

Hindawi Publishing Corporation
 Abstract and Applied Analysis
 Volume 2012, Article ID 489353, 12 pages
 doi:10.1155/2012/489353

Research Article

Monotonic Positive Solutions of Nonlocal Boundary Value Problems for a Second-Order Functional Differential Equation

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Received 13 October 2011; Accepted 5 December 2011

Academic Editor: István Györi

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We study the existence of at least one monotonic positive solution for the nonlocal boundary value problem of the second-order functional differential equation $x''(t) = f(t, x(\phi(t)))$, $t \in (0, 1)$, with the nonlocal condition $\sum_{k=1}^m a_k x(\tau_k) = x_0$, $x'(0) + \sum_{j=1}^n b_j x'(\eta_j) = x_1$, where $\tau_k \in (a, d) \subset (0, 1)$, $\eta_j \in (c, e) \subset (0, 1)$, and $x_0, x_1 > 0$. As an application the integral and the nonlocal conditions $\int_a^d x(t) dt = x_0$, $x'(0) + x(e) - x(c) = x_1$ will be considered.

1. Introduction

The nonlocal boundary value problems of ordinary differential equations arise in a variety of different areas of applied mathematics and physics.

The study of nonlocal boundary value problems was initiated by Il'in and Moiseev [1, 2]. Since then, the non-local boundary value problems have been studied by several authors. The reader is referred to [3–22] and references therein.

In most of all these papers, the authors assume that the function $f : [0, 1] \times R^+ \rightarrow R^+$ is continuous. They all assume that

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{f(x)}{x} &= 0 \quad \text{or } \infty, \\ \lim_{x \rightarrow 0} \frac{f(x)}{x} &= 0 \quad \text{or } \infty. \end{aligned} \tag{1.1}$$

These assumptions are restrictive, and there are many functions that do not satisfy these assumptions.

Here we assume that the function $f : [0, 1] \times R^+ \rightarrow R^+$ is measurable in $t \in [0, 1]$ for all $x \in R^+$ and continuous in $x \in R^+$ for almost all $t \in [0, 1]$ is and there exists an integrable function $a \in L_1[0, 1]$ and a constant $b > 0$ such that

$$|f(t, x)| \leq |a(t)| + b|x|, \quad \forall (t, x) \in [0, 1] \times D. \quad (1.2)$$

Our aim here is to study the existence of at least one monotonic positive solution for the nonlocal problem of the second-order functional differential equation

$$x''(t) = f(t, x(\phi(t))), \quad t \in (0, 1), \quad (1.3)$$

with the nonlocal condition

$$\sum_{k=1}^m a_k x(\tau_k) = x_0, \quad x'(0) + \sum_{j=1}^n b_j x'(\eta_j) = x_1, \quad (1.4)$$

where $\tau_k \in (a, d) \subset (0, 1)$, $\eta_j \in (c, e) \subset (0, 1)$, and $x_0, x_1 > 0$.

As an application, the problem with the integral and nonlocal conditions

$$\int_a^d x(t) dt = x_0, \quad x'(0) + x(e) - x(c) = x_1, \quad (1.5)$$

is studied.

It must be noticed that the nonlocal conditions

$$\begin{aligned} x(\tau) = x_0, \quad \tau \in (a, d), \quad x'(0) + x'(\eta) = x_1, \quad \eta \in (c, e), \\ \sum_{k=1}^m a_k x(\tau_k) = 0, \quad \tau_k \in (a, d), \quad x'(0) + \sum_{j=1}^n b_j x'(\eta_j) = 0, \quad \eta_j \in (c, e), \end{aligned} \quad (1.6)$$

$$\int_a^d x(t) dt = 0, \quad x'(0) + x(e) = x(c)$$

are special cases of our the nonlocal and integral conditions.

2. Integral Equation Representation

Consider the functional differential equation (1.3) with the nonlocal condition (1.4) with the following assumptions.

- (i) $f : [0, 1] \times R^+ \rightarrow R^+$ is measurable in $t \in [0, 1]$ for all $x \in R^+$ and continuous in $x \in R^+$ for almost all $t \in [0, 1]$ and there exists an integrable function $a \in L_1[0, 1]$, and a constant $b > 0$ such that

$$|f(t, x)| \leq |a(t)| + b|x|, \quad \forall (t, x) \in [0, 1] \times D. \quad (2.1)$$

- (ii) $\phi : (0, 1) \rightarrow (0, 1)$ is continuous.
 (iii) $b < 1/(3 - B)$, $B = (\sum_{j=1}^n b_j + 1)^{-1}$.
 (iv)

$$\sum_{k=1}^m a_k > 0, \quad \forall k = 1, 2, \dots, m, \quad \sum_{j=1}^n b_j > 0, \quad \forall j = 1, 2, \dots, n. \quad (2.2)$$

Now, we have the following Lemma.

Lemma 2.1. *The solution of the nonlocal problem (1.3)-(1.4) can be expressed by the integral equation*

$$\begin{aligned} x(t) = & A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\} \\ & + B \left(t - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\} \\ & + \int_0^t (t - s) f(s, x(\phi(s))) ds, \end{aligned} \quad (2.3)$$

where $A = (\sum_{k=1}^m a_k)^{-1}$, $B = (\sum_{j=1}^n b_j + 1)^{-1}$.

Proof. Integrating (1.3), we get

$$x'(t) = x'(0) + \int_0^t f(s, x(\phi(s))) ds. \quad (2.4)$$

Integrating (2.4), we obtain

$$x(t) = x(0) + x'(0)t + \int_0^t (t - s) f(s, x(\phi(s))) ds. \quad (2.5)$$

Let $t = \tau_k$, in (2.5), we get

$$\sum_{k=1}^m a_k x(\tau_k) = \sum_{k=1}^n a_k x(0) + \sum_{k=1}^n a_k \tau_k x'(0) + \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds, \quad (2.6)$$

and we deduce that

$$x(0) = A \left\{ x_0 - \sum_{k=1}^m a_k \tau_k x'(0) - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\}, \quad A = \left(\sum_{k=1}^m a_k \right)^{-1}. \quad (2.7)$$

Substitute from (2.7) into (2.5), we obtain

$$\begin{aligned} x(t) = & A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\} + x'(0) \left(t - A \sum_{k=1}^m a_k \tau_k \right) \\ & + \int_0^t (t - s) f(s, x(\phi(s))) ds. \end{aligned} \quad (2.8)$$

Let $t = \eta_j$, in (2.4), we obtain

$$\begin{aligned} \sum_{j=1}^n b_j x'(\eta_j) &= \sum_{j=1}^n b_j x'(0) + \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds, \\ x_1 - x'(0) &= x'(0) \sum_{j=1}^n b_j + \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds, \end{aligned} \quad (2.9)$$

and we deduce that

$$x'(0) = B \left(x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right), \quad B = \left(\sum_{j=1}^n b_j + 1 \right)^{-1}. \quad (2.10)$$

Substitute from (2.10) into (2.8), we obtain

$$\begin{aligned} x(t) = & A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\} \\ & + B \left(t - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\}, \\ & + \int_0^t (t - s) f(s, x(\phi(s))) ds, \end{aligned} \quad (2.11)$$

which proves that the solution of the nonlocal problem (1.3)-(1.4) can be expressed by the integral equation (2.3). \square

3. Existence of Solution

We study here the existence of at least one monotonic nondecreasing solution $x \in C[0, 1]$ for the integral equation (2.3).

Theorem 3.1. *Assume that (i)–(iv) are satisfied. Then the nonlocal problem (1.3)–(1.4) has at least one solution $x \in C[0, 1]$.*

Proof. Define the subset $Q_r \subset C(0, 1)$ by $Q_r = \{x \in C : |x(t)| \leq r, r = (Ax_0 + Bx_1 + (3 - B)\|a\|)/(1 - (3 - B)b), r > 0\}$. Clear the set Q_r which is nonempty, closed, and convex.

Let H be an operator defined by

$$\begin{aligned} (Hx)(t) = & A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\} \\ & + B \left(t - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\} \\ & + \int_0^t (t - s) f(s, x(\phi(s))) ds. \end{aligned} \quad (3.1)$$

Let $x \in Q_r$, then

$$\begin{aligned} |(Hx)(t)| \leq & A \left\{ x_0 + \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) |f(s, x(\phi(s)))| ds \right\} \\ & + B \left(t - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 + \sum_{j=1}^n b_j \int_0^{\eta_j} |f(s, x(\phi(s)))| ds \right\} \\ & + \int_0^t (t - s) |f(s, x(\phi(s)))| ds \\ \leq & A \left\{ x_0 + \sum_{k=1}^m a_k \int_0^1 [|a(s)| + b|x(\phi(s))|] ds \right\} \\ & + B \left\{ x_1 + \sum_{j=1}^n b_j \int_0^1 [|a(s)| + b|x(\phi(s))|] ds \right\} \\ & + \int_0^1 [|a(s)| + b|x(\phi(s))|] ds \\ \leq & Ax_0 + \|a\| + b \sup_{t \in I} |x(\phi(t))| + Bx_1 + B \sum_{j=1}^n b_j \|a\| \\ & + bB \sum_{j=1}^n b_j \sup_{t \in I} |x(\phi(t))| + \|a\| + b \sup_{t \in I} |x(\phi(t))| \end{aligned}$$

$$\begin{aligned}
&\leq Ax_0 + Bx_1 + 2\|a\| + 2b\|x\| + (1 - B)\|a\| + b(1 - B)\|x\| \\
&\leq Ax_0 + Bx_1 + (3 - B)\|a\| + (3 - B)br \leq r,
\end{aligned} \tag{3.2}$$

then $H : Q_r \rightarrow Q_r$ and $\{Hx(t)\}$ is uniformly bounded in Q_r .

Also for $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$, we have

$$\begin{aligned}
(Hx)(t_2) - (Hx)(t_1) &= B \left(t_2 - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(t))) ds \right\} \\
&\quad + \int_0^{t_2} (t_2 - s) f(s, x(\phi(t))) ds \\
&\quad - B \left(t_1 - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(t))) ds \right\} \\
&\quad - \int_0^{t_1} (t_1 - s) f(s, x(\phi(t))) ds \tag{3.3} \\
&= B(t_2 - t_1) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(t))) ds \right\} \\
&\quad + \int_0^{t_1} (t_2 - t_1) f(s, x(\phi(t))) ds \\
&\quad + \int_{t_1}^{t_2} (t_2 - s) f(s, x(\phi(t))) ds.
\end{aligned}$$

Then

$$\begin{aligned}
|(Hx)(t_2) - (Hx)(t_1)| &\leq B|t_2 - t_1| \left\{ x_1 + \sum_{j=1}^n b_j \int_0^{\eta_j} [|a(s)| + b|x(\phi(s))|] ds \right\} \\
&\quad + |t_2 - t_1| \int_0^{t_1} [|a(s)| + b|x(\phi(s))|] ds \\
&\quad + \int_{t_1}^{t_2} (t_2 - s) [|a(s)| + b|x(\phi(s))|] ds
\end{aligned}$$

$$\begin{aligned} &\leq B|t_2 - t_1|x_1 + \sum_{j=1}^n b_j [\|a\| + br] \\ &\quad + |t_2 - t_1|[\|a\| + br] + \int_{t_1}^{t_2} \|a\| ds + br[t_2 - t_1]. \end{aligned} \tag{3.4}$$

The above inequality shows that

$$|(Hx)(t_2) - (Hx)(t_1)| \rightarrow 0 \quad \text{as } t_2 \rightarrow t_1. \tag{3.5}$$

Therefore $\{Hx(t)\}$ is equicontinuous. By the Arzelà-Ascoli theorem, $\{Hx(t)\}$ is relatively compact.

Since all conditions of the Schauder theorem hold, then H has a fixed point in Q_r which proves the existence of at least one solution $x \in C[0, 1]$ of the integral equation (2.3), where

$$\begin{aligned} \lim_{t \rightarrow 0^+} x(t) &= A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\} \\ &\quad - BA \sum_{k=1}^m a_k \tau_k \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\} = x(0), \\ \lim_{t \rightarrow 1^-} x(t) &= A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\} \\ &\quad + B \left(1 - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\} \\ &\quad + \int_0^1 (1 - s) f(s, x(\phi(s))) ds = x(1). \end{aligned} \tag{3.6}$$

To complete the proof, we prove that the integral equation (2.3) satisfies nonlocal problem (1.3)-(1.4). Differentiating (2.3), we get

$$x'(t) = B \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\} + \int_0^t f(s, x(\phi(s))) ds, \tag{3.7}$$

$$x''(t) = f(t, x(\phi(t))). \tag{3.8}$$

Let $t = \tau_k$ in (2.3), we obtain

$$x(\tau_k) = A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\} + \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds, \quad (3.9)$$

which proves

$$\sum_{k=1}^m a_k x(\tau_k) = x_0. \quad (3.10)$$

Also let $t = \eta_j$ in (3.7), we obtain

$$x'(\eta_j) = B \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\} + \int_0^{\eta_j} f(s, x(\phi(s))) ds, \quad (3.11)$$

then

$$\sum_{j=1}^n b_j x'(\eta_j) = B \sum_{j=1}^n b_j \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\} + \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds. \quad (3.12)$$

Let $t = 0$ in (3.7), we obtain

$$x'(0) = B \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\}. \quad (3.13)$$

Adding (3.12) and (3.13), we obtain

$$x'(0) + \sum_{j=1}^n b_j x'(\eta_j) = x_1. \quad (3.14)$$

This implies that there exists at least one solution $x \in C[0, 1]$ of the nonlocal problem (1.3) and (1.4). This completes the proof. \square

Corollary 3.2. *The solution of the problem (1.3)-(1.4) is monotonic nondecreasing.*

Proof. Let $t_1 < t_2$, we deduce from (2.3) that

$$\begin{aligned}
 x(t_1) &= A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\} \\
 &\quad + B \left(t_1 - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\} \\
 &\quad + \int_0^{t_1} (t_1 - s) f(s, x(\phi(s))) ds \\
 &< A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\} \\
 &\quad + B \left(t_2 - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\} \\
 &\quad + \int_0^{t_2} (t_2 - s) f(s, x(\phi(s))) ds = x(t_2),
 \end{aligned} \tag{3.15}$$

which proves that the solution x of the problem (1.3)-(1.4) is monotonic nondecreasing. \square

3.1. Positive Solution

Let $b_j = 0, j = 1, 2, \dots, n$ and $x_1 = 0$, then the nonlocal problem condition (1.4) will be

$$\sum_{k=1}^m a_k x(\tau_k) = x_0, \quad x'(0) = 0. \tag{3.16}$$

Theorem 3.3. *Let the assumptions (i)–(iv) of Theorem 3.1 be satisfied. Then the solution of the nonlocal problem (1.3)–(3.16) is positive $t \in [d, 1]$.*

Proof. Let $b_j = 0, j = 1, 2, \dots, n$ and $x_1 = 0$ in the integral equation (2.3) and the nonlocal condition (1.4), then the solution of the nonlocal problem (1.3)–(3.16) will be given by the integral equation

$$x(t) = A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\} + \int_0^t (t - s) f(s, x(\phi(s))) ds, \tag{3.17}$$

where $A = (\sum_{k=1}^m a_k)^{-1}$.

Let $t \in [d, 1]$, then

$$\begin{aligned} \int_0^{\tau_k} (\tau_k - s)f(s, x(\phi(s)))ds &\leq \int_0^t (t - s)f(s, x(\phi(s)))ds, \quad \tau_k \leq t, \\ \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)f(s, x(\phi(s)))ds &\leq \sum_{k=1}^m a_k \int_0^t (t - s)f(s, x(\phi(s)))ds. \end{aligned} \quad (3.18)$$

Multiplying by $A = (\sum_{k=1}^m a_k)^{-1}$, we obtain

$$\begin{aligned} A \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)f(s, x(\phi(s)))ds &\leq A \sum_{k=1}^m a_k \int_0^t (t - s)f(s, x(\phi(s)))ds \\ &= \int_0^t (t - s)f(s, x(\phi(s)))ds, \end{aligned} \quad (3.19)$$

and the solution x of the nonlocal problem (1.3) and (3.16), given by the integral equation (3.17), is positive for $t \in [d, 1]$. This complete the proof. \square

Example 3.4. Consider the nonlocal problem of the second-order functional differential equation (1.3) with two-point boundary condition

$$x'(0) = 0, \quad x(\eta) = x_0, \quad \eta \in (a, d) \subset (0, 1). \quad (3.20)$$

Applying our results here, we deduce that the two-point boundary value problem (1.3)–(3.20) has at least one monotonic nondecreasing solution $x \in C[0, 1]$ represented by the integral equation

$$x(t) = x_0 - \int_0^{\eta} (\eta - s)f(s, x(\phi(s)))ds + \int_0^t (t - s)f(s, x(\phi(s)))ds. \quad (3.21)$$

This the solution is positive with $t > \eta$.

4. Nonlocal Integral Condition

Let $x \in C[0, 1]$ be the solution of the nonlocal problem (1.3) and (1.4).

Let $a_k = t_k - t_{k-1}$, $\tau_k \in (t_{k-1}, t_k) \subset (a, d) \subset (0, 1)$ and let $b_j = \xi_j - \xi_{j-1}$, $\eta_j \in (\xi_{j-1}, \xi_j) \subset (c, e) \subset (0, 1)$, then

$$\sum_{k=1}^m (t_k - t_{k-1})x(\tau_k) = x_0, \quad x'(0) + \sum_{j=1}^n (\xi_j - \xi_{j-1})x'(\eta_j) = x_1. \quad (4.1)$$

From the continuity of the solution x of the nonlocal problem (1.3) and (1.4), we obtain

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{k=1}^m (t_k - t_{k-1})x(\tau_k) &= \int_a^d x(s)ds, \\ x'(0) + \lim_{n \rightarrow \infty} \sum_{j=1}^n (\xi_j - \xi_{j-1})x'(\eta_j) &= x'(0) + \int_c^e x'(s)ds, \end{aligned} \tag{4.2}$$

and the nonlocal condition (1.4) transformed to the integral condition

$$\int_a^d x(s)ds = x_0, \quad x'(0) + x(e) - x(c) = x_1, \tag{4.3}$$

and the solution of the integral equation (2.3) will be

$$\begin{aligned} x(t) &= (d - a)^{-1} \left\{ x_0 - \int_a^d \int_0^t (t - s)f(s, x(\phi(s)))ds dt \right\} \\ &\quad + ((b - c) + 1)^{-1}(t - 1) \left\{ x_1 - \int_c^e \int_0^t f(s, x(\phi(s)))ds dt \right\} \\ &\quad + \int_0^t f(s, x(\phi(s)))ds. \end{aligned} \tag{4.4}$$

Now, we have the following theorem.

Theorem 4.1. *Let the assumptions (i)–(iv) of Theorem 3.1 be satisfied. Then the nonlocal problem*

$$\begin{aligned} x''(t) &= f(t, x(\phi(t))), \quad t \in (0, 1), \\ \int_a^d x(s)ds &= x_0, \quad x'(0) + x(e) - x(c) = x_1 \end{aligned} \tag{4.5}$$

has at least one monotonic nondecreasing solution $x \in C[0, 1]$ represented by (4.4).

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