Liu et al. *Advances in Difference Equations* (2015) 2015:254
DOI 10.1186/s13662-015-0568-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

A generalization of Morley's congruence

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Abstract
We establish an explicit formula for q-analog of Morley's congruence.

MSC: Primary 11B68; secondary 11B65; 11B83

Keywords: Morley's congruence; q-analog; binomial sums

1 Introduction

For arbitrary positive integer n, let

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1},$$

which is the q-analog of an integer n since $\lim_{q\to 1}(1-q^n)/(1-q) = n$. Also, for $n, m \in \mathbb{Z}$, define the q-binomial coefficients by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q[n-1]_q \cdots [n-m+1]_q}{[m]_q[m-1]_q \cdots [1]_q}$$

when $m \geq 0$, and if $m < 0$ we set $\begin{bmatrix} n \\ m \end{bmatrix}_q = 0$. It is easy to check that

$$\begin{bmatrix} n+1 \\ m \end{bmatrix}_q = q^m \begin{bmatrix} n \\ m \end{bmatrix}_q + \begin{bmatrix} n \\ m-1 \end{bmatrix}_q.$$

Some combinatorial and arithmetical properties of the binomial sums

$$\sum_{k=0}^{n} \binom{n}{k}^a \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}^a$$

have been investigated by several authors (*e.g.*, Calkin [1], Cusick [2], McIntosh [3], Perlstadt [4]). Indeed, we know (*cf.* [5], equations (3.81) and (6.6)) that

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n} \tag{1.1}$$

and

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\binom{2n}{n}\binom{3n}{n}. \tag{1.2}$$

However, by using asymptotic methods, de Bruijn [6] has showed that no closed form exists for the sum $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^a$ when $a\geq 4$. Wilf proved (in a personal communication with Calkin; see [1]) that the sum $\sum_{k=0}^{n}\binom{n}{k}^a$ has no closed form provided that $3\leq a\leq 9$.

As a q-analog of (1.1), we have

$$\sum_{k=0}^{2n}(-1)^k q^{(n-k)^2}\begin{bmatrix}2n\\k\end{bmatrix}_q^2=(-1)^n\begin{bmatrix}2n\\n\end{bmatrix}_{q^2}. \tag{1.3}$$

Indeed, from the well-known q-binomial theorem (*cf.* Corollary 10.2.2 of [7])

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}}x^k=(x;q)_n,$$

where

$$(x;q)_n=\begin{cases}(1-x)(1-xq)\cdots(1-xq^{n-1}), & \text{if } n\geq 1,\\ 1, & \text{if } n=0,\end{cases}$$

it follows that

$$\begin{aligned}\left(x^2;q^2\right)_{2n}&=(x;q)_{2n}(-x;q)_{2n}=\left(\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}}x^k\right)\left(\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}_q q^{\binom{k}{2}}x^k\right)\\&=\sum_{m=0}^{4n}x^m\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}_q\begin{bmatrix}2n\\m-k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}+\binom{m-k}{2}},\end{aligned}$$

whence (1.3) is derived by comparing the coefficients of x^{2n} in the equation above.

As early as 1895, with the help of De Moivre's theorem, Morley [8] proved that

$$(-1)^{\frac{p-1}{2}}\binom{p-1}{(p-1)/2}\equiv 4^{p-1}\ \left(\operatorname{mod}\ p^3\right). \tag{1.4}$$

In [9], Pan gave a q-analog of Morley's congruence and showed that

$$(-1)^{\frac{n-1}{2}}q^{\frac{n^2-1}{4}}\begin{bmatrix}n-1\\(n-1)/2\end{bmatrix}_{q^2}\equiv(-q;q)_{n-1}^2-\frac{n^2-1}{24}(1-q)^2[n]_q^2\ \left(\operatorname{mod}\ \Phi_n(q)^3\right), \tag{1.5}$$

where

$$\Phi_n(q)=\prod_{\substack{1\leq j\leq n\\(j,n)=1}}\left(q-e^{2\pi ij/n}\right)$$

is the nth cyclotomic polynomial. In this section, we shall establish a generalization of Morley's congruence (1.4) proved by Cai and Granville [10], Theorem 6:

$$\sum_{k=0}^{p-1}(-1)^{(a-1)k}\binom{p-1}{k}^a \equiv 2^{a(p-1)} \pmod{p^3} \tag{1.6}$$

for any prime $p \geq 5$ and positive integer a. We also shall obtain a generalization of (1.5) in view of (1.3).

Theorem 1.1 *Let n be a positive odd integer. Then*

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^{(a-1)k}q^{a\binom{k+1}{2}}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q^a \\ &\quad\equiv (-q;q)_{n-1}^a + \frac{a(a-1)(n^2-1)}{24}(1-q)^2[n]_q^2 \pmod{\Phi_n(q)^3}.\end{aligned} \tag{1.7}$$

Furthermore, we have

$$\begin{aligned}&q^{a(n^2-1)/4}\sum_{k=0}^{n-1}(-1)^k q^{a((n-1)/2-k)^2}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q^{2a} \\ &\quad\equiv (-q;q)_{n-1}^{2a} + \frac{a(a-2)(n^2-1)}{24}(1-q)^2[n]_q^2 \pmod{\Phi_n(q)^3}.\end{aligned} \tag{1.8}$$

Remark Clearly (1.6) is the special case of (1.7) in the limiting case $q->1$ for $n = p$.

2 Some lemmas

In this section, the following lemmas will be used in the proof of Theorem 1.1.

Lemma 2.1

$$q^{kn} \equiv 1 - k(1-q)[n]_q + \frac{k(k-1)}{2}(1-q)^2[n]_q^2 \pmod{[n]_q^3}. \tag{2.1}$$

Proof

$$q^{kn} = \sum_{j=0}^{k}(-1)^j\binom{k}{j}(1-q)^j[n]_q^j \equiv 1 - k(1-q)[n]_q + \frac{k(k-1)}{2}(1-q)^2[n]_q^2 \pmod{[n]_q^3}. \qquad \square$$

Lemma 2.2 *Let n be a positive odd integer. Then*

$$\begin{aligned}\sum_{j=1}^{(n-1)/2}\frac{1}{[2j]_q^2} &= \sum_{j=1}^{(n-1)/2}\frac{q^{2j}}{[2j]_q^2} + (1-q)\sum_{j=1}^{(n-1)/2}\frac{1}{[2j]_q} \\ &\equiv -\frac{n^2-1}{24}(1-q)^2 - \mathrm{Q}_n(2,q)(1-q) \pmod{\Phi_n(q)},\end{aligned} \tag{2.2}$$

where the q-Fermat quotient is defined by

$$\mathrm{Q}_n(m,q) = \frac{(q^m;q^m)_{n-1}/(q;q)_{n-1} - 1}{[n]_q}.$$

Lemma 2.3 *Let n be a positive odd integer. Then*

$$2\sum_{j=1}^{(n-1)/2}\frac{1}{[2j]_q}+2\mathrm{Q}_n(2,q)-\mathrm{Q}_n(2,q)^2[n]_q \equiv \left(\mathrm{Q}_n(2,q)(1-q)+\frac{n^2-1}{8}(1-q)^2\right)[n]_q \pmod{\Phi_n(q)^2}. \tag{2.3}$$

When n is an odd prime, the above two lemmas have been proved in [9], equation (2.7) and [9], Theorem 1.1, respectively. Of course, clearly the same discussions are also valid for general odd n.

3 Proofs of Theorem 1.1

In this section, we shall prove (1.7) and (1.8).

Proof By the properties of the q-binomial coefficients, we know that

$$(-1)^k\begin{bmatrix}n-1\\k\end{bmatrix}_q=\prod_{j=1}^{k}\frac{[j]_q-[n]_q}{q^j[j]_q} \equiv q^{-\binom{k+1}{2}}\left(1-\sum_{j=1}^{k}\frac{[n]_q}{[j]_q}+\sum_{1\le i<j\le k}\frac{[n]_q^2}{[i]_q[j]_q}\right) \pmod{\Phi_n(q)^3}.$$

Thus

$$(-1)^{ak}q^{a\binom{k+1}{2}}\begin{bmatrix}n-1\\k\end{bmatrix}_q^a \equiv 1-a\sum_{j=1}^{k}\frac{[n]_q}{[j]_q}+a\sum_{1\le i<j\le k}\frac{[n]_q^2}{[i]_q[j]_q}+\binom{a}{2}\left(\sum_{j=1}^{k}\frac{[n]_q}{[j]_q}\right)^2 \pmod{\Phi_n(q)^3}.$$

Noting that

$$\left(\sum_{j=1}^{k}\frac{1}{[j]_q}\right)^2=2\sum_{1\le i<j\le k}\frac{1}{[i]_q[j]_q}+\sum_{j=1}^{k}\frac{1}{[j]_q^2},$$

we have

$$\begin{aligned}\sum_{k=1}^{n-1}(-1)^{(a-1)k}q^{a\binom{k+1}{2}}\begin{bmatrix}n-1\\k\end{bmatrix}_q^a &\equiv \sum_{k=1}^{n-1}(-1)^k\left(1-a\sum_{j=1}^{k}\frac{[n]_q}{[j]_q}+a\sum_{1\le i<j\le k}\frac{[n]_q^2}{[i]_q[j]_q}+\binom{a}{2}\left(\sum_{j=1}^{k}\frac{[n]_q}{[j]_q}\right)^2\right)\\ &=\left(-a\sum_{j=1}^{n-1}\frac{[n]_q}{[j]_q}+a^2\sum_{1\le i<j\le n-1}\frac{[n]_q^2}{[i]_q[j]_q}+\binom{a}{2}\sum_{j=1}^{n-1}\frac{[n]_q^2}{[j]_q^2}\right)\sum_{k=j}^{n-1}(-1)^k\\ &=-a\sum_{\substack{j=1\\2\mid j}}^{n-1}\frac{[n]_q}{[j]_q}+a^2\sum_{\substack{1\le i<j\le n-1\\2\mid j}}\frac{[n]_q^2}{[i]_q[j]_q}+\binom{a}{2}\sum_{\substack{j=1\\2\mid j}}^{n-1}\frac{[n]_q^2}{[j]_q^2} \pmod{\Phi_n(q)^3}.\end{aligned} \tag{3.1}$$

Thus letting $a = 1$ in (3.1), we get

$$\sum_{\substack{1\le i<j\le n-1\\ 2|j}} \frac{[n]_q^2}{[i]_q[j]_q} \equiv (-q;q)_{n-1} - 1 + \sum_{\substack{j=1\\ 2|j}}^{n-1} \frac{[n]_q}{[j]_q} \pmod{\Phi_n(q)^3}, \tag{3.2}$$

whence by (2.3) and (3.2)

$$\sum_{\substack{1\le i<j\le n-1\\ 2|j}} \frac{1}{[i]_q[j]_q} \equiv \frac{\mathrm{Q}_n(2,q)^2}{2} + \frac{\mathrm{Q}_n(2,q)}{2}(1-q) + \frac{n^2-1}{16}(1-q)^2 \pmod{\Phi_n(q)}.$$

Recalling (2.2) and (2.3), then we obtain

$$\begin{aligned}
&\sum_{k=0}^{n-1}(-1)^{(a-1)k}q^{a\binom{k+1}{2}} \begin{bmatrix} n-1\\ k\end{bmatrix}_q^a \\
&\quad\equiv 1 + a[n]_q\mathrm{Q}_n(2,q) + \binom{a}{2}[n]_q^2\mathrm{Q}_n(2,q)^2 + \binom{a}{2}\frac{n^2-1}{12}(1-q)^2[n]_q^2 \\
&\quad\equiv \sum_{j=0}^{a}\binom{a}{j}[n]_q^j\mathrm{Q}_n(2,q)^j + \binom{a}{2}\frac{n^2-1}{12}(1-q)^2[n]_q^2 \\
&\quad= (-q;q)_{n-1}^a + \frac{a(a-1)(n^2-1)}{24}(1-q)^2[n]_q^2 \pmod{\Phi_n(q)^3}.
\end{aligned}$$

Let us turn to (1.8). Similarly

$$\begin{aligned}
&\sum_{k=0}^{n-1}(-1)^k q^{2a\binom{k+1}{2}} \begin{bmatrix} n-1\\ k\end{bmatrix}_q^{2a} - q^{a(n^2-1)/4}\sum_{k=0}^{n-1}(-1)^k q^{a((n-1)/2-k)^2}\begin{bmatrix} n-1\\ k\end{bmatrix}_q^{2a} \\
&\quad= 1 - q^{a\binom{n}{2}} + \sum_{k=1}^{n-1}(-1)^k q^{2a\binom{k+1}{2}}\bigl(1-q^{a(\binom{n}{2}-nk)}\bigr)\begin{bmatrix} n-1\\ k\end{bmatrix}_q^{2a} \\
&\quad\equiv 1 - q^{a\binom{n}{2}} + \sum_{k=1}^{n-1}(-1)^k\bigl(1-q^{a(\binom{n}{2}-nk)}\bigr)\left(1-2a\sum_{j=1}^{k}\frac{[n]_q}{[j]_q}\right) \pmod{\Phi_n(q)^3}.
\end{aligned}$$

Recalling (2.1), then we have

$$1-q^{a(\binom{n}{2}-nk)} \equiv a\left(\frac{n-1}{2}-k\right)(1-q^n) + \binom{a((n-1)/2-k)}{2}(1-q^n)^2 \pmod{\Phi_n(q)^3},$$

therefore

$$\begin{aligned}
&\sum_{k=1}^{n-1}(-1)^k\bigl(1-q^{a(\binom{n}{2}-nk)}\bigr)\left(1-2a\sum_{j=1}^{k}\frac{[n]_q}{[j]_q}\right) \\
&\quad\equiv \frac{q^{a\binom{n}{2}}-q^{-a\binom{n}{2}}}{1+q^{an}} - 2a^2\sum_{j=1}^{n-1}\frac{[n]_q}{[j]_q}\sum_{k=j}^{n-1}(-1)^k\left(\frac{n-1}{2}-k\right)(1-q^n) \\
&\quad= \frac{q^{a\binom{n}{2}}-q^{-a\binom{n}{2}}}{1+q^{an}} + a^2(1-q^n)[n]_q\left(\sum_{\substack{j=1\\ 2|j}}^{n-1}\frac{j}{[j]_q} + \sum_{\substack{j=1\\ 2\nmid j}}^{n-1}\frac{n-j}{[j]_q}\right) \pmod{\Phi_n(q)^3}.
\end{aligned}$$

Since

$$q^{an} = \big(1-(1-q^n)\big)^a \equiv 1-a\big(1-q^n\big)+\binom{a}{2}\big(1-q^n\big)^2 \pmod{\Phi_n(q)^3}$$

and

$$\frac{1}{2-a(1-q^n)} = \frac{1}{2}\frac{1}{1-a(1-q^n)/2} \equiv \frac{1}{2}\big(1+a\big(1-q^n\big)/2\big) \pmod{\Phi_n(q)^2},$$

we have

$$\begin{aligned}\frac{q^{a\binom{n}{2}}-q^{-a\binom{n}{2}}}{1+q^{an}} &\equiv -\frac{a(n-1)(1-q^n)+a(n-1)/2\cdot(1-q^n)^2}{2-a(1-q^n)} \\ &\equiv -\frac{a(n-1)}{2}\left(\big(1-q^n\big)+\frac{a+1}{2}\big(1-q^n\big)^2\right) \pmod{\Phi_n(q)^3}.\end{aligned}$$

Noting that

$$\begin{aligned}\sum_{\substack{j=1\\2|j}}^{n-1}\frac{j}{[j]_q}+\sum_{\substack{j=1\\2\nmid j}}^{n-1}\frac{n-j}{[j]_q} &= \sum_{\substack{j=1\\2|j}}^{n-1}\frac{j}{[j]_q}+\sum_{\substack{j=1\\2|j}}^{n-1}\frac{j}{[n-j]_q} \\ &= \sum_{\substack{j=1\\2|j}}^{n-1}\left(\frac{j}{[j]_q}+\frac{jq^j}{[n]_q-[j]_q}\right) \equiv \frac{n^2-1}{4}(1-q) \pmod{\Phi_n(q)},\end{aligned}$$

we have

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^k q^{2a\binom{k+1}{2}}\begin{bmatrix}n-1\\k\end{bmatrix}_q^{2a} - q^{a(n^2-1)/4}\sum_{k=0}^{n-1}(-1)^k q^{a((n-1)/2-k)^2}\begin{bmatrix}n-1\\k\end{bmatrix}_q^{2a} \\ &\quad\equiv\left(-\binom{a(n-1)/2}{2}-\frac{a(a+1)(n-1)}{4}+\frac{a^2(n^2-1)}{4}\right)\big(1-q^n\big)^2 \\ &\quad=\frac{a^2(n^2-1)}{8}\big(1-q^n\big)^2 \pmod{\Phi_n(q)^3}.\end{aligned}$$

In view of (1.7), this concludes the proof of (1.8). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements
The authors thank the anonymous referee for his (her) valuable comments and suggestions. The first and the third authors are supported by the foundation of Jiangsu Educational Committee (No. 14KJB110008) and National Natural Science Foundation of China (Grant No. 11401301). The second author is also supported by NNSF (Grant No. 11271185).

Received: 5 April 2015 Accepted: 8 July 2015 Published online: 16 August 2015

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