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Chern–Simons gravity in four dimensions

Ivan Morales[1,a], **Bruno Neves**[1,b], **Zui Oporto**[1,2,c], **Olivier Piguet**[1,d]

[1] Departamento de Física, Universidade Federal de Viçosa (UFV), Viçosa, MG, Brazil
[2] Present address: Carrera de Física, Universidad Mayor de San Andrés, La Paz, Bolivia

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Abstract Five-dimensional Chern–Simons theory with (anti-)de Sitter SO(1,5) or SO(2,4) gauge invariance presents an alternative to general relativity with cosmological constant. We consider the zero modes of its Kaluza–Klein compactification to four dimensions. Solutions with vanishing torsion are obtained in the cases of a spherically symmetric 3-space and of a homogeneous and isotropic 3-space, which reproduce the Schwarzshild–de Sitter and ΛCDM cosmological solutions of general relativity. We also check that vanishing torsion is a stable feature of the solutions.

1 Introduction

Our present understanding of the fundamental processes in Nature is dominated by two extremely efficient theories: the already half a century old Standard Model (SM) valid in the quantum microscopic realm, and the centenary General Relativity (GR) valid in the classical macroscopic realm, from GPS monitoring in the planetary scale up to the cosmic scale, the evolution of the universe from the Big Bang up to an unforeseeable future. No observation neither experiment have shown any falsification of both theories, up to now.[1]

An important problem, however, is theoretical: the contradiction of GR being classical and SM being quantum. Two cures may be conceived. The more radical one may be the construction of a new framework, beyond "quantum" and "classical", in which GR and SM would stay as approximations of a unique theory, each being valid in its respective domain. String theory represents an effort in this direction.

Another, more obvious and (only apparently) straightforward cure is the direct quantization of GR, along the canonical lines of loop quantum gravity [2,3], for instance. The latter is based on a first order formulation of GR, which has two local symmetries: the invariances under the space-time diffeomorphisms and the local Lorentz transformations. In Dirac's canonical formalism [4,5], a constraint is associated to each local invariance, which has to be solved at the quantum level. Unfortunately, one of these constraints, namely the one associated with the time diffeomorphism invariance – called the hamiltonian or scalar constraint – has resisted to any tentative of solving it, up to now – although significant progress has been made [6–8].

It happens that the de Sitter or anti-de Sitter ((A)dS) gravitation theory in $5D$ space-time defined by a Chern–Simons theory with the gauge groups $SO(n, 6-n)$ for $n = 1$ or 2 [9] shows the remarkable property of its time-diffeomorphism constraint being a consequence of its gauge invariance and its invariance under the space diffeomorphisms [10]. It follows that the scalar constraint is then an automatic consequence of the other ones. This yields a first motivation for studying this particular theory of gravity.

A second motivation is given by the fact that the presence of a cosmological constant, hence of a fundamental scale at the classical level, happens as a necessary feature of this theory, as we shall verify, in contrast with usual GR where its presence or not is the result of an arbitrary choice.

The Chern–Simons (A)dS theory is a special case of the extensions of Einstein theory known as Lovelock theories [11] which, in spite of containing higher powers of the curvature, obey second order field equations. There exists a vast literature[2] on Lovelock theories, beginning with the historical papers [12–15]. Reference [16] already gives explicit solutions of the Schwarzschild, Reissner–Nordström and Kerr type in higher dimension Einstein theory with cosmo-

[1] See, however, [1] for an experimental result hinting to a possible problem with the Standard Model.

[2] Only a few references are given here. A rather complete list may be found in the book [9], which offers an up-to-date review on Lovelock and Chern–Simons theories of gravitation.

[a] e-mail: mblivan@gmail.com
[b] e-mail: bruno.lqg@gmail.com
[c] e-mail: azurnasirpal@gmail.com
[d] e-mail: opiguet@pq.cnpq.br

logical constant. More recent work may be divided into general Lovelock models [17–21], Chern–Simons models based on (A)dS gauge invariance [22], and Chern–Simons models based on larger gauge groups; see in particular [23–26]. It is worth noticing the work of [25], where the choice of the gauge group extension leads to a theory which reduces to 5D Einstein theory with cosmological constant in the case of a vanishing torsion. We may also mention genuinely four-dimensional models together with the search for physically reliable solutions of them, such as the Chamseddine model [27,28] obtained from the 5D (A)dS Chern–Simons by dimensional reduction and truncation of some fields, or the model of [33] obtained by adding to the Einstein–Hilbert action the coupling of a scalar field with the 4D Euler density.

The aim of the present work is an investigation of the classical properties of the 4D theory obtained from the 5D (A)dS Chern–Simons theory by a Kaluza–Klein compactification, find solutions of the field equations with spherical symmetry and solutions of the cosmological type, and comparisons of these solutions with the results of usual GR. This is intended to be a preliminary step to any attempt of quantization, the latter deserving future care.

(A)dS theory and its reduction to four dimensions are reviewed in Sect. 2, solutions with spherical symmetry and cosmological solutions are showed in Sects. 3 and 4. Conclusions are presented in Sect. 5. Appendices present details omitted in the main text.

2 (A)dS Chern–Simons theory for 5D and 4D gravity

2.1 (A)dS Chern–Simons theory as a 5D gravitation theory

Apart of some considerations from the authors, the content of this subsection is not new. A good review may be found in the book [9] together with references to the original literature.[3]

Chern–Simons theories are defined in odd-dimensional space-times, we shall concentrate to the five-dimensional case. We first define the gauge group as the pseudo-orthogonal group SO(1,5) or SO(2,4), the de Sitter or anti-de Sitter group in five dimensions, generically denoted by (A)dS. These are the matrix groups leaving invariant the quadratic forms

$$\eta_{(\mathrm{A})\mathrm{dS}} = \mathrm{diag}\ (-1, 1, 1, 1, s),$$
$$s = 1 \text{ for de Sitter}, \quad s = -1 \text{ for anti-de Sitter}.$$

A convenient basis of the Lie algebra of (A)dS is given by 10 Lorentz SO(1,4) generators $M_{AB} = -M_{BA}$ and five "translation" generators P_A, where A, B, etc., are Lorentz indices taking the values 0, …, 4. These generators obey the commutation rules

$$\begin{aligned}[M_{AB}, M_{CD}] &= \eta_{BD} M_{AC} + \eta_{AC} M_{BD} - \eta_{AD} M_{BC} \\ &\quad - \eta_{BC} M_{AD}, \\ [M_{AB}, P_C] &= \eta_{AC} P_B - \eta_{BC} P_A, \quad [P_A, P_B] = -s\, M_{AB},\end{aligned} \tag{2.1}$$

where $\eta_{AB} := \mathrm{diag}(-1, 1, 1, 1, 1)$ is the $D = 5$ Minkowski metric.

We then define the (A)dS connection 1-form, expanded in this basis as

$$A(x) = \frac{1}{2}\omega^{AB}(x) M_{AB} + \frac{1}{l}\, e^A(x) P_A, \tag{2.2}$$

where l is a parameter of dimension of a length. ω^{AB} will play the role of the 5D Lorentz connection form and e^A of the "5-bein" form in the corresponding gravitation theory. We may already note that the presence of the parameter l, which will be related to the cosmological constant (see Eq. (2.14)), is necessary in order to match the dimension of the 5-bein form e^A, which is that of a length, to that of the dimensionless Lorentz connection form ω^{AB}.

The (A)dS gauge transformations of the connection read, in infinitesimal form,

$$\delta A = \mathrm{d}\epsilon - [A, \epsilon],$$

where the infinitesimal parameter ϵ expands as

$$\epsilon(x) = \frac{1}{2}\epsilon^{AB}(x) M_{AB} + \frac{1}{l}\, \beta^A(x) P_A.$$

From this follows the transformations rules of the fields ω and e:

$$\begin{aligned}\delta\omega^{AB} &= \mathrm{d}\epsilon^{AB} + \omega^A{}_C\, \epsilon^{CB} + \omega^B{}_C\, \epsilon^{AC} \\ &\quad + \frac{s}{l^2}\left(e^A \beta^B - e^B \beta^A\right), \\ \delta e^A &= e_C\, \epsilon^{CA} + \mathrm{d}\beta^A + \omega^A{}_C\, \beta^C.\end{aligned} \tag{2.3}$$

Desiring to construct a background independent theory, we assume a dimension 5 manifold $\mathcal{M}_5$ without an a priori metric. Then the unique (A)dS gauge invariant action – up to boundary terms – which may constructed with the given connection is the Chern–Simons action for the group (A)dS, which in our notation reads

$$\begin{aligned}S_{\mathrm{CS}} = \frac{1}{8\kappa}\int_{\mathcal{M}_5} \varepsilon_{ABCDE}\Big(& e^A \wedge R^{BC} \wedge R^{DE} \\ & - \frac{2s}{3l^2} e^A \wedge e^B \wedge e^C \wedge R^{DE} \\ & + \frac{1}{5l^4} e^A \wedge e^B \wedge e^C \wedge e^D \wedge e^E\Big),\end{aligned} \tag{2.4}$$

where κ is a dimensionless[4] coupling constant and

$$R^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B \tag{2.5}$$

is the Riemann curvature 2-form associated to the Lorentz connection ω. We may add to the action a part S_{matter} describing matter and its interactions with the geometric fields ω^{AB}

[3] Notations and conventions are given in Appendix A.

[4] In our units $c = 1$.

and e^A, which leads to a total action $S = S_{\text{CS}} + S_{\text{matter}}$. The resulting field equations read

$$\begin{aligned}\frac{\delta S}{\delta e^A} &= \frac{1}{8\kappa}\varepsilon_{ABCDE}F^{BC}\wedge F^{DE} + \mathcal{T}_A = 0,\\ \frac{\delta S}{\delta\omega^{AB}} &= \frac{1}{2\kappa}\varepsilon_{ABCDE}T^C\wedge F^{DE} + \mathcal{S}_{AB} = 0,\end{aligned}\tag{2.6}$$

where

$$T^A = De^A = de^A + \omega^A{}_B\wedge e^B \tag{2.7}$$

is the torsion 2-form,

$$F^{AB} = R^{AB} - \frac{s}{l^2}e^A\wedge e^B \tag{2.8}$$

is the (A)dS curvature, and

$$\mathcal{T}_A := \frac{\delta S_{\text{matter}}}{\delta e^A},\quad \mathcal{S}_{AB} := \frac{\delta S_{\text{matter}}}{\delta\omega^{AB}} \tag{2.9}$$

are the energy-momentum 4-form, related to the energy-momentum components $\mathcal{T}^A{}_B$ in the 5-bein frame, by

$$\mathcal{T}_A = \frac{1}{4!}\varepsilon_{BCDEF}\mathcal{T}^B{}_A\, e^C\wedge e^D\wedge e^E\wedge e^F, \tag{2.10}$$

and the spin 4-form $\mathcal{S}_{AB}$.

A generalized continuity equation for energy, momentum and spin results from the field equations (2.6), the zero-torsion condition $De^A = 0$ and the Bianchi identity $DR^{AB} = 0$:

$$D\mathcal{T}_A + \frac{s}{l^2}\mathcal{S}_{AB}\wedge e^B = 0, \tag{2.11}$$

which reduces to the energy-momentum continuity equation in the case of spinless matter:

$$D\mathcal{T}_A = 0. \tag{2.12}$$

We observe that the sum of the second and third term of the action (2.4) is proportional to the 5D Einstein–Palatini action with cosmological constant, which is equivalent to the more familiar 5D Einstein–Hilbert action in the metric formulation:

$$S_{\text{EH}} = \frac{1}{16\pi G_{(5D)}}\int_{\mathcal{M}_5} d^5x\sqrt{-g}(R - 2\Lambda),$$

with $G_{(5D)}$ the 5D gravitation constant, Λ the cosmological constant, R the Ricci scalar and $g = -\det(e^A{}_\alpha)^2$ the determinant of the 5D metric

$$g_{\alpha\beta} = \eta_{AB}e^A{}_\alpha e^B{}_\beta. \tag{2.13}$$

This allows us to express the parameters κ and l in terms of the 5D physical parameters $G_{(5D)}$ and Λ as

$$\frac{3s}{l^2} = \Lambda,\quad \kappa = -\frac{4\pi}{9}\Lambda\, G_{(5D)}. \tag{2.14}$$

The coefficient of the first term in the action (2.4) – the so-called Gauss–Bonnet term – is of course fixed by (A)dS gauge invariance in terms of the two parameters of the theory. This is a special case of the more general Lanczos–Lovelock or Lovelock–Cartan theory [9,29].

2.1.1 *A trivial solution*

In the vacuum defined by the absence of matter, a special class of solutions of the field equations (2.6) is that of the solutions of the stronger equations

$$F^{AB} = 0,$$

with the (A)dS curvature 2-form F^{AB} given by (2.8). In fact the solution is unique up to an arbitrary torsion as is readily seen by inspection of the second of the field equations (2.6). This is a solution of constant curvature and corresponds to an empty de Sitter or anti-de Sitter 5D space-time with a 5-bein form

$$\begin{aligned}e^A &= \sqrt{1-\frac{\Lambda}{3}r^2}\,\mathrm{d}t + \frac{1}{\sqrt{1-\frac{\Lambda}{3}r^2}}\,\mathrm{d}r\\ &\quad + r\left(\mathrm{d}\theta + \sin\theta\,\mathrm{d}\phi + \sin\theta\sin\phi\,\mathrm{d}\psi\right),\end{aligned}\tag{2.15}$$

or its Lorentz transforms, leading to the metric

$$\begin{aligned}\mathrm{d}s^2 &= -\left(1-\frac{\Lambda}{3}r^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{\Lambda}{3}r^2}\\ &\quad + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \sin^2\theta\sin^2\phi\,\mathrm{d}\psi^2\right),\end{aligned}\tag{2.16}$$

in spherical 4-space coordinates t, r, θ, ϕ, ψ. This metric has the symmetry O(4) of 4-space rotations.

2.2 Compactification to four dimensions

In order to connect the theory with four-dimensional physics, we choose to implement a Kaluza–Klein type of compactification [30], considering the fourth spatial dimension to be compact. In other words, we consider a 5D space-time with the topology of $\mathcal{M}_5 = \mathcal{M}_4 \times S^1$, the first factor being a four-dimensional manifold and S^1 the circle representing the compactified dimension.[5] Any space-time function admits a Fourier expansion in the S^1 coordinate χ, its coefficients – the Kaluza–Klein modes – being functions in $\mathcal{M}_4$. In the applications presented in Sects. 3 and 4 only the zero mode is considered, which amounts to consider all functions as constant in χ.

Note that the zero (A)dS curvature solution (2.15, 2.16) is not a solution of the compactified theory.

[5] The coordinates of $\mathcal{M}_5$ are denoted by x^α ($\alpha = 0,\ldots,4$) and those of $\mathcal{M}_4$ by x^μ ($\mu = 0,\ldots,3$). The coordinate of S^1 is denoted by χ, with $0 \le \chi < 2\pi$.

2.3 Solutions with zero torsion

2.3.1 On the number of degrees of freedom

The number of local physical degrees of freedom of the theory is best calculated by means of a canonical analysis. It is known [10] that the present theory has 75 constraints of first and second class. The number n_2 of the latter is equal, in the weak sense,[6] to the rank r of the matrix formed by the Poisson brackets of the constraints. The number of first class constraints – which generate the gauge transformations – is thus equal to $n_1 = 75 - n_2$. Moreover, the number of generalized coordinates[7] is equal to 60. Thus the number $n_{\text{d.o.f.}}$ of physical degrees of freedom, at each point of 4-space, is given by

$$n_{\text{d.o.f.}} = \frac{1}{2}(2 \times 60 - 2n_1 - n_2) = \frac{r}{2} - 15.$$

The authors of [10] have shown that the result for the rank r, hence for $n_{\text{d.o.f.}}$, depends on the region of phase space where the state of the system lies. They have computed it in the "generic" case, *i.e.,* the case where the rank r is maximal, corresponding to the situation with the minimal set of local invariances, namely that of the fifteen (A)dS gauge invariances and the four 4-space diffeomorphism invariances. This results in $r = 56$, *i.e.,* in 13 physical degrees of freedom.

The case with zero torsion is non-generic in the sense given above. We have checked by numerical tests that the rank r is then at most equal to 40, which shows that $n_{\text{d.o.f.}} \le 5$ in the case of a zero torsion.

2.3.2 On the stability of solutions with zero torsion

The second of the field equations (2.6) is identically solved by assuming zero torsion. We would like to know in which extent solutions with zero torsion are stable under small perturbations. More precisely, considering a field configuration with a torsion of order ϵ, we will look for conditions ensuring its vanishing as a consequence of the equations.

The second of Eq. (2.6), written in 5-bein components as

$$\frac{1}{8}\varepsilon_{ABCDE}\,\varepsilon^{XYZTU}\,T^C{}_{XY}\,F^{DE}{}_{ZT} = 0,$$

can be rewritten in the form

$$T^i\,M_i{}^j = 0, \tag{2.17}$$

with

$$M_i{}^j = \frac{1}{8}\varepsilon_{ABCDE}\,\varepsilon^{XYZTU}\,F^{DE}{}_{ZT}, \tag{2.18}$$

the index i standing for $(C, [XY])$ and j for $(U, [AB])$. If the 50×50 matrix M is invertible, then (2.17) implies the vanishing of the torsion.

Let us write the infinitesimal torsion as $T^A{}_{BC} = \epsilon\, t^A{}_{BC}$. We note that the connection ω (B.2) constructed from the 5-bein and the torsion is linear in the torsion components, thus in ϵ, hence F is a polynomial in ϵ, and so is the matrix M. This implies that its inverse M^{-1} exists and is analytic in ϵ in a neighborhood of $\epsilon = 0$, if the matrix

$$M_i{}^{j\,(0)} = M_i^j\Big|_{\epsilon=0} \tag{2.19}$$

is regular. It then follows, under the latter assumption, that the torsion vanishes. We can summarize this result as follows.

Stability criterion: A sufficient condition for the stability of the solutions at zero torsion under possible fluctuations of the torsion is that the matrix (2.18) restricted to zero torsion, $M^{(0)}$, be regular.

This criterion is important in view of the difference between the number of physical degrees of freedom for states with zero torsion and this number for generic states,[8] as discussed in Sect. 2.3.1. Indeed, if the state of the system lies in the sub-phase space of zero-torsion states, the fulfillment of the condition of the criterion guarantees that the state will evolve staying in that subspace.

3 Solutions with 3D rotational symmetry

The most general metric and torsion tensor components compatible with the rotational symmetry of 3-space are calculated in Appendix C, with the metric given by (C.1) and the torsion by (C.5) in a system of coordinates $t,\ r,\ \theta,\ \phi,\ \chi$. All component fields depend on $t,\ r,\ \chi$. But we shall restrict ourselves here to look for stationary solutions, neglecting also the higher Kaluza–Klein modes. Thus only a dependence on the radial coordinate r is left. In this situation the metric takes the simpler form (C.4) with only one non-diagonal term, thanks to some suitable coordinate transformations, as explained in Appendix C.

Through the definition (2.13), this metric leads to the 5-bein $e^A = e^A{}_\alpha dx^\alpha$, up to local Lorentz transformations $e'^A = \Lambda^A{}_B e^B$, with

$$(e^A{}_\alpha) = \begin{pmatrix} n(r) & 0 & 0 & 0 & c(r) \\ 0 & a(r) & 0 & 0 & 0) \\ 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & r\sin\theta & 0 \\ 0 & 0 & 0 & 0 & b(r) \end{pmatrix}, \tag{3.1}$$

and the relations

$$g_{tt}(r) = -n^2(r), \quad g_{rr}(r) = a^2(r), \quad g_{t\chi}(r) = -n(r)c(r),$$

[6] A "weak equality" is an equality valid up to the constraints.

[7] The generalized coordinates are the space components of the connection and 5-bein fields: $\omega^{AB}{}_a$ and $e^A{}_a$, with $a = 1, \ldots, 4$.

[8] We thank Jorge Zanelli for pointing out this problem to us.

$$g_{\chi\chi}(r) = b^2(r) - c^2(r).$$

Beyond the spherical symmetry of 3-space, the stationarity and the restriction to the zero KK mode, we still make the following hypotheses:

(i) The torsion (2.7) is zero: $T^A = 0$, hence the second of the field equations (2.6) is trivially satisfied.
(ii) We look for static solutions, hence $g_{t\chi}(r) = 0$, and $c(r) = 0$ in (3.1).
(iii) We restrict the discussion to the de Sitter case, *i.e.,* with a positive cosmological constant: $s = 1$, which corresponds to the present data [31].

Consistently with the symmetry requirements and the hypotheses above, the tensor $\mathcal{T}^A{}_B$ appearing in the definition (2.10) of the energy-momentum 4-form reads[9]

$$\mathcal{T}^A{}_B = \mathrm{diag}\,(-\hat{\rho}(r), \hat{p}(r), \hat{p}(r), \hat{p}(r), \hat{\lambda}(r)), \tag{3.2}$$

We also assume that the spin current 4-form $\mathcal{S}_{AB}$ in (2.9) is vanishing. In the present setting, the continuity equation (2.11) takes then the form

$$p'(r) + p(r)\left(\frac{n'(r)}{n(r)} - \frac{b'(r)}{b(r)}\right) - \lambda(r)\frac{b'(r)}{b(r)} + \rho(r)\frac{n'(r)}{n(r)} = 0. \tag{3.3}$$

We shall consider the case of an empty physical 3-space, which means zero energy density and pressure, *i.e.,* $\rho(r) = p(r) = 0$, keeping only the "compact dimension pressure" $\lambda(r) \neq 0$ (We shall see that the solution of interest indeed has a non-vanishing λ). The continuity equation thus implies the 5-bein component $b(r)$ to be a constant:

$$b(r) = R = \text{constant}. \tag{3.4}$$

The parameter R, which has the dimension of a length, defines the compactification scale.

With all of this, the field equations (first of (2.6)) reduce to the three independent equations

$$\begin{aligned}
&\left(1 - \frac{3r^2}{l^2}\right) a(r)^2 n(r) - 2rn'(r) - n(r) = 0,\\
&\quad -2ra'(r) + a(r)^3\left(\frac{3r^2}{l^2} - 1\right) + a(r) = 0,\\
&\kappa l^2 r^2 a(r)^5 n(r)\lambda(r) - \left(l^2 a(r) + (r^2 - l^2)a(r)^3\right) n''(r)\\
&\quad + \left(3l^2 + (r^2 - l^2)a(r)^2\right) a'(r)n'(r) + 2ra(r)^2 n(r)a'(r)\\
&\quad -2ra(r)^3 n'(r) - \left(a(r)^3 + \left(\frac{3r^2}{l^2} - 1\right) a(r)^5\right) n(r) = 0.
\end{aligned} \tag{3.5}$$

Note that these equations do not depend on the compactification scale R. The second equation solves for $a(r)$, and then the first one yields $n(r)$:

$$n(r) = \sqrt{1 - \frac{2\mu}{r} - \frac{r^2}{l^2}}, \qquad a(r) = 1/n(r), \tag{3.6}$$

after a time coordinate re-scaling is made. The Schwarzschild mass μ is an integration constant as in GR. The third equation yields $\lambda(r)$ in terms of the functions $a(r)$ and $n(r)$, with the final result

$$\lambda(r) = \frac{6\mu^2}{\kappa}\frac{1}{r^6}. \tag{3.7}$$

The final 5-bein and metric thus read

$$(e^A{}_\alpha) = \begin{pmatrix} \sqrt{1 - \frac{2\mu}{r} - \frac{r^2}{l^2}} & 0 & 0 & 0 & 0 \\ 0 & \left(\sqrt{1 - \frac{2\mu}{r} - \frac{r^2}{l^2}}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & r\sin\theta & 0 \\ 0 & 0 & 0 & 0 & R \end{pmatrix},$$
$$\begin{aligned} ds^2 &= -(1 - \frac{2\mu}{r} - \frac{r^2}{l^2})dt^2 + \frac{dr^2}{1 - \frac{2\mu}{r} - \frac{r^2}{l^2}} \\ &\quad + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + R^2 d\chi^2. \end{aligned} \tag{3.8}$$

This result is just the generalization of the Schwarzschild solution in a space-time which is asymptotically de Sitter, with cosmological constant $\Lambda = 3/l^2$. One remembers that we have described the "vacuum" as described by an energy-momentum tensor (3.2) with one possibly non-zero component: the "compact dimension pressure" $\lambda(r)$. Our result is that this "pressure" is indeed non-vanishing, singular at the origin and decaying as the inverse of the sixth power of the radial coordinate as shown in Eq. (3.7).

We must emphasize that this result follows uniquely from the hypotheses we have made.

Finally, we have checked the condition of stability of the zero-torsion solutions of the model according to the criterion proved in Sect. 2.3.2: a computation of the matrix $M_i{}^{j(0)}$ (2.19) using the 5-bein (3.1) (with the non-diagonal component $c(r) = 0$) indeed shows that its rank takes the maximum value, 50, hence it is regular. We have also computed its determinant for the case of the solution (3.8):

$$\mathrm{Det}\,(M_i{}^{j(0)}) = -\frac{4608\mu^6\left(2r^3 + l^2\mu\right)^3}{l^{88}r^{27}},$$

which is clearly not vanishing as long as the mass μ is not equal to zero.

[9] The hats on $\hat{\rho}$, etc., mean energy density, etc. in 4-space.

4 Cosmological solutions

We turn now to the search for cosmological solutions, again the case of a positive cosmological constant Λ, *e.g.,* taking the parameter s equal to 1.

This search is based on the hypotheses of isotropy and homogeneity of the physical 3-space. The space-time coordinates are taken as t, r, θ, ϕ, χ as in Sect. 3, r, θ, ϕ being spherical coordinates for the 3-space and $\chi \in (0, 2\pi)$ the compact subspace S^1 coordinate. We shall only consider here the zero modes of Kaluza–Klein, *i.e.,* all functions will only depend on the time coordinate t.

The most general metric satisfying our symmetry requirements, up to general coordinate transformations, is given by Eq. (D.1) of Appendix D. In the present case of χ-independence, we can perform another time coordinate transformation in order to eliminate the factor in front of $\mathrm{d}t^2$, which yields the metric

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{a^2(t)}{1-kr^2}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \\ +b^2(t)\mathrm{d}\chi^2, \end{aligned} \tag{4.1}$$

which is of the FLRW type in what concerns the 4D subspacetimes at constant χ. We shall restrict on the case of a flat 3-space, *i.e.,* $k = 0$. This metric can then be obtained, using (2.13), from the 5-bein

$$\left(e^A{}_\alpha\right) = \operatorname{diag}\left(1,\ a(t),\ a(t)r,\ a(t)r\sin\theta,\ b(t)\right), \tag{4.2}$$

up to a 5D Lorentz transformation. We shall not assume from the beginning a null torsion T^A (2.7). Due to the isotropy and homogeneity conditions, the torsion depends on five independent functions $\tilde f(t)$, $h(t)$, $\tilde h(t)$, $u(t)$, and $\tilde u(t)$, as shown in Eq. (D.2) of Appendix D. The equations resulting from the field equations (2.6) are also displayed in this appendix.

Let us now show that two components of the torsion, namely u and $\tilde u$, can be set to zero by a partial gauge fixing condition. The two gauge invariances which are fixed in this way are the ones generated by P_0 and P_4, *i.e.,* the transformations (2.3) for the parameters $\beta^0(t)$ and $\beta^4(t)$. The torsion components which transform non-trivially are $T^0_{t\chi}$ and $T^4_{t\chi}$:

$$\delta T^0_{t\chi} = F^{04}_{t\chi}\beta_4, \quad \delta T^4_{t\chi} = -F^{04}_{t\chi}\beta_0.$$

These transformations are non-trivial as a consequence of the non-vanishing of the F-curvature component occurring here:

$$F^{04}_{t\chi} = \left(-\ddot b + \partial_t(\dot b\tilde u) + \frac{s}{l^2}\right)b,$$

as can be read off from (D.5). It follows therefore that the gauge fixing conditions

$$u = 0, \quad \tilde u = 0, \tag{4.3}$$

are permissible.

We shall describe matter with the perfect fluid energy-momentum tensor (D.7) with zero pressure, $p = 0$, a non-vanishing energy density and a possibly non-vanishing "compact dimension pressure":

$$\mathcal{T}^A{}_B = \operatorname{diag}\left(-\frac{\rho(t)}{2\pi b(t)}, 0, 0, 0, \frac{\lambda(t)}{2\pi b(t)}\right). \tag{4.4}$$

The first entry here is the energy density $\hat\rho(t)$ in 4-space, written in terms of the effective 3-space energy density $\rho(t)$. We have correspondingly redefined λ for the sake of homogeneity in the notation (see Appendix A). With this form of the energy-momentum tensor and the assumptions made at the beginning of this section, the continuity equation (D.19) is trivially satisfied, whereas (D.18) reads

$$\dot\rho + 3\rho\frac{\dot a}{a} + \lambda\frac{\dot b}{b} = 0, \tag{4.5}$$

which reduces to the usual continuity equation for dust in the case of a constant compactification scale b.

Let us now solve the field equations (D.8–D.17). A key observation is that none of both expressions $(\mathbb{K} - \frac{s}{l^2})$ and $(\mathbb{B} - \frac{s}{l^2})$ can vanish, since we assume a non-zero energy density. (Remember the gauge conditions (4.3), and that all derivatives in χ vanish since we only consider the zero KK modes.) Then, Eqs. (D.17), (D.13), and (D.16), taken in that order, imply

$$\tilde h(t) = 0, \quad h(t) = 0, \quad \tilde f(t) = \text{constant},$$

respectively. Now, solving (D.15) leads to two possibilities: $\tilde f$ vanishing or not. Let us first show that the latter case leads to a contradiction. Equation (D.15) with $\tilde f \neq 0$ implies the equation $\ddot b/b - 1/l^2 = 0$, which solves in $b(t) = b_0\exp(\pm t/l)$. Then Eq. (D.12) reads $(\ddot a - 1/l^2)(\dot a - 1/l)$, which is solved by $a(t) = a_0\exp(\pm t/l)$. Inserting this into Eq. (D.8) yields $\rho(t) = 0$, which contradicts the hypothesis of a non-vanishing energy density. We thus conclude that $\tilde f(t) = 0$, which finally means a vanishing torsion[10]:

$$T^A = 0.$$

In order to solve now for the remaining field equations, we make the simplifying hypothesis that the compactification scale is constant:

$$b(t) = R. \tag{4.6}$$

[10] The attentive reader may – correctly – find that the gauge fixing conditions (4.3) are not necessary in order to achieve the result in the case $\tilde f = 0$: Their are in fact consequences, together with $h = \tilde h = 0$, of the field equations (D.13–D.17). But the result above for $\tilde f \neq 0$ indeed does need these gauge fixing conditions.

At this stage, the field equations reduce to the system

$$\begin{aligned} &\frac{\dot{a}(t)^2}{a(t)^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho(t) \\ &\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}(t)^2}{2a(t)^2} - \frac{\Lambda}{2} = 0, \\ &\lambda(t) = \frac{-3a^2(t)\ddot{a}(t) - 3a(t)\dot{a}^2(t) + \frac{9}{\Lambda}\dot{a}^2(t)\ddot{a}(t) + \Lambda a^3(t)}{16\pi^2 G R\, a^3(t)}, \end{aligned} \tag{4.7}$$

where we have expressed the parameters κ and l in terms of the Newton constant G and the cosmological constant Λ as

$$\kappa = -\frac{16\pi^2}{3} G R \Lambda, \quad l = \sqrt{\frac{3}{\Lambda}}. \tag{4.8}$$

We recognize in the first two equations the Friedmann equations for dust. The third equation gives the "compact dimension pressure" λ.

With the Big Bang boundary conditions a(0) = 0 , the solution of the system reads

$$\begin{aligned} &a(t) = C \sinh^{\frac{2}{3}}\left(\tfrac{\sqrt{3\Lambda}}{2}t\right), \quad \rho(t) = \frac{\Lambda}{8\pi G}\left(\frac{C}{a(t)}\right)^3, \\ &\lambda(t) = -\frac{\Lambda}{32\pi^2 G R}\left(\frac{C}{a(t)}\right)^6. \end{aligned} \tag{4.9}$$

where C is an integration constant. The first line of course reproduces the ΛCDM solution for dust matter, whereas the second line shows a decreasing of λ as the sixth inverse power of the scale parameter a.

As it should, the solution obeys the continuity equation (D.18), which now reads

$$\dot{\rho} + 3\rho\frac{\dot{a}}{a} = 0.$$

The continuity equation (D.19) is trivially satisfied.

We recall that we have made the assumption of a constant scale parameter b for the compact dimension. This assumption is not necessary, but it is interesting to note, as can easily be checked, that solving the equation in which we insert the ΛCDM expression of (4.9) for the 3-space scale parameter $a(t)$, implies the constancy of b.

We have also explicitly checked the validity of the condition for stability according to the criterion of Sect. 2.3.2: the matrix $M_i{}^{j(0)}$ (2.19) calculated using the 5-bein (4.2) has its maximum rank, 50, hence it is regular. We have also computed its determinant for the case of the solution (4.9):

$$\mathrm{Det}\,(M_i{}^{j(0)}) = -\frac{439453125\left(4 - 9\coth^2\left(\frac{3t}{2l}\right)\right)^{16}\left(16 + 9\coth^2\left(\frac{3t}{2l}\right)\right)^3}{1152921504606846976\, l^{100}},$$

which is generically not vanishing as a function of t.

5 Conclusions

After recalling basic facts on the five-dimensional Chern–Simons gravity with the five-dimensional (anti)-de Sitter ((A)dS) gauge group, we have studied some important aspects of this theory in comparison with the results of general relativity with cosmological constant.

First of all, the cosmological constant is here a necessary ingredient due to the (A)dS algebraic structure, although it remains a free parameter. It cannot be set to zero.

We have shown that, for a spherically symmetrical 3-space, the "vacuum" Schwarzschild–de Sitter solution (3.8) follows uniquely from the hypotheses of a zero-torsion, stationary, and static geometry. However, the existence of this solution implies the presence of a non-vanishing "compact dimension pressure" $\lambda(r)$ as given by (3.7), a fact of not-so-easy interpretation, in particular due to the expected smallness of the compactification scale.

For the other physically interesting case of a cosmological model based on an homogeneous and isotropic 3-space, where we have restricted ourselves to the observationally favored flatness of 3-space, we have shown that the equations for the Friedmann scale parameter $a(t)$ and the energy density $\rho(t)$ are identical to the well-known Friedmann equations of general relativity under the hypothesis that the compact scale parameter $b(t)$ be a constant. Conversely, only this constancy is compatible with the Friedmann equations. There is also a non-vanishing "compact dimension pressure", decreasing in time as the sixth inverse power of the scale parameter $a(t)$. We have also seen that the vanishing of the torsion follows from the full (A)dS gauge invariance and of the field equations.

An important aspect of this work is the establishment of a criterion guaranteeing the stability of the zero-torsion solutions if a certain condition based on zero-torsion geometrical quantities is fulfilled. We have also checked that this condition is indeed met in the two situations considered in this paper.

Summarizing all these considerations, we can conclude that the two families of solutions investigated here coincide with the corresponding solutions of general relativity in the presence of a (positive) cosmological constant. However, we recall that we have only examined the Kaluza–Klein zero modes of the theory. Possible deviations from the results of Einstein general relativity could follow from the consideration of higher modes. Also, solutions with torsion would be interesting in view of its possible physical effects.

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Appendices

A Notations and conventions

- Units are such that $c = 1$.
- Indices $\alpha, \beta, \ldots = 0, \ldots, 4$, also called t, r, θ, ϕ, χ, are 5D space-time coordinates.
- Indices. $A, B, \ldots = 0, \ldots, 4$ are 5-bein frame indices.
- Indices $A, \ldots$, are raised or lowered with the Minkowski metric $(\eta_{AB}) = \mathrm{diag}(-1, 1, 1, 1, 1)$.
- Indices $\alpha, \ldots$ may be exchanged with indices $A, \ldots$ using the 5-bein $e^A{}_\alpha$ or its inverse $e^\alpha{}_A$.
- A hat on a symbol means a 5D quantity, like *e.g.,* $\hat{\rho}(t, x^1, x^2, x^3, x^4)$ for the energy density in 4-space.

B Construction of the spin connection

We recall here how the spin connection ω can be constructed from the 5-bein e and the torsion T [33]. First, given the 5-bein, one constructs the torsion-free connection $\bar{\omega}$, solution of the zero-torsion equation $de^A + \bar{\omega}^A{}_B \wedge e^B = 0$. The result [34] is

$$\bar{\omega}^{AB}{}_\mu = \frac{1}{2}\left(\xi_C{}^{AB} + \xi^B{}_C{}^A - \xi^{AB}{}_C\right) e^C{}_\mu,$$

with

$$\xi_{AB}{}^C = e^\mu{}_A e^\nu{}_B (\partial_\mu e^C{}_\nu - \partial_\nu e^C{}_\mu).$$

One then defines the contorsion 1-form $\mathcal{C}^A{}_B$ by the equation $T^A = \mathcal{C}^A{}_B \wedge e^B$, which solves in

$$\mathcal{C}^{AB} = -\frac{1}{2}\left(T^{AB}{}_C + T_C{}^{AB} - T^B{}_C{}^A\right) e^C, \tag{B.1}$$

where $T^A{}_{BC} = e^A{}_\mu T^\mu{}_{\nu\rho} e^\nu{}_B e^\rho{}_C$ are the torsion components in the 5-bein basis.[11] From this we get the full connection form as

$$\omega^A{}_B = \bar{\omega}^A{}_B + \mathcal{C}^A{}_B, \tag{B.2}$$

obeying the full torsion equation $T^A = de^A + \omega^A{}_B \wedge e^B$.

[11] In the 5-bein basis: $T^A = \frac{1}{2} T^A{}_{BC} e^B \wedge e^C$.

C Metric and torsion for 3-space spherical symmetry

In this appendix, we derive the metric $g_{\mu\nu}$ and torsion tensors $T^\rho{}_{\mu\nu}$ in the case of a 3-space with spherical symmetry around the origin $r = 0$. Accordingly, observables such as the metric and the torsion components in the coordinate basis must satisfy Killing equations, which are the vanishing of the Lie derivatives of the fields along the vectors ξ which generate the symmetries.

The set of Killing vectors ξ are the generators J_i ($i = 1, 2, 3$) of $SO(3)$, which generate the spatial rotations. In the coordinate system t, r, θ, ϕ, χ, where r, θ, ϕ are spherical coordinates for 3-space, and χ the compact subspace coordinate, these vectors read

$$J_1 = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi,$$
$$J_2 = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi,$$
$$J_3 = \partial_\phi,$$

and obey the commutation rules

$$\left[J_i, J_j\right] = \varepsilon_{ijk} J_k.$$

The Killing equations for the metric and the torsion read, for $\xi = J_1, J_2, J_3$,

$$\pounds_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\mu}\partial_\nu \xi^\rho + g_{\nu\rho}\partial_\mu \xi^\rho = 0,$$
$$\pounds_\xi T^\delta{}_{\mu\nu} = \xi^\rho \partial_\rho T^\delta{}_{\mu\nu} - T^\rho{}_{\mu\nu}\partial_\nu \xi^\delta + T^\delta{}_{\rho\nu}\partial_\mu \xi^\rho + T^\delta{}_{\mu\rho}\partial_\nu \xi^\rho = 0.$$

This yields, for the metric,

$$\begin{aligned} ds^2 = {} & g_{tt}(t,r,\chi)\mathrm{d}t^2 + 2g_{tr}(t,r,\chi)\mathrm{d}t\mathrm{d}r \\ & + 2g_{t\chi}(t,r,\chi)\mathrm{d}t\mathrm{d}\chi + 2g_{r\chi}(t,r,\chi)\mathrm{d}r\mathrm{d}\chi \\ & + g_{rr}(t,r,\chi)\mathrm{d}r^2 + g_{\chi\chi}(t,r,\chi)\mathrm{d}\chi^2 \\ & + g_{\theta\theta}(t,r,\chi)\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right). \end{aligned} \tag{C.1}$$

If we perform a change of radial coordinate r to $r' = (g_{\theta\theta}(t,r,\chi))^{1/2}$, and after that drop the primes, the line element becomes

$$\begin{aligned} ds^2 = {} & g_{tt}(t,r,\chi)\mathrm{d}t^2 + 2g_{tr}(t,r,\chi)\mathrm{d}t\mathrm{d}r \\ & + 2g_{t\chi}(t,r,\chi)\mathrm{d}t\mathrm{d}\chi + 2g_{r\chi}(t,r,\chi)\mathrm{d}r\mathrm{d}\chi \\ & + g_{rr}(t,r,\chi)\mathrm{d}r^2 + g_{\chi\chi}(t,r,\chi)\mathrm{d}\chi^2 \\ & + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right). \end{aligned} \tag{C.2}$$

We shall consider the stationary case, *i.e.,* where the components of the metric are independent of the coordinate t. For this case we can consider the differential $g_{\chi\chi}(r,\chi)\mathrm{d}\chi + g_{r\chi}(r,\chi)\mathrm{d}r$, and from the theory of partial differential equations we know that we can multiply it by an integrating factor $I_1 = I_1(r,\chi)$ which makes it an exact differential. Using this result to define a new coordinate χ' by requiring $\mathrm{d}\chi' = I_1(r,\chi)(g_{\chi\chi}(r,\chi)\mathrm{d}\chi + g_{r\chi}(r,\chi)\mathrm{d}r)$, substituting this in the latter expression of the line element and again dropping the prime, the line element simplifies to,[12]

$$ds^2 = g_{tt}(r,\chi)\mathrm{d}t^2 + 2g_{tr}(r,\chi)\mathrm{d}t\mathrm{d}r + 2g_{t\chi}(r,\chi)\mathrm{d}t\mathrm{d}\chi + g_{rr}(r,\chi)\mathrm{d}r^2 + g_{\chi\chi}(r,\chi)\mathrm{d}\chi^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right). \quad \text{(C.3)}$$

Now we go to the special case where the components of the metric depend only on the radial variable r, which amounts to restrict to the Kaluza–Klein zero modes. We can now consider the differential form $g_{tt}(r)\mathrm{d}t + g_{tr}(r)\mathrm{d}r$ and multiply it by an integral factor $I_2(t,r)$ that permits one to write it as a perfect differential, $\mathrm{d}t' = I_2(t,r)(g_{tt}(r)\mathrm{d}t + g_{tr}(r)\mathrm{d}r)$. Substituting in the line element and dropping the prime we finally get

$$ds^2 = g_{tt}(r)\mathrm{d}t^2 + 2g_{t\chi}(r)\mathrm{d}t\mathrm{d}\chi + g_{rr}(r)\mathrm{d}r^2 + g_{\chi\chi}(r)\mathrm{d}\chi^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right). \quad \text{(C.4)}$$

For the torsion, the Killing equations leave the following non-vanishing components:

$$\begin{aligned}
&T^t_{tr} = h_1(t,r,\chi), && T^t_{t\chi} = q_1(t,r,\chi), && T^t_{r\chi} = q_2(t,r,\chi),\\
&T^r_{tr} = h_2(t,r,\chi), && T^r_{t\chi} = q_3(t,r,\chi), && T^r_{r\chi} = q_4(t,r,\chi),\\
&T^\chi_{tr} = h_5(t,r,\chi), && T^\chi_{t\chi} = q_5(t,r,\chi), && T^\chi_{r\chi} = q_6(t,r,\chi),\\
&T^t_{\theta\phi} = \sin(\theta) f_1(t,r,\chi, && T^r_{\theta\phi} = \sin(\theta) f_2(t,r,\chi), && T^\chi_{\theta\phi} = \sin(\theta) f_5(t,r,\chi),\\
&T^\theta_{t\theta} = h_3(t,r,\chi) = T^\phi_{t\phi}, && T^\theta_{t\phi} = \sin(\theta) f_3(t,r,\chi), && T^\phi_{t\theta} = -\frac{f_3(t,r,\chi)}{\sin(\theta)},\\
&T^\theta_{r\theta} = h_4(t,r,\chi) = T^\phi_{r\phi}, && T^\theta_{r\phi} = \sin(\theta) f_4(t,r,\chi), && T^\phi_{r\theta} = -\frac{f_4(t,r,\chi)}{\sin(\theta)},\\
&T^\theta_{\theta\chi} = h_6(t,r,\chi) = T^\phi_{\phi\chi}, && T^\theta_{\phi\chi} = \sin(\theta) f_6(t,r,\chi), && T^\phi_{\theta\chi} = -\frac{f_6(t,r,\chi)}{\sin(\theta)}.
\end{aligned} \quad \text{(C.5)}$$

A derivation of the connection $\omega(t,r,\chi)$ from the general metric (C.2) and the torsion (C.5) following the lines of Appendix B, hence of the curvature forms and the field equations, may be found in [36]. In the present work we shall restrict to solutions which are independent of t (stationary) and independent of χ (Kaluza–Klein zero modes). The metric (C.4) will be used.

D Equations in the case of an isotropic and homogeneous 3-space

We give here the derivation of the general set of field equations with full dependence on the compact dimension coordinate χ, in the case of a 5D space-time with an isotropic and homogeneous 3D subspace. All fields are functions of the time coordinate t and the compact coordinate χ. The 3-space coordinates are spherical: $r,\ \theta,\ \phi$.

[12] This well-known argument may be found in the textbook [35].

D.1 Metric, 5-bein, torsion, and curvature

The cosmological principle requires that the 3D spatial section of space-time be isotropic and homogeneous. Therefore the fields involved in the model must be compatible with this assumption. Isotropy of space-time means that the same observational evidence is available by looking in any direction in the universe, *i.e.,* all the geometric properties of the space remain invariant after a rotation. Homogeneity means that at any random point the universe looks exactly the same. These two assumptions are translated in Killing equations, which are the vanishing of the Lie derivatives of the fields along the vectors ξ which generate the symmetries.

The set of Killing vectors ξ are the generators J_i ($i = 1,\ 2,\ 3$) of $SO(3)$, which generate the spatial rotations, and the generators of spatial translations P_i, satisfying the commutation rules

$$\left[J_i, J_j\right] = \varepsilon_{ijk}J_k \quad \left[J_i, P_j\right] = \varepsilon_{ijk}P_k \quad \left[P_i, P_j\right] = -k\varepsilon_{ijk}J_k,$$

where k is the 3-space curvature parameter: $k = 0,\ 1,\ -1$ for plane, closed or open 3-space, respectively. In our coordinate system, these vectors read

$$J_1 = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi,$$
$$J_2 = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi,$$
$$J_3 = \partial_\phi,$$

and

$$P_1 = \sqrt{1-kr^2}\left(\sin\theta\cos\phi\,\partial_r + \frac{\cos\theta\cos\phi}{r}\partial_\theta - \frac{\sin\phi}{r\sin\theta}\partial_\phi\right),$$
$$P_2 = \sqrt{1-kr^2}\left(\sin\theta\sin\phi\,\partial_r + \frac{\cos\theta\sin\phi}{r}\partial_\theta + \frac{\cos\phi}{r\sin\theta}\partial_\phi\right),$$
$$P_3 = \sqrt{1-kr^2}\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right).$$

The Killing conditions must hold for (A)dS gauge invariant tensors. We are interested here in these conditions for the metric tensor $g_{\alpha\beta} = \eta_{AB}e^A{}_\alpha e^B{}_\beta$ and the torsion tensor $T^\gamma{}_{\alpha\beta} = e^\gamma{}_A T^A{}_{\alpha\beta}$:

$$
\begin{aligned}
&\pounds_\xi g_{\mu\nu} = \xi^\gamma \partial_\gamma g_{\mu\nu} + g_{\gamma\mu}\partial_\nu \xi^\gamma + g_{\nu\gamma}\partial_\mu \xi^\gamma = 0,\\
&\pounds_\xi T^\delta{}_{\mu\nu} = \xi^\gamma \partial_\gamma T^\delta{}_{\mu\nu} - T^\gamma{}_{\mu\nu}\partial_\nu \xi^\delta + T^\delta{}_{\gamma\nu}\partial_\mu \xi^\gamma\\
&\qquad + T^\delta{}_{\mu\gamma}\partial_\nu \xi^\gamma = 0,
\end{aligned}
$$

with $\xi = J_1,\ J_2,\ J_3,\ P_1,\ P_2,\ P_3$. The Killing conditions for the metric yield the line element

$$
\begin{aligned}
ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta\\
&= g_{tt}(t,\chi)\mathrm{d}t^2 + g_{\chi\chi}(t,\chi)\mathrm{d}\chi^2\\
&\quad + \alpha(t,\chi)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right)\\
&\quad + 2g_{t\chi}(t,\chi)\mathrm{d}t\mathrm{d}\chi.
\end{aligned}
$$

In the same way as we did in Appendix C, we can eliminate the cross term in $\mathrm{d}t\,\mathrm{d}\chi$ through a change of the time coordinate defined by [35]

$$
\mathrm{d}t' = I(t,\chi)\left(g_{tt}(t,\chi) + g_{t\chi}(t,\chi)\right),
$$

where $I(t,\chi)$ is an integrating factor turning the right-hand side into an exact differential. Dropping the prime and redefining the coefficients we write the resulting line element as

$$
\begin{aligned}
ds^2 &= -n^2(t,\chi)\mathrm{d}t^2 + b^2(t,\chi)\mathrm{d}\chi^2 + a^2(t,\chi)\\
&\quad\times\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right).
\end{aligned} \tag{D.1}
$$

The non-vanishing components of the torsion left by the Killing conditions are:

$$
\begin{aligned}
&T^t{}_{t\chi} = u(t,\chi),\ T^r{}_{tr} = T^\theta{}_{t\theta} = T^\phi{}_{t\phi} = -h(t,\chi)\\
&T^\chi{}_{t\chi} = \tilde{u}(t,\chi),\ T^r{}_{r\chi} = T^\theta{}_{\theta\chi} = T^\phi{}_{\phi\chi} = \tilde{h}(t,\chi)\\
&T^r{}_{\theta\phi} = r^2\sqrt{1-kr^2}\sin\theta\,\tilde{f}(t,\chi),\quad T^\theta{}_{r\phi} = -\frac{\sin\theta\,\tilde{f}(t,\chi)}{\sqrt{1-kr^2}},\\
&T^\phi{}_{r\theta} = \frac{\tilde{f}(t,\chi)}{\sin\theta\sqrt{1-kr^2}}.
\end{aligned} \tag{D.2}
$$

The 5-bein $e^A{}_\alpha$ corresponding to the metric(D.1) may be written in diagonal form by fixing the 10 local invariances generated by the Lorentz generators M_{AB} (see Eq. (2.1)). The result is

$$
\left(e^A{}_\alpha\right) = \begin{pmatrix}
n(t,\chi) & 0 & 0 & 0 & 0\\
0 & \frac{a(t,\chi)}{\sqrt{1-kr^2}} & 0 & 0 & 0\\
0 & 0 & a(t,\chi)r & 0 & 0\\
0 & 0 & 0 & a(t,\chi)r\sin\theta & 0\\
0 & 0 & 0 & 0 & b(t,\chi)
\end{pmatrix}. \tag{D.3}
$$

The 5-bein forms $e^A = e^A{}_\alpha dx^\alpha$ read

$$
\begin{aligned}
&e^1 = \frac{a(t,\chi)}{\sqrt{1-kr^2}}\mathrm{d}r,\ e^2 = a(t,\chi)r\mathrm{d}\theta,\ e^3 = a(t,\chi)r\sin\theta\mathrm{d}\varphi,\\
&e^0 = n(t,\chi)\mathrm{d}t,\qquad e^4 = b(t,\chi)\mathrm{d}\chi.
\end{aligned}
$$

To find the connection compatible with the 5-bein (D.3)] and the torsion T^A (see (D.2)), *i.e.,* a connection ω^{AB} such that (2.7) holds, is a lengthy but well-known procedure (see, *e.g.,* [33]), summarized in Appendix B. The result reads, in the 5-bein basis,

$$
\begin{aligned}
\omega^{04} &= Qe^0 - \tilde{Q}e^4,\quad \omega^{0i} = \mathbb{U}e^i,\\
\omega^{i4} &= \tilde{\mathbb{U}}e^i\quad (i = 1,2,3),\\
\omega^{12} &= -\frac{\sqrt{1-kr^2}}{ar}e^2 - \frac{\tilde{f}}{2a}e^3,\\
\omega^{13} &= -\frac{\sqrt{1-kr^2}}{ar}e^3 + \frac{\tilde{f}}{2a}e^2,\\
\omega^{23} &= -\frac{\cot\theta}{ar}e^3 - \frac{\tilde{f}}{2a}e^1.
\end{aligned}
$$

where

$$
\begin{aligned}
&Q = \frac{u}{b} + \frac{\partial_\chi n}{b\,n},\quad \tilde{Q} = \frac{\tilde{u}}{n} - \frac{\partial_t b}{b\,n},\\
&\mathbb{U} = \frac{1}{n}(H + h),\quad \tilde{\mathbb{U}} = \frac{1}{b}(\tilde{H} + \tilde{h})),\\
&H = \frac{\partial_t a}{a},\quad \tilde{H} = \frac{\partial_\chi a}{a}.
\end{aligned} \tag{D.4}
$$

From the connection and the 5-bein we can calculate the Riemann curvature R^{AB} (2.5) and the (A)dS curvature $F^{AB} = R^{AB} - \frac{s}{l^2}e^A\wedge e^B$(2.8). The result for the latter reads, in the 5-bein basis,

$$
\begin{aligned}
F^{04} &= \left(\frac{\mathbb{Q}}{b\,n} - \frac{s}{l^2}\right)e^0e^4,\\
F^{0i} &= \left(\mathbb{A} - \frac{s}{l^2}\right)e^0e^i - \mathbb{A}_1\,e^ie^4 + \mathbb{U}\frac{\tilde{f}}{2a}\,\varepsilon^i{}_{jk}\,e^je^k,\\
F^{i4} &= \left(\mathbb{B} - \frac{s}{l^2}\right)e^ie^4 + \mathbb{B}_1e^0e^i + \tilde{\mathbb{U}}\frac{\tilde{f}}{2a}\,\varepsilon^i{}_{jk}\,e^je^k,\\
F^{ij} &= \left(\mathbb{K} - \frac{s}{l^2}\right)e^ie^j - \frac{1}{2an}\partial_t\tilde{f}\,\varepsilon^{ij}{}_k\,e^0e^k\\
&\quad + \frac{1}{2ab}\partial_\chi\tilde{f}\,\varepsilon^{ij}{}_k\,e^ke^4,
\end{aligned} \tag{D.5}
$$

with

$$
\begin{aligned}
&\mathbb{Q} = -\partial_\chi(nQ) - \partial_t(b\tilde{Q}),\ \mathbb{A} = \frac{1}{a\,n}\partial_t(a\mathbb{U}) - Q\tilde{\mathbb{U}},\\
&\mathbb{A}_1 = \frac{1}{a\,b}\partial_\chi(a\mathbb{U}) + \tilde{Q}\tilde{\mathbb{U}},\ \mathbb{B} = -\frac{1}{a\,n}\partial_t(a\tilde{\mathbb{U}}) - \tilde{Q}\mathbb{U},\\
&\mathbb{B}_1 = \frac{1}{a\,n}\partial_t(a\tilde{\mathbb{U}}) - Q\mathbb{U},\quad \mathbb{K} = \frac{k}{a^2} + \mathbb{U}^2 - \tilde{\mathbb{U}}^2 - \left(\frac{\tilde{f}}{2a}\right)^2.
\end{aligned}
$$

We will also need the torsion components $T^A{}_{BC} = e^A{}_\alpha e^\beta{}_B e^\gamma{}_C T^\alpha{}_{\beta\gamma}$ in the 5-bein basis:

$$
\begin{aligned}
&T^0{}_{04} = \frac{u}{b},\quad T^4_{04} = \frac{\tilde{u}}{n},\\
&T^i{}_{0i} = \frac{h}{n},\quad T^i{}_{i4} = \frac{\tilde{h}}{b},\quad T^i{}_{jk} = \varepsilon^i{}_{jk}\frac{\tilde{f}}{2a},\\
&\quad (i,\ j,\ \ldots, = 1,2,3).
\end{aligned}
$$

As a -2-form, the torsion reads

$$
\begin{aligned}
T^0 &= \frac{u}{b} e^0 \wedge e^4, \\
T^4 &= \frac{\tilde{u}}{n} e^0 \wedge e^4, \\
T^i &= \frac{h}{n} e^0 \wedge e^i + \frac{\tilde{h}}{b} e^i \wedge e^4 + \frac{\tilde{f}}{2a} e^j \varepsilon^i{}_{jk} \wedge e^k.
\end{aligned} \tag{D.6}
$$

D.2 Field equations

Matter will be assumed to consist in a spinless perfect fluid described by the energy-momentum 4-form

$$\mathcal{T}_A = \frac{1}{4!} \varepsilon_{BCDEF} \mathcal{T}^B{}_A \, e^C \wedge e^D \wedge e^E \wedge e^F,$$

with[13]

$$\mathcal{T}^A{}_B = \mathrm{diag}\left(-\hat{\rho}(t,\chi), \hat{p}(t,\chi), \hat{p}(t,\chi), \hat{p}(t,\chi), \hat{\lambda}(t,\chi)\right), \tag{D.7}$$

and the spin 4-form $\mathcal{S}_{AB} = 0$.

With the expressions above for the curvature and torsion components and for the matter content, we can now write the explicit form of the field equations (2.6):

$$\left(\mathbb{K} - \frac{s}{l^2}\right)\left(\mathbb{B} - \frac{s}{l^2}\right) + \frac{1}{2a^2 b} \tilde{\mathbb{U}} \tilde{f} \partial_\chi \tilde{f} - \frac{l}{24\kappa} \hat{\rho} = 0, \tag{D.8}$$

$$\left(\mathbb{K} - \frac{s}{l^2}\right)\mathbb{B}_1 - \frac{1}{2a^2 n} \tilde{\mathbb{U}} \tilde{f} \partial_t \tilde{f} = 0, \tag{D.9}$$

$$\left(\mathbb{K} - \frac{s}{l^2}\right)\left(\mathbb{A} - \frac{s}{l^2}\right) - \frac{1}{2a^2 n} \mathbb{U} \tilde{f} \partial_t \tilde{f} + \frac{l}{24\kappa} \hat{\lambda} = 0, \tag{D.10}$$

$$\left(\mathbb{K} - \frac{s}{l^2}\right)\mathbb{A}_1 - \frac{1}{2a^2 b} \mathbb{U} \tilde{f} \partial_\chi \tilde{f} = 0, \tag{D.11}$$

$$
\begin{aligned}
&\left(\mathbb{K} - \frac{s}{l^2}\right)\left(\frac{1}{bn}\mathbb{Q} - \frac{s}{l^2}\right) + 2\left(\mathbb{A} - \frac{s}{l^2}\right)\left(\mathbb{B} - \frac{s}{l^2}\right) \\
&\quad + 2\mathbb{A}_1 \mathbb{B}_1 + \frac{l}{8\kappa} \hat{p} = 0,
\end{aligned} \tag{D.12}
$$

$$\left(\mathbb{K} - \frac{s}{l^2}\right)\frac{\tilde{u}}{n} + 2\left(\mathbb{B} - \frac{s}{l^2}\right)\frac{h}{n} - 2\mathbb{B}_1 \frac{\tilde{h}}{b} = 0, \tag{D.13}$$

$$\left(\mathbb{K} - \frac{s}{l^2}\right)\frac{u}{b} + 2\left(\mathbb{A} - \frac{s}{l^2}\right)\frac{\tilde{h}}{b} + 2\mathbb{A}_1 \frac{h}{n} = 0, \tag{D.14}$$

$$\frac{\tilde{f}}{a}\left(\frac{u}{b}\tilde{\mathbb{U}} + \frac{\tilde{u}}{n}\mathbb{U} - \frac{1}{bn}\mathbb{Q} + \frac{s}{l^2}\right) = 0, \tag{D.15}$$

$$\left(\mathbb{K} - \frac{s}{l^2}\right)\frac{h}{n} - \frac{1}{2a^2 n} \tilde{f} \partial_t \tilde{f} = 0, \tag{D.16}$$

$$\left(\mathbb{K} - \frac{s}{l^2}\right)\frac{\tilde{h}}{b} + \frac{1}{2a^2 b} \tilde{f} \partial_\chi \tilde{f} = 0. \tag{D.17}$$

[13] The hats on $\hat{\rho}$, etc., mean energy density, etc in 4-space.

D.3 Continuity equations

For the spinless perfect fluid considered in the previous section, the continuity equation (2.12), consequence of the field equations, takes the form of a system of two equations:

$$\partial_t \hat{\rho} + 3\left(\hat{\rho} + \hat{p}\right)\frac{\partial_t a}{a} + \left(\hat{\rho} + \hat{\lambda}\right)\frac{\partial_t b}{b} + 3\hat{p}h - \hat{\lambda}\tilde{u} = 0, \tag{D.18}$$

$$\partial_\chi \hat{\lambda} + 3\left(\hat{\lambda} - \hat{p}\right)\frac{\partial_\chi a}{a} + \left(\hat{\rho} + \hat{\lambda}\right)\frac{\partial_\chi n}{n} - 3\hat{p}\tilde{h} + \hat{\rho}u = 0. \tag{D.19}$$

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