Research Article
Rational Generalized Offsets of Rational Surfaces

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The rational surfaces and their offsets are commonly used in modeling and manufacturing. The purpose of this paper is to present relationships between rational surfaces and orientation-preserving similarities of the Euclidean 3-space. A notion of a similarity surface offset is introduced and applied to different constructions of rational generalized offsets of a rational surface. It is shown that every rational surface possesses a rational generalized offset. Rational generalized focal surfaces are also studied.

1. Introduction

Surfaces with a rational parametrization play an important role in computer-aided design (CAD) and computer-aided manufacturing (CAM). The computation of the offset to a surface is another significant topic in CAD/CAM applications. The offset to a rational surface, in general, does not possess a rational parametrization. A generalization of a surface offset is proposed in this paper such that a rationality of the original surface implies a rationality of the generalized offset.

Pythagorean normal vector surfaces (PN surfaces for brevity) form a special class of rational surfaces. These surfaces can be characterized by a rational representation of their unit normal vector fields. In other words, the PN surfaces possess rational offsets. The notion of a PN surface was introduced by Pottmann [1]. Later, these surfaces have been studied by many authors (for a complete bibliography, see [2–4]). Linear normal vector surfaces form a subset of the set of all rational surfaces possessing rational offsets. These surfaces have been introduced by Jüttler [5] and systematically studied in a series of papers (see [6–9]). Offsetting quadratic surfaces is another topic which is intensively developed. Various cases of offsets of quadrics are considered by Maekawa [10], Patrikalakis and Maekawa [11], Bastl et al. [9], Aigner et al. [12], and Bastl et al. [13]. NURBS surfaces form a class of piecewise rational surfaces. Algorithms for offsetting NURBS surfaces are obtained by Piegl and Tiller [14],

Ravi Kumar et al. [15], and Sun et al. [16]. Note that the offsets of NURBS surfaces are not piecewise rational, and existing algorithms give approximations of these offsets. Algebraic properties of the offsets are studied in [17–19]. Different constructions of generalized offset surfaces with variable distance functions depending on principal curvatures are considered by Hagen et al. [20], Hahmann et al. [21], and Moon [22], but a rationality of these offsets is not discussed. The most recent investigations of offset surfaces are devoted to their practical use (see, e.g., [2–4, 9, 23, 24]).

Any Euclidean motion of the Euclidean 3-space $\mathbb{E}^3$ is an affine transformation that preserves the distances. Any similarity of $\mathbb{E}^3$ is an affine transformation that preserves the angles. Therefore, the group of the similarities is the smallest extension of the Euclidean motion group. Offset surfaces are closely related to the Euclidean motions. In fact, any Euclidean motion maps the pair formed by a regular surface and its offset of a distance d into another pair of surfaces that are also a regular surface and its offset of the same distance d. This statement is not valid in a case of a similarity different from a Euclidean motion. It is well known that the similarities preserve any pair of a PN surface and its rational offset. This property is a motivation for a definition of generalized surface offsets such that offsetting becomes a closed operation for the class of all rational surfaces. The aim of the present paper is to give constructions for rational generalized offsets to an arbitrary rational surface.

The paper is organized as follows. A brief description of the orientation-preserving similarities is given in Section 2. After that, a relationship between PN surfaces and similarities is discussed. In Section 4, the notion of a similarity surface offset is introduced. Then, a construction of a rational similarity offset corresponding to an arbitrary rational surface is presented. Section 5 is devoted to rational generalized focal surfaces which are also similarity offsets. The paper concludes with final remarks.

2. Preliminaries

We start with a short overview of some well-known facts and notations concerning the three-dimensional vector algebra and orientation-preserving similarity transformations of the Euclidean 3-space.

Let $\mathbf{a} = (a_1, a_2, a_3)^T$, $\mathbf{b} = (b_1, b_2, b_3)^T$, and $\mathbf{c} = (c_1, c_2, c_3)^T$ be three arbitrary vectors in $\mathbb{R}^3$. The scalar (or dot) product of $\mathbf{a}$ and $\mathbf{b}$ can be represented by a matrix multiplication $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$. Then, the norm of the vector $\mathbf{a}$ is expressed by $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. The vector cross product $\mathbf{a} \times \mathbf{b}$ is a binary operation on $\mathbb{R}^3$ with the property $\|\mathbf{a} \times \mathbf{b}\| = \sqrt{\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2}$. The vector triple product $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is related to the scalar and vector cross products and can be calculated as follows: $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det(\mathbf{a}, \mathbf{b}, \mathbf{c})$.

We consider the Euclidean three-dimensional space $\mathbb{E}^3$ as an affine space with an associated vector space $\mathbb{R}^3$. This means that we identify the points of $\mathbb{E}^3$ with their position vectors.

A map $\Phi : \mathbb{E}^3 \longrightarrow \mathbb{E}^3$ is called an orientation-preserving (or direct) similarity if for any point $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{E}^3$, the image $\Phi(\mathbf{x}) \in \mathbb{E}^3$ is determined by the following matrix equation:

$$\Phi(\mathbf{x}) = \varrho \mathbf{A} \mathbf{x} + \mathbf{t}, \tag{2.1}$$

where $\varrho > 0$ is a constant, $\mathbf{A}$ is a fixed orthogonal 3×3 matrix with $\det(\mathbf{A}) = 1$, and $\mathbf{t} = (t_1, t_2, t_3)^T \in \mathbb{R}^3$ is a translation vector. Every direct similarity is an affine transformation of $\mathbb{E}^3$

that preserves the orientation and the angles. We denote by $\text{Sim}^+(\mathbb{E}^3)$ the group of all direct similarities of $\mathbb{E}^3$. The direct similarity Φ given by (2.1) induces a linear map $\Phi^{\text{ind}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $\Phi^{\text{ind}}(\mathbf{a}) = \varrho\mathbf{A}\mathbf{a}$ for an arbitrary vector $\mathbf{a} = (a_1, a_2, a_3)^T \in \mathbb{R}^3$. Since $\|\Phi^{\text{ind}}(\mathbf{a})\| = \varrho\|\mathbf{a}\|$, the positive number ϱ is called *a similarity ratio*. In the remaining part of the paper, we will write Φ in place of Φ^{ind} because this shorter notation could not lead to a confusion.

If the map Φ given by (2.1) is a Euclidean motion, that is, $\varrho = 1$, then $\Phi(\mathbf{a} \times \mathbf{b}) = \Phi(\mathbf{a}) \times \Phi(\mathbf{b})$, $\|\Phi(\mathbf{a} \times \mathbf{b})\| = \|\Phi(\mathbf{a}) \times \Phi(\mathbf{b})\| = \|\mathbf{a} \times \mathbf{b}\|$ and $[\Phi(\mathbf{a})\ \Phi(\mathbf{b})\ \Phi(\mathbf{c})] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ for any vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$. In other words, both the vector cross product and the vector triple product are compatible with Euclidean motions.

If $\varrho \neq 1$, then the direct similarity Φ given by (2.1) is not a Euclidean motion and

$$\Phi(\mathbf{a} \times \mathbf{b}) = \frac{1}{\varrho}(\Phi(\mathbf{a}) \times \Phi(\mathbf{b})),$$

$$\|\Phi(\mathbf{a} \times \mathbf{b})\| = \frac{1}{\varrho}\|\Phi(\mathbf{a}) \times \Phi(\mathbf{b})\| = \varrho\|\mathbf{a} \times \mathbf{b}\|, \tag{2.2}$$

$$[\Phi(\mathbf{a})\ \Phi(\mathbf{b})\ \Phi(\mathbf{c})] = \varrho^3[\mathbf{a}\ \mathbf{b}\ \mathbf{c}].$$

Hence, both the vector cross product and the vector triple product are not invariant under of a direct similarity different from a Euclidean motion.

Note that differential-geometric invariants of curves and surfaces with respect to the group of direct similarities can be used for an analysis of the local shape of curves and surfaces (see [25, 26]).

3. Similarity Invariance of Pythagorean Normal-Vector Surfaces

3.1. Rational Unit Normal-Vector Fields and Rational Offset Surfaces

A PN surface defined on a certain domain $D \subset \mathbb{R}^2$ is a rational surface patch $S : D \to \mathbb{E}^3$ which admits the so-called PN parametrization

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))^T, \quad (u, v) \in D, \tag{3.1}$$

with the following property: if $\mathbf{r}_u = (\partial/\partial u)\mathbf{r}(u, v)$ and $\mathbf{r}_v = (\partial/\partial v)\mathbf{r}(u, v)$, then the norm of the normal vector field $\mathbf{n}(u, v) = \mathbf{r}_u \times \mathbf{r}_v$ is a rational function. For such a parametrization of the PN surface, the unit normal vector field

$$\mathbf{n}_1(u, v) = \frac{1}{\|\mathbf{r}_u \times \mathbf{r}_v\|}(\mathbf{r}_u \times \mathbf{r}_v) \tag{3.2}$$

has rational coordinate functions and the offset surface S_d at a certain distance d possesses a rational parametrization

$$\mathbf{r}_{\text{d}}(u, v) = \mathbf{r}(u, v) + d\mathbf{n}_1(u, v) = \mathbf{r}(u, v) + \frac{d}{\|\mathbf{r}_u \times \mathbf{r}_v\|}(\mathbf{r}_u \times \mathbf{r}_v). \tag{3.3}$$

A detailed description of the PN surfaces is given in [1, 2, 4, 27].

3.2. PN Surfaces and Direct Similarities

The purpose of this subsection is to point out that the PN surfaces in $\mathbb{E}^3$ and direct similarities are closely connected.

Let $\Phi : \mathbb{E}^3 \to \mathbb{E}^3$ be the direct similarity given by (2.1), and let S be a PN surface with a PN parametrization (3.1). Since the coordinate functions $x(u,v)$, $y(u,v)$, $z(u,v)$ are rational, the tangent vector fields to isoparametric lines $\mathbf{r}_u = (\partial/\partial u)\mathbf{r}(u,v)$, $\mathbf{r}_v = (\partial/\partial v)\mathbf{r}(u,v)$ have rational coordinate functions, and it fulfills the equality

$$\mathbf{n}(u,v) \cdot \mathbf{n}(u,v) = (\mathbf{r}_u \times \mathbf{r}_v) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \sigma^2(u,v) \tag{3.4}$$

for some rational function $\sigma(u,v)$. Then, the image $\Phi(S) = \widetilde{S}$ is the rational surface given by

$$\widetilde{\mathbf{r}}(u,v) = \varrho \mathbf{A}\mathbf{r}(u,v) + \mathbf{t} \tag{3.5}$$

and the tangent vectors to the isoparametric lines of $\widetilde{S}$ at the point $\widetilde{\mathbf{r}}(u,v) \in \widetilde{S}$ are $\widetilde{\mathbf{r}}_u = \varrho \mathbf{A}\mathbf{r}_u$ and $\widetilde{\mathbf{r}}_v = \varrho \mathbf{A}\mathbf{r}_v$. From this, the well-known fact that $\widetilde{S}$ is a PN surface with a PN parametrization (3.5) follows. As a consequence, if S_d denotes a rational offset of the surface S at certain distance d, then the image $\Phi(S_d)$ is a rational offset surface of the PN surface $\Phi(S)$ at distance $\widetilde{d} = \varrho d$. These properties of the PN surfaces can be considered as a motivation for a construction of rational generalized offset to an arbitrary rational surface.

4. A Class of Generalized Surface Offsets

Now we will study generalized surface offsets. Their parametric representations are similar to the parametrization (3.3) of ordinary surface offsets. The difference is that the distance d is replaced by a variable distance function.

Let S be a regular surface of class C^3, and let

$$\mathbf{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)^T, \quad (u,v) \in D \subseteq \mathbb{R}^2, \tag{4.1}$$

be its parametrization. This means that

(i) the coordinate functions $x(u,v)$, $y(u,v)$, $z(u,v)$ are defined on the same domain $D \subseteq \mathbb{R}^2$ and have continuous derivatives up to order 3; that is, the vector function $\mathbf{r}(u,v)$ possesses continuous partial derivatives $\mathbf{r}_u = (\partial/\partial u)\mathbf{r}(u,v)$, $\mathbf{r}_v = (\partial/\partial v)\mathbf{r}(u,v)$, $\mathbf{r}_{uu} = (\partial/\partial u)\mathbf{r}_u$, $\mathbf{r}_{uv} = (\partial/\partial v)\mathbf{r}_u$, $\mathbf{r}_{vv} = (\partial/\partial v)\mathbf{r}_v$, $\mathbf{r}_{uuu} = (\partial/\partial u)\mathbf{r}_{uu}$, $\mathbf{r}_{uuv} = (\partial/\partial v)\mathbf{r}_{uu}$, $\mathbf{r}_{uvv} = (\partial/\partial v)\mathbf{r}_{uv}$, and $\mathbf{r}_{vvv} = (\partial/\partial v)\mathbf{r}_{vv}$,

(ii) the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ at any point of the surface are linearly independent, or equivalently, the normal vector $\mathbf{n}(u,v) = \mathbf{r}_u \times \mathbf{r}_v$ is nonzero everywhere.

Then, the coefficients of the first fundamental form of the surface

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \qquad F = \mathbf{r}_u \cdot \mathbf{r}_v, \qquad G = \mathbf{r}_v \cdot \mathbf{r}_v, \tag{4.2}$$

the coefficients of the second fundamental form of the surface

$$L = \frac{[\mathbf{r}_u \ \mathbf{r}_v \ \mathbf{r}_{uu}]}{\sqrt{EG - F^2}}, \qquad M = \frac{[\mathbf{r}_u \ \mathbf{r}_v \ \mathbf{r}_{uv}]}{\sqrt{EG - F^2}}, \qquad N = \frac{[\mathbf{r}_u \ \mathbf{r}_v \ \mathbf{r}_{vv}]}{\sqrt{EG - F^2}}, \tag{4.3}$$

and the components of the unit normal vector field

$$\mathbf{n}_1(u, v) = \frac{1}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \ (\mathbf{r}_u \times \mathbf{r}_v) = \frac{1}{\sqrt{EG - F^2}} (\mathbf{r}_u \times \mathbf{r}_v) \tag{4.4}$$

are differentiable functions on the surface S.

All differentiable functions on the surface S form a real algebra, in which division by a nonvanishing function is always possible. This algebra is denoted by $\mathcal{F}(S)$. Moreover, any function $f \in \mathcal{F}(S)$ can be considered as a differentiable function of two variables, u and v, which is defined on the domain D.

All differentiable functions on S, which can be expressed as rational functions of E, F, G, L, M, and N, form a subalgebra of $\mathcal{F}(S)$. We denote this subalgebra by $\mathcal{F}_{\mathrm{I,II}}(S)$. Any direct similarity $\Phi : \mathbb{E}^3 \to \mathbb{E}^3$, as a linear map, transforms the surface S into a regular surface $\widetilde{S}$ of class C^3 defined on the same domain D. Furthermore, if $\widetilde{E}, \widetilde{F}, \widetilde{G}$, and $\widetilde{L}, \widetilde{M}, \widetilde{N}$ are the coefficients of the first and the second fundamental forms of $\widetilde{S}$, respectively, then there exists an isomorphism

$$i_S^\Phi : \mathcal{F}_{\mathrm{I,II}}(S) \longrightarrow \mathcal{F}_{\mathrm{I,II}}\left(\widetilde{S}\right) \tag{4.5}$$

determined by $i_S^\Phi(E) = \widetilde{E}$, $i_S^\Phi(F) = \widetilde{F}$, $i_S^\Phi(G) = \widetilde{G}$, $i_S^\Phi(L) = \widetilde{L}$, $i_S^\Phi(M) = \widetilde{M}$, and $i_S^\Phi(N) = \widetilde{N}$.

Definition 4.1. The function $f \in \mathcal{F}_{\mathrm{I,II}}(S)$ is called *a similarity function* if the following condition is satisfied:

$$\widetilde{f} = i_S^\Phi(f) = \varrho f \tag{4.6}$$

for any $\Phi \in \mathrm{Sim}^+(\mathbb{E}^3)$ with similarity ratio ϱ.

Let us notice basic properties of similarity functions.

Lemma 4.2. *Let g_1 and g_2 be arbitrary similarity functions, and let g_3 be a nonvanishing similarity function. Then*

(i) *$\lambda_1 g_1 \pm \lambda_2 g_2$ is a similarity function for any $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$;*

(ii) *$g_1 g_2 / g_3$ is a similarity function.*

Proof. The statements follow from the equalities

$$i_S^\Phi(\lambda_1 g_1 \pm \lambda_2 g_2) = \varrho(\lambda_1 g_1 \pm \lambda_2 g_2), \qquad i_S^\Phi\left(\frac{g_1 g_2}{g_3}\right) = \varrho\left(\frac{g_1 g_2}{g_3}\right). \tag{4.7}$$

$\square$

A composition of two direct similarities with similarity ratios ϱ_1 and ϱ_2 is a direct similarity with a similarity ratio $\varrho_1\varrho_2$. Consequently, if $f \in \mathcal{F}_{I,II}(S)$ is a similarity function, then the function $\widetilde{f} = i_S^\Phi(f) \in \mathcal{F}_{I,II}(\widetilde{S})$ is also a similarity function.

First, we show that the similarity functions exist.

Proposition 4.3. *Let S be a surface of class C^3 given by (4.1). Then, the function $f_0 \in \mathcal{F}_{I,II}(S)$ defined by*

$$f_0(u,v) = L - 2M + N, \quad (u,v) \in D, \tag{4.8}$$

is a similarity function.

Proof. Suppose that an arbitrary direct similarity Φ of $\mathbb{E}^3$ with a similarity ratio ϱ is presented by (2.1). Then, a parametrization of the surface $\widetilde{S} = \Phi(S)$ can be written in the form $\widetilde{\mathbf{r}}(u,v) = \varrho\mathbf{Ar}(u,v) + \mathbf{t}$, $(u,v) \in D$. For the derivatives of the vector function $\widetilde{\mathbf{r}}(u,v)$ we have $\widetilde{\mathbf{r}}_u = \varrho\mathbf{Ar}_u$, $\widetilde{\mathbf{r}}_v = \varrho\mathbf{Ar}_v$, $\widetilde{\mathbf{r}}_{uu} = \varrho\mathbf{Ar}_{uu}$, $\widetilde{\mathbf{r}}_{uv} = \varrho\mathbf{Ar}_{uv}$ and $\widetilde{\mathbf{r}}_{vv} = \varrho\mathbf{Ar}_{vv}$ Using (2.2), (4.2), and (4.3), we obtain that the coefficients of the second fundamental form of the surface $\widetilde{S} = \Phi(S)$ are $\widetilde{L} = \varrho L$, $\widetilde{M} = \varrho M$, and $\widetilde{N} = \varrho N$. Hence, $\widetilde{f}_0(u,v) = i_S^\Phi(f_0) = \widetilde{L} - 2\widetilde{M} + \widetilde{N} = \varrho f_0(u,v)$. Furthermore, the product

$$\frac{1}{\sqrt{EG - F^2}} f_0(u,v) = \frac{[\mathbf{r}_u \ \mathbf{r}_v \ \mathbf{r}_{uu}] - 2[\mathbf{r}_u \ \mathbf{r}_v \ \mathbf{r}_{uv}] + [\mathbf{r}_u \ \mathbf{r}_v \ \mathbf{r}_{vv}]}{EG - F^2} \tag{4.9}$$

is a rational function of the first and second derivatives. $\qquad\square$

Second, we introduce a special kind of generalized surface offsets.

Definition 4.4. Let S be a regular surface of class C^3 given by (4.1), and let d be a nonzero real constant. Then, for any similarity function $f \in \mathcal{F}_{I,II}(S)$, the generalized surface offset S_{df} with a parametrization

$$\begin{aligned}
\mathbf{r}_{df}(u,v) &= \mathbf{r}(u,v) + df(u,v)\mathbf{n}_1(u,v) \\
&= \mathbf{r}(u,v) + \frac{df(u,v)}{\sqrt{EG - F^2}} \ (\mathbf{r}_u \times \mathbf{r}_v)
\end{aligned} \tag{4.10}$$

is called *a similarity offset*.

It is clear that the product $df(u,v)$ in (4.10) plays the role of a variable distance function.

Theorem 4.5. *Let S be an arbitrary surface of class C^3 with a rational parametrization (4.1); that is, $x(u,v)$, $y(u,v)$, and $z(u,v)$ denote rational functions, and let $f \in \mathcal{F}_{I,II}(S)$ be a similarity function. Then the similarity offset S_{df} given by (4.10) is a rational surface. Moreover, for any direct similarity $\Phi : \mathbb{E}^3 \rightarrow \mathbb{E}^3$ with similarity ratio ϱ, the rational surface $\Phi(S_{df})$ is the similarity offset of the rational surface $\widetilde{S} = \Phi(S)$ defined by the similarity function $i_S^\Phi(f) = \varrho f$.*

Proof. Since S is a rational surface, all vector functions $\mathbf{r}_u$, $\mathbf{r}_v$, $\mathbf{r}_{uu}$, $\mathbf{r}_{uv}$, $\mathbf{r}_{vv}$, and $\mathbf{r}_u \times \mathbf{r}_v$ have rational coordinate functions. This implies that E, F, and G are also rational functions. As in

the proof of Proposition 4.3, we can see that the functions L, M, N, E/L, E/M, E/N, F/L, F/M, F/N, G/L, G/M, G/N are similarity functions. Let n be a positive integer, and i_1, i_2, i_3, j_1, j_2, j_3, k_1, k_2, k_3, l_1, l_2, l_3 denote nonnegative integers such that

$$2(i_1 + i_2 + i_3) + j_1 + j_2 + j_3 = n, \qquad 2(k_1 + k_2 + k_3) + l_1 + l_2 + l_3 = n - 1. \tag{4.11}$$

Then, using Lemma 4.2 and applying induction, we obtain that every similarity function can be written in the form

$$f(u,v) = \frac{\sum_{2(i_1+i_2+i_3)+j_1+j_2+j_3=n} a_{j_1,j_2,j_3}^{i_1,i_2,i_3} E^{i_1} F^{i_2} G^{i_3} L^{j_1} M^{j_2} N^{j_3}}{\sum_{2(k_1+k_2+k_3)+l_1+l_2+l_3=n-1} b_{l_1,l_2,l_3}^{k_1,k_2,k_3} E^{k_1} F^{k_2} G^{k_3} L^{l_1} M^{l_2} N^{l_3}}, \tag{4.12}$$

where

   (i) either the sum $j_1 + j_2 + j_3$ is odd for each term in the numerator and the sum $l_1 + l_2 + l_3$ is even for each term in the denominator, or the sum $j_1 + j_2 + j_3$ is even for each term in the numerator and the sum $l_1 + l_2 + l_3$ is odd for each term in the denominator,

   (ii) at least one of the coefficients $a_{j_1,j_2,j_3}^{i_1,i_2,i_3} \in \mathbb{R}$ and at least one of the coefficients $b_{l_1,l_2,l_3}^{k_1,k_2,k_3} \in \mathbb{R}$ is not equal to zero.

From here, it follows that any similarity function can be represented as

$$f(u,v) = \frac{1}{\sqrt{EG - F^2}} g(u,v), \tag{4.13}$$

where $g(u,v)$ is a rational function of u and v. Consequently, the similarity offset S_{df} is a rational surface. The second assertion is obvious. $\qquad \square$

Corollary 4.6. *For any rational surface S with a parametrization* (4.1), *the similarity offset S_{d0} given by*

$$\mathbf{r}_{d0}(u,v) = \mathbf{r}(u,v) + df_0(u,v)\mathbf{n}_1(u,v) \tag{4.14}$$

is a rational surface.

The tangent vector $\mathbf{m} = \mathbf{r}_u - \mathbf{r}_v$ is nonzero at any point of a regular surface S. Clearly, the similarity function f_0 can be expressed as $f_0 = \mathbf{II}(\mathbf{m}, \mathbf{m})$, where $\mathbf{II}$ denotes the second fundamental form of the surface. If S is a plane, then $f_0(u,v)$ vanishes identically and S_{d0} coincides with S. If the tangent vector $\mathbf{m}$ at any point of the surface S is not asymptotic, then $f_0(u,v) = \mathbf{II}(\mathbf{m}, \mathbf{m}) \neq 0$ for any $(u,v) \in D$, or equivalently, the surfaces S and S_{d0} have no common points. As an illustration we consider similarity offsets of rational patches lying on quadratic and cubic surfaces.

Example 4.7. Let S be a rational bilinear Bézier patch

$$\mathbf{r}(u,v) = \frac{\sum_{i=0}^{1} \sum_{j=0}^{1} \binom{1}{i} \binom{1}{j} (1-u)^i u^{1-i} (1-v)^j v^{1-j} w_{ij} \mathbf{b}_{ij}}{\sum_{i=0}^{1} \sum_{j=0}^{1} \binom{1}{i} \binom{1}{j} (1-u)^i u^{1-i} (1-v)^j v^{1-j} w_{ij}}, \quad u,v \in [0,1], \qquad (4.15)$$

with control points $\mathbf{b}_{00} = (1,0,0)$, $\mathbf{b}_{01} = (0,1,0)$, $\mathbf{b}_{10} = (1,0,1)$, and $\mathbf{b}_{11} = (0,0,1)$ and weights $w_{00} = 1$, $w_{01} = 1$, $w_{10} = 1$, and $w_{11} = 2$. This patch possesses the following rational parametrization:

$$\mathbf{r}(u,v) = \left(\frac{1-v}{1+uv}, \frac{v-uv}{1+uv}, \frac{u(1+v)}{1+uv} \right)^T, \quad u \in [0,1], \ v \in [0,1], \qquad (4.16)$$

and lies on the hyperboloid of one sheet $y^2 + xy - xz + yz + x + z - 1 = 0$. Then, we calculate

$$\mathbf{n}_1(u,v) = \left(\frac{(-1+u)(1+v)}{\sqrt{2}\sqrt{p(u,v)}}, \frac{-1+u(-1+v)-v}{\sqrt{2}\sqrt{p(u,v)}}, -\frac{\sqrt{2}\,v}{\sqrt{p(u,v)}} \right)^T \qquad (4.17)$$

and $L = 0$, $M = \sqrt{2}/(1+uv)\sqrt{p(u,v)}$, $N = 0$, where

$$p(u,v) = 1 + u^2 - 2(-1+u)v + 3v^2 + (-2+u)uv^2 > 0. \qquad (4.18)$$

This implies $f_0(u,v) = -2M < 0$ for any pair $(u,v) \in [0,1] \times [0,1]$. Assuming $d = 1$, we obtain that the parametrization (4.14) of the similarity offset S_{d0} is

$$\mathbf{r}_{d0}(u,v) = \begin{pmatrix} \dfrac{(1-v)p(u,v) + 2(-1+u)(1+v)}{(1+uv)p(u,v)} \\[2mm] \dfrac{(1-u)vp(u,v) + 2(-1+u(-1+v)-v)}{(1+uv)\,p(u,v)} \\[2mm] \dfrac{u(1+v)p(u,v) - 4v}{(1+uv)p(u,v)} \end{pmatrix}. \qquad (4.19)$$

The Bézier surface patch S and its similarity offset S_{d0} are plotted in Figure 1.

Example 4.8. Consider the surface patch S with polynomial parametrization

$$\mathbf{r}(u,v) = \left(uv, u - v^2, v \right)^T, \quad u \in [0,1], \ v \in [0.4,1], \qquad (4.20)$$

lying on the cubic surface $yz - x + z^3 = 0$ (see [28]). The unit normal vector to the surface S is

$$\mathbf{n}_1(u,v) = \frac{1}{\sqrt{1 + v^2 + (u+2v^2)^2}} \left(1, -v, -u - 2v^2 \right)^T, \qquad (4.21)$$
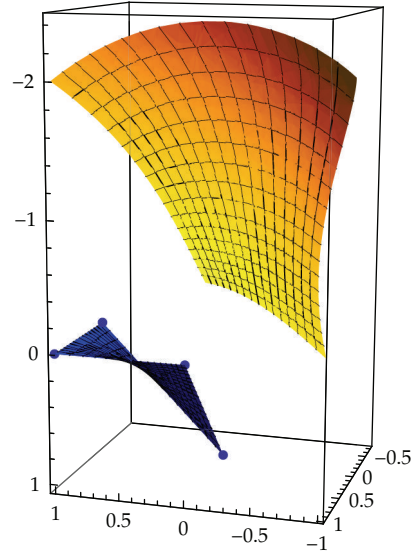
Figure 1: The rational Bézier patch (blue) and its rational similarity offset (yellow).
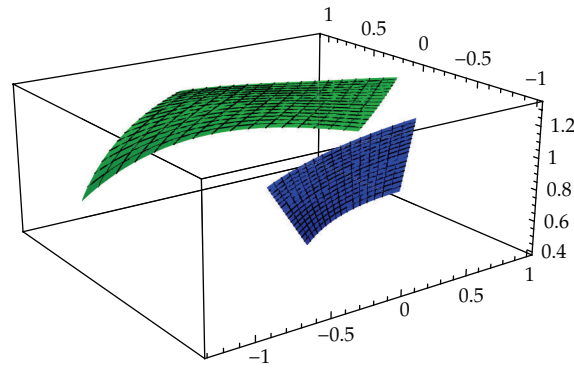

Figure 2: The polynomial surface patch (blue) and its rational similarity offset (green).

and the similarity function f_0 corresponding to S is

$$f_0(u,v) = \frac{(v-2)}{\sqrt{1+v^2+(u+2v^2)^2}} < 0. \tag{4.22}$$

Thus, for $d = 1$ the parametrization (4.14) of the similarity offset S_{d0} can be written in the form

$$\left(\frac{uvq(u,v)+v-2}{q(u,v)}, \frac{(u-v^2)q(u,v)-(v-2)v}{q(u,v)}, \frac{vq(u,v)-(v-2)(u+2v^2)}{q(u,v)} \right)^T, \tag{4.23}$$

where $u \in [0, 1]$, $v \in [0.4, 1]$, and $q(u, v) = 1 + v^2 + (u + 2v^2)^2$. The polynomial surface patch S and its similarity offset S_{d0} are plotted in Figure 2.

There are three reasons for considering the similarity offset S_{d0}. First, this generalized offset is determined for any regular surface of class C^3. Second, the computation of the similarity offset S_{d0} takes a bit more time than the computation of the ordinary surface offset. Third, a rationality of the original surface S implies a rationality of S_{d0}. Thus, we can conclude that every rational regular surface S of class C^3 possesses a rational generalized offset S_{d0}.

5. Rational Generalized Focal Surfaces

Generalized surface offsets with variable distance functions depending only on the Gaussian curvature K and the mean curvature H are presented in this section. Since K and H determine locally the surface up to a Euclidean motion, such a kind of surface offsets are closely related to the local shape of the surface. In other words these generalized offsets can be considered as a tool for studying the local shape of the original surface.

Let S be a regular surface of class C^3 given by (4.1), and let $f(u, v)$ be a function defined on S. A generalized surface offset S_{df} with a parametrization

$$\mathbf{r}_{df}(u, v) = \mathbf{r}(u, v) + df(u, v)\mathbf{n}_1(u, v) \tag{5.1}$$

is called *a generalized focal surface* if d is a nonzero constant, $\mathbf{n}_1(u, v)$ is the unit normal-vector field, and the variable distance function $df(u, v)$ can be expressed as a function of the principal curvatures k_1 and k_2 of the original surface S. Such a kind of generalized offsets will be studied in this section. It is well known that the Gaussian curvature K and the mean curvature H at any point $\mathbf{r}(u, v)$ of the surface S can be represented in terms of the first- and second-order partial derivatives as follows:

$$K = \frac{[\mathbf{r}_{uu} \ \mathbf{r}_u \ \mathbf{r}_v][\mathbf{r}_{vv} \ \mathbf{r}_u \ \mathbf{r}_v] - [\mathbf{r}_{uv} \ \mathbf{r}_u \ \mathbf{r}_v]^2}{\left(\|\mathbf{r}_u\|^2\|\mathbf{r}_v\|^2 - (\mathbf{r}_u \cdot \mathbf{r}_v)^2\right)^2}, \tag{5.2}$$

$$H = \frac{[\mathbf{r}_{uu} \ \mathbf{r}_u \ \mathbf{r}_v]\|\mathbf{r}_v\|^2 + [\mathbf{r}_{vv} \ \mathbf{r}_u \ \mathbf{r}_v]\|\mathbf{r}_u\|^2 - 2[\mathbf{r}_{uv} \ \mathbf{r}_u \ \mathbf{r}_v](\mathbf{r}_u \cdot \mathbf{r}_v)}{2\left(\|\mathbf{r}_u\|^2\|\mathbf{r}_v\|^2 - (\mathbf{r}_u \cdot \mathbf{r}_v)^2\right)^{3/2}}$$

(see [29, page 405]). Recall that a point $\mathbf{r}(u, v) \in S$ at which $K = 0$ and $H = 0$, or equivalently $L = M = N = 0$, is called planar. Every plane in $\mathbb{E}^3$ is a surface which consists of planar points. A point $\mathbf{r}(u, v) \in S$ is called parabolic if $K = 0$ and $H \neq 0$ at this point. A developable surface is a regular surface whose Gaussian curvature is everywhere zero. A minimal surface is a regular surface for which the mean curvature vanishes identically. Our investigations are limited to regular surface patches which are not a part of a plane.

Let us examine two special similarity functions.

Proposition 5.1. *Let S be a regular surface of class C^3, and let* (4.1) *be its parametrization.*

(i) *If S is a nondevelopable surface, then the function $f_1(u, v) = H/K$ defined on the set of all nonplanar and nonparabolic points is a similarity function.*

(ii) *If S is not a minimal surface, then the function $f_2(u,v) = 1/H$, defined on the set of all points at which $H \neq 0$, is a similarity function.*

Proof. From $K = (LN - M^2)/(EG - F^2)$ and $H = (EN + GL - 2FM)/2(EG - F^2)$, it follows that $f_1(u,v) \in \mathcal{F}_{I,II}(S)$ and $f_2(u,v) \in \mathcal{F}_{I,II}(S)$. Suppose that a direct similarity Φ of $\mathbb{E}^3$ with similarity ratio ϱ is given by (2.1). Then, the surface $\widetilde{S} = \Phi(S)$ has a parametrization $\widetilde{\mathbf{r}}(u,v) = \varrho \mathbf{A}\mathbf{r}(u,v) + \mathbf{t}$. Since $\widetilde{S} = \Phi(S)$ is also a surface of class C^3, the Gaussian curvature $\widetilde{K}$ and the mean curvature $\widetilde{H}$ are well defined at any point of the image surface $\widetilde{S}$. Using (5.2), we have

$$\widetilde{K} = \frac{1}{\varrho^2}K, \qquad \widetilde{H} = \frac{1}{\varrho}H . \tag{5.3}$$

This means that $i_S^{\Phi}(f_1(u,v)) = \varrho f_1(u,v)$ and $i_S^{\Phi}(f_2(u,v)) = \varrho f_2(u,v)$. Hence, according to Definition 4.1, both functions $f_1(u,v) = H/K$ and $f_2(u,v) = 1/H$ are similarity functions. $\square$

Now, we can introduce two constructions for similarity offset surfaces. Let S be a regular nondevelopable surface of class C^3 given by (4.1), and let d be a nonzero constant. Then the similarity offset surface S_{d1} with a parametrization

$$\mathbf{r}_{d1}(u,v) = \mathbf{r}(u,v) + d\frac{f_1(u,v)}{\|\mathbf{r}_u \times \mathbf{r}_v\|}(\mathbf{r}_u \times \mathbf{r}_v) \tag{5.4}$$

is well defined for any parametric value $(u,v) \in D$ whose corresponding point $\mathbf{r}(u,v) \in S$ is neither planar nor parabolic. Such surfaces were introduced by Rando and Roulier [30]. If the regular nonminimal surface S of class C^3 is given by (4.1) and if d is a nonzero constant, then the similarity offset surface S_{d2} with a parametrization

$$\mathbf{r}_{d2}(u,v) = \mathbf{r}(u,v) + d\frac{f_2(u,v)}{\|\mathbf{r}_u \times \mathbf{r}_v\|}(\mathbf{r}_u \times \mathbf{r}_v) \tag{5.5}$$

is well defined for any $(u,v) \in D$ such that the mean curvature H is nonzero at the point $\mathbf{r}(u,v) \in S$.

The similarity functions $f_1(u,v)$ and $f_2(u,v)$ can be expressed in terms of principal curvatures κ_1 and κ_2 of the surface S, that is, $f_1(u,v) = (\kappa_1 + \kappa_2)/2\kappa_1\kappa_2$ and $f_2(u,v) = 2/(\kappa_1 + \kappa_2)$. Therefore, the similarity offsets S_{d1} and S_{d2} can be considered as generalized focal surfaces. Other examples of generalized focal surfaces are described and studied by Hagen et al. [20] and Hahmann et al. [21]. Recently, Moon [22] investigated the so-called equivolumetric offset surfaces with variable distance functions depending on K and H.

In the remaining part of this section, we assume that the regular surface S of class C^3 has a rational parametric representation

$$\mathbf{r}(u,v) = \left(\frac{p_1(u,v)}{q(u,v)}, \frac{p_2(u,v)}{q(u,v)}, \frac{p_3(u,v)}{q(u,v)}\right), \quad (u,v) \in D, \tag{5.6}$$

where $p_1(u,v), p_2(u,v), p_3(u,v)$, and $q(u,v)$, are polynomials and $q(u,v) \neq 0$ for any $(u,v) \in D$. The Gaussian curvature of such a surface is a rational function of u and v. But the mean

curvature is a rational function if and only if $\|\mathbf{r}_u \times \mathbf{r}_v\|^2 = \|\mathbf{r}_u\|^2\|\mathbf{r}_v\|^2 - (\mathbf{r}_u \cdot \mathbf{r}_v)^2$ is a perfect square of a rational function; that is, S is a PN surface with PN parametrization (5.6).

Combining Theorem 4.5 and Proposition 5.1, we immediately obtain two types of rational generalized focal surfaces. If S is a rational surface patch (5.6) of class C^3 without planar and parabolic points, then the similarity surface offset S_{d1} given by (5.4) is rational for any nonzero real constant d. If S is a rational surface patch (5.6) with a nonzero mean curvature at any of its point, then the similarity surface offset S_{d2} given by (5.5) is rational for any nonzero real constant d.

The last considerations give direct constructions for a rational generalized offset of any rational surface. If the regular surface patch S given by (5.6) is not a part of a plane, then there are a subdomain $D_1 \subseteq D$ such that $K \neq 0$ on D_1 and a subdomain $D_2 \subseteq D$ such that $H \neq 0$ on D_2. Thus we may consider the exact rational generalized offsets S_{d1} and S_{d2}, which are determined by (5.4) and (5.5), respectively. The intersecting points of the surfaces S and S_{d1} are those at which $H = 0$. There are no intersecting points of the surfaces S and S_{d2}. The following algorithm is based on the above observations.

Algorithm 5.2. Parametrization of rational generalized offsets S_{d1} and S_{d2} of a rational surface S gave by (5.6)

(1) Calculate the first- and second-order partial derivatives $\mathbf{r}_u$, $\mathbf{r}_v$, $\mathbf{r}_{uu}$, $\mathbf{r}_{uv}$, and $\mathbf{r}_{vv}$.

(2) Find the unit normal vector field $\mathbf{n}_1(u, v)$.

(3) Express the Gaussian curvature K and the mean curvature H of the surface as functions of the parameters u and v.

(4) Determine the open subsets D_1 and D_2 of the domain D as follows: $D_1 = \{(u, v) \in D \mid K(u, v) \neq 0\}$, $D_2 = \{(u, v) \in D \mid H(u, v) \neq 0\}$.

(5) In the case of a nonempty set D_1, define for an arbitrary nonzero real constant d the rational generalized surface offset S_{d1} by (5.4).

(6) In the case of a nonempty set D_2, define for an arbitrary nonzero real constant d the rational generalized surface offset S_{d2} with a parametrization (5.5).

Example 5.3. Consider the surface S given by $x^2 + y^2 - z^2 = 1$, which is a hyperboloid of revolution of one sheet. This surface possesses a rational parametrization

$$\mathbf{r}(u, v) = \left(\frac{2(1 + u^2)v}{2u(1 + v^2)}, \frac{(1 + u^2)(1 - v^2)}{2u(1 + v^2)}, \frac{(1 - u^2)(1 + v^2)}{2u(1 + v^2)} \right)^T, \quad u > 0, \ v \in \mathbb{R}. \qquad (5.7)$$

Then,

$$\mathbf{n}_1(u, v) = \left(\frac{-\sqrt{2}(1 + u^2)v}{\sqrt{1 + u^4}(1 + v^2)}, \frac{(1 + u^2)(-1 + v^2)}{\sqrt{2}\sqrt{1 + u^4}(1 + v^2)}, \frac{(1 - u^2)(1 + v^2)}{\sqrt{2}\sqrt{1 + u^4}(1 + v^2)} \right)^T, \qquad (5.8)$$

$K = -4u^4/(1 + u^4)^2$, and $H = u(-1 + u^2)^2/\sqrt{2}(1 + u^4)^{3/2}$. Hence, for $d = 1$, the rational generalized offset S_{d1} with a parametrization

$$\mathbf{r}_{d1}(u, v) = \begin{pmatrix} \dfrac{2(1 + u^2)^3 v}{8u^3(1 + v^2)} \\[3mm] -\dfrac{(1 + u^2)^3(-1 + v^2)}{8u^3(1 + v^2)} \\[3mm] \dfrac{(-1 + u^2)\left((-1 + u^2)^2 - 4u^2\right)(1 + v^2)}{8u^3(1 + v^2)} \end{pmatrix} \tag{5.9}$$

is defined on the domain $D = \{0 < u < \infty,\ -\infty < v < \infty\}$. Similarly, for $d = 1$ the rational generalized offset S_{d2} with a parametrization

$$\mathbf{r}_{d2}(u, v) = \begin{pmatrix} -\dfrac{2(1 + u^2)^3 v}{2u(-1 + u^2)^2(1 + v^2)} \\[3mm] \dfrac{(1 + u^2)^3(-1 + v^2)}{2u(-1 + u^2)^2(1 + v^2)} \\[3mm] \dfrac{(3 - 2u^2 + 3u^4)(-1 + u^2)(1 + v^2)}{2u(-1 + u^2)^2(1 + v^2)} \end{pmatrix} \tag{5.10}$$

is defined on the domain $D_2 = \{0 < u < 1,\ -\infty < v < \infty\} \bigcup \{1 < u < \infty,\ -\infty < v < \infty\}$. The three rational surfaces S, S_{d1}, S_{d2} are well defined on the subdomain $\overline{D} = \{1.7 < u < 3,\ 0 < v < 3\}$. Then the original surface S is placed between S_{d1} and S_{d2} (see Figure 3).

Let us summarize the considerations in this section. For any nondevelopable rational surface S, we construct a surface S_{d1}, and for any nonminimal rational surface S, we construct a surface S_{d2}. These new rational surfaces have the following properties:

(i) both S_{d1} and S_{d2} are similarity surface offsets;

(ii) both S_{d1} and S_{d2} are generalized focal surfaces.

An additional advantage of the rational offsets S_{d1} and S_{d2} is that there is no need to change the rational parametrization of the original surface S.

6. Conclusion

The class of the PN surfaces is a good illustration for the relationship between rationality and similarity. The image of any pair of a PN surface and its rational offset under a direct similarity is also a pair of a PN surface and its rational offset. The differential-geometric invariants of such a surface as Gaussian and mean curvatures are rational functions. The notion of a similarity offset gives another natural connection between rational surfaces and similarity transformations. Every similarity offset is a rational generalized offset if the original surface is rational. Moreover, any rational surface and its similarity offset form a pair which is invariant
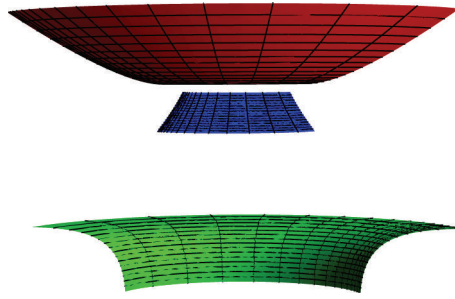
Figure 3: The hyperboloid S (blue) placed between its rational offsets S_{11} (green) and S_{12} (red).

under an arbitrary direct similarity. It is shown that there are similarity offsets which are generalized focal surfaces. Consequently, there exist rational generalized focal surfaces corresponding to rational surfaces. In particular, these rational generalized focal surfaces can be considered as an additional tool for a local shape analysis of rational surfaces.
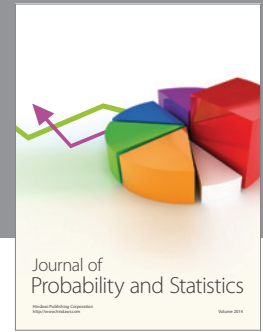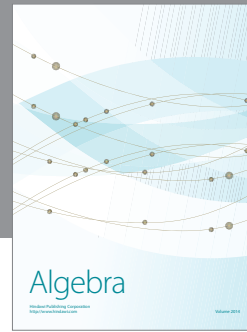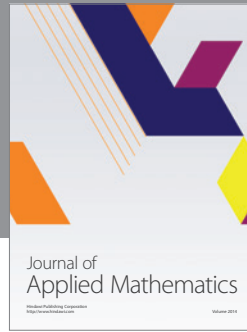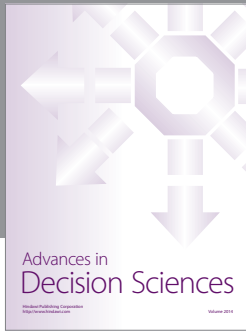
Acknowledgments

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