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Research Article

Monotonic Positive Solutions of Nonlocal Boundary Value Problems for a Second-Order Functional Differential Equation

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We study the existence of at least one monotonic positive solution for the nonlocal boundary value problem of the second-order functional differential equation $x''(t) = f(t, x(\phi(t))), t \in (0, 1)$, with the nonlocal condition $\sum_{k=1}^m a_k x(\tau_k) = x_0, x'(0) + \sum_{j=1}^n b_j x'(\eta_j) = x_1$, where $\tau_k \in (a, d) \subset (0, 1)$, $\eta_j \in (c, e) \subset (0, 1)$, and $x_0, x_1 > 0$. As an application the integral and the nonlocal conditions $\int_a^d x(t)dt = x_0, x'(0) + x(e) - x(c) = x_1$ will be considered.

1. Introduction

The nonlocal boundary value problems of ordinary differential equations arise in a variety of different areas of applied mathematics and physics.

The study of nonlocal boundary value problems was initiated by Il'in and Moiseev [1, 2]. Since then, the non-local boundary value problems have been studied by several authors. The reader is referred to [3–22] and references therein.

In most of all these papers, the authors assume that the function $f:[0,1]\times R^+\to R^+$ is continuous. They all assume that

$$\lim_{x \to \infty} \frac{f(x)}{x} = 0 \quad \text{or } \infty,$$

$$\lim_{x \to 0} \frac{f(x)}{x} = 0 \quad \text{or } \infty.$$
(1.1)

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These assumptions are restrictive, and there are many functions that do not satisfy these assumptions.

Here we assume that the function $f:[0,1]\times R^+\to R^+$ is measurable in $t\in[0,1]$ for all $x\in R^+$ and continuous in $x\in R^+$ for almost all $t\in[0,1]$ is and there exists an integrable function $a\in L_1[0,1]$ and a constant b>0 such that

$$|f(t,x)| \le |a(t)| + b|x|, \quad \forall (t,x) \in [0,1] \times D.$$
 (1.2)

Our aim here is to study the existence of at least one monotonic positive solution for the nonlocal problem of the second-order functional differential equation

$$x''(t) = f(t, x(\phi(t))), \quad t \in (0, 1), \tag{1.3}$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \qquad x'(0) + \sum_{j=1}^{n} b_j x'(\eta_j) = x_1, \tag{1.4}$$

where $\tau_k \in (a, d) \subset (0, 1), \eta_i \in (c, e) \subset (0, 1), \text{ and } x_0, x_1 > 0.$

As an application, the problem with the integral and nonlocal conditions

$$\int_{a}^{d} x(t)dt = x_{0}, \qquad x'(0) + x(e) - x(c) = x_{1}, \tag{1.5}$$

is studied.

It must be noticed that the nonlocal conditions

$$x(\tau) = x_0, \quad \tau \in (a, d) , \qquad x'(0) + x'(\eta) = x_1, \quad \eta \in (c, e),$$

$$\sum_{k=1}^{m} a_k x(\tau_k) = 0, \quad \tau_k \in (a, d), \qquad x'(0) + \sum_{j=1}^{n} b_j x'(\eta_j) = 0, \quad \eta_j \in (c, e),$$

$$\int_{a}^{d} x(t) dt = 0, \qquad x'(0) + x(e) = x(c)$$
(1.6)

are special cases of our the nonlocal and integral conditions.

2. Integral Equation Representation

Consider the functional differential equation (1.3) with the nonlocal condition (1.4) with the following assumptions.

(i) $f:[0,1]\times R^+\to R^+$ is measurable in $t\in[0,1]$ for all $x\in R^+$ and continuous in $x\in R^+$ for almost all $t\in[0,1]$ and there exists an integrable function $a\in L_1[0,1]$, and a constant b>0 such that

$$|f(t,x)| \le |a(t)| + b|x|, \quad \forall (t,x) \in [0,1] \times D.$$
 (2.1)

(ii) $\phi:(0,1)\to(0,1)$ is continuous.

(iii)
$$b < 1/(3-B)$$
, $B = (\sum_{i=1}^{n} b_i + 1)^{-1}$.

(iv)

$$\sum_{k=1}^{m} a_k > 0, \quad \forall k = 1, 2, \dots, m, \qquad \sum_{j=1}^{n} b_j > 0, \quad \forall j = 1, 2, \dots, n.$$
 (2.2)

Now, we have the following Lemma.

Lemma 2.1. The solution of the nonlocal problem (1.3)-(1.4) can be expressed by the integral equation

$$x(t) = A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\}$$

$$+ B \left(t - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\}$$

$$+ \int_0^t (t - s) f(s, x(\phi(s))) ds,$$
(2.3)

where $A = (\sum_{k=1}^{m} a_k)^{-1}$, $B = (\sum_{j=1}^{n} b_j + 1)^{-1}$.

Proof. Integrating (1.3), we get

$$x'(t) = x'(0) + \int_0^t f(s, x(\phi(s))) ds.$$
 (2.4)

Integrating (2.4), we obtain

$$x(t) = x(0) + x'(0)t + \int_0^t (t - s)f(s, x(\phi(s)))ds.$$
 (2.5)

Let $t = \tau_k$, in (2.5), we get

$$\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{n} a_k x(0) + \sum_{k=1}^{n} a_k \tau_k x'(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds, \tag{2.6}$$

and we deduce that

$$x(0) = A\left\{x_0 - \sum_{k=1}^m a_k \tau_k x'(0) - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds\right\}, \quad A = \left(\sum_{k=1}^m a_k\right)^{-1}. \quad (2.7)$$

Substitute from (2.7) into (2.5), we obtain

$$x(t) = A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\} + x'(0) \left(t - A \sum_{k=1}^m a_k \tau_k \right)$$

$$+ \int_0^t (t - s) f(s, x(\phi(s))) ds.$$
(2.8)

Let $t = \eta_j$, in (2.4), we obtain

$$\sum_{j=1}^{n} b_{j} x'(\eta_{j}) = \sum_{j=1}^{n} b_{j} x'(0) + \sum_{j=1}^{n} b_{j} \int_{0}^{\eta_{j}} f(s, x(\phi(s))) ds,$$

$$x_{1} - x'(0) = x'(0) \sum_{j=1}^{n} b_{j} + \sum_{j=1}^{n} b_{j} \int_{0}^{\eta_{j}} f(s, x(\phi(s))) ds,$$
(2.9)

and we deduce that

$$x'(0) = B\left(x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds\right), \quad B = \left(\sum_{j=1}^n b_j + 1\right)^{-1}.$$
 (2.10)

Substitute from (2.10) into (2.8), we obtain

$$x(t) = A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\}$$

$$+ B \left(t - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\},$$

$$+ \int_0^t (t - s) f(s, x(\phi(s))) ds,$$
(2.11)

which proves that the solution of the nonlocal problem (1.3)-(1.4) can be expressed by the integral equation (2.3).

3. Existence of Solution

We study here the existence of at least one monotonic nondecreasing solution $x \in C[0,1]$ for the integral equation (2.3).

Theorem 3.1. Assume that (i)–(iv) are satisfied. Then the nonlocal problem (1.3)-(1.4) has at least one solution $x \in C[0,1]$.

Proof. Define the subset $Q_r \subset C(0,1)$ by $Q_r = \{x \in C : |x(t)| \le r, r = (Ax_0 + Bx_1 + (3 - B)\|a\|)/(1 - (3 - B)b), r > 0\}$. Clear the set Q_r which is nonempty, closed, and convex. Let H be an operator defined by

$$(Hx)(t) = A \left\{ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \right\}$$

$$+ B \left(t - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\}$$

$$+ \int_0^t (t - s) f(s, x(\phi(s))) ds.$$
(3.1)

Let $x \in Q_r$, then

$$|(Hx)(t)| \leq A \left\{ x_0 + \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) | f(s, x(\phi(s))) | ds \right\}$$

$$+ B \left(t - A \sum_{k=1}^m a_k \tau_k \right) \left\{ x_1 + \sum_{j=1}^n b_j \int_0^{\eta_j} | f(s, x(\phi(s))) | ds \right\}$$

$$+ \int_0^t (t - s) | f(s, x(\phi(s))) | ds$$

$$\leq A \left\{ x_0 + \sum_{k=1}^m a_k \int_0^1 [|a(s)| + b|x(\phi(s))|] ds \right\}$$

$$+ B \left\{ x_1 + \sum_{j=1}^n b_j \int_0^1 [|a(s)| + b|x(\phi(s))|] ds \right\}$$

$$+ \int_0^1 [|a(s)| + b|x(\phi(s))|] ds$$

$$\leq Ax_0 + ||a|| + b \sup_{t \in I} |x(\phi(t))| + Bx_1 + B \sum_{j=1}^n b_j ||a||$$

$$+ bB \sum_{j=1}^n b_j \sup_{t \in I} |x(\phi(t))| + ||a|| + b \sup_{t \in I} |x(\phi(t))|$$

$$\leq Ax_0 + Bx_1 + 2\|a\| + 2b\|x\| + (1 - B)\|a\| + b(1 - B)\|x\|$$

$$\leq Ax_0 + Bx_1 + (3 - B)\|a\| + (3 - B)br \leq r,$$
(3.2)

then $H: Q_r \to Q_r$ and $\{Hx(t)\}$ is uniformly bounded in Q_r . Also for $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$, we have

$$(Hx)(t_{2}) - (Hx)(t_{1}) = B\left(t_{2} - A\sum_{k=1}^{m} a_{k}\tau_{k}\right) \left\{x_{1} - \sum_{j=1}^{n} b_{j} \int_{0}^{\eta_{j}} f(s, x(\phi(t)))ds\right\}$$

$$+ \int_{0}^{t_{2}} (t_{2} - s)f(s, x(\phi(t)))ds$$

$$- B\left(t_{1} - A\sum_{k=1}^{m} a_{k}\tau_{k}\right) \left\{x_{1} - \sum_{j=1}^{n} b_{j} \int_{0}^{\eta_{j}} f(s, x(\phi(t)))ds\right\}$$

$$- \int_{0}^{t_{1}} (t_{1} - s)f(s, x(\phi(t)))ds$$

$$= B(t_{2} - t_{1}) \left\{x_{1} - \sum_{j=1}^{n} b_{j} \int_{0}^{\eta_{j}} f(s, x(\phi(t)))ds\right\}$$

$$+ \int_{0}^{t_{1}} (t_{2} - t_{1})f(s, x(\phi(t)))ds$$

$$+ \int_{t_{1}}^{t_{2}} (t_{2} - s)f(s, x(\phi(t)))ds.$$

$$(3.3)$$

Then

$$\begin{aligned} |(Hx)(t_2) - (Hx)(t_1)| &\leq B|t_2 - t_1| \left\{ x_1 + \sum_{j=1}^n b_j \int_0^{\eta_j} \left[|a(s)| + b \big| x(\phi(s)) \big| \right] ds \right\} \\ &+ |t_2 - t_1| \int_0^{t_1} \left[|a(s)| + b \big| x(\phi(s)) \big| \right] ds \\ &+ \int_{t_1}^{t_2} (t_2 - s) \left[|a(s)| + b \big| x(\phi(s)) \big| \right] ds \end{aligned}$$

$$\leq B|t_{2} - t_{1}|x_{1} + \sum_{j=1}^{n} b_{j}[\|a\| + br] + |t_{2} - t_{1}|[\|a\| + br] + \int_{t_{1}}^{t_{2}} \|a\| ds + br[t_{2} - t_{1}].$$

$$(3.4)$$

The above inequality shows that

$$|(Hx)(t_2) - (Hx)(t_1)| \longrightarrow 0 \quad \text{as } t_2 \longrightarrow t_1. \tag{3.5}$$

Therefore $\{Hx(t)\}\$ is equicontinuous. By the Arzelà-Ascoli theorem, $\{Hx(t)\}\$ is relatively compact.

Since all conditions of the Schauder theorem hold, then H has a fixed point in Q_r which proves the existence of at least one solution $x \in C[0,1]$ of the integral equation (2.3), where

$$\lim_{t \to 0^{+}} x(t) = A \left\{ x_{0} - \sum_{k=1}^{m} a_{k} \int_{0}^{\tau_{k}} (\tau_{k} - s) f(s, x(\phi(s))) ds \right\}$$

$$-BA \sum_{k=1}^{m} a_{k} \tau_{k} \left\{ x_{1} - \sum_{j=1}^{n} b_{j} \int_{0}^{\eta_{j}} f(s, x(\phi(s))) ds \right\} = x(0),$$

$$\lim_{t \to 1^{-}} x(t) = A \left\{ x_{0} - \sum_{k=1}^{m} a_{k} \int_{0}^{\tau_{k}} (\tau_{k} - s) f(s, x(\phi(s))) ds \right\}$$

$$+ B \left(1 - A \sum_{k=1}^{m} a_{k} \tau_{k} \right) \left\{ x_{1} - \sum_{j=1}^{n} b_{j} \int_{0}^{\eta_{j}} f(s, x(\phi(s))) ds \right\}$$

$$+ \int_{0}^{1} (1 - s) f(s, x(\phi(s))) ds = x(1).$$
(3.6)

To complete the proof, we prove that the integral equation (2.3) satisfies nonlocal problem (1.3)-(1.4). Differentiating (2.3), we get

$$x'(t) = B\left\{x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds\right\} + \int_0^t f(s, x(\phi(s))) ds, \tag{3.7}$$

$$x''(t) = f(t, x(\phi(t))).$$
 (3.8)

Let $t = \tau_k$ in (2.3), we obtain

$$x(\tau_k) = A\left\{x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds\right\} + \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds,$$
 (3.9)

which proves

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0. {(3.10)}$$

Also let $t = \eta_j$ in (3.7), we obtain

$$x'(\eta_j) = B\left\{x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds\right\} + \int_0^{\eta_j} f(s, x(\phi(s))) ds, \tag{3.11}$$

then

$$\sum_{j=1}^{n} b_j x'(\eta_j) = B \sum_{j=1}^{n} b_j \left\{ x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds \right\} + \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds.$$
 (3.12)

Let t = 0 in (3.7), we obtain

$$x'(0) = B\left\{x_1 - \sum_{j=1}^n b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds\right\}.$$
 (3.13)

Adding (3.12) and (3.13), we obtain

$$x'(0) + \sum_{j=1}^{n} b_j x'(\eta_j) = x_1.$$
(3.14)

This implies that there exists at least one solution $x \in C[0,1]$ of the nonlocal problem (1.3) and (1.4). This completes the proof.

Corollary 3.2. *The solution of the problem* (1.3)-(1.4) *is monotonic nondecreasing.*

Proof. Let $t_1 < t_2$, we deduce from (2.3) that

$$x(t_{1}) = A \left\{ x_{0} - \sum_{k=1}^{m} a_{k} \int_{0}^{\tau_{k}} (\tau_{k} - s) f(s, x(\phi(s))) ds \right\}$$

$$+ B \left(t_{1} - A \sum_{k=1}^{m} a_{k} \tau_{k} \right) \left\{ x_{1} - \sum_{j=1}^{n} b_{j} \int_{0}^{\eta_{j}} f(s, x(\phi(s))) ds \right\}$$

$$+ \int_{0}^{t_{1}} (t_{1} - s) f(s, x(\phi(s))) ds$$

$$< A \left\{ x_{0} - \sum_{k=1}^{m} a_{k} \int_{0}^{\tau_{k}} (\tau_{k} - s) f(s, x(\phi(s))) ds \right\}$$

$$+ B \left(t_{2} - A \sum_{k=1}^{m} a_{k} \tau_{k} \right) \left\{ x_{1} - \sum_{j=1}^{n} b_{j} \int_{0}^{\eta_{j}} f(s, x(\phi(s))) ds \right\}$$

$$+ \int_{0}^{t_{2}} (t_{2} - s) f(s, x(\phi(s))) ds = x(t_{2}),$$

$$(3.15)$$

which proves that the solution x of the problem (1.3)-(1.4) is monotonic nondecreasing.

3.1. Positive Solution

Let $b_i = 0$, j = 1, 2, ... n and $x_1 = 0$, then the nonlocal problem condition (1.4) will be

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \qquad x'(0) = 0.$$
(3.16)

Theorem 3.3. Let the assumptions (i)–(iv) of Theorem 3.1 be satisfied. Then the solution of the nonlocal problem (1.3)–(3.16) is positive $t \in [d,1]$.

Proof. Let $b_j = 0$, j = 1, 2, ...n and $x_1 = 0$ in the integral equation (2.3) and the nonlocal condition (1.4), then the solution of the nonlocal problem (1.3)–(3.16) will be given by the integral equation

$$x(t) = A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds\right\} + \int_0^t (t - s) f(s, x(\phi(s))) ds, \qquad (3.17)$$

where $A = (\sum_{k=1}^{m} a_k)^{-1}$.

Let $t \in [d, 1]$, then

$$\int_{0}^{\tau_{k}} (\tau_{k} - s) f(s, x(\phi(s))) ds \leq \int_{0}^{t} (t - s) f(s, x(\phi(s))) ds, \quad \tau_{k} \leq t,
\sum_{k=1}^{m} a_{k} \int_{0}^{\tau_{k}} (\tau_{k} - s) f(s, x(\phi(s))) ds \leq \sum_{k=1}^{m} a_{k} \int_{0}^{t} (t - s) f(s, x(\phi(s))) ds.$$
(3.18)

Multiplying by $A = (\sum_{k=1}^{m} a_k)^{-1}$, we obtain

$$A\sum_{k=1}^{m} a_{k} \int_{0}^{\tau_{k}} (\tau_{k} - s) f(s, x(\phi(s))) ds \leq A\sum_{k=1}^{m} a_{k} \int_{0}^{t} (t - s) f(s, x(\phi(s))) ds$$

$$= \int_{0}^{t} (t - s) f(s, x(\phi(s))) ds,$$
(3.19)

and the solution x of the nonlocal problem (1.3) and (3.16), given by the integral equation (3.17), is positive for $t \in [d,1]$. This complete the proof.

Example 3.4. Consider the nonlocal problem of the second-order functional differential equation (1.3) with two-point boundary condition

$$x'(0) = 0,$$
 $x(\eta) = x_0, \quad \eta \in (a, d) \subset (0, 1).$ (3.20)

Applying our results here, we deduce that the two-point boundary value problem (1.3)–(3.20) has at least one monotonic nondecreasing solution $x \in C[0,1]$ represented by the integral equation

$$x(t) = x_0 - \int_0^{\eta} (\eta - s) f(s, x(\phi(s))) ds + \int_0^t (t - s) f(s, x(\phi(s))) ds.$$
 (3.21)

This the solution is positive with $t > \eta$.

4. Nonlocal Integral Condition

Let $x \in C[0,1]$ be the solution of the nonlocal problem (1.3) and (1.4). Let $a_k = t_k - t_{k-1}, \tau_k \in (t_{k-1}, t_k) \subset (a, d) \subset (0, 1)$ and let $b_j = \xi_j - \xi_{j-1}, \eta_j \in (\xi_{j-1}, \xi_j) \subset (c, e) \subset (0, 1)$, then

$$\sum_{k=1}^{m} (t_k - t_{k-1}) x(\tau_k) = x_0, \qquad x'(0) + \sum_{j=1}^{n} (\xi_j - \xi_{j-1}) x'(\eta_j) = x_1.$$
 (4.1)

From the continuity of the solution x of the nonlocal problem (1.3) and (1.4), we obtain

$$\lim_{m \to \infty} \sum_{k=1}^{m} (t_k - t_{k-1}) x(\tau_k) = \int_{a}^{d} x(s) ds,$$

$$x'(0) + \lim_{n \to \infty} \sum_{j=1}^{n} (\xi_j - \xi_{j-1}) x'(\eta_j) = x'(0) + \int_{c}^{e} x'(s) ds,$$
(4.2)

and the nonlocal condition (1.4) transformed to the integral condition

$$\int_{a}^{d} x(s)ds = x_{0}, \qquad x'(0) + x(e) - x(c) = x_{1}, \tag{4.3}$$

and the solution of the integral equation (2.3) will be

$$x(t) = (d-a)^{-1} \left\{ x_0 - \int_a^d \int_0^t (t-s) f(s, x(\phi(s))) ds dt \right\}$$

$$+ ((b-c)+1)^{-1} (t-1) \left\{ x_1 - \int_c^e \int_0^t f(s, x(\phi(s))) ds dt \right\}$$

$$+ \int_0^t f(s, x(\phi(s))) ds.$$
(4.4)

Now, we have the following theorem.

Theorem 4.1. Let the assumptions (i)–(iv) of Theorem 3.1 be satisfied. Then the nonlocal problem

$$x''(t) = f(t, x(\phi(t))), \quad t \in (0, 1) ,$$

$$\int_{0}^{d} x(s)ds = x_{0}, \qquad x'(0) + x(e) - x(c) = x_{1}$$
(4.5)

has at least one monotonic nondecreasing solution $x \in C[0,1]$ represented by (4.4).

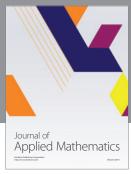
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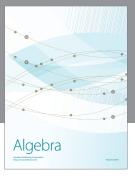
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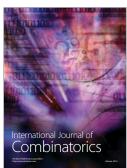








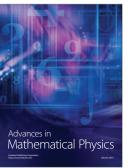


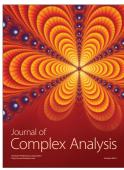




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