Time-Partitioning Heuristics: Application to One Warehouse, Multiitem, Multiretailer Lot-Sizing Problems

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Abstract: We describe effective time partitioning heuristics for dynamic lot-sizing problems in multiitem and multilocation production/distribution systems. In a time-partitioning heuristic, the complete horizon of (say) N periods, is partitioned into smaller intervals. An instance of the problem is solved, to optimality, on each of these intervals, and the resulting solution coalesced into a solution for the complete horizon. The intervals are selected to be of a size which permits the use of exact and effective solution methods (e.g., branch-and-bound methods). Each interval's problem is specified to include options for starting conditions which adequately complement the solutions obtained for prior intervals. The heuristics can usually be designed to be of low polynomial complexity as well as to guarantee ϵ-optimality for any desired precision $\epsilon > 0$, and asymptotic optimality as N goes to infinity. We first give a general description of the design of time-partitioning heuristics for dynamic lot-sizing problems. We subsequently develop such a heuristic in detail, for the one warehouse multiretailer model representing a two-echelon distribution network with m retailers, selling J distinct items. A comprehensive numerical study exhibits that the partitioning heuristics are very efficient and close-to-optimal. Even problems with a planning horizon of up to 150 periods can be solved within 1.5% of optimality, employing intervals of 5–10 periods only and in a matter of CPU seconds, or up to a few minutes, using longer intervals and when the number of items and retailers is large. These CPU times refer to a SUN 4M (SPARC) workstation. © 1999 John Wiley & Sons, Inc. Naval Research Logistics 46: 463–486, 1999

1. INTRODUCTION

We describe effective time-partitioning heuristics for dynamic lot-sizing problems in one-warehouse, multiitem multiretailer systems. In a time-partitioning heuristic, the complete horizon of (say) N periods is partitioned into smaller intervals. An instance of the problem is solved, to optimality, in each of these intervals, and the resulting solutions are coalesced into a solution of the complete horizon. This approach is motivated by recent forecast horizon results for single-item models (see Chand and Morton [4], Bensoussan, Proth, and Queyranne [3], and Federgruen and Tzur [8]), which suggest that optimal or close to optimal initial decisions can be determined on the basis of relatively short horizons. The intervals are selected to be of a size which permits the use of exact and effective solution (e.g., branch-and-bound) methods. Each

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interval's problem is specified to include options for starting conditions which adequately complement the solutions obtained for prior intervals. The heuristics can usually be designed to be of low polynomial complexity, to guarantee ϵ-optimality for any precision $\epsilon > 0$, and asymptotic optimality as $N \to \infty$.

Time-partitioning heuristics bear similarity to the *space*-partitioning heuristics, successfully employed for various combinatorial problems in the plane or three-dimensional space. The first such heuristic was introduced in Karp's [16] seminal paper on the Euclidean Traveling Salesman Problem (TSP). There, the *plane* is partitioned into regions, a smaller TSP is solved on each, and the resulting tours are adequately combined into one overall travelling salesman tour. There, too, the heuristic can be designed to guarantee ϵ-optimality for any $\epsilon > 0$ and asymptotic optimality under mild probabilistic conditions. Many other routing problems have subsequently been addressed via space-partitioning heuristics (e.g., Haimovich and Rinnooy Kan [13] and Federgruen and Simchi-Levi [6]).

Surprisingly enough, horizon partitioning heuristics do not appear to have been attempted for other types of complex dynamic programs (see, e.g., Morin's [18] survey of approximation methods). To our knowledge, the idea closest to *time*-partitioning is the frequently employed approach to solve dynamic programs on a *rolling* horizon: A large, or perhaps infinite, planning horizon is truncated and replaced by a relatively short one. As each period passes, it is eliminated and replaced by a new period, appended at the end of the horizon. Only the first period decisions are retained from the solution of any given instance, and these decisions are put together over time to generate a complete solution. Such rolling horizon procedures have not been designed to guarantee any specific optimality gap, for finite N or asymptotically as $N \to \infty$. Since the basic steps of our time-partitioning heuristics described in Section 2 are generally applicable, we anticipate that similar time-partitioning heuristics will prove to be effective in solving other types of planning problems over time.

We first (Section 2) give a general description of the design of time-partitioning heuristics for dynamic lot-sizing problems. We subsequently (Section 3) develop such a heuristic in detail, for the one-warehouse multiretailer model with m retailers, selling J distinct items. All items are shipped to the retailers via a common warehouse, where they may be stored. The cost structure consists of fixed as well as variable order costs along with variable holding costs.

For the case of a single item ($J = 1$) we obtain a partitioning heuristic of complexity $O(mN^2 \log \log N)$ which can be designed to guarantee an ϵ-optimal solution for any desired $\epsilon > 0$. The heuristic is also shown to be asymptotically optimal as N tends to infinity or as N and m simultaneously tend to infinity, under some mild conditions regarding the model parameters. In the case of multiple items ($J \geq 2$) we restrict ourselves to the most prevalent case where inventories are only held to exploit economies of scale and not out of speculative motives (i.e., extreme fluctuations in the variable production or purchase cost rates). Here, a similar partitioning heuristic is obtained of complexity $O(JmN^2 \log N)$. Once again, this heuristic can be designed to guarantee ϵ-optimality, as well as asymptotic optimality. The partitioning heuristics employ effective and novel branch-and-bound methods. Section 4 reports on a comprehensive numerical study. This study exhibits that the partitioning heuristics are very efficient and close-to-optimal. Even problems with a planning horizon of up to 150 periods can be solved within 1.5% of optimality, employing intervals of 5–10 periods only and in a matter of CPU seconds, or up to a few minutes, using longer intervals and when the number of items and retailers is large. These CPU times refer to a SUN 4M (SPARC) workstation.

We complete this introduction with a brief literature review. Until recently the literature on multiitem or multifacility dynamic lot-sizing problems has focused on exact methods. With the exception of only a few simple network structures, these tend to result in exponential algorithms (see, e.g., Erickson, Monma, and Veinott [5]). A serial system model is, to our knowledge, the only type of general network topology for which a polynomial time algorithm exists (see Zangwill [27] and Love [17]). Dynamic lot-sizing models for many other basic network structures are NP-complete (see e.g., Arkin, Joneja, and Roundy [2] and the discussion in Section 2).

More recently, the focus has shifted towards cost-effective heuristics. We refer to Salomon [23] for an excellent review of this part of the literature. Except for limited numerical studies of test problems, most of these heuristics lack a rigorous assessment of their computational complexity or optimality gap. The first heuristics with known worst case bounds are due to Joneja [14, 15], for the Joint Replenishment Problem (see Section 2) and a general assembly system that results in a single finished product with external demands. In both cases a heuristic is developed whose cost under stationary cost parameters is guaranteed to be within a factor 3 of the optimal value. In Federgruen and Tzur [9, 10], we have designed time-partitioning heuristics of the above type for two specific models, the Joint Replenishment Problem and the Multiitem Capacitated model; see Section 2 for details.

2. THE GENERAL FRAMEWORK

In this section we describe the elements required in the design of a time partitioning heuristic, as well as those needed to ensure polynomial complexity, asymptotic optimality and ϵ-optimality for any arbitrarily small optimality gap ϵ. Different choices can be made regarding most or all of these elements, with different impacts on the heuristics' feasibility and performance.

We demonstrate the design and analytical evaluation of time-partitioning heuristics with the help of three basic and related multiitem dynamic lot-sizing models: the Joint Replenishment Problem (JRP), the Multiitem Capacitated Problem (MCP), and the One-Warehouse Multiretailer problem (OWMR). The first two models were analyzed in detail in Federgruen and Tzur [9, 10]. A full analysis of the OWMR model follows in Section 3. Each of these multiitem models assumes that arbitrary demands are specified for m distinct items in each of N periods and that all cost and capacity parameters may be time-dependent. Stockouts are not permitted. In all three models there are variable order and holding costs which are proportional with the order and end-of-period inventory sizes at item-dependent cost rates. The models differ in the specification of the setup cost structure, in the existence of capacity limits for the orders in each period, and as to whether the items are directly procured from an outside source with ample supply or via an intermediate facility with limited inventory. The objective, in all three models, is to minimize total (discounted) costs over the planning horizon:

The Joint Replenishment Problem (JRP): In this model, a joint setup cost is incurred whenever an order is placed (regardless of its composition) in addition to any item-specific setup costs charged for each specific item included in the order. Order quantities are unconstrained, and all items are procured directly from an outside source with ample supply.

The Multiitem Capacitated Problem (MCP): The MCP applies to settings where the different items are produced in a common facility or distributed via a common transportation mode of limited capacity. Thus, the aggregate order size in each period is constrained by a capacity limit, and a joint (but no item-specific) setup cost is incurred in each order period.

The One-Warehouse Multiretailer Problem (OWMR): In the OWMR model, items are procured from an outside source via a warehouse, where they may be stocked. Thus, in addition to the holding and variable order costs of the items at their point-of-sale, similar item-specific costs arise at the warehouse. In each period, a warehouse setup cost is incurred when the warehouse places an order, in addition to item-specific (variable and fixed) costs for each item which is shipped from the warehouse. The quantities ordered into and out of the warehouse are all unconstrained. The m items may represent the same physical product, differentiated only by their point-of-sale or "retailer" location, as in the classical OWMR model (see, e.g., Roundy [22]); in this case the holding cost and variable order cost rates at the warehouse are identical for all items. More generally, the m items may represent different finished goods held at a variety of geographical locations as, e.g., in Muckstadt and Roundy [19].

All these models are NP-complete: For the JRP and OWMR models this is shown in Arkin, Joneja, and Roundy [2]; Florian, Lenstra, and Rinnooy Kan [12] proved this for the MCP model, even for the *single-item* case. A model generalizing all three of them is the General Two-Echelon (GTE) model. It allows for a general joint cost structure for the setups associated with the orders to and from the warehouse as well as individual and aggregate capacity constraints for these orders.

The design of a time partitioning heuristic consists of the following four steps:

STEP 1: Identify the collection of intervals into which the full horizon is to be partitioned.
STEP 2: Define initial conditions for the intervals.
STEP 3: Apply or develop an exact procedure to solve the subproblem associated with each interval and solve the subproblems sequentially.
STEP 4: (Optional): Construct a solution which minimizes variable costs, while maintaining the decisions regarding fixed costs (e.g., the order periods for all items and facilities) according to the solution obtained in Step 3.

Step 1: Identify the Intervals

The intervals are chosen to be consecutive collections of periods. Initially we assume that they are nonoverlapping; see, however, the discussion below. In later steps, a subproblem is associated with each interval obtained from the restriction of the complete problem to its periods, i.e., treating the first and last period of the interval as the first and last period of the subproblem and leaving all other parameters unchanged. In Step 2 we add for each interval options to choose alternative starting conditions that appropriately complement the solutions obtained in prior intervals. The subproblems are thus solved sequentially, with the solution of the first h (say) subproblems possibly used in the specification of the initial conditions for the next subproblem; see Step 2. Several issues need to be considered.

First, assuming that the complete problem is feasible, intervals need to be chosen to ensure that each of the associated subproblems is feasible as well. This can most easily be done by ensuring that each subproblem has a feasible solution without any of the options for alternative starting conditions, as created in Step 2. In some models, every interval is feasible, i.e., it results in a feasible subproblem, e.g., the JRP and OWMR models. This fails to apply to the MCP problem, but a simple (linear time) test can be applied to verify whether an interval is feasible or not. In other models, a more complex procedure is required, e.g., the GTE problem where feasibility can be verified with the help of a max-flow algorithm.

In settings where some intervals may be associated with infeasible subproblems, we obtain a complete partition of feasible intervals by identifying a shortest path in an acyclic network which has a node for each of the N periods, an extra node $N + 1$ and an arc between every pair of nodes (i,j) with $i < j$. The arc costs $\{f_{ij}: 1 \le i < j \le N + 1\}$ are to be chosen such that $f_{ij} = \infty$ whenever the interval $[i, i + 1, \ldots, j)$ is infeasible. Note that a path of finite length, e.g., with the single arc $[1, N + 1]$, exists whenever the complete problem is feasible. For feasible intervals, f_{ij} can be chosen arbitrarily, perhaps to induce other desired properties of the partition, such as interval lengths that are as close as possible to a desired target length.

The number of intervals and their lengths have a major impact on the complexity of the overall procedure, as well as the quality of the resulting solution. Computational requirements are in general reduced by selecting more intervals of smaller length, in particular since each subproblem needs to be solved by an (exact) procedure whose complexity is at least exponential in its horizon length. On the other hand, selecting more intervals of smaller length, in general, results in inferior solutions since an additional set of restrictions is imposed by each interval.

To ensure that the overall complexity of the procedure is polynomial requires that the length of each interval be asymptotically bounded by an appropriate function of N which can be derived from the complexity bound of the exact procedure used to solve the subproblem. For example, in all three of the above discussed models, an exact branch-and-bound procedure exists, capable of solving problems for intervals of length n in $O(2^n P(n,m))$ time with $P(n,m)$ polynomial in n and m. Thus choosing all interval lengths $n \sim \log_2 N$ (as $N \to \infty$), results in an overall complexity of $O(N^2 P(\log_2 N, m)/\log_2 N)$ since the number of intervals $I \sim N/\log_2 N$ under this choice. [We write $f(N) \sim g(N)$ if $\lim_{N\to\infty} f(N)/g(N) = 1$.] For the JRP and OWMR models we have $P(n,m) = O(mn \log n)$, hence resulting in an overall complexity of $O(mN^2 \log_2 \log_2 N)$; for the MCP model, $P(n,m) = O((\log_2 \text{cap*})m^2 n^2)$ with cap* the largest capacity value, which results in an overall complexity of $O((\log_2 \text{cap*})m^2 N^2 \log_2 N)$.

For the GTE model, problems with an interval of length n can be solved in $O(2^{nm} P(n,m))$ time (again by a branch-and-bound procedure). Thus, in order to ensure that the overall procedure be polynomial, it is now necessary to choose $n \sim \log_2(N/m)$, which remains practical as long as m is not too large compared with N. (In the alternative case, an effective procedure may call for the partitioning of the product line into families, along with the partitioning of time into intervals.)

Often we may desire to design the heuristic so as to guarantee an ϵ-optimal solution for a prespecified optimality gap ϵ. As will be discussed later, such designs are often possible for *any* $\epsilon > 0$, i.e., the heuristic can be designed as a fully polynomial approximation scheme; ϵ-optimality can often be guaranteed by choosing the interval lengths to be bounded from below by a constant $Y_1(\epsilon)$, which can easily be computed upfront, as a function of the model parameters.

To simultaneously ensure that feasible intervals are chosen and that their length is $\sim \log_2 N$ or any other function of N ensuring polynomial complexity, one may, e.g., specify that the interval lengths be bounded from below by $\underline{n} = \lceil(\log_2 N)\rfloor$ and from above by $\bar{n} = \underline{n} + Y_2$, with Y_2 an arbitrary integer. In this case, the above shortest path problem is solved with $f_{ij} = \infty$ when the interval $[i,j)$ is infeasible, or its length outside these bounds. (In a highly constrained problem, it is advisable to choose a large value for Y_2. If even with this choice of Y_2, no path with finite length is found, remove the upper bound $\bar{n}$, thus ensuring that a feasible partition is generated, albeit that asymptotic polynomial complexity is no longer guaranteed. Note, however, that such highly constrained problems have relatively few feasible solutions and are thus more easily solvable by methods based on implicit enumeration such as branch-and-bound or column generation.) To further ensure that an ϵ-optimal solution is obtained, it suffices to replace $\underline{n}$ as described above, by $\underline{n} = \max\{Y_1(\epsilon), \log_2 N\}$.

Step 2: Define Initial Conditions for the Intervals

A subproblem is associated with each interval, obtained from the restriction of the complete problem to its periods. As such, since in each subproblem it is assumed that the system starts out empty, the partition forces each of the interval's demands to be satisfied from orders placed *within* the interval. In other words, order cycles are not allowed to cross interval boundaries, even though these boundaries are created in Step 1 without any consideration as to where they would most economically be located. Recall that the subproblems are solved sequentially. To create significantly superior solutions, it may therefore be desirable to allow some of the demands in a given interval to be satisfied from orders placed in a few selected periods preceding the interval. Especially beneficial in this respect are periods which, in the solution of the preceding interval problems, have been used as attractive order (production/distribution) periods, in particular when economies of scale can be exploited (e.g., due to fixed costs, quantity discounts, etc.) by enlarging those order volumes, if possible.

For example, in a time partitioning heuristic for the JRP model, it makes sense to specify initial conditions of the hth interval problem so as to allow demands for a given item to be satisfied either (i) by enlarging *its* last order in the first $h - 1$ intervals, without incurring an additional joint or an additional item-specific setup cost, or (ii) by including an order for this item to the last overall order placed in the first $h - 1$ intervals, without incurring an additional joint setup cost, i.e., incurring only an additional *item*-specific setup cost. With appropriate choices of the cost parameters, these options can be made available by adding only two dummy periods, with zero demands, at the beginning of the hth interval. In the OWMR model, since no *joint* setup costs prevail, only one dummy period is needed. In the MCP model, on the other hand, the required dummy periods are the last order periods in the first $h - 1$ intervals, for the different items, since different residual capacity limits may prevail for these last order periods. Thus, up to m dummy periods are added to each interval; however, the addition of these dummy periods does not affect the size of the branch-and-bound tree by which each subproblem is solved, since no (additional) setup costs are incurred in the dummy periods. Thus, the addition of the dummy periods has minimal impact on the worst case complexity bound of the entire procedure and even less on its practical performance.

Step 3: Apply or Develop an Exact Procedure To Solve the Subproblem Associated with Each Interval and Solve the Subproblems Sequentially

Any exact solution method may be employed. For example, for the JRP problem, several efficient methods have been developed recently, capable of solving subproblems with up to 40 periods and 100 products (see Raghavan and Rao [21], Federgruen and Tzur [9], and Stowers and Palekar [24]). The former consists of a tailored cutting plane method, and the latter two propose branch-and-bound methods. In designing a branch and bound method for dynamic lot-sizing problems, it is noteworthy that only some of the binary variables need to be fixed to result in a significantly simplified problem. For example, for the JRP it is only necessary to branch on the n variables describing in which periods the major setup cost is incurred, thanks to the availability of very efficient solution methods for uncapacitated *single-item* problems. On the other hand, in the GTE problem, to obtain a branch and bound tree in which the leaves are easily solvable, it is necessary to branch on all nm binary variables describing for which of the m products or facilities an order is placed in each of the n periods involved. [It is this distinction which explains the difference in the complexity bounds, $O(2^n P(n,m))$ and $O(2^{nm} P(n,m))$ for the two problems, respectively; see the discussion in Step 1.] For the GTE model or network models of even larger complexity, it may no longer be possible to solve the interval problems exactly for reasonable choices of interval lengths. We refer to Tzur [25, Chap. 8] for a discussion of two approaches (network restriction and decomposition) to solve the interval problems approximately but efficiently, instead.

Step 4 (Optional): Construct a Solution Which Minimizes Variable Costs, While Maintaining the Decisions Regarding Fixed Costs (i.e., the Order Periods for All Items and Facilities) According to the Solution Obtained in Step 3

The need to partition the overall planning horizon into smaller intervals, arises almost invariably because of the complexity of determining some of the *binary* variables (see also our discussion in Step 3). With these binary variables being fixed separately for each of the subintervals (by the solution method chosen in Step 3), all other variables are easily determined so as to optimize the remaining costs over the *entire* planning horizon. (When a branch and bound method is used in Step 3, the latter problem is usually of the type required to evaluate its leaves.) Note that Step 4 results in a solution at least as good as the one obtained after solving the subproblems sequentially.

As a simple example, consider again the JRP. Once the set of order periods with orders is determined by the sequential solution of all subproblems, the optimal composition of these orders (i.e., the items included and their specific order quantities) are easily determined to minimize the remaining costs over the *entire* planning horizon; the latter problem reduces to the solution of m independent single-item dynamic lot-sizing problems.

Step 1 specifies that the planning horizon be partitioned into nonoverlapping intervals. Instead we may wish to allow for considerable overlap between consecutive intervals, e.g., by including some of the last periods (perhaps the second half) of a given interval in the next one. This variant would continue to append dummy periods at the beginning of each interval to allow for starting conditions that appropriately complement the solutions obtained in prior intervals.

The advantage of this variant is that *all* order decisions are determined by looking ahead at future cost and demand parameters pertaining to a desired number of future periods (at least). Note that the order of the complexity of this variant remains unchanged. For example, the complexity increases by a factor of two (approximately) if each interval includes the second half of the prior interval (and the total interval length is determined as described above).

Finally, to characterize the optimality gap incurred by the partitioning heuristic, it is convenient to bound the additional cost required to transform an optimal solution into one achievable by the heuristic. These transformations "truncate" procurement decisions which cut across interval boundaries in ways not considered by the partitioning heuristic. This bound for the *absolute* optimality gap is therefore related to (and often linear in) the number of intervals I. Together with a comparable *lower bound* for the optimal value and a desired *relative* optimality gap ϵ, this allows for the determination of the required number of intervals $I(\epsilon)$; see Step 1.

3. THE ONE-WAREHOUSE MULTIRETAILER PROBLEM

In the general OWMR model, there are m retailers selling J distinct items. All items are shipped to the retailers via a common warehouse. We denote the warehouse as facility O and retailer i as facility i, $i = 1, \ldots, m$. Whenever one of the retailers or the warehouse places an order, a fixed setup cost is incurred which is specific to the facility involved but independent of its specific composition.

This cost structure applies to many settings. The fixed costs incurred when a retailer places an order often consist of the cost of processing a purchase order or that of a shipment by a common carrier or company owned truck. On the other hand, sometimes more complex joint setup cost structures prevail, e.g., when, in addition to the basic setup cost for a retailer order, additional setup costs need to be added for each specific item included in the order (as assumed, for example, in the Muckstadt and Roundy model [19]). Even more general, nonseparable setup cost structures are needed in other settings, e.g., where the total setup cost associated with the retailer orders in a given period is given by a general set function of the set of retailers and items involved (see Federgruen and Zheng [11] and Federgruen and Tzur [9]). Any such more complex setup cost structures, possibly combined with individual and aggregate capacity constraints for the orders, result in special cases of the more complex GTE model (see Section 2).

To simplify the exposition, we initially consider the case of a *single* item, i.e., where $J = 1$; see, however, subsection 3.6 for a treatment of the general case.

3.1. The Model

For $J = 1$, the (OWMR) problem is specified by the following parameters:

N = number of periods,
m = number of retailers.

For $i = 0, \dots, m$ and $t = 1, \dots, N$:

d_{it} = demand at retailer i in period t (we assume, without loss of generality, that $d_{it} \geq 0$),
d_{0t} = total system-wide demand in period t $(= \sum_{i=1}^{m} d_{it})$,
K_{it} = fixed cost for an order placed by facility i in period t,
c_{it} = variable per unit order cost for facility i in period t,
h_{it} = cost of carrying a unit of inventory at facility i at the end of period t.

We first describe a mathematical programming formulation for this problem. For all $t = 1, \dots, N$ and $i = 0, \dots, m$, let

X_{it} = the number of units ordered by facility i in period t,
I_{it} = the inventory at retailer i at the end of period t,
I_{0t} = the echelon inventory (i.e., the total inventory in the *system*) at the end of period t,

$$Y_{it} = \begin{cases} 1, & \text{if facility } i \text{ places an order in period } t, \\ 0, & \text{otherwise.} \end{cases}$$

$$(P)\ z^* = \min \sum_{t=1}^{N} \{ \sum_{i=0}^{m} K_{it} Y_{it} + \sum_{i=0}^{m} c_{it} X_{it} + h_{0t} I_{0t} + \sum_{i=1}^{m} (h_{it} - h_{0t}) I_{it} \} \tag{1}$$

$$\text{s.t. } I_{i,t-1} + X_{it} = d_{it} + I_{it} \qquad (i = 0, \dots, m, \quad t = 1, \dots, N), \tag{2}$$

$$\sum_{s=1}^{t} X_{0s} \geq \sum_{s=1}^{t} \sum_{i=1}^{m} X_{is} \qquad (t = 1, \ldots, N), \tag{3}$$

$$X_{it} \leq (\sum_{s=t}^{N} d_{is})\, Y_{it} \qquad (i = 0, \ldots, m, \quad t = 1, \ldots, N). \tag{4}$$

Constraints (2) are the usual inventory balance equations, specified for each retailer as well as for the system as a whole. The second set of constraints ensures that the cumulative amount ordered by the warehouse up to any period t is at least as large as the cumulative amount ordered by the retailers. The last set of constraints specifies the standard relationship between the X- and Y-variables. The four terms in the objective function denote the total setup costs, the variable order costs, the basic holding costs charged against systemwide inventory, and the incremental holding costs incurred for inventories stored at the retailers, respectively. It is easily verified that an optimal solution exists with zero inventory ordering, i.e., $X_{it}I_{it-1} = 0$ for all i, t.

3.2. Models with Prespecified Warehouse Order Periods

In this subsection we address the special case of the model which arises when S^+, the set of warehouse order periods, is prespecified. This special case is of interest by itself: It includes settings where no warehouse setup costs prevail, i.e., $K_{0t} = 0$ for all $t = 1, \ldots, N$ so that $S^+ = \{1, \ldots, N\}$. It is also used in the lower bounds described in subsections 3.3 and 3.4.

With a prespecified set of warehouse order periods, the system decomposes into m independent single item dynamic lot sizing models (one for each retailer), maintaining the retailers' fixed and holding cost parameters as well as demands, and substituting their variable order cost rates $\{c_{it}: i = 1, \ldots, m, t = 1, \ldots, N\}$ by

$$c'_{it} = c_{it} + \min\{c_{0s} + \sum_{r=s}^{t-1} h_{0r}: 1 \leq s \leq t \text{ and } s \in S^+\} \qquad i = 1, \ldots, m, \quad t = 1, \ldots, N. \tag{5}$$

The expression within curled brackets in (5) represents the variable warehouse order and holding costs per unit, if this unit reaches the warehouse in period $s \leq t$, and is kept there until the end of period $t - 1$. Therefore c'_{it} represents the total variable order and warehouse holding cost rate for any unit received by retailer i in period t, considering all possible periods in which the unit may be ordered by the warehouse. The optimum value of the model is obtained by adding $\sum_{t \in S^+} K_{0t}$ to the sum of the optimum values of the individual single-item dynamic lot sizing problems.

We conclude that while the OWMR model is NP-complete, an optimal schedule may be computed in $O(mN \log N)$ time when the set of warehouse order periods is prespecified (see Aggarwal and Park [1], Federgruen and Tzur [7], and Wagelmans, Van Hoesel, and Kolen [26]).

3.3. Lower Bounds

We first develop a lower bound for the minimum cost value z^*, to be used in the branch-and-bound procedure, or to evaluate the heuristic described below.

The lower bound Z_{LB} is obtained by complete or Lagrangean relaxation of the coupling constraints (3) in (P). Note that the relaxed problem decomposes into $m + 1$ independent single-item dynamic lot sizing models. The relaxed problem can thus be solved in $O(mN \log N)$ time (for given values of the Lagrange multipliers). Let $\lambda_t^{(k)}$ represent the Lagrange multiplier for the tth constraint in the kth iteration. These values may be recursively updated via

$$\lambda_t^{(k)} = \lambda_t^{(k-1)} + k^{-1}\left[\sum_{s=1}^{t} \sum_{i=1}^{m} X_{is}^{*(k-1)} - \sum_{s=1}^{t} X_{0s}^{*(k-1)}\right],$$

where $X_{it}^{*(k)}$ denotes the optimal value of X_{it} after the kth iteration.

An alternative type of lower bound may be developed as follows: Since demands are deterministically known, it is possible, without loss of optimality, to allocate every incoming order of the warehouse to the individual retailers. The warehouse may thus be viewed as consisting of m separate depots, each designed to serve a specific retailer. Assume now that the warehouse setup costs are allocated in some arbitrary way to these depots (so that the sum of the allocated costs equals the true warehouse costs). This allocation scheme results in a lower bound, since the cost charged under any given strategy is lower than or equal to the costs incurred under the true cost structure. Moreover, under the allocated costs scheme the system decomposes into m independent tandem systems for which an optimal procurement strategy can be computed in polynomial time (see Zangwill [27] and Love [17]). The best lower bound of this type may be obtained by maximizing over all feasible setup cost allocations (see Tzur [25] for details). Indeed, a lower bound of this specific type was successfully used in the branch and bound procedure and partitioning heuristic for the JRP (see Federgruen and Tzur [9] for details).

3.4. An Exact Branch-and-Bound Method

We now describe an exact branch-and-bound procedure. This procedure can be used by itself for problems of moderate size; more importantly, it is used to solve the subproblems which arise in the partitioning heuristic described below.

Given a choice for S^+, the set of periods in which the warehouse places an order, the remaining problem reduces to the special case discussed in subsection 3.2. This implies that the problem may be solved by enumerating all 2^{N-1} possible sets of warehouse order periods, evaluating for each the associated m single item lot sizing problems. The complexity of a *full* enumeration scheme is therefore $O(m2^N N \log N)$, which is prohibitive for all but small values of N. In its stead, we show that a branch-and-bound procedure can be used as an attractive *implicit* enumeration method.

Each node of the branch-and-bound tree is characterized by a partition of the set $\{1, \ldots, N\}$ into three sets S^+, S^-, and S^0, where S^+ is the set of periods in which the warehouse is committed to placing an order, S^- is the set of periods in which no warehouse order is allowed, and $S^0 = \{1, \ldots, N\} \backslash (S^+ \cup S^-)$. Note that $1 \in S^+$ in any feasible partition.

To evaluate a node in the tree, we compute the lower bound, appropriately modified to incorporate the restrictions implied by the sets S^+ and S^- as follows: Replace the parameters $\{K_{0t}\}$ by $K'_{0t} = 0$ if $t \in S^+$; $K'_{0t} = \infty$ if $t \in S^-$; $K'_{0t} = K_{0t}$ otherwise, and add $\sum_{t \in S^+} K_{0t}$ to the Lagrangean dual value. The Lagrangean dual is itself an *upper* bound for z^* if its associated solution $\{X_{it}^*: i = 0, \ldots, m,\ t = 1, \ldots, N\}$ is feasible in (P). If this solution fails to be feasible, an upper bound for z^* is obtained by resolving the single item dynamic lot sizing model associated with the warehouse, replacing the demand values $\{d_{0t}\}$ by $\{\sum_{i=1}^{m} X_{it}^*\}$. At any stage of the branch-and-bound procedure, the best available upper bound may be used to eliminate parts of the tree. Nodes at the bottom of the tree have $S^0 = \emptyset$ so that S^+ represents the set of periods in which the warehouse places an order. The optimum value for this choice of S^+ is determined as described in subsection 3.2.

Every node in the tree has two successor nodes; the first (second) successor node has an additional period in S^0 shifted to S^- (S^+). The branch-and-bound procedure is thus completely specified by the choice of the branching rule, i.e., a rule to select a period from the set S^0. Here we confine ourselves to the description of one branching rule, which employs a fixed ranking of the periods $\{2, \ldots, N\}$ and chooses the highest ranked period in S^0: For any set of periods $S \subseteq \{2, \ldots, N\}$ let $\Phi(S)$ denote the minimum cost when a warehouse order is placed in each of the periods in $S \cup \{1\}$ (and no order is placed in any other period). Evaluation of the function $\Phi(\cdot)$ reduces to solving a one warehouse multiretailer problem with *zero* warehouse setup costs, which can be solved in $O(mN \log N)$ time as described in subsection 3.2. Our branching rule starts with $S^0 = \{2, \ldots, N\}$ and $S^+ = \{1\}$ and branches any given node (specified by the triple $\{S^+, S^-, S^0\}$) into successor nodes by shifting a period j from S^0 into S^+ with $\Phi(S^+ \cup \{j\}) = \min_{i \in S^0} \Phi(S^+ \cup \{i\})$. Alternatively, one may shift a period j from S^0 into S^- with $\Phi(S^+ \cup S^0 \backslash \{j\}) = \min_{i \in S^0} \Phi(S^+ \cup S^0 \backslash \{i\})$.

3.5. The Partitioning Heuristic

We now develop and analyze our proposed time-partitioning heuristic, following the steps described in Section 2. We partition the complete horizon $\{1, \ldots, N\}$ into I intervals of lengths $n_1, n_2, \ldots, n_I$ (i.e., $\sum_{h=1}^{I} n_h = N$). OWMR$_h$ denotes the one-warehouse multiretailer problem associated with the hth interval. Let $N_h = \sum_{k=1}^{h} n_k$, $h = 1, \ldots, I$. To specify OWMR$_h$ for some $h = 2, \ldots, I$, let $\ell_i(N_{h-1})$ denote the last order period for facility i in the partial solution constructed thus far, i.e., the solution constructed from OWMR$_1$ up to OWMR$_{h-1}$, $i = 0, \ldots, m$. OWMR$_h$ consists of $(n_h + 1)$ periods: the periods $N_{h-1} + 1, \ldots, N_h$ preceded by a dummy period -1 with zero demands, which is appended to allow for starting conditions that appropriately complement the solutions obtained in prior intervals.

An order in period -1 for a facility i ($i = 0, \ldots, m$) represents an addition to the order placed in period $\ell_i(N_{h-1})$ so as to cover demands of some of the initial (or possibly all) periods in the hth interval. We therefore specify the setup costs of period -1 to be zero.

A unit ordered by retailer i in period $\ell_i(N_{h-1})$ and kept there until the hth interval, incurs until the end of the $(h - 1)$st interval a total systemwide variable cost which is denoted by $c_{i,-1}$. It follows from the optimality of a zero inventory ordering policy (see subsection 3.1) that such a unit reaches the warehouse in period $\ell_i^0(N_{h-1})$, the last warehouse order period preceding period $\ell_i(N_{h-1})$ (possibly $\ell_i(N_{h-1})$ itself). Thus $c_{i,-1} = c_{i,\ell_i(N_{h-1})} + \sum_{t=\ell_i^0(N_{h-1})}^{N_{h-1}} (h_{it} - h_{0t})$ is the variable order cost for retailer i in period -1, so that the holding cost rate $h_{i,-1} = 0$ for all $i = 1, \ldots, m$.

A unit ordered by the warehouse in period $\ell_0(N_{h-1})$ and kept there until the hth interval, incurs until the end of the $(h - 1)$st interval a total variable cost of $c_{0,-1} = c_{0,\ell_0(N_{h-1})} + \sum_{t=\ell_0(N_{h-1})}^{N_{h-1}} h_{0t}$. We use $c_{0,-1}$ as the variable order cost rate for the warehouse in period -1, and set $h_{0,-1} = 0$.

After solving OWMR$_h$ *exactly* (e.g., via the branch-and-bound method described in subsection 3.4), we update the partial solution for the entire horizon. This can be done as follows: For all $i = 0, \ldots, m$ and $t = 1, \ldots, N$ let

$X_{it} =$ the order quantity for facility i in period t in the partial solution obtained at the end of the $(h - 1)$st iteration.

For the optimal solution of OWMR_h, let

$$X_{it}^{(h)} = \text{order quantity for facility } i \text{ in period } t \ (t = -1, N_{h-1} + 1, N_{h-1} + 2, \ldots, N_h).$$

Set

$$\ell_i(N_h) = \begin{cases} \ell_i(N_{h-1}) & \text{if } X_{i,-1}^{(h)} > 0 = X_{it}^{(h)} \text{ for } t \neq -1, \\ \max\{t\colon X_{it}^{(h)} > 0\} & \text{otherwise,} \end{cases}$$

$$X_{it} := X_{it}^{(h)} \qquad t = N_{h-1} + 1, \ldots, N_h,$$

$$X_{i,\ell_i(N_{h-1})} := X_{i,\ell_i(N_{h-1})} + X_{i,-1}^{(h)}$$

Finally, let $z(\mathrm{OWMR}_h)$ denote the optimal cost for problem OWMR_h. We conclude:

LEMMA 1: The solution obtained by the partitioning heuristic is feasible and has a cost value $z^H \equiv \sum_{h=1}^{I} z(\mathrm{OWMR}_h)$.

We now derive a worst case bound for the optimality gap which arises when I intervals are used in the partitioning heuristic. We do so under mild conditions with respect to the cost and demand parameters. We first need to derive a lower bound for z^*.

THEOREM 1 (The lower bound theorem): Assume there exist for all $i = 0, \ldots, m$ constants K_{i*}, $d_{i*} > 0$, h_{i*} and c_{i*} such that for all $t \geq 1$ and all $i = 0, \ldots, m$, $d_{it} \geq d_{i*}$, $K_{it} \geq K_{i*}$, $h_{it} \geq h_{i*}$, $c_{it} \geq c_{i*}$.

Then, $z^* \geq \gamma_1 N$, where γ_1 is Roundy's [22] lower bound for the long run average cost in the one-warehouse multiretailer system with *stationary* demand rate d_{i*}, holding cost rate h_{i*}, variable order cost rate c_{i*}, and fixed order cost K_{i*} for retailer i ($i = 1, \ldots, m$) and *stationary* holding cost rate h_{0*}, variable cost rate c_{0*} and fixed order cost K_{0*} for the warehouse.

PROOF: Let $\underline{z}(N)$ denote the minimum cost over the planning horizon of N periods when all cost and demand parameters are replaced by their stationary lower bounds. Clearly $z^* \geq \underline{z}(N) \geq \gamma_1 N$ as follows from the proof of the lower bound theorem in Roundy [22]. □

REMARK 1: The value of γ_1 can be computed in $O(m \log m)$ time, or even in $O(m)$ time when implemented as in Queyranne [20]. This lower bound is guaranteed to come within 6% of the minimum cost value of the stationary version of the one-warehouse multiretailer system.

REMARK 2: For items with a highly seasonal demand, the assumption of a positive uniform lower bound for all periods' demands may be too restrictive. It may be relaxed to assuming that integers $M_i \geq 1$ exist for all $i = 1, \ldots, m$ such that

$$(d_{it} + \cdots + d_{i,t+M_i}) \geq M_i d_{i*} \qquad \text{and} \qquad \sum_{t=1}^{N} d_{it} \geq N d_{i*} \qquad (i = 1, \ldots, m, \quad t = 1, \ldots, N).$$

Under the relaxed assumption, Theorem 1 continues to hold (see Tzur [25]), i.e., $z^* \geq \gamma_2 N$, where

$$\gamma_2 = \max\left\{ \sum_{i\geq 0: M_i=1} \sqrt{2K_{i*}h_{i*}d_{i*}} + \sum_{i\geq 0: M_i\geq 2} \psi_i(K_{i*}) \right\} + \sum_{i=1}^{m} c_{i*}d_{i*},$$

$$\psi_i(K_{i*}) = \begin{cases} K_{i*}/2M_i & \text{if} \quad 0 \leq K_{i*} \leq M_i^2 h_{i*} d_{i*}, \\ \sqrt{2(K_{i*} + h_{i*}M_i^2 d_{i*})h_{i*}d_{i*}} - 1.5 h_{i*} M_i d_{i*} & \text{if} \quad K_{i*} > M_i^2 h_{i*} d_{i*} \end{cases}.$$

We are now ready to derive a worst case bound for the heuristic's optimality gap.

THEOREM 2: Let I denote the number of intervals employed by the partitioning heuristic. Assume there exist for all $i = 1, \ldots, m$ an integer $M_i \geq 1$ and constants K_0^*, K_{0*}, K_i^*, K_{i*}, d_i^*, d_{i*}, c_i^*, c_{i*}, c_0^*, c_{0*}, and h_{0*} such that, for all $t \geq 1$ and all $i = 1, \ldots, m$,

$$(d_{it} + \cdots + d_{i,t+M_i}) \geq M_i d_{i*}, \qquad d_{it} \leq d_i^*, \qquad \sum_{t=1}^{N} d_{it} \geq N d_{i*},$$

$$K_{0*} \leq K_{0t} \leq K_0^*, \qquad K_{i*} \leq K_{it} \leq K_i^*, \qquad c_{i*} \leq c_{it} \leq c_i^*, \qquad c_{0*} \leq c_{0t} \leq c_0^*, \qquad h_{0*} \leq h_{0t}.$$

Let $\eta = \max_{i\geq 1} \{c_i^* + c_0^* - c_{i*} - c_{0*} - h_{0*}\}$. Then

$$\frac{z^H - z^*}{z^*} \leq \frac{(I-1)}{N} \frac{\rho}{\gamma},$$

where

$$\rho = \sum_{i=0}^{m} K_i^* + \eta \left[L \sum_{i=1}^{m} d_i^* + \left(\sum_{i=0}^{m} K_i^*\right) \Big/ h_{0*} \right]$$

$$L = \lceil [(c_0^* - c_{0*}) + \max_i \{c_i^* - c_{i*}\}]/h_{0*} \rceil \quad \text{and} \quad \gamma = \begin{cases} \gamma_1 & \text{if all } M_i = 1, \\ \gamma_2 & \text{otherwise.} \end{cases}$$

PROOF: We show that $(z^H - z^*) \leq (I - 1)\rho$. The theorem then follows from Theorem 1. Consider an optimal solution of the one-warehouse multiretailer problem on the entire N-period horizon. We show that this solution can be transformed into one which is achievable by the partitioning heuristic, adding at most $(I - 1)\rho$ to the total cost.

If the optimal solution to the complete OWMR problem fails to be achievable by the partitioning heuristic, there must be an interval (say interval h) in which, in some periods, units are demanded at some retailers which enter the warehouse *before* the start of the hth interval. We refer to those units as the carry-over units of the hth interval. We transform the solution by postponing the ordering of *all* such units by the warehouse till period $N_{h-1} + 1$ (the first period of the hth interval); likewise, we postpone till period $N_{h-1} + 1$ the transfer from the warehouse to the relevant retailer of those carryover units which in the (original) optimal solution are transferred *before* the beginning of the hth interval, and maintain the transfer period for all other carryover units. We show that the additional cost incurred due to the transformations for the carryover units of this interval is bounded by ρ. Since there are carryover units in at most $(I - 1)$ intervals, the total incremental cost due to *all* transformations is then bounded by $(I - 1)\rho$.

Let $S = \{i_1, \ldots, i_{|s|}\}$ denote the set of retailers with carry-over units in the hth interval and let t_ℓ denote the last period in the hth interval for which some of the units demanded at retailer i_ℓ are carried over. Assume the retailers in S are numbered such that $t_1 \leq t_2 \leq \cdots \leq t_{|s|}$. Let D^i $(i \in S)$ denote the total number of carryover units for retailer $i \in S$. Renumber the periods in the hth interval from $1, \ldots, n_h$ and let $S_\ell = \{i_\ell, \ldots, i_{|S|}\}$. The transformations for the carryover units of interval h add at most

$$\sum_{i=0}^{m} K_i^* + \sum_{i \in S} (c_i^* + c_0^* - c_{i*} - c_{0*} - h_{0*})\, D^i \tag{6}$$

in setup and variable cost, since for each of the D^i carryover units of retailer i at most $(c_i^* + c_0^* - c_{i*} - c_{0*})$ in additional order cost is incurred and at least one period's echelon holding cost is saved. It thus suffices to obtain bounds for the quantities D^i $i \in S$. We derive in Lemma 2 in the Appendix bounds of the type

$$\sum_{i \in S_\ell} D^i \leq b_\ell, \qquad \ell = 1, \ldots, |S| \tag{7}$$

and show that ρ is an upper bound for the maximum of (6) over the bounded polyhedron described by (7). □

In many practical settings, no speculative motives prevail for carrying inventories at any of the retailers or at the warehouse, i.e.,

$$c_{it} \leq c_{is} + h_{is} + \cdots + h_{it-1} \qquad \text{for all } s < t \text{ and all } i = 0, \ldots, m. \tag{8}$$

The expression for ρ simplifies considerably in this case and fewer conditions need to be imposed on the demand and cost parameters. This follows immediately from the proof of Theorem 2.

COROLLARY 1: Let I denote the number of intervals employed by the partitioning heuristic. Assume no speculative motives prevail for carrying inventories at any of the retailers or the warehouse. Assume there exist for all $i = 1, \ldots, m$ an integer $M_i \geq 1$ and constants K_0^*, K_{0*}, K_i^*, K_{i*}, $d_{i*} > 0$, and c_{i*} such that for all $t \geq 1$ and all $i = 1, \ldots, m$:

$$(d_{it} + \cdots + d_{i,t+M_i}) \geq M_i d_{i*}, \qquad \sum_{t=1}^{N} d_{it} \geq Nd_{i*},$$

$$K_{0*} \leq K_{0t} \leq K_0^*, \qquad K_{i*} \leq K_{it} \leq K_i^*, \qquad 0 < c_{i*} \leq c_{it}.$$

Then, $(z^H - z^*)/z^* \leq (I - 1)\ \rho'/(N\gamma)$, where $\rho' = \sum_{i=0}^{m} K_i^*$.

Possible Choices of Interval Lengths for the Partitioning Heuristic

Theorem 2 and Corollary 1 along with our discussion regarding Step 1 in Section 2 suggest the following choice for the interval lengths n_h $(h = 1, \ldots, I)$ to be employed in the partitioning heuristic:

$$n_h = \max\{Y, \log N\}, \qquad h = 1, \ldots, I - 1, \tag{9}$$

$$n_I = N - \sum_{h=1}^{I-1} n_h \tag{10}$$

with Y an arbitrary integer.

Theorem 2 shows that an ϵ-optimal solution may be guaranteed by choosing $Y = \min\{\lceil \rho/\epsilon\gamma \rceil, N\}$.

COROLLARY 2: Assume the parameter conditions of Theorem 2 are satisfied. The partitioning heuristic results in an ϵ-optimal solution for any given $\epsilon > 0$ if the intervals n_h $(h = 1, \ldots, I)$ are specified as in (9) and (10) and $Y = \min\{\lceil \rho/\epsilon\gamma \rceil, N\}$.

PROOF: If $Y = N$, then $I = 1$ and an optimal solution is achieved. Otherwise, it follows from (9) that $(I - 1) \leq N/Y \leq N\epsilon\gamma/\rho$; ϵ-optimality then follows from Theorem 2. $\square$

Theorem 2 also allows us to conclude that the partitioning heuristic is asymptotically optimal as N increases to infinity.

COROLLARY 3: Consider the partitioning heuristic with interval lengths specified by (9) and (10).

(a) The heuristic has complexity $O(mN^2 \log \log N)$.
(b) Assume the parameter conditions of Theorem 2 are satisfied. For fixed m, the heuristic is asymptotically optimal as N increases to infinity.
(c) Assume the parameter conditions of Theorem 2 are satisfied with values K_i^*, d_i^*, c_i^* uniformly bounded in i, and K_{i*}, d_{i*}, c_{i*} uniformly bounded away from zero. The heuristic is asymptotically optimal as $m \to \infty$ and $N \to \infty$.

PROOF:

(a) The partitioning heuristic requires, to compute its solution for the hth interval, at most 2^{n_h} solutions of m single-item dynamic lot-sizing models, each of which can be solved in

$O(n_h \log n_h)$ time. The above complexity bound then follows because $n_h = O(\log N)$ and $I = O(N/\log N)$.

(b) Asymptotic optimality for fixed m is immediate from Theorem 2.

(c) Note that ρ is $O(m)$ while $\gamma = \Omega(m)$, i.e., there exists a constant $a > 0$ such that $\gamma \geq am$. Thus, $\rho/\gamma = O(1)$ as $m \to \infty$, and part (c) follows from part (b). □

3.6. The General (OWMR) Problem

We now discuss what modifications are required in the OWMR model, the branch and bound method, the partitioning heuristic, and its analysis when dealing with an arbitrary number of items, J, carried in the system, i.e., $J \geq 2$. With multiple items we confine ourselves to the most prevalent case where no speculative motives for carrying inventories prevail [see (8)].

In terms of the model itself, it is necessary to use three indices when specifying the demands d and variable cost parameters c and h. (On the other hand, the fixed costs K remain as specified, since they are incurred for any warehouse or retailer order, regardless of its specific composition.) Similarly, in the mathematical programming formulation (1)–(4), it is now necessary to disaggregate the continuous order- and inventory-variables (X and I) by item type, i.e., to use three indices for both types of variables. (Once again, the Y variables remain unaltered.) All three of the constraint sets (2)–(4) now need to be specified on an item-by-item basis.

With prespecified warehouse order periods (see subsection 3.2), the problem again decomposes into m separate lot-sizing problems, one for each retailer. Each of these retailer problems deals with joint replenishments for J different items; however, since no item-specific setup costs are incurred and since no speculative motives for carrying inventories exist, it is optimal to place orders only when *all* items' inventories equal zero. As in the single item case, the dynamic lot-sizing problem can thus be embedded on order periods only, and hence be solved by a standard shortest path method. Each retailer's lot-sizing decisions can thus be determined in $O(JN^2)$ time; with prespecified warehouse order periods, the remaining problem is thus solvable with $O(mJN^2)$ effort. [As in subsection 3.2, it is necessary to adjust the variable order cost rates of the m retailers and the J items, to include the variable costs optimally incurred at the warehouse level, see (5).]

A lower bound Z_{LB} is again obtained by complete or Lagrangean relaxation of the coupling constraints (3) in (P). Given our discussion above, the relaxed problem can be solved in $O(mJN^2)$ time, for given values of the Lagrange multipliers. Note, however, that there are now JN coupling constraints, and hence as many Lagrange multipliers, an essential complication in solving the Lagrangean dual to optimality. The branch and bound method of subsection 3.3 can be employed with this modification of the lower bound to evaluate its nodes.

A time-partitioning heuristic can be designed in close analogy to the one specified in subsection 3.5 for the single-item case. In particular, when specifying OWMR$_h$, the problem for

Table 1. Problem set 1: $m = 2$, $J = 5$.

N	10 ($n = 5$)	15 ($n = 5$)	20	
			$n = 5$	$n = 10$
z^H/z^*	1.0056	1.0048	1.0054	1.0004
% opt	20	10	10	50
CPU z^H	0.03 s	0.038 s	0.051 s	0.47 s
CPU z^*	0.2 s	7.6 s	2.25 min	
z^*/z_{LB}	1.0004	1.00055	1.0013	

Table 2. Problem set 2: $m = 5$, $J = 2$.

N	10 ($n = 5$)	15 ($n = 5$)	20 $n = 5$	20 $n = 10$
z^H/z^*	1.013	1.013	1.01	1.001
% opt	20	10	0	20
CPU z^H	0.027 s	0.04 s	0.069	0.75
CPU z^*	0.33 s	7.6 s	2.51 min	
z^*/z_{LB}	1.005	1.0047	1.0051	

the hth interval, $h = 2, \ldots, I$, it continues to suffice to append a single dummy period -1 at the beginning of the interval, so as to allow for starting conditions that appropriately complement the solutions obtained in prior intervals. This dummy period has again zero demands, setup cost parameters, and holding cost rates. Its variable order cost rates should again be specified as above, straightforwardly disaggregated on an item-by-item basis. The same applies to the updates of the partial solutions to the full planning horizon at the completion of each interval's problem.

The lower bound theorem (Theorem 1) continues to apply with the uniform positive lower bounds for the d, h, and c parameters now assumed to apply on an item-by-item basis. The constant γ is now selected as the long run average cost rate in the stationary one-warehouse, multiitem multiretailer system obtained when all parameters are replaced by their stationary lower bounds. The proof of this generalization of Theorem 1 is identical, replacing the lower bound theorem of Roundy [22] by that of Roundy (1986). The value of γ can be computed in $O(mJ \log mJ)$ with the method of Muckstadt and Roundy [19].

Finally, the upper bound for the heuristic's optimality gap in Corollary 1 continues to apply, without any modifications, provided that the lower bounds for the d- and c-parameters are again specified on an item-by-item basis. This permits us once again to design an ϵ-optimal heuristic for any $\epsilon > 0$ and to show that the partitioning heuristic with interval lengths specified by (9) and (10) (i) has complexity $O(JmN^2 \log_2 N)$ and (ii) is asymptotically optimal as $N \to \infty$ for fixed m and J, or as N, m, and J jointly increase to ∞ under straightforward item-by-item specifications of the parameter conditions in parts (b) and (c) of Corollary 3.

4. NUMERICAL STUDY

In this section we report on a numerical study with 300 problem instances, conducted to gauge the performance potential of the partitioning heuristic. The investigated instances vary in terms of the horizon length (N), the number of items (J), the number of retailers (m), and the type

Table 3. Problem set 3: $m = 5$, $J = 5$.

N	10 ($n = 5$)	15 ($n = 5$)	20 $n = 5$	20 $n = 10$
z^H/z^*	1.0046	1.0042	1.0036	1.0
% opt	0	0	0	100
CPU z^H	0.06 s	0.1 s	0.16 s	3.32 s
CPU z^*	1.57 s	51.5 s	17.85 min	
z^*/z_{LB}	1.00384	1.0036	1.0043	

Table 4. Problem set 4: $m = 10$, $J = 10$.

N	10 ($n = 5$)	15 ($n = 5$)	20	
			$n = 5$	$n = 10$
z^H/z^*	1.0	1.0	1.0	1.0
% opt	100	100	100	100
CPU z^H	1.13 s	1.9 s	2.74 s	45.1 s
CPU z^*	13.83 s	4.95 min	101.7 min	
z^*/z_{LB}	1.0153	1.0144	1.0147	

of demand pattern. As assumed in many forecasting systems, the demand values follow an autoregressive pattern of the first order, i.e., for some $0 \le \alpha \le 1$:

$$d_{ij1} = e^d_{ij1} \qquad (i = 1, \ldots, m;\ j = 1, \ldots, J),$$

$$d_{ijt} = \alpha\, d_{ijt-1} + (1 - \alpha)\, e^d_{ijt} \qquad (i = 1, \ldots, m;\ j = 1, \ldots, J;\ t = 2, \ldots, N),$$

where for all i,j the sequence $\{e^d_{ijt}: t = 1, \ldots, N\}$ consists of independent random variables that are uniformly distributed on a prespecified interval. The parameter α guides the degree of variability of the demands. [In the extreme case where $\alpha = 0$, demands are independent and identically distributed (i.i.d.) while demands are constant over time, if $\alpha = 1$.] The setup cost parameters follow a similar pattern:

$$K_{01} = e^K_{01}, \qquad K_{0t} = \beta K_{0t-1} + (1 - \beta) e^K_{0t}, \tag{11}$$

$$K_{ij1} = e^K_{ijt}, \qquad K_{ijt} = \beta K_{ijt-1} + (1 - \beta) e^K_{ijt}, \tag{12}$$

where the series $\{e^K_{0t}\}$ and $\{e^K_{ijt}\}$ are i.i.d. and uniform on the intervals [8, 12] and [5, 15] respectively. We choose $\beta = 0.5$ in all instances. The remaining parameters are specified as follows: $c_{0jt} = 1$ (all j, t), $h_{0jt} = 0.2$ (all j, t), $c_{ijt} = 0.5$ (all i, j, t), and $h_{ijt} = 0.25$ (all i, j, t).

In problem sets 1–5, we evaluate instances with a planning horizon of up to $N = 20$ periods, via an exact branch and bound method as well as the partitioning heuristic. For instances with horizon $N = 20$ (18), we apply the partitioning heuristic both with intervals of length $n = 5$ (6), and length $n = 10$ (9). In sets 1–4, we fix $\alpha = 0.5$. These sets differ from each other, in terms of the number of items (J) and retailers (m). Problem sets 5–8 consider 18-period instances, all with 5 items and 5 retailers. In set 5 we vary the demand variability parameter α,

Table 5. Problem set 5: $N = 18$, $m = 5$, $J = 5$.

α	0.0		0.25		0.5		0.75		1.0	
	$n = 6$	$n = 9$	$n = 6$	$n = 9$	$n = 6$	$n = 9$	$n = 6$	$n = 9$	$n = 6$	$n = 9$
z^H/z^*	1.00001	1.0027	1.0006	1.0012	1.0002	1.0019	1.0003	1.0004	1.0	1.0038
% opt	90	10	70	20	80	20	60	30	100	0
CPU z^H	0.37 s	1.83 s	0.35 s	1.69 s	0.35 s	1.66 s	0.35 s	1.68 s	0.27 s	1.935 s
CPU z^*	5.2 min		5.1 min		5.74 min		4.75 min		4.59 min	
z^*/z_{LB}	1.00376		1.0043		1.0042		1.0052		1.0034	

Table 6. Problem set 6: K_i constant; $N = 18$, $J = m = 5$.

K_i	10		30		90	
	$n = 6$	$n = 9$	$n = 6$	$n = 9$	$n = 6$	$n = 9$
z^H/z^*	1.0001	1.0013	1.012	1.005	1.019	1.0061
% opt	80	20	10	20	0	10
CPU z^H	0.33 s	1.43 s	0.2 s	1.08 s	0.11 s	0.46 s
CPU z^*	4.51 min		1.24 min		13.4 s	
z^*/z_{LB}	1.006		1.03		1.107	

from 0 to 1 in increments of 0.25. Note that in (2), $EK_{ijt} = 10$. Problem sets 6–8 are designed to investigate the impact of larger setup costs for the retailers, resulting in larger reorder intervals. In these sets, we replace the first order autoregressive time series for the setup cost parameters by one in which the setup costs are constant over time (set 6), increase from 50% below their time-average value to 50% above this value, in constant increments (set 7), and decrease from 50% above the time average under to 50% below this value, again in constant decrements (set 8). In each of the sets 6–8, we consider three values for the average retailer setup cost $\kappa_i = 10, 30, 90$. To provide some insight into the implications of these parameter choices, note that if the demands across different items could be aggregated and if all parameter values were constant over time at their time-average value, the model would reduce to a periodic review version of Roundy [22] in which case all reorder intervals are optimally set, before rounding to power-of-two values, between 1.26 and 2.83 ($K = 10$), 2.19 and 4.9 ($K = 30$), and 3.78 and 8.49 ($K = 90$).

Tables 1–8 correspond to sets 1–8. In each column we report for a sample of 10 instances, generated with the corresponding parameters: (i) the average optimality gap, (ii) the % of instances for which the partitioning heuristic identifies an optimal solution, (iii) the average CPU time required by the heuristic, (iv) the average CPU time required by the branch-and-bound method, and (v) the average ratio of the heuristic value and the lower bound LB.

Problem sets 9 and 10 consider instances with $N = 50$ and $N = 150$, respectively. Since the exact branch-and-bound method is intractable when $N \gg 20$, we evaluate the optimality gap of the partitioning heuristic only via the ratio of the cost value of the heuristic solution and the lower bound LB. As before, each column reports on the average performance of the heuristic across a sample of 10 instances, generated with the specified parameters and solved with the specified interval lengths.

Our overall conclusion is that the partitioning heuristics are very efficient and highly accurate even beyond our experience with a similar heuristic for Joint Replenishment Problems (see Federgruen and Tzur [9]. Even problems with a planning horizon of up to 150 periods can be

Table 7. Problem set 7: K_i increasing from 50% below to 50% above specified levels; $N = 18$, $J = m = 5$.

K_i	10		30		90	
	$n = 6$	$n = 9$	$n = 6$	$n = 9$	$n = 6$	$n = 9$
z^H/z^*	1.0002	1.0009	1.009	1.0003	1.0002	1.005
% opt	70	20	10	60	70	0
CPU z^H	0.31 s	1.23 s	0.22 s	1.07 s	0.18 s	0.53 s
CPU z^*	3.57 min		1.11 min		9.5 s	
z^*/z_{LB}	1.0065		1.034		1.13	

Table 8. Problem set 8: K_i decreasing from 50% above to 50% below specified levels; $N = 18$, $J = m = 5$.

K_i	10		30		90	
	$n = 6$	$n = 9$	$n = 6$	$n = 9$	$n = 6$	$n = 9$
z^H/z^*	1.0003	1.001	1.009	1.004	1.008	1.006
% opt	60	30	0	0	0	10
CPU z^H	0.29 s	1.59 s	0.17 s	0.63 s	0.13 s	0.5 s
CPU z^*	5.46 min		1.77 min		10.2 s	
z^*/z_{LB}	1.0042		1.031		1.16	

solved within 1.5% of optimality, employing intervals of 5–10 periods only and in a matter of CPU seconds, or up to a few minutes, using longer intervals and when the number of items and retailers is large. These CPU times refer to a SUN 4M (SPARC) workstation. The perfect performance of the heuristic in set 4 appears to arise because, with 10 items sold at 10 retailers, it becomes optimal for the warehouse to place an order in almost all, if not every period, so that time partitioning can be implemented without loss of optimality. As can be expected, the computational effort increases significantly with the choice of the interval length, reaching over 100 CPU minutes when intervals of length $n = 15$ are chosen (and $N = 150$). We also notice that, in accordance with our asymptotic optimality results, the performance of the heuristic continues to improve as N, the length of the planning horizon, increases. In general, the optimality gaps decrease as the interval length is increased, i.e., as fewer intervals are employed. This is in particular true when an interval length n is replaced by a multiple thereof. The monotonicity is less clear when comparing two interval lengths $n_1 < n_2$, where n_2 fails to be a multiple of n_1 (see, e.g., Table 5).

5. CONCLUDING REMARKS

We have described a general methodology for the design of time-partitioning heuristics for dynamic lot-sizing problems. We have shown how different elements can be chosen to ensure polynomial complexity, asymptotic optimality as N, the length of the horizon, tends to infinity, as well as ϵ-optimality for fixed N and any arbitrarily small optimality gap ϵ. We have shown how different choices can be made regarding most or all of these elements, with different impacts on the heuristics' feasibility and performance. Finally, we have used the general approach to develop an efficient heuristic for a one-warehouse, multiretailer and multiitem model, which can be tailored to satisfy each of the above optimality criteria. A comprehensive numerical study shows that the method is highly efficient and generates solutions that are very close to optimal. The methodology can, in principle, be applied to more general production/ distribution networks. On the other hand, for such more elaborate systems, additional heuristic adaptations are needed to ensure practical efficiency, due to the complexity involved in solving

Table 9. Problem set 9: $m = 5$, $J = 5$.

N	50		150		
	$n = 5$	$n = 10$	$n = 5$	$n = 10$	$n = 15$
z^H/z_{LB}	1.0088	1.0048	1.0087	1.0052	1.0054
CPU z^H	0.45 s	14.88 s	1.55 s	49.0 s	14.5 min

Table 10. Problem set 10: $m = 10$, $J = 10$.

N	50		150		
	$n = 5$	$n = 10$	$n = 5$	$n = 10$	$n = 15$
z^H/z_{LB}	1.015	1.0148	1.0149	1.0149	1.0149
CPU z^H	8.15 s	2.3 min	22.7 s	7.1 min	113.9 min

each of the intervals' subproblems. Tzur [25] describes two possible approaches to achieve this objective, but additional future research needs to be carried out to evaluate how well these approaches perform in various settings.

APPENDIX

LEMMA 2: The quantities D^i ($i = 1, \ldots, m$), defined in the proof of Theorem 2, satisfy bounds of the type:

$$\sum_{i \in S_\ell} D^i \leq b_\ell, \qquad \ell = 1, \ldots, |S| \tag{13}$$

Moreover,

$$\rho = \max\left\{ \sum_{i=0}^{m} K_i^* + \sum_{i \in S} (c_i^* + c_0^* - c_{i*} - c_{0*} - h_{0*}) D^i \right\}$$

$$\text{s.t.} \qquad \sum_{i \in S_\ell} D^i \leq b_\ell, \qquad \ell = 1, \ldots, |S|.$$

PROOF: Note that for all $1 \leq r \leq t_\ell$,

$$rh_{0*} \sum_{i \in S_\ell} (D^i - X_r^i) \leq \sum_{i \in S_\ell} K_i^* + K_0^* + \sum_{i \in S_\ell} (c_i^* + c_0^* - c_{i*} - c_{0*})(D^i - X_r^i)$$
$$\leq \sum_{i \in S_\ell} K_i^* + K_0^* + Lh_{0*} \sum_{i \in S_\ell} (D^i - X_r^i), \tag{14}$$

where X_r^i denotes the cumulative demand for retailer i in the first $(r - 1)$ periods of the hth interval.

The second inequality in (14) is immediate from the definition of L. To verify the first inequality in (14), we assume without loss of generality that, at each retailer, units are demanded on a FIFO basis, i.e., in the sequence in which they enter the warehouse. This implies that the number of carryover units for retailer i, left in the system at the beginning of period r, equals $(D^i - X_r^i)$. Thus, if the first inequality in (14) is violated then a strict cost improvement could be achieved by postponing till period r the ordering by the warehouse of *all* the carryover units in stock (somewhere in the system) at the beginning of that period and by postponing their transfer to the appropriate retailer till period r, if this transfer was originally scheduled to take place prior to period r. Note that the additional setup costs due to these postponements is bounded by $\Sigma_{i \in S_\ell} K_i^* + K_0^*$. The additional variable order costs are bounded by $\Sigma_{i \in S_\ell} (c_i^* + c_0^* - c_{i*} - c_{0*})(D^i - X_r^i)$ since the warehouse orders of *exactly* $(D^i - X_r^i)$ units for retailer $i (i \in S_\ell)$ are postponed as well as the retailer orders of *at most* as many units. (Note that the per unit increase in warehouse order costs is bounded by $(c_0^* - c_{0*})$ and the per unit increase in retailer order costs is bounded by $(c_i^* - c_{i*})$. On the other hand, the postponements reduce the echelon stock for $(D^i - X_r^i)$ units at retailer i in at least one period prior to the hth interval and *all* periods $1, \ldots, r - 1$, i.e., in at least r periods resulting in inventory cost savings of at least $rh_{0*} \Sigma_{i \in S_\ell} (D^i - X_r^i)$.

Let $\bar{X}_r^i = D^i - X_r^i$ It follows from (14) that

$$(r - L)h_{0*} \sum_{i \in S_\ell} \bar{X}_r^i \leq \sum_{i \in S_\ell} K_i^* + K_0^*. \tag{15}$$

Thus, for $r > L$,

$$\sum_{i\in S_\ell} \bar{X}_r^i \le \frac{\sum_{i\in S_\ell} K_i^* + K_0^*}{(r-L)h_{0*}}, \tag{16}$$

$$\sum_{i\in S_\ell} X_r^i \le (r-1) \sum_{i\in S_\ell} d_i^* \tag{17}$$

Adding (16) and (17) and taking the minimum over r on the interval $[L+1, t_\ell]$, we get

$$\sum_{i\in S_\ell} D^i \le \min_{L+1\le r\le t_\ell} \left\{ (r-1)\left(\sum_{i\in S_\ell} d_i^*\right) + \frac{\sum_{i\in S_\ell} K_i^* + K_0^*}{(r-L)h_{0*}} \right\} \tag{18}$$

The expression within curled brackets is clearly a convex function of r which decreases for

$$L+1 \le r \le r^* \equiv L + \left\lfloor \sqrt{\frac{2(\sum_{i\in S_\ell} K_i^* + K_0^*)}{(h_{0*} \sum_{i\in S_\ell} d_i^*)}} \right\rfloor \quad \text{or} \quad r^* \equiv L + \left\lceil \sqrt{\frac{2(\sum_{i\in S_\ell} K_i^* + K_0^*)}{(h_{0*} \sum_{i\in S_\ell} d_i^*)}} \right\rceil.$$

Assume first that $t_\ell \ge L+1$. It follows that the minimum in (18) is achieved for $r = \min(t_\ell, r^*)$, i.e.,

$$\sum_{i\in S_\ell} D^i \le [\min(t_\ell, r^*) - 1]\left(\sum_{i\in S_\ell} d_i^*\right) + \frac{\sum_{i\in S_\ell} K_i^* + K_0^*}{(\min(t_\ell, r^*) - L)h_{0*}}, \qquad t_\ell \ge L+1. \tag{19}$$

If $t_\ell \le L$, we clearly have the bound

$$\sum_{i\in S_\ell} D^i \le t_\ell \sum_{i\in S_\ell} d_i^* \tag{20}$$

Let $b_\ell(t_\ell)$ denote the right-hand side of (19) when $t_\ell \ge L+1$ and of (20) when $t_\ell \le L$. For each demand unit of retailer $i \in S$ whose procurement costs are increased due to the transformation, the increase is clearly bounded by $(c_i^* + c_0^* - c_{i*} - c_{0*} - h_{0*})$.

A bound for the increase in variable costs due to the transformation is thus given by the value of the following linear program:

$$\phi(t_1, \ldots, t_{|S|}) = \max \sum_{i\in S} (c_i^* + c_0^* - c_{i*} - c_{0*} - h_{0*})\, D^i$$

$$\text{s.t.} \qquad \sum_{i\in S_\ell} D^i \le b_\ell(t_\ell), \qquad l = 1, \ldots, |S|, \quad D^i \ge 0. \tag{21}$$

Note that the function $b_\ell(\cdot)$ achieves its maximum for $t_\ell = L+1$, since it is increasing for $t_\ell \le L$, [see (20)], decreasing for $t_\ell \ge L+1$, and

$$b_\ell(L+1) = L \sum_{i\in S_\ell} d_i^* + \left(\sum_{i\in S_\ell} K_i^* + K_0^*\right)/h_{0*} > L \sum_{i\in S_\ell} d_i^* = b_\ell(L).$$

The optimum value of the linear program is clearly nondecreasing in the values $\{b_\ell: \ell = 1, \ldots, |S|\}$. It follows that $\phi(L+1, L+1, \ldots, L+1) = \max\{\phi(t_1, \ldots, t_{|S|}): t_1 \le t_2 \le \cdots \le t_{|S|}\}$ since the function ϕ achieves its *unconstrained* maximum in the point $(L+1, \ldots, L+1)$. Thus, substituting $t_\ell = L+1$ in (21), we obtain the following bound for the total increase in variable costs due to the transformation:

$$\max \sum_{i\in S} (c_i^* + c_0^* - c_{i*} - c_{0*} - h_{0*}) \tag{22}$$

$$\text{s.t.} \qquad \sum_{i \in S_\ell} D^i \le L \sum_{i \in S_\ell} d_i^* + \left(\sum_{i \in S_\ell} K_i^* + K_0^* \right) / h_{0*}, \qquad \ell = 1, \ldots, |S|, \quad D^i \ge 0.$$

The objective function of (22) can clearly be bounded from above by $\eta \sum_{i \in S} D^i$. The latter objective is clearly maximized when

$$\sum_{i \in S} D^i = L \sum_{i \in S} d_i^* + \left(\sum_{i \in S} K_i^* + K_0^* \right) / h_{0*}.$$

Thus, the resulting bound is maximized when $S = \{1, \ldots, m\}$ in which case it equals to

$$\eta \left\{ L \sum_{i \in S} d_i^* + \left(\sum_{i \in S} K_i^* + K_0^* \right) / h_{0*} \right\}. \qquad \square$$

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