x) + \lambda_{21}H_M^T G_{21}\widehat{P}(x) + \lambda_{22}N_M^T G_{22}\widehat{P}(x) &= F_2\widehat{P}(x), \end{aligned} \tag{20}$$

which can be written as

$$\begin{pmatrix} H_M^T\widehat{P}(x) \\ N_M^T\widehat{P}(x) \end{pmatrix} + \begin{pmatrix} \lambda_{11}H_M^T G_{11}\widehat{P}(x) \\ \lambda_{22}N_M^T G_{22}\widehat{P}(x) \end{pmatrix} + \begin{pmatrix} \lambda_{12}N_M^T G_{12}\widehat{P}(x) \\ \lambda_{21}H_M^T G_{21}\widehat{P}(x) \end{pmatrix} = \begin{pmatrix} F_1\widehat{P}(x) \\ F_2\widehat{P}(x) \end{pmatrix}. \tag{21}$$

The transpose of the above system is given by

$$\begin{aligned} &\left(H_M^T\widehat{P}(x) \quad N_M^T\widehat{P}(x)\right) + \left(\lambda_{11}H_M^T G_{11}\widehat{P}(x) \quad \lambda_{22}N_M^T G_{22}\widehat{P}(x)\right) \\ &\quad + \left(\lambda_{12}N_M^T G_{12}\widehat{P}(x) \quad \lambda_{21}H_M^T G_{21}\widehat{P}(x)\right) \\ &\quad = \left(F_1\widehat{P}(x) \quad F_2\widehat{P}(x)\right) \end{aligned} \tag{22}$$

which can further be written as

$$\begin{aligned} &\left(H_M^T \quad N_M^T\right)A + \left(H_M^T \quad N_M^T\right)\begin{pmatrix} \lambda_{11}G_{11} & 0 \\ 0 & \lambda_{22}G_{22} \end{pmatrix}A \\ &\quad + \left(H_M^T \quad N_M^T\right)\begin{pmatrix} 0 & \lambda_{21}G_{21} \\ \lambda_{12}G_{12} & 0 \end{pmatrix}A = \left(F_1 \quad F_2\right)A, \end{aligned} \tag{23}$$

where

$$A = \begin{pmatrix} \widehat{P}(x) & 0 \\ 0 & \widehat{P}(x) \end{pmatrix}. \tag{24}$$

Hence it follows that

$$\left(H_M^T \quad N_M^T\right) + \left(H_M^T \quad N_M^T\right)\begin{pmatrix} \lambda_{11}G_{11} & \lambda_{21}G_{21} \\ \lambda_{12}G_{12} & \lambda_{22}G_{22} \end{pmatrix} - \left(F_1 \quad F_2\right) = 0, \tag{25}$$

which is a generalized Sylvester type equation and can easily be solved for the unknown H_M and N_M by any computational software.

4. Illustrative Examples

Example 1. Consider the following system of Fredholm integral equation:

$$\begin{aligned} U(x) - \frac{1}{3}\int_0^1 (x+t)\,U(t)\,dt - \frac{1}{3}\int_0^1 (x+t)\,V(t)\,dt &= \frac{x}{18} + \frac{17}{36}, \\ V(x) - \int_0^1 (xt)\,U(t)\,dt - \int_0^1 (xt)\,V(t)\,dt &= x^2 - \frac{19}{12}x + 1. \end{aligned} \tag{26}$$

The exact solutions of the system are $U(x) = 1 + x$ and $V(x) = x^2$. The solutions $(U(x), V(x))$ obtained via our technique for

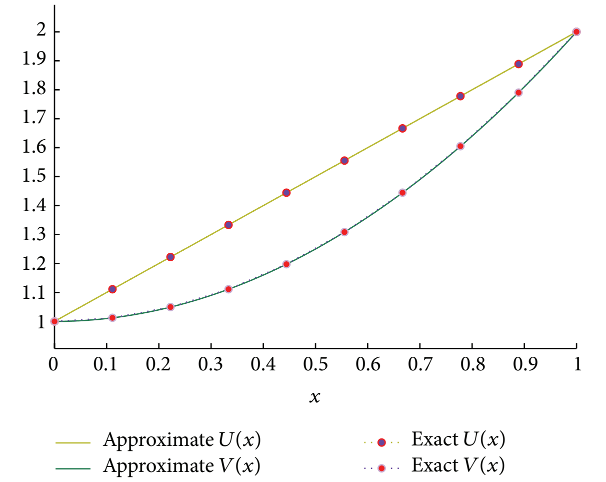

FIGURE 1: Comparison between the exact solutions and the solutions obtained via the new method for $M = 3$. Dots represent the exact solution and the approximate solutions are represented by curved lines.

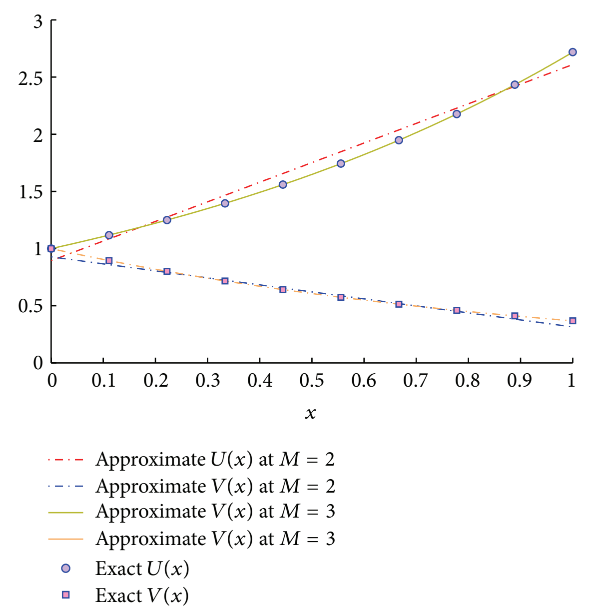

FIGURE 2: Comparing exact solutions with the solutions obtained by our method at different values of M.

$M = 3$ (small enough) are compared with the exact solutions of the problem in Figure 1, where dots represent the exact solutions and the curves are for the solutions obtained via the new method. From Figure 1, it follows that our solutions matchs very well with the exact solution of the problem even for small value M, which shows the effectiveness of our technique.

Example 2. For comparison purposes, consider the following coupled system of Fredholm integral equations:

$$U(x) + \int_0^1 e^{(x-t)} U(t)\,dt + \int_0^1 e^{(xt+2t)} V(t)\,dt = 2e^x + \frac{1}{(x+1)}\left(e^{(x+1)} - 1\right),$$

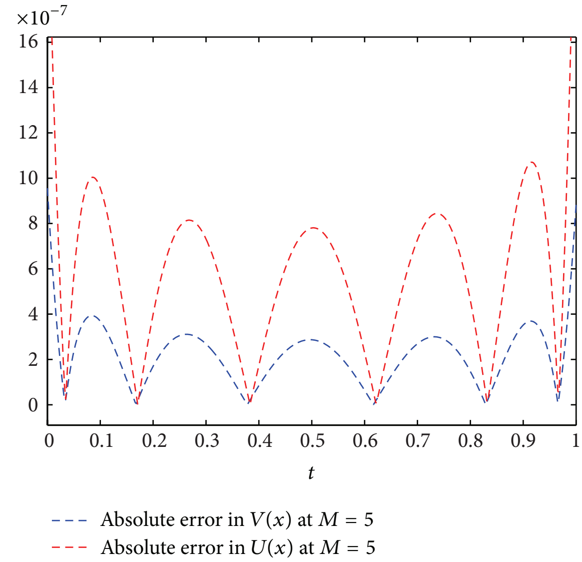

FIGURE 3: Error analysis in $U(x)$ and $V(x)$ for $M = 5$.

$$V(x) + \int_0^1 e^{(xt)} U(t)\,dt + \int_0^1 e^{(x+t)} V(t)\,dt = e^x + e^{-x} + \frac{1}{(x+1)}\left(e^{(x+1)} - 1\right). \tag{27}$$

The exact solutions of the system are $U(x) = e^x$ and $V(x) = e^{-x}$. We obtain the approximate solutions of the system for different values of M and compare the results with the exact solutions of the system. For $M = 2$ and $M = 3$, the comparison is shown in Figure 2, where dots represent the exact solutions of the system and doted curves (red and yellow) represent the approximate solution ($U(x)$ and $V(x)$) obtained via our technique for $M = 2$ while Blue and orange dots represent the approximate solutions ($U(x)$ and $V(x)$) obtained via our technique for $M = 3$. It is clear that the approximate solutions approach rapidly the exact solutions as the values of M increase. It also shows that the approximate solutions are very close to the exact ones for $M = 3$. For example, error of approximation in both $U(x)$ (red doted curve) and $V(x)$ (blue doted curve) is less than 10^{-6} for $M = 5$ as shown in Figure 3, which is much more acceptable number and demonstrates high accuracy of the new technique. Further, we compare our results with some other available results in the literature. We compare the absolute errors (red line) with the absolute error obtained in [19] using Taylor series approximation and also with absolute error obtained in [16] using numerical solution with block pulses. The results are shown in Figures 4 and 5. From these analyses, it is clear that the absolute error in our method even for small value of $M = 4$ is much smaller than those obtained in [16, 19] even for much larger values of m such as $m = 16, 32$. It is a clear indication that the new techniques provide highly accurate solutions as compared to Taylor series approximation method and block pulse method.

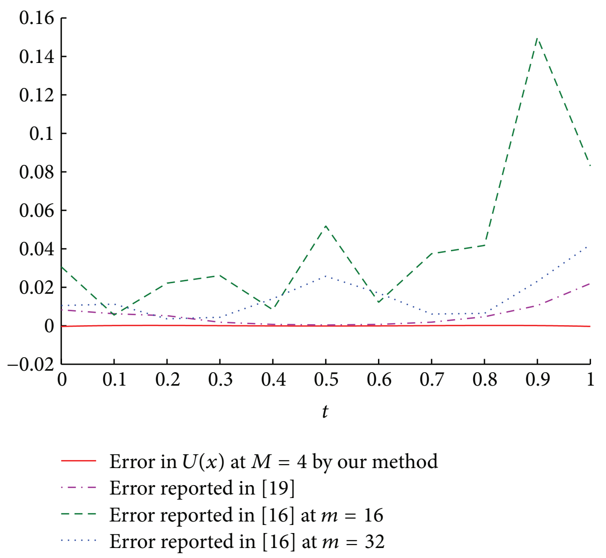

FIGURE 4: Comparing the error estimates in $U(x)$ by our method with error found with Taylor series approximation method (purple dots) and block pulse method (green and blue dots).

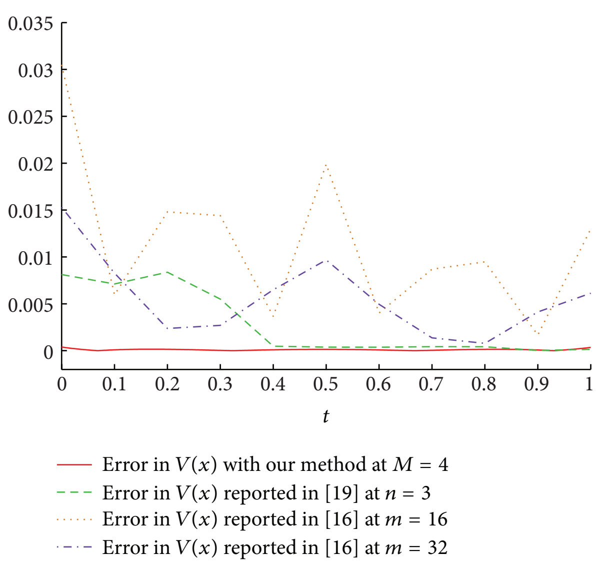

FIGURE 5: Comparing the error estimates in $V(x)$ by our method with error found with taylor series approximation method (green dots) and block pulse method (orange and purple dots).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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