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Actuarial transform pricing

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Actuarial Transform Pricing

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Actuarial Transform Pricing

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Actuarial Transform Pricing

Abstract

This article studies four transform pricing methods in the context of general equilibrium (GE) framework. The four methods, viz. the Esscher transform, indifference pricing, the Wang transform, and the standard deviation loading, are popular among actuarial literature and practice. The transform pricing methods offer a convenient solution to contingent claim pricing problem when the underlying risk exposure cannot be fully hedged. We show analytically that these four methods are similar and close to the GE approach if the utility has an exponential function, and the underlying distribution is Normal. When the payoff distribution is non-gaussian, prices produced by the four methods vary widely. Moreover, some transform methods may lead to prices that are not coherent, violating one or more of the following properties; additivity, homogeneity, scale invariance and monotonicity. We discuss the implications of our findings on incomplete market pricing.

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Actuarial Transform Pricing

1 Introduction

Asset prices can be uniquely determined when its payoffs can be replicated by traded securities. This no-arbitrage principle is the cornerstone of well known models such as Black and Scholes (1973) and Cox, Ross and Rubinstein (1979). However, uncertain payoffs often cannot be fully replicated.[1] The result is that there is no unique price but, rather, a range of no-arbitrage prices exist[2] resulting in a situation known as incomplete markets. Typical examples of incomplete markets include catastrophe and mortality bonds, weather derivatives, wage-indexed pension fund liabilities, equity gap risk, and GDP linked bonds. The actuaries have been dealing with such a pricing problem for centuries since insurance contracts came into existence.[3] In this article we compare four well known transform pricing methods popular among actuaries; viz. the Esscher transform, indifference pricing, the Wang transform and the standard deviation loading method that are potential solutions for pricing contingent claims in incomplete markets. In particular, we relate all four methods to the general equilibrium (GE) approach that has a deep foundation in Economics.

All four transform pricing methods have been used widely: the Esscher transform by Gerber and Shiu (1994), the indifference price method by Musiella and Za-

[1] Replication failure could be due to the number of independent securities being smaller than the number of future states, or the fact that the underlying asset and its associated contingent claims are not tradable.

[2] See for instance Merton 1973; Perrakis and Ryan 1984; Perrakis 1986; Bernardo and Ledoit 2000; Bizid and Jouini 2005.

[3] Karl Borch (1963, p.322) claims that "actuarial mathematics and the scientific basis of insurance were developed into a self-contained and fairly complete theory long before economists could claim the name of science for their subject."

riphopoulou (2004), the Wang transform by Wang (2004) and Lin and Cox (2008), and the standard deviation loading method by Roustant, Laurent, Bay and Carraro (2004). We show that under specific conditions, all four methods will give the same price, but these conditions will not always be fulfilled. In a case study in Section 5, we show substantial differences in prices produced by the four methods. We examine some of the properties of these pricing methods against the conditions required for a coherent pricing measure, such as additivity, homogeneity, scale invariance and monotonicity. We find that the four methods will satisfy all these conditions again only in special cases. The potential violation of coherent pricing properties makes it difficult to apply these methods to price, for example, liquidly traded contingent claims. If claims are liquidly traded, market participants expect pricing additivity (i.e. the price of a portfolio of claims is equal to the sum of the prices of the individual claims) even if the cash flows or assets underlying these contingent claims are not market traded. As such, the prices produced by these transform pricing method could present arbitrage opportunities.

The rest of this paper proceeds as follows: Section 2 reviews the four pricing methods and examine their connections with the GE approach. Section 3 considers the special case of normally distributed cash flows. In Section 4, we examine how well each of the four pricing methods adhere to the concept of coherent pricing measures. Section 5 applies the four methods to price Argentina's GDP linked bonds and compare the pricing results. Finally, Section 6 provides some discussion on the conceptual issues related to the four transform pricing methods and concludes.

2 Equilibrium vs. Transform Pricing

Broadly, there are two non-mutually exclusive approaches for pricing contingent claims that involve unhedgeable underlying cash flows. The first approach is the general equilibrium approach where prices are derived based on the preference of a single representative agent given the joint distributions of the state variables and underlying cash flows. The second approach is to specify the set of properties that a coherent pricing measure must satisfy and choose the transform pricing methods that are consistent with these properties. All transform pricing methods are based on some heuristic adjustments of cash flow to reflect risk bearing. In this section, we provide an overview of these two main pricing approaches. We will show that if utility function is exponential and cash flows are normally distributed, not only that the two main approaches give very similar results, the pricing formulae produced by the four transform methods are also very close to each other. Unfortunately, the exponential utility assumption has many criticisms, and in reality asset cash flows and insurance liability typically have heavy tails. This is when the different approaches and the different transform methods start to depart and produce markedly different prices as will be illustrated in our case study later in Section 5.

2.1 The general equilibrium approach

The general equilibrium (GE) approach to pricing contingent claim has a long history (see e.g. Samuelson and Merton 1969; Rubinstein 1976; and Brennan 1979). In this framework, the risk averse representative agent seeks to maximise expected utility of terminal wealth. At equilibrium, the forward price of uncertain payoff x_t can be expressed as

$$F_{GE}(x_t) = \frac{E^{\mathbb{P}}\left[U'(W_t)x_t\right]}{E^{\mathbb{P}}\left[U'(W_t)\right]} = E^{\mathbb{P}}\left[\phi_{GE}(W_t)x_t\right], \tag{1}$$

where W_t is the aggregate wealth at time t and $U'(\cdot)$ is the representative agent's marginal utility function. The superscript $\mathbb{P}$ of $E(\cdot)$ means the expectation is taken with respect to the physical probability. The relative marginal utility of wealth of the representative investor, written as

$$\phi_{GE}(W_t) = \frac{U'(W_t)}{E^{\mathbb{P}}\left[U'(W_t)\right]}, \tag{2}$$

is called the (forward) pricing kernel, and is key to GE pricing theory. Equation (1), which requires the solution of double integrals, can be further simplified by taking conditional expectations yielding

$$F_{GE}(x_t) = E^{\mathbb{P}}\left[\psi_{GE}(x_t)x_t\right], \tag{3}$$

where

$$\psi_{GE}(x_t) = E^{\mathbb{P}}\left[\phi_{GE}(W_t) \mid x_t\right]. \tag{4}$$

The expectation in (1) is calculated using the joint probability distribution of W_t and x_t, whereas the expectation in (4) is calculated using the probability distribution of x_t and the conditional distribution of W_t given x_t. The conditional pricing kernel $\psi_{GE}(x_t)$ is termed the *asset specific pricing kernel* by Camara (2003), the *conditional expected relative marginal utility function* by Brennan (1979) or the *stochastic discount factor* by Cochrane (2001). Given the functional form of $U'(\cdot)$ and the distributions of W_t and x_t, $\psi_{GE}(x_t)$ can be deduced and used for pricing all claims whose payoffs are functions of x_t. If the solution for $\psi_{GE}(x_t)$ is unique, the market is said to be complete.

The classical results under the GE approach were presented by Rubinstein (1976) and Brennan (1979). Assuming that aggregate consumption and the underlying asset are bivariate lognormally distributed, Rubinstein (1976) obtains the Black-Scholes model with constant proportional risk aversion preferences. Brennan (1979) derives preference free pricing formulae by assuming a representative agent who has a negative exponential utility function and a bivariate normal distribution for aggregate wealth and the underlying asset. These results were generalised by Camara (2003), who shows that these pricing relationships also hold under more general conditions; namely, transformed normal distributions for the underlying asset and wealth (which need not belong to the same family of distributions) and exponential marginal utility. Later, Camara (2005) extended these results further to allow for multivariate state variables and payoffs of several assets, while Vitiello and Poon (2008 and forthcoming) provides solutions for payoffs that have a mixture of two g distributions and transformed gamma distribution. While the GE framework could lead to a single pricing kernel and a unique price, there is no consensus on the treatment of utility function (see e.g. Friedman and Savage 1948; Rubinstein 1976; Epstein and Zin 1989; and Abel 1990). Also the GE approach is the most effective if the joint distrubutions of the state variables and uncertain payoffs have well defined functional forms. This will facilitate the conditional expectation in (4). Otherwise, there is no straightforward way of calculating contingent claim prices since the state variables are usually unobservable. Lately, Vitiello and Poon (2010) extend the distributions, in which solutions exist, to a mixture of N transformed normal distributions significantly widen the shapes of distributions that uncertain payoffs may have. So the GE approach is indeed very flexible. The fact that the theory is firmly based on preference makes the GE approach very attractive.

2.2 Properties of exponential utility

The exponential utility plays an important role in most of the transform pricing methods and is an important link between the GE approach and the transform pricing approach. Specifically, an exponential utility function has the form

$$U(W_t) = 1 - e^{-\eta W_t}, \tag{5}$$

where η is the risk aversion parameter. It follows that

$$U'(W_t) = \eta e^{-\eta W_t} > 0$$

and

$$U''(W_t) = -\eta^2 e^{-\eta W_t} < 0.$$

The coefficient of absolute risk aversion, given by

$$\alpha(W_t) = -\frac{U''(W_t)}{U'(W_t)} = \frac{\eta^2 e^{-\eta W_t}}{\eta e^{-\eta W_t}} = \eta$$

is constant, i.e. under exponential utility the representative investor does not change the amount invested in risky assets as wealth increases.

The coefficient of relative risk aversion is given by

$$r(W_t) = \alpha(W_t)W_t = \eta W_t.$$

That is the exponential utility function is characterised by increasing relative risk aversion.

Exponential utility has been criticised on several grounds. The implication of increasing relative risk aversion is that investors will put the same amount of money in risky assets and hence the ratio invested in risky asset gets smaller as investors get wealthier. Empirical studies, however, suggest that relative risk aversion among individuals with different levels of wealth is roughly constant (e.g. Friend and Blume 1975). Another shortcoming is that exponential utility implies that optimal consumption is linear in wealth, but not proportional to wealth, meaning that the consumption to wealth ratio is not stationary, which is inconsistent with a balanced growth path (Merton 1992).

Nevertheless, given the exponential utility function in (5), we can express the pricing kernel in equation (2) as

$$\phi_{GE,Exp}(W_t) = \frac{e^{-\eta W_t}}{E^{\mathbb{P}}\left[e^{-\eta W_t}\right]}, \tag{6}$$

and the forward price in equation (1) becomes

$$F_{GE,Exp}(x_t) = \frac{E^{\mathbb{P}}\left[x_t e^{-\eta W_t}\right]}{E^{\mathbb{P}}\left[e^{-\eta W_t}\right]}. \tag{7}$$

2.3 Transform Pricing methods

2.3.1 Esscher transform

The Esscher transform has been studied in the actuarial literature for a long time and its most heavily cited reference is Gerber and Shiu (1994). If X is an uncertain insurable loss, the premium of an insurance written on X is

$$H[X] = \frac{E^{\mathbb{P}}\left[X e^{\eta Z}\right]}{E^{\mathbb{P}}\left[e^{\eta Z}\right]}, \tag{8}$$

where Z is the aggregate risk of the exchange market. Note the sign change in (8) compared with equation (7) as we are now dealing with insurable losses and negative cash flows. If we apply (8) on positive cash flows x_t, we have

$$F_{Ess,W}(x_t) = \frac{E^{\mathbb{P}}\left[x_t e^{-\eta W_t}\right]}{E^{\mathbb{P}}\left[e^{-\eta W_t}\right]} = E^{\mathbb{P}}\left[x_t \psi_{Ess,W}(x_t)\right] \tag{9}$$

which is exactly the same as the GE pricing result in (7) with Z now being replaced by the aggregate wealth of the economy.

As equation (8) involves two random variables, X and Z, Bühlmann (1980) shows, provided that X and $(Z - X)$ are independent and that utility is exponential, equation (8) can be rewritten as

$$H[X] = \frac{E^{\mathbb{P}}\left[X e^{\eta X}\right]}{E^{\mathbb{P}}\left[e^{\eta X}\right]}, \tag{10}$$

which Bühlmann calls the Esscher principle due to its resemblance to the original Esscher transform. Following the same principle, and provided that x_t and $(W_t - x_t)$ are independent, equation (9) can also be rewritten as

$$F_{Ess,x}(x_t) = E^{\mathbb{P}}\left[x_t \frac{e^{-\eta x_t}}{E^{\mathbb{P}}\left[e^{-\eta x_t}\right]}\right] = E^{\mathbb{P}}\left[x_t \psi_{Ess,x}(x_t)\right]. \tag{11}$$

While (11) has the same functional form as (3), their conceptual difference could not be more further apart. First, $\psi_{GE}(x_t)$ in (3) is a conditional expectation of W_t given x_t, whereas the $\psi_{Ess,x}(x_t)$ in (11) is fully defined given η and the distribution of x_t, and assuming that utility is exponential; information on W_t is not needed in (11). Second, equation (11) is built on the basis that $(W - x)$ and x are independent. But if $(W - x)$ and x are independent, then $\psi_{GE}(x_t) = 1$ for all x and $F_{GE}(x_t) = E^{\mathbb{Q}}[x_t] = E^{\mathbb{P}}[x_t \psi_{GE}(x_t)] = E^{\mathbb{P}}[x_t]$, that is x_t would be treated as if it is risk free in equation (3).

Following Landsman (2004), if x_t and W_t have a finite variance-covariance structure, equation (9) can be expressed as

$$F_{Ess,W}(x_t) = E^{\mathbb{P}}(x_t) - \eta Cov^{\mathbb{P}}(x_t, W_t) + o(\eta).$$

Hence, the Esscher transform is asymptotically equivalent to a covariance pricing principle. If wealth and cash flow are joint normally distributed, this asymptotic relationships holds exactly. However, if the payoff distribution deviates from normal by having, for example, heavy tail marginals, omission of the higher order terms will seriously underprice the cash flows (or insurable losses).

2.3.2 Indifference pricing

The indifference forward price $F(x_t)$ solves the equation

$$U(W_t) = E^{\mathbb{P}}[U(W_t + x_t - F(x_t))].$$

It is a price at which an investor with wealth W_t and utility function $U(\cdot)$ is indifferent to doing nothing and buying an asset in the forward market (at $F(x_t)$) with a simultaneous sale in the spot market (at x_t). Musiela and Zariphopolou (2004) derive the indifference price for the case of exponential utility. They show that if the value of the payoff is independent of the value of the traded assets under the physical measure $\mathbb{P}$ then the forward price of claims on the non-traded asset can be written as

$$F_{IP}(x_t) = \frac{1}{-\eta} \log E^{\mathbb{P}} \left[e^{-\eta x_t} \right]. \tag{12}$$

This expression is the same as the actuarial exponential premium principle. Defining

$$J(x_t) = \frac{E^{\mathbb{P}}[\exp(-\eta x_t)]}{\exp(E^{\mathbb{P}}[-\eta x_t])} = E^{\mathbb{P}} \left[e^{\eta(E[x_t]-x_t)} \right].$$

We can then write (12) as

$$F_{IP}(x_t) = E^{\mathbb{P}}[x_t] - \frac{\log J(x_t)}{\eta},$$

i.e. the forward price is the mean value of x_t minus an exponential premium. Hence, $F_{IP}(x_t) \leq E^{\mathbb{P}}(x_t)$ and that as $\eta \to 0$, $F_{IP}(x_t) \to E^{\mathbb{P}}(x_t)$. Furthermore, noting that $K(h) = \log E^{\mathbb{P}} \left[e^{hx_t} \right]$ is the cumulant generating function (cgf), we can re-write equation (12) by using the Taylor series expansion of the cgf

$$F_{IP}(x_t) = \frac{K(-\eta)}{-\eta} = \frac{1}{-\eta} \left(-\eta E^{\mathbb{P}}[x_t] + \frac{(-\eta)^2}{2} Var^{\mathbb{P}}(x_t) + ... \right),$$

which shows again that indifference pricing with exponential utility also resembles variance loading up to a second order.

2.3.3 Wang transform

Suppose that the uncertain payoff x_t has a physical probability density function $f(x_t)$ and a cumulative distribution function (cdf) $\mathcal{F}(x_t)$. The Wang transform is applied directly to the cumulative probability distribution $\mathcal{F}(x_t)$, resulting in a new cumulative distribution function $\mathcal{F}^*(x_t)$

$$\mathcal{F}^*(x_t) = \Phi \left[\Phi^{-1} \left[\mathcal{F}(x_t) \right] + \lambda \right], \tag{13}$$

where Φ is a standard normal distribution and λ is the market price of risk (Wang 2002). Wang (2006) shows that for a continuous distribution the asset specific pricing kernel can be written as

$$\psi_{Wang}(x_t) = \frac{f^*(x_t)}{f(x_t)} = ce^{-\lambda V}, \tag{14}$$

where $V = \Phi^{-1}[\mathcal{F}(x_t)]$ and $c = e^{-0.5\lambda^2}$. As Wang transform assumes comonotonicity between risks, it can be related to Yaari's (1987) dual theory of risk (Wang 1996).

2.3.4 Standard Deviation Loading

Actuarial literature often uses standard deviation loading (Sdl) to calculate premia for insurance risks. This idea is related to the concept of market price of risk in finance. Given a (normally distributed) risky cash flow, x_t, standard deviation loading produces the forward price as

$$F_{Sdl}(x_t) = E^{\mathbb{P}}[x_t] - \beta\sigma_x^{\mathbb{P}}. \tag{15}$$

Assume that $t = 1$ (i.e. the payoff x_t occurs one period from now) and let x_0 be the spot price of x_t. Assuming that the forward price can be hedged or replicated using the spot with no cost of carrying other than the risk free rate of interest r_f, then we can write $F(x_t) = x_0(1 + r_f)$. Moreover, it is possible to express the random return of x_t as $R_t = (x_t/x_0) - 1$. Using these definitions, equation (15) becomes

$$\begin{aligned} x_0(1 + r_f) &= E^{\mathbb{P}}[x_0(1 + R_t)] - \beta\sqrt{Var^{\mathbb{P}}[x_0(1 + R_t)]} \\ r_f &= E^{\mathbb{P}}[R_t] - \beta\sigma_R^{\mathbb{P}} \\ \beta &= \frac{E^{\mathbb{P}}[R_t] - r_f}{\sigma_R^{\mathbb{P}}}. \end{aligned}$$

This risk load, β, representing the excess return per unit of standard deviation, is a familiar expression for the market price of risk in the finance literature. Standard deviation loading was used in the pricing of weather derivatives (e.g. Roustantet al. 2004).[4] If x_t is normally distributed then the Wang transform is the same as

[4] A more sophisticated version of standard deviation loading has been proposed by Schweizer (2001), who uses an indifference argument to compute the expected value under the variance

standard deviation loading.[5]

3 Normally distributed cash flows

For the special case of normally distributed cash flows, we can express equation (1) as

$$F_{GE}(x_t) = E^{\mathbb{P}}[x_t] + cov^{\mathbb{P}}[x_t, \phi(x_t)]$$

given $E^{\mathbb{P}}[\phi(W_t)] = 1$. From Stein's lemma for normally distributed x_t and W_t, we can write $cov[x_t, \phi(W_t)] = E^{\mathbb{P}}[\phi'(W_t)]\, cov^{\mathbb{P}}(x_t, W_t)$. Hence,

$$F_{GE}(x_t) = E^{\mathbb{P}}[x_t] + E^{\mathbb{P}}[\phi'(W_t)]\, cov^{\mathbb{P}}(x_t, W_t). \tag{16}$$

This fundamental equation will be used to derive the prices corresponding to the four transform pricing methods.

From the pricing kernel in equation (6), we obtain, under exponential utility,

$$E^{\mathbb{P}}\left[\phi'_{GE,Exp}(W_t)\right] = -\eta.$$

So equation (16) can be rewritten as

$$F_{GE,Exp}(x_t) = E^{\mathbb{P}}[x_t] - \eta cov^{\mathbb{P}}(x_t, W_t). \tag{17}$$

We show in the following subsections that when x_t (and W_t) is normally distributed and when utility has an exponential form, the Esscher transform, Wang

optimal martingale measure. This scheme assigns an expected value of insurer's utility to each outcome of insurer's final wealth, while the loading factor is a function of the variance of the unhedgeable risk.

[5] Young (1999) shows that this is true in the more general case of location-scale family Ω, a family of univariate probability distributions parametrised by a location parameter μ and a scale parameter $\sigma \geq 0$. If F is the cdf of a member of Ω, then $G(x) = F(\mu + \sigma x)$ is also a cdf of a member of Ω.

transform and standard deviation loading approach are closely related to the GE approach and produce similar pricing results. The indifference price, on the other hand, is larger by $\frac{1}{2}\eta\sigma^2$.

3.1 The Esscher transform

In the special case where x_t and $W_t - x_t$ are independent, so that the asset specific pricing kernel has the form of (11) and x_t has a Normal (μ, σ^2) distribution, the Esscher transform gives another normal distribution with $\mu^* = \mu - \eta\sigma^2$ and $\sigma^* = \sigma$. From (11), we see that

$$E^{\mathbb{P}}\left[\psi'_{Ess,x}(x_t)\right] = -\eta$$

and equation (17) can be written as

$$F_{Ess,x}(x_t) = E^{\mathbb{P}}[x_t] - \eta\sigma^2. \tag{18}$$

3.2 The Wang transform

The Wang transform also results in another normal distribution with $\mu^* = \mu - \lambda\sigma$ and $\sigma^* = \sigma$. From (14), when x_t has a Normal (μ, σ^2) distribution, the asset specific pricing kernel takes the form

$$\psi_{Wang}(x_t) = c\exp\left[-\lambda\Phi^{-1}(\Phi_{\mu,\sigma^2})\right] = ce^{-\lambda(x-\mu)/\sigma},$$

where Φ_{μ,σ^2} is the cumulative density function of a normally distributed variable with mean μ and variance σ^2. Using this result,

$$E^{\mathbb{P}}\left[\psi'_{Wang}(x_t)\right] = -\frac{\lambda}{\sigma}e^{\lambda\mu/\sigma-0.5\lambda^2}e^{-(\lambda\mu/\sigma-0.5\lambda^2)} = -\frac{\lambda}{\sigma},$$

and hence the forward price is given by

$$F_{Wang}(x_t) = \mu + \left[-\frac{\lambda}{\sigma}\right]\sigma^2 = \mu - \lambda\sigma.$$

As mentioned in Section 2.3.4, the Wang transform has the same functional form as the standard deviation loading when the cash flows are normally distributed. Furthermore, Wang (2003) shows that Wang transform can be derived from Bühlmann's (1980) equilibrium pricing model if W_t has a normal distribution and $x_t = \mathcal{F}^{-1}\left[\Phi(V)\right]$, where $\{V, W_t\}$ have a bivariate normal distribution with correlation coefficient ρ.

3.3 Indifference price with exponential utility

In the case of indifference price with exponential utility, for a normally distributed $x_t \sim N(\mu, \sigma)$ we have

$$F_{IP}(x_t) = \frac{1}{-\eta}\log\left(e^{-\eta\mu+0.5\eta^2\sigma^2}\right) = \mu - \frac{1}{2}\eta\sigma^2. \tag{19}$$

Under exponential utility and a normally distributed cash flow x_t, $F_{IP}(x_t)$ in (19) will be greater than $F_{Ess}(x_t)$ in (18) if the parameter value η is the same for both. Specifically

$$F_{IP}(x_t) - F_{Ess}(x_t) = \frac{1}{2}\eta\sigma^2.$$

4 Coherent Pricing

There are several properties that are desirable for a pricing measure. These are related to the idea of coherent measures, discussed in detail in Artzner et al. (1999).[6] A coherent measure satisfies the properties of monotonicity, (sub-)additivity, scale invariance (homogeneity) and translation invariance. We examine these properties below and consider the degree to which the four pricing methods are consistent with

[6] Strictly Artzner et al. (1999) mainly concern with risks and measures for risk. Since price is a direct function of risk, we conjecture that all the conditions required for a coherent risk measure would carry forward to conditions required for a coherent pricing measure.

these properties. The prices produced by the GE approach will always satisfy the conditions for coherent measure.

4.1 Properties of a coherent pricing measure

4.1.1 Risk Loading

Due to risk aversion, prices of risky assets are rarely equal to the expected value of the underlying assets under the physical measure $\mathbb{P}$. The risk-neutral probability distribution $\mathbb{Q}$ changes the original probability distribution and gives more weight to unfavourable events for a risk averse investor. One desirable property of a pricing method is, therefore, *risk loading.* If x is a risky cash flow, whose covariance with wealth is positive $cov(x, W) > 0$, then[7]

$$F(x) = E^{\mathbb{Q}}[x] \leq E^{\mathbb{P}}[x].$$

This means that $E^{\mathbb{P}}[\phi'(W_t)]$ in equation (16) is negative. All four pricing methods considered here satisfy this property.

Further, if we know that if x is equal to a constant c, then

$$F(x) = c.$$

That is, when there is no uncertainty in the cash flow, there should not be a risk load. This is also satisfied by all four pricing methods.

4.1.2 Monotonicity

A coherent risk measure should also be *monotonic.* That is, provided covariance with wealth is positive for the appropriate cases, if the risky cash flows

$$x_i < y_i$$

[7] If covariance with wealth is negative, e.g. x is an insurable loss, then risk loading requires that $E^{\mathbb{P}}[\phi'(W_t)] > 0$ and hence $F(x) \geq E^{\mathbb{P}}[x]$.

for all possible states of the world i, then their forward prices must satisfy

$$F(x) < F(y).$$

Of the pricing principles that we considered, this is always satisfied by the indifference pricing method with exponential utility and the Wang transform. Wang (2003) shows that it can be violated for the Esscher transform unless different η is used for different asset.[8]

4.1.3 Translation and Scale Invariance

Translation invariance means that for any constant c

$$F(x+c) = F(x) + c.$$

Scale invariance means that

$$F(cx) = cF(x).$$

A stronger form of scale invariance, also justified by no arbitrage, is *additivity* or *linear pricing*, which states that for two sets of cash flows x and y,

$$F(x+y) = F(x) + F(y).$$

That is investors cannot make instantaneous profits by repackaging portfolio.

It is desirable for translation invariance, scale invariance and additivity to hold at least in the case of liquidly traded contingent claims. As noted in Cont and Tankov (2004, p.330) "nonlinear pricing may be acceptable for over the counter (OTC) structured products but for "vanilla" instruments, linear pricing is implicitly assumed by market participants."

[8]See Wamg (2003) example 3.1 on page 63.

4.2 Coherent Transform Method

As we will show below, the Esscher transform in (9) appears to be the only transform method that passed all the coherency tests. Unfortunately, this version of Esscher transform relies on an unobservable quantity W_t which is hard to measure in practice.

4.2.1 Esscher Transform

The Esscher transform in (9) satisfies translation invariance, scale invariance and additivity as shown below:

$$\begin{aligned} F_{Ess,W}(x+y) &= \frac{E^{\mathbb{P}}\left[(x+y)e^{-\eta W}\right]}{E^{\mathbb{P}}\left[e^{-\eta W}\right]} \\ &= \frac{E^{\mathbb{P}}\left[xe^{-\eta W}\right]}{E^{\mathbb{P}}\left[e^{-\eta W}\right]} + \frac{E^{\mathbb{P}}\left[ye^{-\eta W}\right]}{E^{\mathbb{P}}\left[e^{-\eta W}\right]} \\ &= F_{Ess}(x) + F_{Ess}(y) \end{aligned}$$

and

$$F_{Ess,W}(cx) = \frac{E^{\mathbb{P}}\left[cxe^{-\eta W}\right]}{E^{\mathbb{P}}\left[e^{-\eta W}\right]} = c\frac{E^{\mathbb{P}}\left[xe^{-\eta W}\right]}{E^{\mathbb{P}}\left[e^{-\eta W}\right]} = cF_{Ess}(x).$$

4.2.2 Indifference Pricing

The indifference price method with exponential utility does not satisfy scale invariance, while additivity is satisfied only if x and y are independent. It does, however, satisfy translation invariance. Using (12), we have

$$F_{IP}(cx) = \frac{1}{-\eta}\log E^{\mathbb{P}}\left[e^{-\eta cx}\right] \neq cF_{IP}(x).$$

If $Cov\left[e^{-\eta x}e^{-\eta y}\right] = 0$, then

$$F_{IP}(x+y) = \frac{1}{-\eta}\log E^{\mathbb{P}}\left[e^{-\eta x}e^{-\eta y}\right] = F_{IP}(x) + F_{IP}(y),$$

and

$$F_{IP}(x+c) = \frac{1}{-\eta}\log E^{\mathbb{P}}\left[e^{-\eta x}e^{c}\right] = F_{IP}(x) + c.$$

4.2.3 Wang transform

The Wang transform satisfies translation and scale invariance, but does not always satisfy additivity. For instance, as shown by Wang, Young and Panjer (1997), the Wang transform is additive for only comonotone risks.[9]

The Wang transform, with its student-t extension, is popular in the pricing of catastrophe and mortality linked claims. However, it is not always convenient to use it for pricing liquidly traded claims, since it may not satisfy additivity.

4.2.4 Standard Deviation Loading

The standard deviation loading satisfies translation and scale invariance, but does not always satisfy additivity. Also with standard deviation loading,

$$F_{Sdl}(x+y) = E^{\mathbb{P}}(x) + E^{\mathbb{P}}(y) - \beta\sqrt{Var^{\mathbb{P}}(x) + Var^{\mathbb{P}}(y) + 2cov^{\mathbb{P}}(x,y)}.$$

It is clear that additivity will only be satisfied in the special case with $|\rho| = 1$, where $cov^{\mathbb{P}}(x,y) = \sigma_x^{\mathbb{P}}\sigma_y^{\mathbb{P}}$. In general for $|\rho| < 1$,

$$\left|cov^{\mathbb{P}}(x,y)\right| \leq \sigma_x^{\mathbb{P}}\sigma_y^{\mathbb{P}}. \tag{20}$$

5 Example: GDP linked bonds

This section presents a cash study where the four transform methods studied above are used to price the Argentina's GDP (Gross Domestic Product) warrants. The results are compared with the market prices of these warrants. The GDP linked securities are derivative instruments where the cash flows—coupon, principal or both—are tied to the country's GDP. As GDP is not traded, these securities cannot

[9] Two risks x and y are comonotone if $(x_i - x_j)(y_i - y_j) \geq 0$ for all scenarios i and j. The condition in equation (20) is much stronger than this. In the Gaussian case, equation (20) implies $\rho = 1$, but here it is only required that $\rho > 0$.

be priced through replication arguments, and we need an approach to price them in relation to other instruments whose market prices are known, such as vanilla bonds of the same sovereign.

5.1 Outline of the problem

Currently, Argentina has GDP linked warrants trading alongside its vanilla external debt. Therefore, we can use the vanilla bonds to calibrate the pricing parameters η_{Ess}, η_{IP}, λ_{Wang} and β_{Sdl} and use them to price the GDP warrants using formulas (11), (12), (13) and (15). For this purpose, we use a US dollar denominated vanilla bond, paying a coupon of 8.28 percent, maturing on June 30, 2033. Besides the market price of vanilla debt, we also need a model of the distribution of the vanilla and GDP-linked cash flows under the physical measure $\mathbb{P}$. For this we use the structural model outlined in Ruban, Poon and Vonatsos (2008), which is based on the dynamics of actual and potential GDP and the real exchange rate.[10] Both the vanilla bond and GDP warrant can be interpreted as contingent claims on Argentina's sovereign assets and GDP. Since both debt instruments are subject to the same underlying risk, it is reasonable to assume that investor risk aversion or the market price of risk for both securities would be the same. The plan is to use parameter values calibrated to the vanilla bond to price the GDP warrants.[11]

Both the vanilla bond and the GDP warrant have more than 20 years maturity. Equations (9), (12), (13) and (15) can give forward prices for the cash flows that occur at each time period (i.e. 1 year from now, 2 years from now, etc). To arrive at the spot prices for these instruments, the forward prices $F(x_t)$ for cash flows at different time periods t need to be discounted to the present time and added

[10] Please refer to Ruban et al. for a detailed description of the calibration of the model to Argentinean data, as well as the terms of Argentina's GDP warrants.

[11] Indeed, until 2005, the vanilla bond and the GDP warrant were traded as a GDP linked bond. They are decoupled in 2005 and have been traded separately since.

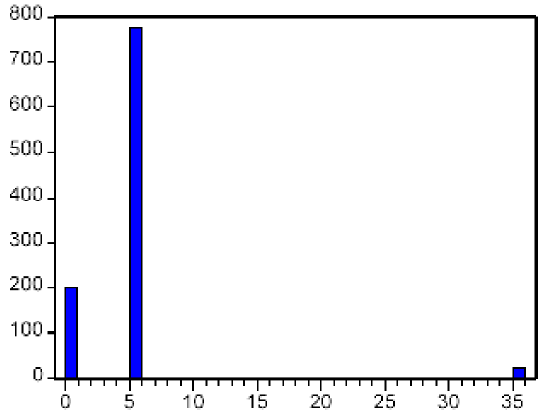

Figure 1: Vanilla bond cashflow disribution in a representative period

together

$$P = \sum_{t=1}^{T} e^{-rt} F(x_t), \tag{21}$$

where P is the spot price of the instruments (i.e. the vanilla bond or the GDP warrant) and r is the risk free rate. Following Ruban, Poon and Vonatsos (2008), we assume that the term structure is flat.

The physical cash flow distributions x_t of both securities are shown in Figures 1 and 2, for a representative period in the middle of the vanilla bond's and warrant's life. The vanilla bond has three spikes in Figure 1 corresponding to three possible cash flows: in 20 percent of the cases it will pay nothing (i.e. this will be the case if the bond has already defaulted before the current time period), in 78 percent of of the cases it will pay the coupon amount (i.e. this will be the case if there has been no default up to and including this time period) and in 2 percent of the cases it will pay the recovery amount (i.e. this happens if the bond defaults in this time period). There is a wider range of possible cash flows for the GDP warrant as shown in Figure 2. The payoff for each state is contingent on the terms of the warrant, the evolution of potential GDP, the real exchange rate and the output gap. The

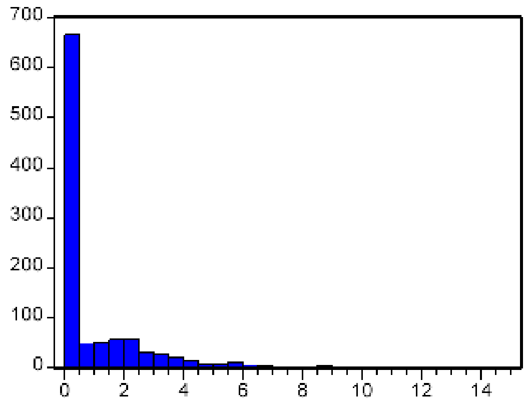

Figure 2: GDP warrant cashflow distribution in a representative period

simulated cash flows show that, the GDP warrant pays nothing in approximately 66 percent of the time, where the sovereign has either formally defaulted on the warrants, or the conditions for the GDP linked payments are not met. However, when the payment conditions are met, there is also a range of payment patterns contingent on the simulated values of actual GDP.

5.2 Calibration

To apply Esscher transform in equation (9), we need to have information on aggregate wealth W_t, which is an unobservable variable. Here, we assume that an orthogonal subspace S_t within W_t that captures all the risk characteristics of the cash flows x_t. With the assumption that S_t and $(W_t - S_t)$ are independent, we may write

$$\begin{aligned} F_{Ess}(x_t) &= \frac{E^{\mathbb{P}}\left[x_t e^{-\eta S_t} e^{-\eta(W_t - S_t)}\right]}{E^{\mathbb{P}}\left[e^{-\eta S_t} e^{-\eta(W_t - S_t)}\right]} \\ &= \frac{E^{\mathbb{P}}\left[x_t e^{-\eta S_t}\right] E^{\mathbb{P}}\left[e^{-\eta(W_t - S_t)}\right]}{E^{\mathbb{P}}\left[e^{-\eta S_t}\right] E^{\mathbb{P}}\left[e^{-\eta(W_t - S_t)}\right]} \\ &= \frac{E^{\mathbb{P}}\left[x_t e^{-\eta S_t}\right]}{E^{\mathbb{P}}\left[e^{-\eta S_t}\right]} \end{aligned} \quad (22)$$

where

$$S_t = x_{1t} + x_{2t} + \cdots + x_{nt}$$

is the sum of all relevant risky cash flows whose contingent claims pricing must pass the linear pricing rule. Since the GDP linked bond was subsequently split into a plain vanilla bond and a GDP warrant, S_t is the sum of the cash flows from both the vanilla bond and the GDP warrant.

For the representative period cash flow above that occurs in the middle of the warrant's life, Table 1 illustrates the parameter values and the resulting forward prices of this warrant cash flow. Selecting $\eta_{Ess} = 0.21$, $\eta_{IP} = 0.42$, $\lambda_{Wang} = 0.77$ and $\beta_{Std} = 0.43$ all give the vanilla cash flow a forward price $F_{vanilla}$ of \$3.12. The forward prices $F_{warrant}$ of the warrant cash flow, however, vary substantially. The warrant prices produced using utility based approaches (i.e. Esscher transform and indifference price) are both higher and close to each other, while the prices produced by Wang transform and standard deviation loading are also close to each other but considerably much lower.

Table 1: Representative period cash flow valuation results

	Parameter value	Vanilla price	Warrant price
Indifference pricing, η_{IP}	0.42	3.12	0.53
Esscher transform, η_{Ess}	0.21	3.12	0.51
Wang transform, λ_{Wang}	0.77	3.12	0.24
Standard deviation loading, β_{Sdl}	0.43	3.12	0.19

5.3 Pricing results

The calibration exercise in Section 5.2 uses the cash flow from one representative period only. To calibrate to market price of vanilla bond, we need to sum up cash flows for every period till maturity as in equation (21). Then, the whole

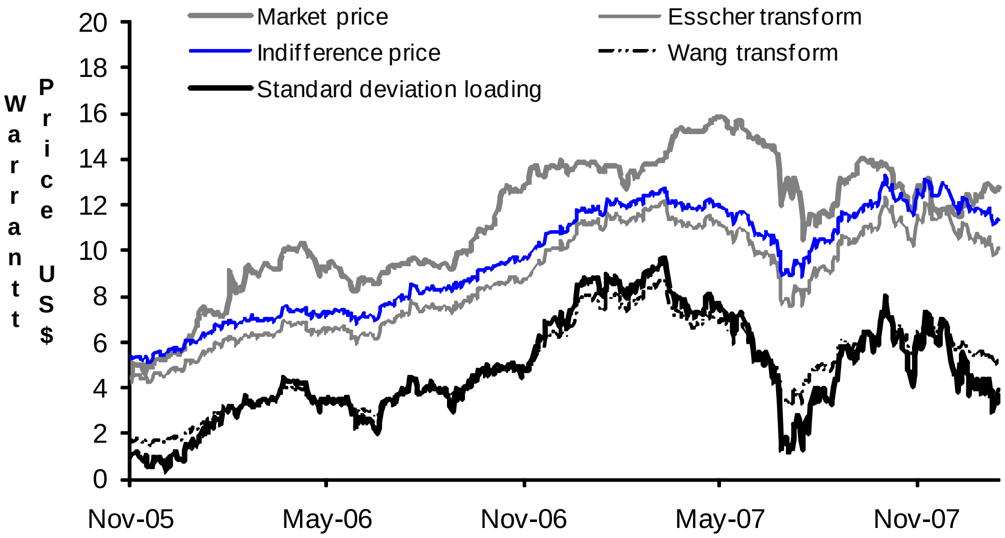

Figure 3: Market and model prices of Argentina's GDP warrants under different transform pricing methods

process is repeated for each day in the sample period from 30 November 2005 to 14 February 2008. Figure 3 illustrates the model calibrated prices and market observed prices of Argentina's GDP warrant over the entire sample period. As before, the prices given by Esscher transform price and utility indifference price are closer to the observed market price, and both of them are consistently higher than the prices given by Wang transform and standard deviation loading. We noted in Section 4.2 that Esscher transform is additive but not indifference pricing. Since Argentina's vanilla bond and GDP warrants are liquidly traded, the results suggest that the Esscher transform in (22) is preferred to all the other transform methods for pricing the GDP warrants. Figure 4 shows the evolution of parameter values of the four pricing methods. As noted Section 3.3, the η_{IP} parameter from the indifference price approach is approximately twice as high as the η_{Ess} parameter from the Esscher transform.

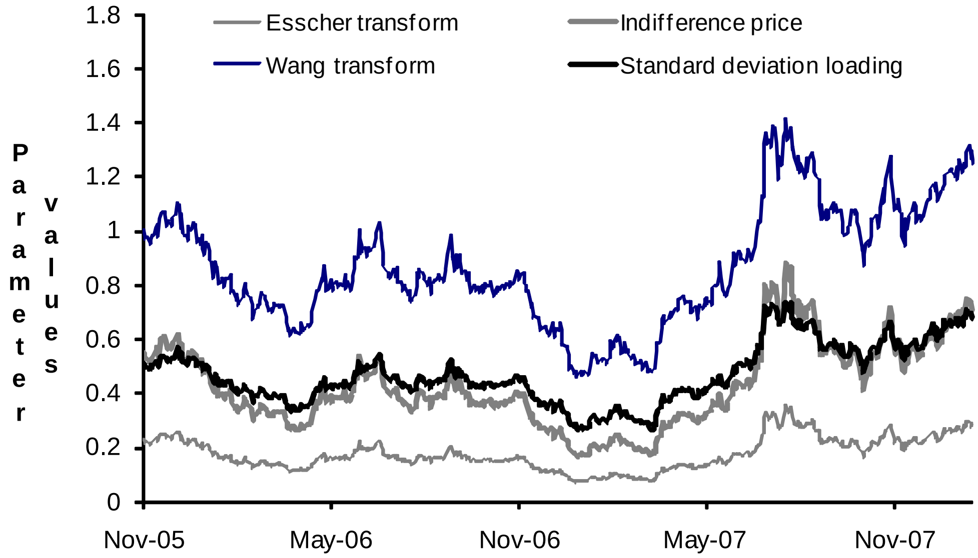

Figure 4: Implied parameter values of different pricing approaches

6 Conclusion

In this paper, we studied four transform methods for pricing claims on uncertain payoffs that are not hedgeable or replicable. A theoretically robust approach to asset pricing in this situation is to use the General Equilibrium (GE) approach, where the parameters of the asset specific pricing kernel can be implied from related instruments, whose prices are observable. When the utility function is exponential, the transform methods that are closest to the GE approach are the Esscher transform and the indifference pricing approach with exponential utility. With additional assumption on Normally distributed variables, the Wang transform and the standard deviation loading can be made comparable to the GE result also.

All transform pricing methods are heuristic rules of adjusting the physical distribution in order to produce a 'risk adjusted' price. All four transform methods are easy to use and do not require information that is not observable. A key issue is what happens if the risky cash flows and the state variable(s) are non-Gaussian and possibly strongly dependent in the extreme. In this case, the high order terms not captured in the mean and covariance terms become important, and the four methods produce prices that differ substantially. In our case study of Argentina's GDP warrants, with parameters calibrated from Argentina's vanilla debt, we find GDP warrant prices can differ by more than two and a half times by using these four methods.

We also study the characteristics of different pricing methods. In particular, we find that certain desirable properties necessary for a coherent risk measure may not be satisfied by all methods. For example, while intuitively appealing, the indifference price approach with exponential utility does not satisfy the additivity property, except in the case where cash flows are independent. The same is also true for the Wang transform, and standard deviation loading. This makes it difficult to justify using these methods for pricing liquidly traded instruments, where market participants expect linear pricing property, i.e. the sum of components in the package should produce the same price. This is important if there is no arbitrage in a liquid market. Given its strong theoretical foundations and appealing properties we recommend that the modified version of Esscher transform in (22) is used for pricing liquidly traded assets where linear pricing property is an important consideration.

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