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Research Article

On the Dimension of the Pullback Attractors for g-Navier-Stokes Equations

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We consider the asymptotic behaviour of nonautonomous 2D g-Navier-Stokes equations in bounded domain Ω. Assuming that $f \in L^2_{\text{loc}}$, which is translation bounded, the existence of the pullback attractor is proved in $L^2(\Omega)$ and $H^1(\Omega)$. It is proved that the fractal dimension of the pullback attractor is finite.

1. Introduction

In this paper, we study the behavior of solutions of the nonautonomous g-Navier-Stokes equations in spatial dimension 2. These equations are a variation of the standard Navier-Stokes equations, and they assume the form

$$\begin{gathered}\frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in } \Omega, \\ \frac{1}{g}(\nabla \cdot gu) = \frac{\nabla g}{g} \cdot u + \nabla \cdot u = 0 \quad \text{in } \Omega,\end{gathered} \tag{1.1}$$

where $g = g(x_1, x_2)$ is a suitable smooth real-valued function defined on $(x_1, x_2) \in \Omega$ and Ω is a suitable bounded domain in $\mathbb{R}^2$. Notice that if $g(x_1, x_2) = 1$, then (1.1) reduce to the standard Navier-Stokes equations.

In addition, we assume that the function $f(\cdot,t) =: f(t) \in L^2_{\text{loc}}(\mathbb{R};E)$ is translation bounded, where $E = L^2(\Omega)$ or $H^{-1}(\Omega)$. This property implies that

$$\|f\|^2_{L^2_b} = \|f\|^2_{L^2_b(\mathbb{R};E)} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s)\|^2_E ds < \infty. \tag{1.2}$$

We consider this equation in an appropriate Hilbert space and show that there is a pullback attractor $\mathfrak{A}$. This is the basic idea of our construction, which is motivated by the works of [1].

Let $\Omega = (0,1) \times (0,1)$. We assume that the function $g(x) = g(x_1, x_2)$ satisfies the following properties:

(1) $g(x) \in C^\infty_{\text{per}}(\Omega)$,

(2) there exist constants $m_0 = m_0(g)$ and $M_0 = M_0(g)$ such that, for all $x \in \Omega$, $0 < m_0 \leq g(x) \leq M_0$. Note that the constant function $g \equiv 1$ satisfies these conditions.

We denote by $L^2(\Omega, g)$ the space with the scalar product and the norm given by

$$(u,v)_g = \int_\Omega (u \cdot v) g dx, \qquad |u|^2_g = (u,u)_g, \tag{1.3}$$

as well as $H^1(\Omega, g)$ with the norm

$$\|u\|_{H^1(\Omega,g)} = \left[(u,u)_g + \sum_{i=1}^2 (D_i u, D_i u)_g \right]^{1/2}, \tag{1.4}$$

where $\partial u/\partial x_i = D_i u$.

Then for the functional setting of the problems (1.1), we use the following functional spaces:

$$\begin{aligned} H_g &= Cl_{L^2_{\text{per}}(\Omega,g)} \left\{ u \in C^\infty_{\text{per}}(\Omega) : \nabla \cdot g u = 0, \int_\Omega u dx = 0 \right\}, \\ V_g &= \left\{ u \in H^1_{\text{per}}(\Omega, g) : \nabla \cdot g u = 0, \int_\Omega u dx = 0 \right\}, \end{aligned} \tag{1.5}$$

where H_g is endowed with the scalar product and the norm in $L^2(\Omega, g)$ and V_g is the spaces with the scalar product and the norm given by

$$((u,v))_g = \int_\Omega (\nabla u \cdot \nabla v) g dx, \qquad \|u\|_g = ((u,u))_g. \tag{1.6}$$

Also, we define the orthogonal projection P_g as

$$P_g : L^2_{\text{per}}(\Omega, g) \longrightarrow H_g, \tag{1.7}$$

and we have that $Q \subseteq H_g^{\perp}$, where

$$Q = Cl_{L^2_{\rm per}(\Omega,g)}\left\{\nabla\phi : \phi \in C^1\left(\overline{\Omega}, \mathbb{R}\right)\right\}. \tag{1.8}$$

Then, we define the g-Laplacian operator

$$-\Delta_g u \equiv \frac{1}{g}(\nabla \cdot g\nabla)u = -\Delta u - \frac{1}{g}(\nabla g \cdot \nabla)u \tag{1.9}$$

to have the linear operator

$$A_g u = P_g\left[-\frac{1}{g}(\nabla \cdot (g\nabla u))\right]. \tag{1.10}$$

For the linear operator A_g, the following hold (see [1]).

(1) A_g is a positive, self-adjoint operator with compact inverse, where the domain of A_g is $D(A_g) = V_g \cap H^2(\Omega, g)$.

(2) There exist countable eigenvalues of A_g satisfying

$$0 < \lambda_g \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots, \tag{1.11}$$

where $\lambda_g = 4\pi^2 m_0/M_0$ and λ_1 is the smallest eigenvalue of A_g. In addition, there exists the corresponding collection of eigenfunctions $\{e_1, e_2, e_3, \ldots\}$ which forms an orthonormal basis for H_g.

Next, we denote the bilinear operator $B_g(u,v) = P_g(u\cdot\nabla)v$ and the trilinear form

$$b_g(u,v,w) = \sum_{i,j=1}^{2}\int_\Omega u_i(D_i v_j)w_j g\,dx = \left(P_g(u\cdot\nabla)v, w\right)_g, \tag{1.12}$$

where u, v, and w lie in appropriate subspaces of $L^2(\Omega, g)$. Then, the form b_g satisfies

$$b_g(u,v,w) = -b_g(u,w,v) \quad \text{for } u,v,w \in H_g. \tag{1.13}$$

We denote a linear operator R on V_g by

$$Ru = P_g\left[\frac{1}{g}(\nabla g\cdot\nabla)u\right] \quad \text{for } u \in V_g \tag{1.14}$$

and have R as a continuous linear operator from V_g into H_g such that

$$|(Ru,u)| \le \frac{|\nabla g|_\infty}{m_0}\|u\|_g |u|_g \le \frac{|\nabla g|_\infty}{m_0\lambda_g^{1/2}}\|u\|_g \quad \text{for } u \in V_g. \tag{1.15}$$

We now rewrite (1.1) as abstract evolution equations:

$$\frac{du}{dt} + \nu A_g u + B_g u + \nu R u = P_g f,$$
$$u(\tau) = u_\tau. \tag{1.16}$$

In [1] the author established the global regularity of solutions of the g-Navier-Stokes equations. The Navier-Stokes equations were investigated by many authors, and the existence of the attractors for 2D Navier-Stokes equations was first proved in [2] and independently in [3]. The finite-dimensional property of the global attractor for general dissipative equations was first proved in [4]. For the analysis of the Navier-Stokes equations, one can refer to [5], specially [6] for the periodic boundary conditions.

The theory of pullback (or cocycle) attractors has been developed for both the nonautonomous and random dynamical systems (see [7–13]) and has shown to be very useful in the understanding of the dynamics of nonautonomous dynamical systems.

The understanding of the asymptotic behaviour of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem for a system having some dissipativity properties is to analyse the existence and structure of its global attractor, which, in the autonomous case, is an invariant compact set which attracts all the trajectories of the system, uniformly on bounded sets. This set has, in general, a very complicated geometry which reflects the complexity of the long-time behaviour of the system (see [14–17] and the references therein). However, nonautonomous systems are also of great importance and interest as they appear in many applications to natural sciences. In this situation, there are various options to deal with the problem of attractors for nonautonomous systems (kernel sections [18], skew-product formalism [16, 19], etc.); for our particular situation we have preferred to choose that of pullback attractor (see [9, 10, 13, 20]) which has also proved extremely fruitful, particularly in the case of random dynamical systems (see [11, 13]).

In this paper, we study the existence of compact pullback attractor for the nonautonomous g-Navier-Stokes equations in bounded domain Ω with periodic boundary condition. It is proved that the fractal dimension of the pullback attractor is finite.

Hereafter c will denote a generic scale invariant positive constant, which is independent of the physical parameters in the equation and may be different from line to line and even in the same line.

2. Abstract Results

We now recall the preliminary results of pullback attractors, as developed in [8–10, 13].

The semigroup $S(t)$ property is replaced by the process $U(t,\tau)$ composition property

$$U(t,\tau)U(\tau,s) = U(t,s) \quad \forall t \geq \tau \geq s, \tag{2.1}$$

and, obviously, the initial condition implies that $U(\tau,\tau) = \mathrm{Id}$. As with the semigroup composition $S(t)S(\tau) = S(t+\tau)$, this just expresses the uniqueness of solutions.

It is also possible to present the theory within the more general framework of cocycle dynamical systems. In this case the second component of U is viewed as an element of some parameter space J, so that the solution can be written as $U(t,p)\varrho$, and a shift map $\theta_t : J \to J$ is defined so that the process composition becomes the cocycle property

$$U(t+\tau,p) = U(t,\theta_\tau p)U(\tau,p). \tag{2.2}$$

However, when one tries to develop a theory under a unified abstract formulation, the context of cocycle (or skew-product flows) may not be the most appropriate to deal with the problem. In this paper, we apply a process $U(t,\tau)$ to (1.16) by using the concept of measure of noncompactness to obtain pullback attractors.

By $\mathcal{B}(E)$ we denote the collection of the *bounded* sets of E.

Definition 2.1. Let U be a process on a complete metric space E. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a pullback attractor for U if, for all $\tau \in \mathbb{R}$, it satisfies

(i) $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \ge \tau$,

(ii) $\lim_{s\to\infty} \operatorname{dist}(U(t,t-s)D,\mathcal{A}(t)) = 0$, for $D \in \mathcal{B}(E)$.

The pullback attractor is said to be uniform if the attraction property is uniform in time, that is,

$$\lim_{s\to\infty}\sup_{t\in\mathbb{R}} \operatorname{dist}(U(t,t-s)D,\mathcal{A}(t)) = 0, \quad \text{for } D \in \mathcal{B}(E). \tag{2.3}$$

Definition 2.2. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a forward attractor for U if, for all $\tau \in \mathbb{R}$, it satisfies

(i) $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \ge \tau$,

(ii) $\lim_{t\to\infty} \operatorname{dist}(U(t,\tau)D,\mathcal{A}(t)) = 0$, for $D \in \mathcal{B}(E)$.

The forward attractor is said to be uniform if the attraction property is uniform in time, that is,

$$\lim_{t\to\infty}\sup_{\tau\in\mathbb{R}} \operatorname{dist}(U(t+\tau,\tau)D,\mathcal{A}(t+\tau)) = 0, \quad \text{for } D \in \mathcal{B}(E). \tag{2.4}$$

In the definition, $\operatorname{dist}(A,B)$ is the Hausdorff semidistance between A and B, defined as

$$\operatorname{dist}(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b), \quad \text{for } A,B \subseteq E. \tag{2.5}$$

Property (i) is a generalization of the invariance property for autonomous dynamical systems. The pullback attracting property (ii) considers the state of the system at time t when the initial time $t-s$ goes to $-\infty$.

The notion of an attractor is closely related to that of an absorbing set.

Definition 2.3. The family $\{B(t)\}_{t\in\mathbb{R}}$ is said to be (pullback) absorbing with respect to the process U if, for all $t \in \mathbb{R}$ and $D \in \mathcal{B}(E)$, there exists $S(D,t) > 0$ such that for all $s \geq S(D,t)$

$$U(t,t-s)D \subset B(t). \tag{2.6}$$

The absorption is said to be uniform if $S(D,t)$ does not depend on the time variable t.

Now we recall the abstract results in [21].

Definition 2.4. The family of processes $\{U(t,t-s)\}$ is said to be satisfying pullback Condition (C) if, for any fixed $B \in \mathcal{B}(E)$ and $\varepsilon > 0$, there exist $s_0 = s(B,t,\varepsilon) \geq 0$ and a finite dimensional subspace E_1 of E such that

(i) $\{\|P(\bigcup_{s\geq s_0} U(t,t-s)B)\|_E\}$ is bounded,

(ii) $\|(I-P)(\bigcup_{s\geq s_0} U(t,t-s)B)\|_E \leq \varepsilon$,

where $P : E \to E_1$ is a bounded projector.

Theorem 2.5. *Let the family of processes $\{U(t,\tau)\}$ acting in E be continuous and possess compact pullback attractor $\mathcal{A}(t)$ satisfying*

$$\mathcal{A}(t) = \overline{\bigcup_{B\in\mathcal{B}} \omega(B,t)}, \quad \text{for } t \in \mathbb{R}, \tag{2.7}$$

if it

(i) *has a bounded (pullback) absorbing set B,*

(ii) *satisfies pullback Condition (C).*

Moreover if E is a uniformly convex Banach space, then the converse is true.

3. Pullback Attractor of Nonautonomous g-Navier-Stokes Equations

This section deals with the existence of the attractor for the two-dimensional nonautonomous g-Navier-Stokes equations in a bounded domain Ω with periodic boundary condition.

In [1], the author has shown that the semigroup $S(t) : H_g \to H_g$ $(t \geq 0)$ associated with the autonomous systems (1.16) possesses a global attractor in H_g and V_g. The main objective of this section is to prove that the nonautonomous system (1.16) has uniform attractors in H_g and V_g.

To this end, we first state the following results of existence and uniqueness of solutions of (1.16).

Proposition 3.1. *Let $f \in V'$ be given. Then for every $u_\tau \in H_g$ there exists a unique solution $u = u(t)$ on $[0,\infty)$ of (1.16), satisfying $u(\tau) = u_\tau$. Moreover, one has*

$$u(t) \in C[\tau,T;H_g) \cap L^2(\tau,T;V_g), \quad \forall T > \tau. \tag{3.1}$$

Finally, if $u_\tau \in V_g$*, then*

$$u(t) \in C[\tau, T; V_g) \cap L^2(\tau, T; D(A_g)), \quad \forall T > \tau. \tag{3.2}$$

Proof. The Proof of Proposition 3.1 is similar to autonomous case in [1, 17]. □

Proposition 3.2. *The process* $\{U(t, t-s)\} : V_g \to V_g$ *associated with the system* (1.16) *possesses (pullback) absorbing sets, that is, there exists a family* $\{B(t)\}_{t\in R}$ *of bounded (pullback) absorbing sets in* H_g *and* V_g *for the process* U*, which is given by*

$$\begin{aligned} \mathcal{B}_0 = B(t) &= \left\{u \in H_g \mid |u|_g \le \rho_0\right\}, \\ \mathcal{B}_1 = B(t) &= \left\{u \in V_g \|u\|_g \le \rho_1\right\}, \end{aligned} \tag{3.3}$$

which absorb all bounded sets of H_g*. Moreover* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *absorb all bounded sets of* H_g *and* V_g *in the norms of* H_g *and* V_g*, respectively.*

Proof. The proof of Proposition 3.2 is similar to autonomous g-Navier-Stokes equation. We can obtain absorbing sets in H_g and V_g from [1]. □

The main results in this section are as follows.

Theorem 3.3. *If* $f(x,t) \in L_b^2(R; V')$ *and* $u_\tau \in H_g$*, then the processes* $\{U(t, t-s)\}$ *corresponding to problem* (1.16) *possess compact pullback attractor* $\mathcal{A}_0(t)$ *in* H_g *which coincides with the pullback attractor:*

$$\mathcal{A}_0(t) = \overline{\bigcup_{\mathcal{B}_0 \in \mathcal{B}} \omega(\mathcal{B}_0, t)}, \tag{3.4}$$

where $\mathcal{B}_0$ *is the (pullback) absorbing set in* H_g*.*

Proof. As in the previous section, for fixed N, let H_1 be the subspace spanned by $w_1, \dots, w_N$, and H_2 the orthogonal complement of H_1 in H_g. We write

$$u = u_1 + u_2, \quad u_1 \in H_1,\ u_2 \in H_2 \text{ for any } u \in H_g. \tag{3.5}$$

Now, we only have to verify Condition (C). Namely, we need to estimate $|u_2(t)|_g$, where $u(t) = u_1(t) + u_2(t)$ is a solution of (1.16) given in Proposition 3.1.

Multiplying (1.16) by u_2, we have

$$\left(\frac{du}{dt}, u_2\right)_g + (\nu A_g u, u_2)_g + (B_g(u,u), u_2)_g = (f, u_2)_g - (\nu R u, u_2)_g. \tag{3.6}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}|u_2|_g^2 + \nu\|u_g\|_g^2 \le \left|(B(u,u), u_2)_g\right| + \left|(f, u_2)_g\right| + (Ru, u_2)_g. \tag{3.7}$$

Since b_g satisfies the following inequality (see [6]):

$$\left|b_g(u,v,w)\right| \le c|u|_g^{1/2}\|u\|_g^{1/2}\|v\|_g|w|_g^{1/2}\|w\|_g^{1/2}, \quad \forall u,v,w \in V_g, \tag{3.8}$$

thus,

$$\begin{aligned}\left|(B(u,u),u_2)_g\right| &\le c|u|_g^{1/2}\|u\|_g^{3/2}|u_2|_g^{1/2}\|u_2\|_g^{1/2} \\ &\le \frac{c}{\lambda_{m+1}}|u|_g^{1/2}\|u\|_g^{3/2}\|u_2\|_g \\ &\le \frac{\nu}{6}\|u_2\|_g^2 + c\rho_0\rho_1^3.\end{aligned} \tag{3.9}$$

Next, using the Cauchy inequality,

$$\begin{aligned}\left|(\nu Ru,u_2)_g\right| &= \left|\left(\frac{\nu}{g}(\nabla g\cdot\nabla)u,u_2\right)_g\right| \\ &\le \frac{\nu}{m_0}\left|\nabla g\right|_\infty\|u\|_g|u_2|_g \\ &\le \frac{\nu}{6}\|u_2\|_g^2 + \frac{3\nu\rho_1^2\left|\nabla g\right|_\infty^2}{2m_0^2\lambda_g\lambda_{m+1}}.\end{aligned} \tag{3.10}$$

Finally, we have

$$\left|(f,u_2)_g\right| \le \left|f\right|_{V_g'}\|u_2\| \le \frac{\nu}{6}\|u_2\|_g^2 + \frac{3}{2\nu}\left|f\right|_{V_g'}^2. \tag{3.11}$$

Putting (3.9)–(3.11) together, there exists constant $M_1 = M_1(m_0,|\nabla g|_\infty,\rho_0,\rho_1)$ such that

$$\frac{1}{2}\frac{d}{dt}|u_2|_g^2 + \frac{1}{2}\nu\|u_2\|_g^2 \le \frac{3\left|f\right|_{V_g'}}{2\nu} + M_1. \tag{3.12}$$

Therefore, we deduce that

$$\frac{d}{dt}|u_2|_g^2 + \nu\lambda_{m+1}|u_2|_g^2 \le 2M_1 + \frac{3}{\nu}\left|f\right|_{V_g'}^2. \tag{3.13}$$

Here, M_1 depends on λ_{m+1}, is not increasing as λ_{m+1} increasing.

By the Gronwall inequality, the above inequality implies that

$$\begin{aligned}|u_2(t)|_g^2 \le{}& |u_2(\tau)|_g^2 e^{-\nu\lambda_{m+1}(t-\tau)} + \frac{2M_1}{\nu\lambda_{m+1}} \\ &+ \frac{3}{\nu}\int_\tau^t e^{-\nu\lambda_{m+1}(t-s)}\left|f\right|_{V'}^2 ds.\end{aligned} \tag{3.14}$$

If we consider the time $t-s$ instead of τ (so that we can use more easily the definition of pullback attractors), we have

$$\frac{3}{\nu}\int_{\tau}^{t} e^{-\nu\lambda_{m+1}(t-\sigma)}\left|f(\sigma)\right|_{V'}^{2}d\sigma = \frac{3}{\nu}\int_{t-s}^{t} e^{-\nu\lambda_{m+1}(t-\sigma)}\left|f(\sigma)\right|_{V'}^{2}d\sigma. \tag{3.15}$$

Applying continuous integral and Lemma II 1.3 in [18] for any ε, there exists $\eta = \eta(\varepsilon) > 0$ such that

$$\int_{t-\eta}^{t}\left|f(\sigma)\right|_{V'}^{2}d\sigma < \frac{\nu\varepsilon}{18}; \tag{3.16}$$

thus, we have

$$\frac{3}{\nu}\int_{t-\eta}^{t} e^{-\nu\lambda_{m+1}(t-\sigma)}\left|f(\sigma)\right|_{V'}^{2}d\sigma \leq \frac{\varepsilon}{6}, \tag{3.17}$$

$$\begin{aligned}
&\frac{3}{\nu}\int_{t-s}^{t-\eta} e^{-\nu\lambda_{m+1}(t-\sigma)}\left|f(\sigma)\right|_{V'}^{2}d\sigma \\
&\quad\leq \frac{3}{\nu}\int_{t-\eta-1}^{t-\eta} e^{-\nu\lambda_{m+1}(t-\sigma)}\left|f(\sigma)\right|_{V'}^{2}d\sigma \\
&\qquad+ \frac{3}{\nu}\int_{t-\eta-2}^{t-\eta-1} e^{-\nu\lambda_{m+1}(t-\sigma)}\left|f(\sigma)\right|_{V'}^{2}d\sigma + \cdots \\
&\quad\leq \frac{3}{\nu}e^{-\nu\lambda_{m+1}\eta}\left(\int_{t-\eta-1}^{t-\eta}\left|f(\sigma)\right|_{V'}^{2}d\sigma + e^{-\nu\lambda_{m+1}}\int_{t-\eta-2}^{t-\eta-1}\left|f(\sigma)\right|_{V'}^{2}d\sigma + \cdots\right) \\
&\quad\leq \frac{3}{\nu}e^{-\nu\lambda_{m+1}\eta}\left(1 + e^{-\nu\lambda_{m+1}} + \cdots\right)\sup_{s\in R}\int_{s-1}^{s}\left|f(\sigma)\right|_{V'}^{2}d\sigma \\
&\quad\leq \frac{(3/\nu)e^{-\nu\lambda_{m+1}\eta}}{1-e^{-\nu\lambda_{m+1}}}\|f\|_{L_b^2}^{2}.
\end{aligned} \tag{3.18}$$

Using (1.11) and letting $s_1 = (1/\nu\lambda_{m+1})\ln(3\rho_0^2/\varepsilon)$, then $s \geq s_1$ implies that

$$\frac{3}{\nu}\int_{t-s}^{t-\eta} e^{-\nu\lambda_{m+1}(t-\sigma)}\left|f(\sigma)\right|_{V'}^{2}d\sigma \leq \frac{(3/\nu)e^{-\nu\lambda_{m+1}\eta}}{1-e^{-\nu\lambda_{m+1}}}\|f\|_{L_b^2(R;V')}^{2} \leq \frac{\varepsilon}{6}, \tag{3.19}$$

$$\begin{gathered}
\frac{2M_1}{\nu\lambda_{m+1}} \leq \frac{\varepsilon}{3}, \\
|u_2(\tau)|_g^2 e^{-\nu\lambda_{m+1}(t-\tau)} \leq \rho_0^2 e^{-\nu\lambda_{m+1}s_1} \leq \frac{\varepsilon}{3}.
\end{gathered} \tag{3.20}$$

Therefore, we deduce from (3.14) that

$$|u_2|_g^2 \leq \varepsilon, \quad \forall s \geq s_1, \tag{3.21}$$

which indicates $\{U(t,\tau)\}$ satisfying pullback Condition (C) in H_g. Applying Theorem 2.5, the proof is complete. □

According to Propositions 3.1-3.2, we can now regard that the families of processes $\{U(t,\tau)\}$ are defined in V_g and $\mathcal{B}_1$ is a pullback absorbing set in V_g.

Theorem 3.4. *If $f(x,t) \in L_b^2(R; H_g)$, then the processes $\{U(t,\tau)\}$ corresponding to problem* (1.16) *possess compact pullback attractor $\mathcal{A}_1(t)$ in V_g:*

$$\mathcal{A}_1(t) = \overline{\bigcup_{\mathcal{B}_1 \in \mathcal{B}} \omega(\mathcal{B}_1, t)}, \tag{3.22}$$

where $\mathcal{B}_1$ is the absorbing set in V_g.

Proof. Using Proposition 3.2, we have that the family of processes $\{U(t,\tau)\}$ corresponding to (1.16) possess the pullback absorbing set in V_g.

Now we testify that the family of processes $\{U(t,\tau)\}$ corresponding to (1.16) satisfies pullback Condition (C).

Multiplying (1.16) by $A_g u_2(t)$, we have

$$\left(\frac{dv}{dt}, A_g u_2\right) + (\nu A_g u, A_g u_2) + \left(B_g(u,u), A_g u_2\right)_g = (f, A_g u_2) - (\nu R u, A_g u_2)_g. \tag{3.23}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|u_2\|_g^2 + \nu\left|A_g u_2\right|_g^2 \leq \left|\left(B_g(u,u), A_g u_2\right)_g\right| + \left|(f, A_g u_2)_g\right| + \left|(\nu R u, A_g u_2)_g\right|. \tag{3.24}$$

To estimate $(B_g(u,u), Au_2)_g$, we recall some inequalities (see [22]), for every $u, v \in D(A_g)$,

$$\left|B_g(u,v)\right| \leq c \begin{cases} |u|_g^{1/2}\|u\|_g^{1/2}\|v\|_g^{1/2}\left|A_g v\right|_g^{1/2}, \\ |u|_g^{1/2}\left|A_g u\right|_g^{1/2}\|v\|_g, \end{cases} \tag{3.25}$$

$$|w|_{L^\infty(\Omega)^2} \leq c\|w\|_g\left(1 + \log\frac{\left|A_g w\right|}{\lambda_g\|w\|_g^2}\right)^{1/2}, \tag{3.26}$$

from which we deduce that

$$\left|B_g(u,v)\right| \leq c|u|_{L^\infty(\Omega)}|\nabla v|_g|u|_g|\nabla v|_{L^\infty(\Omega)}, \tag{3.27}$$

and using (3.26),

$$\left|B_g(u,v)\right| \le c\begin{cases} \|u\|_g\|v\|_g\left(1+\log\dfrac{|A_gu|^2}{\lambda_g\|u\|_g^2}\right)^{1/2}, \\ |u|_g|A_gv|_g\left(1+\log\dfrac{|A_g^{3/2}v|^2}{\lambda_g\|A_gv\|_g^2}\right)^{1/2}. \end{cases} \tag{3.28}$$

Expanding and using Young's inequality, together with the first one of (3.28) and the second one of (3.25), we have

$$\begin{aligned} \left|\left(B_g(u,u),A_gu_2\right)\right| &\le \left|\left(B_g(u_1,u_1+u_2),A_gu_2\right)\right| + \left|\left(B_g(u_2,u_1+u_2),A_gu_2\right)\right| \\ &\le cL^{1/2}\|u_1\|_g\left|A_gu_2\right|_g\left(\|u_1\|_g+\|u_2\|_g\right) + c|u_2|_g^{1/2}\left|A_gu_2\right|_g^{3/2} \\ &\le \frac{\nu}{6}\left|A_gu_2\right|_g^2 + \frac{c}{\nu}\rho_1^4L + \frac{c}{\nu^3}\rho_0^2\rho_1^4, \quad t \ge t_0+1, \end{aligned} \tag{3.29}$$

where we use

$$\left|A_gu_1\right|_g^2 \le \lambda_m\|u_1\|_g^2 \tag{3.30}$$

and set

$$L = 1+\log\frac{\lambda_{m+1}}{\lambda_g}. \tag{3.31}$$

Next, using the Cauchy inequality,

$$\begin{aligned} \left|\left(Ru,A_gu_2\right)_g\right| &= \left|\left(\frac{\nu}{g}(\nabla g\cdot\nabla)u,A_gu_2\right)_g\right| \\ &\le \frac{\nu}{m_0}\left|\nabla g\right|_\infty\|u\|_g\left|A_gu_2\right|_g \\ &\le \frac{\nu}{6}\left|A_gu_2\right|_g^2 + \frac{3\nu}{2}\left|\nabla g\right|_\infty^2\rho_1^2. \end{aligned} \tag{3.32}$$

Finally, we estimate $|(f,A_gu_2)|$ by

$$\begin{aligned} \left|\left(f,A_gu_2\right)\right| &\le \left|f\right|_g\left|A_gu_2\right|_2 \\ &\le \frac{\nu}{6}\left|A_gu_2\right|_g^2 + \frac{3}{2\nu}\left|f\right|_g^2. \end{aligned} \tag{3.33}$$

Putting (3.29)–(3.33) together, there exists a constant M_2 such that

$$\frac{d}{dt}\|u_2\|_g^2 + \nu\lambda_{m+1}\|u_2\|_g^2 \leq \frac{3}{\nu}\left|f\right|_g + M_2. \tag{3.34}$$

Here, $M_2 = M_2(\rho_0, \rho_1, L, \nu, |\nabla g|)$ depends on λ_{m+1}, is not increasing as λ_{m+1} increasing. Therefore, by the Gronwall inequality, the above inequality implies that

$$\|u_2\|_g^2 \leq \|u_2(\tau)\|_g^2 e^{-\nu\lambda_{m+1}(t-\tau)} + \frac{2M_2}{\nu\lambda_{m+1}} + \frac{3}{\nu}\int_\tau^t e^{-\nu\lambda_{m+1}(t-s)}\left|f\right|_g^2 ds. \tag{3.35}$$

We consider the time $t-s$ instead of τ. The following result is similar to (3.17)–(3.19), for any ε:

$$\frac{2c}{\nu}\int_\tau^t e^{-\nu\lambda_{m+1}(t-\sigma)/2}\left|f\right|_g^2 d\sigma \leq \frac{\varepsilon}{3}. \tag{3.36}$$

Using (1.11) and letting $s_2 = (2/\nu\lambda_{m+1})\ln(3\rho_1^2/\varepsilon)$, then $s \geq s_2$ implies that

$$\begin{gathered} \frac{2M_2}{\nu\lambda_{m+1}} \leq \frac{\varepsilon}{3}, \\ \|u_2(\tau)\|_g^2 e^{-\nu\lambda_{m+1}(t-\tau)} \leq \rho_1^2 e^{-\nu\lambda_{m+1}s} < \frac{\varepsilon}{3}. \end{gathered} \tag{3.37}$$

Therefore, we deduce from (3.35) that

$$\|u_2\|_g^2 \leq \varepsilon, \quad \forall s \geq s_1, \tag{3.38}$$

which indicates $\{U(t,\tau)\}$ satisfying pullback Condition (C) in V_g. □

4. The Dimension of the Pullback Attractor

To estimate the dimension of the pullback attractor $\mathcal{A}_0(t)$, we will apply the abstract machinery in [18, 23]. Let $F : V_g \times \mathbb{R} \to V_g'$ be a given family of nonlinear operators such that, for all $\tau \in \mathbb{R}$ and any $u_\tau \in H_g$, there exists a unique function $u(t) = u(t;\tau,u_0)$ satisfying

$$\begin{gathered} u \in L^2(\tau,T;V_g) \cap C[\tau,T;H_g), \quad F(u(t),t) \in L^1\left(\tau,T;V_g'\right), \quad \forall T > \tau, \\ \frac{du}{dt} = F(u(t),t), \quad t > \tau, \\ u(\tau) = u_\tau, \end{gathered} \tag{4.1}$$

where $F(u) = -\nu A_g u - B_g u - \nu R u + P_g f$.

Using the standard methods (see [17, 18]), we can show that $\{U(t,\tau)\}$ is uniformly quasidifferentiable on $\{B(t)\}_{t\in\mathbb{R}}$. Then, for all $\tau \le T$ and any $u_\tau, v_\tau \in H_g$, there exists a unique $v(t) = v(t;\tau,u_\tau,v_\tau)$, which is a solution of

$$v \in L^2(\tau,T;V_g) \cap C[\tau,T;H_g),$$
$$\frac{dv}{dt} = F'(U(t,\tau)u_\tau,t)v, \tag{4.2}$$
$$v(\tau) = v_\tau.$$

For all $\tau < T$, we define the linear operator $L(t,u_\tau) : H_g \to H_g$ by

$$v(t;\tau,u_\tau,v_\tau) = L(t,\tau,u_\tau)v_\tau. \tag{4.3}$$

Theorem 4.1. *Suppose that $f(t)$ satisfies the assumptions of Theorem 3.3. Then, if $\gamma = 1 - (2|\nabla g|_\infty/m_0\lambda_g^{1/2}) > 0$, the Pullback attractor (uniformly in the past) $\mathcal{A}_0$ defined by* (3.4) *satisfies*

$$d_F(\mathcal{A}_0) \le \sqrt{\frac{\beta}{\alpha}}, \tag{4.4}$$

where

$$\alpha = \frac{c_2\nu m_0\lambda_1'\gamma}{2M_0},$$
$$\beta = \frac{c_1 d_1}{2\nu^3 m_0\gamma}\sup_{\substack{\varphi_j\in H_g,|\varphi_j|\le 1\\ j=1,2,\dots,m}}\frac{1}{T}\int_{\tau-T}^{\tau}\|f(s)\|_{V_g'}^2\,ds, \tag{4.5}$$

with the constant c_1, c_2 of (3.29) and (3.32) of Chapter VI in [17], λ_1' is the first eigenvalue of the Stokes operator and $d_1 = |\nabla g|_\infty^2/4m_0 + |\nabla g|_\infty + M_0$.

Proof. With Theorem 3.3 at our disposal we may apply the abstract framework in [17, 18, 23, 24].

For $\xi_1,\xi_2,\dots,\xi_m \in H_g$, let $v_j(t) = L(t,u_\tau)\cdot\xi_j$, where $u_\tau \in H_g$. Let $\{\varphi_j(s);\ j=1,2,\dots,m\}$ be an orthonormal basis for span $\{v_j;\ j = 1,2,\dots,m\}$. Since $v(s;\tau,u_\tau,v_\tau^j) \in V_g$ almost everywhere $s \ge \tau$, we can also assume that $\varphi_j(s) \in V_g$ almost everywhere $s \ge \tau$. Then, similar to the proof process of Theorems 3.3 and 3.4, we may obtain

$$\sum_{i=1}^{m}\langle F'(U(s,\tau)u_\tau,s)\varphi_i,\varphi_i\rangle$$
$$= -\nu\sum_{i=1}^{m}\|\varphi_j\|_g^2 - \sum_{i=1}^{m}b_g(\varphi_j,U(s,\tau)u_\tau,\varphi_j) - \sum_{i=1}^{m}\left(\frac{\nu}{g}(\nabla g\cdot\nabla)\varphi_j,\varphi_j\right)_g, \tag{4.6}$$

almost everywhere $s \geq \tau$. From this equality, and in particular using the Schwarz and Lieb-Thirring inequality (see [17, 18, 23, 24]), one obtains

$$\begin{aligned}
&\sum_{i=1}^{m} \|\varphi\|_g^2 \geq \lambda_1 + \cdots + \lambda_m \geq \frac{m_0}{M_0}\left(\lambda_1' + \cdots + \lambda_m'\right) \geq \frac{m_0}{M_0} c_2 \lambda_1' m^2, \\
Tr_j\left(F'(U(s,\tau)u_\tau, s\right) &\leq -\nu\left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right)\sum_{i=1}^{m}\|\varphi_j\|_g^2 + \|U(s,\tau)u_\tau\|\left(\frac{c_1 d_1}{m_0}\sum_{i=1}^{m}\|\varphi_j\|_g^2\right)^{1/2} \\
&\leq -\frac{\nu}{2}\left(1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right)\sum_{i=1}^{m}\|\varphi_j\|_g^2 + \frac{c_1 d_1}{2\nu m_0}\|U(s,\tau)u_\tau\|_g^2 \\
&\leq -\frac{\nu m_0}{2M_0}\left(1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right) c_2 \lambda_1' m^2 + \frac{c_1 d_1}{2\nu m_0}\|U(s,\tau)u_\tau\|_g^2.
\end{aligned} \tag{4.7}$$

On the other hand, we can deduce that

$$\frac{d}{dt}|U(s,\tau)u_\tau|_g^2 + \nu\|U(s,\tau)u_\tau\|_g^2 \leq \frac{\|f\|_{V_g'}^2}{\nu} + \frac{2\nu}{m_0 \lambda_g^{1/2}}|\nabla g|_\infty \|U(s,\tau)u_\tau\|_g^2 \tag{4.8}$$

for $\lambda_g = 4\pi^2 m_0/M_0$, and then

$$\int_\tau^t \|U(s,\tau)u_\tau\|_g^2 ds \leq \left(\frac{1}{\nu^2}\int_\tau^t \|f(s)\|_{V_g'}^2 ds + \frac{|u_\tau|^2}{\nu}\right)\left(1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_g^{1/2}}\right)^{-1}, \quad t \geq \tau. \tag{4.9}$$

Now we define

$$\begin{aligned}
q_m &= \sup_{\substack{\varphi_j \in H_g, |\varphi_j| \leq 1 \\ j=1,2,\ldots,m}} \left(\frac{1}{T}\int_{\tau-T}^{\tau} Tr_j\left(F'(U(s,\tau)u_\tau, s)ds\right)\right), \\
\widetilde{q}_m &\leq -\frac{\nu m_0}{2M_0}\left(1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right) c_2 \lambda_1' m^2 + \frac{c_1 d_1}{2\nu m_0}\left(\sup_{\substack{\varphi_j \in H_g, |\varphi_j| \leq 1 \\ j=1,2,\ldots,m}}\left(\frac{1}{T}\int_{\tau-T}^{\tau}\|U(s,\tau)u_\tau\|_g^2 ds\right)\right) \\
&\leq -\frac{\nu m_0}{2M_0}\left(1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right) c_2 \lambda_1' m^2 \\
&\quad + \frac{c_1 d_1}{2\nu m_0}\left(\frac{1}{\nu^2}\sup_{\substack{\varphi_j \in H_g, |\varphi_j| \leq 1 \\ j=1,2,\ldots,m}}\frac{1}{T}\int_{\tau-T}^{\tau}\|f(s)\|_{V_g'}^2 ds + \frac{|u_\tau|^2}{\nu T}\right)\left(1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_g^{1/2}}\right)^{-1}), \\
q_m &= \limsup_{T \to \infty} \widetilde{q}_m \leq -\alpha m^2 + \beta.
\end{aligned} \tag{4.10}$$

Hence,

$$\dim_F \mathcal{A}_0(\tau) \leq \sqrt{\frac{\beta}{\alpha}}. \tag{4.11}$$

□

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