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Reverse arithmetic-harmonic mean and mixed mean operator inequalities

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Abstract
This note aims to present some scalar inequalities and operator inequalities on a Hilbert space. Firstly, the direct reverse weighted arithmetic-harmonic mean inequalities for scalars are obtained. Secondly, based on these scalar inequalities, the corresponding operator inequalities are established. Finally, we present the mixed arithmetic-geometric and geometric-harmonic means inequalities for two positive operators.

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1 Introduction

Let $\mathcal{B}(H)$ be the C^*-algebra of all bounded linear operators on a complex separable Hilbert space H. I stands for the identity operator. $\mathcal{B}^{++}(H)$ denotes the cone of all positive invertible operators on H. As a matter of convenience, we use the following notations to define the μ-weighted arithmetic mean (AM), geometric mean (GM), and harmonic mean (HM) for scalars and operators:

$$a\nabla_\mu b = (1-\mu)a + \mu b, \qquad a!_\mu b = \left((1-\mu)a^{-1} + \mu b^{-1}\right)^{-1},$$
$$A\nabla_\mu B = (1-\mu)A + \mu B, \qquad A\#_\mu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\mu A^{\frac{1}{2}},$$
$$A!_\mu B = \left((1-\mu)A^{-1} + \mu B^{-1}\right)^{-1},$$

where $a, b > 0$, $A, B \in \mathcal{B}^{++}(H)$, and $\mu \in [0,1]$. When $\mu = \frac{1}{2}$, we write $a\nabla b$, $a!b$, $A\nabla B$, $A\#B$ and $A!B$ for brevity, respectively.

It is well known that the famous μ-weighted A-G-H mean inequalities hold,

$$a\nabla_\mu b \geq a^{1-\mu}b^\mu \geq a!_\mu b \tag{1.1}$$

for $a, b > 0$ and $\mu \in [0,1]$ with equalities if and only if $a = b$. The first inequality of (1.1) is the classical Young inequality.

An operator version of (1.1) proved in [1] says that if $A, B \in \mathcal{B}^{++}(H)$ and $\mu \in [0,1]$, then

$$A\nabla_\mu B \geq A\#_\mu B \geq A!_\mu B.$$

In recent years, the study of the A-G-H mean inequalities has received increasing attention in the literature (see [2–5]).

Zuo *et al.* (see [5]) refined the Young inequality with the Kantorovich constant and obtained the following results:

$$a\nabla_\mu b \geq \mathrm{K}(h,2)^r a^{1-\mu} b^\mu, \tag{1.2}$$

where $a, b > 0$, $\mu \in [0,1]$, $r = \min\{\mu, 1-\mu\}$, and $h = \frac{b}{a}$. In addition, they also refined the μ-weighted arithmetic-harmonic mean inequality and extended it to an operator version as follows:

$$a\nabla_\mu b \geq a!_\mu b + 2r(a\nabla b - a!b), \tag{1.3}$$

$$A\nabla_\mu B \geq A!_\mu B + 2r(A\nabla B - A!B), \tag{1.4}$$

where $a, b > 0$, $A, B \in \mathcal{B}^{++}(H)$, $\mu \in [0,1]$, and $r = \min\{\mu, 1-\mu\}$. By (1.3) and (1.4), we are encouraged to investigate whether there exist reverse forms of the μ-weighted arithmetic-harmonic mean inequality, so we give an affirmative answer to this question in our paper.

Moreover, mixed mean inequalities are also extremely attractive. Sagae and Tanabe [6] establish a mixed A-G mean inequality for a finite number of strictly positive operators. Mond and Pečarić [7] establish a mixed A-G and G-H mean inequalities for two noncommutative strictly positive operators as follows: Let A and B be positive invertible operators. The mixed arithmetic-geometric and geometric-harmonic mean inequalities are valid:

$$A\#(A\nabla B) \geq A\nabla(A\#B), \tag{1.5}$$

$$A\#(A!B) \leq A!(A\#B). \tag{1.6}$$

In this paper, we are concerned with the weighted arithmetic-harmonic, mixed arithmetic-geometric and mixed geometric-harmonic mean inequalities. In Section 2, we present direct reverse weighted arithmetic-harmonic mean inequalities by the Kantorovich constant and deduce some auxiliary results. In Section 3, we extend inequalities proved in Section 2 from the scalars setting to a Hilbert space operator setting. In Section 4, we establish mixed weighted arithmetic-geometric and geometric-harmonic means inequalities for two positive operators which are the refinements of (1.5) and (1.6).

2 Reverse arithmetic-harmonic mean inequalities

In this section, we mainly present the direct reverse forms of the μ-weighted arithmetic-harmonic mean inequality for two positive numbers a, b.

First of all, we recall the classical Kantorovich inequality in [8].

Lemma 2.1 *Let $0 < a = x_1 < x_2 < \cdots < x_n = b$ be given positive numbers, $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. Then*

$$\left(\sum_{i=1}^n \lambda_i x_i\right)\left(\sum_{i=1}^n \lambda_i x_i^{-1}\right) \leq A^2 G^{-2}, \tag{2.1}$$

where $A = \frac{1}{2}(a+b)$, $G = \sqrt{ab}$ are the arithmetic and geometric means, respectively.

The inequality (2.1) is the Kantorovich inequality and the number $\frac{(a+b)^2}{4ab}$ is called the Kantorovich constant. For convenience, we write the Kantorovich constant as $\mathrm{K}(t,2) = \frac{(t+1)^2}{4t}$ ($t = \frac{b}{a}$), which has the properties $\mathrm{K}(1,2) = 1$, $\mathrm{K}(t,2) = \mathrm{K}(\frac{1}{t},2) \geq 1$ ($t > 0$), and $\mathrm{K}(t,2)$ is monotone increasing on $[1,\infty)$, and monotone decreasing on $(0,1]$.

When $n = 2$, we can get a special form for the inequality (2.1) as follows.

Proposition 2.1 *Let $a,b > 0$ and $\mu \in [0,1]$. Then*

$$a\nabla_\mu b \leq \mathrm{K}(h,2)a!_\mu b, \tag{2.2}$$

where $h = \frac{b}{a}$. Equality holds if and only if $a = b$.

Remark 2.1 The inequality (2.2) is a direct reverse of the μ-weighted arithmetic-harmonic mean inequality. It is very interesting that by the inequality (2.2) and (1.2) we can get a reverse of Young inequality (it is also a reverse of the inequality (1.2)):

$$a\nabla_\mu b \leq \mathrm{K}(h,2)a!_\mu b \leq \mathrm{K}(h,2)^{1-r}a^{1-\mu}b^\mu = \mathrm{K}(h,2)^R a^{1-\mu}b^\mu, \tag{2.3}$$

where $a,b > 0$, $\mu \in [0,1]$, $r = \min\{\mu, 1-\mu\}$, and $R = \max\{1-\mu,\mu\}$. Replacing a, b by a^{-1}, b^{-1}in the above inequalities, respectively, we have

$$a^{1-\mu}b^\mu \leq \mathrm{K}(h,2)^R a!_\mu b. \tag{2.4}$$

Next, we deduce a direct reverse of the inequality (1.3) by the following lemma.

Lemma 2.2 [9] *Let x_i ($i = 1,2,\ldots,n$) belong to a fixed closed interval $\mathbb{I}$, $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$ and $\bar{p} = \max\{p_1,p_2,\ldots,p_n\}$. If f is a convex function on $\mathbb{I}$, then*

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \leq n\bar{p}\left[\sum_{i=1}^n \frac{1}{n} f(x_i) - f\left(\sum_{i=1}^n \frac{1}{n} x_i\right)\right]. \tag{2.5}$$

It is easy to see that if we take $f(x) = x^{-1}$ in the inequality (2.5), then we have the following.

Proposition 2.2 *If $x_1,x_2,\ldots,x_n > 0$ and $p_i \geq 0$ ($i = 1,2,\ldots,n$) with $\sum_{i=1}^n p_i = 1$, then*

$$\sum_{i=1}^n p_i x_i^{-1} - \left(\sum_{i=1}^n p_i x_i\right)^{-1} \leq n\bar{p}\left[\sum_{i=1}^n \frac{1}{n} x_i^{-1} - \left(\sum_{i=1}^n \frac{1}{n} x_i\right)^{-1}\right], \tag{2.6}$$

where $\bar{p} = \max\{p_1,p_2,\ldots,p_n\}$.

In particular, when $n = 2$ in the inequality (2.6), we can get

$$a\nabla_\mu b \leq a!_\mu b + 2R(a\nabla b - a!b), \tag{2.7}$$

where $a,b > 0$, $\mu \in [0,1]$, $R = \max\{1-\mu,\mu\}$. Equality holds if and only if $a = b$.

Note that (2.7) is a reverse of the inequality (1.3) with a similar form.

The following three theorems are our main reverse forms of the μ-weighted arithmetic-harmonic mean inequality for scalars.

Theorem 2.1 *Let $a, b > 0$, and $\mu \in [0,1]$. Then the inequality*

$$a\nabla_\mu b \leq 2r(a\nabla b - a!b) + \mathrm{K}(\sqrt{h},2)^{R'}\mathrm{K}(h,2)^{R} a!_\mu b \tag{2.8}$$

holds, where $h = \frac{b}{a}$, $r = \min\{\mu, 1-\mu\}$, $R = \max\{\mu, 1-\mu\}$, and $R' = \max\{2r, 1-2r\}$. Equality holds if and only if $a = b$.

Proof By the inequalities (2.3) and (2.4), firstly, we consider the case $\mu \in [0, \frac{1}{2}]$,

$$\begin{aligned} a\nabla_\mu b - 2\mu(a\nabla b - a!b) &\leq a\nabla_\mu b - 2\mu(a\nabla b - \sqrt{ab}) \\ &= (1-2\mu)a + 2\mu\sqrt{ab} \\ &\leq \mathrm{K}(\sqrt{h},2)^{R'} a^{1-\mu} b^{\mu} \\ &\leq \mathrm{K}(\sqrt{h},2)^{R'}\mathrm{K}(h,2)^{R} a!_\mu b. \end{aligned}$$

If $\mu \in (\frac{1}{2}, 1]$, then we have

$$\begin{aligned} a\nabla_\mu b - 2(1-\mu)(a\nabla b - a!b) &\leq a\nabla_\mu b - 2(1-\mu)(a\nabla b - \sqrt{ab}) \\ &= (2\mu-1)b + 2(1-\mu)\sqrt{ab} \\ &\leq \mathrm{K}(\sqrt{h},2)^{R'} a^{1-\mu} b^{\mu} \\ &\leq \mathrm{K}(\sqrt{h},2)^{R'}\mathrm{K}(h,2)^{R} a!_\mu b. \end{aligned}$$

By the above discussion, for any $\mu \in [0,1]$, the inequality (2.8) always holds. □

Note that (2.8) can also be considered as a reverse ratio inequality of (1.3):

$$0 < a!_\mu b \leq a\nabla_\mu b - 2r(a\nabla b - a!b) \leq \mathrm{K}(\sqrt{h},2)^{R'}\mathrm{K}(h,2)^{R} a!_\mu b.$$

Theorem 2.2 *Let $a, b > 0$, $\mu \in [0,1]$, and $h = \frac{b}{a}$.*

(I) *If $0 \leq \mu \leq \frac{1}{2}$, then*

$$a\nabla_\mu b - 2\mu(a\nabla b - a!b) \leq \mathrm{K}(\sqrt{h},2) a!_{2\mu}\sqrt{ab}.$$

Equality holds if and only if $a = b$.

(II) *If $\frac{1}{2} < \mu \leq 1$, then*

$$a\nabla_\mu b - 2(1-\mu)(a\nabla b - a!b) \leq \mathrm{K}(\sqrt{h},2) b!_{2-2\mu}\sqrt{ab}.$$

Equality holds if and only if $a = b$.

Proof By the inequality (2.2), firstly, we consider the case $\mu \in [0, \frac{1}{2}]$, then we have

$$\begin{aligned} a\nabla_\mu b - 2\mu(a\nabla b - a!b) &\leq a\nabla_\mu b - 2\mu(a\nabla b - \sqrt{ab}) \\ &= (1-2\mu)a + 2\mu\sqrt{ab} \\ &\leq \mathrm{K}(\sqrt{h},2) a!_{2\mu}\sqrt{ab}. \end{aligned}$$

If $\mu \in (\frac{1}{2}, 1]$, then we have

$$\begin{aligned} a\nabla_\mu b - 2(1-\mu)(a\nabla b - a!b) &\leq a\nabla_\mu b - 2(1-\mu)(a\nabla b - \sqrt{ab}) \\ &= (2\mu - 1)b + 2(1-\mu)\sqrt{ab} \\ &\leq \mathrm{K}(\sqrt{h}, 2) b!_{2-2\mu}\sqrt{ab}. \end{aligned}$$

The proof is completed. □

Theorem 2.3 *Let $a, b > 0$ and $\mu \in [0,1]$.*

(I) *If $0 \leq \mu \leq \frac{1}{2}$, then*

$$a\nabla_\mu b - 2\mu(a\nabla b - a!b) \leq a!_{2\mu}\sqrt{ab} + 2R'(a\nabla\sqrt{ab} - a!\sqrt{ab}).$$

Equality holds if and only if $a = b$.

(II) *If $\frac{1}{2} < \mu \leq 1$, then*

$$a\nabla_\mu b - 2(1-\mu)(a\nabla b - a!b) \leq b!_{2-2\mu}\sqrt{ab} + 2R'(b\nabla\sqrt{ab} - b!\sqrt{ab}).$$

Equality holds if and only if $a = b$.

Proof Firstly, we consider the case $\mu \in [0, \frac{1}{2}]$, by the inequality (2.7), then we have

$$\begin{aligned} a\nabla_\mu b - 2\mu(a\nabla b - a!b) &\leq (1-2\mu)a + 2\mu\sqrt{ab} \\ &\leq a!_{2\mu}\sqrt{ab} + 2\max\{1-2\mu, 2\mu\}(a\nabla\sqrt{ab} - a!\sqrt{ab}). \end{aligned}$$

If $\mu \in (\frac{1}{2}, 1]$, by the inequality (2.7), then we have

$$\begin{aligned} &a\nabla_\mu b - 2(1-\mu)(a\nabla b - a!b) \\ &\quad \leq (2\mu - 1)b + 2(1-\mu)\sqrt{ab} \\ &\quad \leq b!_{2-2\mu}\sqrt{ab} + 2\max\{2-2\mu, 2\mu - 1\}(b\nabla\sqrt{ab} - b!\sqrt{ab}). \end{aligned}$$

The proof is completed. □

Note that the inequalities proved in Theorem 2.1-2.3 are all the reverses of the inequality (1.3).

3 Reverse arithmetic-harmonic mean operator inequalities

In this section, we present the operator versions of these reverse arithmetic-harmonic mean inequalities proved in Section 2. The techniques are based on the monotonicity property of operator functions described in the following lemma (for more details, see [1, 10]).

Lemma 3.1 *Let $X \in \mathcal{B}(H)$ be self-adjoint operator and if f and g are both continuous functions with $f(t) \geq g(t)$ for $t \in \mathrm{Sp}(X)$ (the spectrum of X), then $f(X) \geq g(X)$ with equality if and only if $f(t) = g(t)$ for all $t \in \mathrm{Sp}(X)$.*

Based on Proposition 2.1, we can deduce the operator inequality which is also the non-commutative Kantorovich inequality proved by Furuta *et al.* [1], but our method is different.

Proposition 3.1 *Let $A, B \in \mathcal{B}^{++}(H)$ and positive real numbers m, M satisfy $0 < mI \le A, B \le MI$. Then*

$$A\nabla_\mu B \le \mathrm{K}(h,2)A!_\mu B, \tag{3.1}$$

where $\mu \in [0,1]$ and $h = \frac{M}{m}$. Equality holds if and only if $A = B$ and $m = M$.

Proof By the inequality (2.2), we have

$$(1-\mu) + \mu x \le \mathrm{K}(x,2)\big((1-\mu) + \mu x^{-1}\big)^{-1}$$

for any $x > 0$, and hence

$$(1-\mu)I + \mu X \le \max_{\frac{1}{h}\le x\le h} \mathrm{K}(x,2)\big((1-\mu)I + \mu X^{-1}\big)^{-1}$$

for the positive operator X such that $0 < \frac{1}{h}I \le X \le hI$.

Since $0 < \frac{1}{h}I \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le hI$ and the Kantorovich constant $\mathrm{K}(t,2)$ is an increasing function for $t > 1$ and $\mathrm{K}(\frac{1}{h},2) = \mathrm{K}(h,2)$, substituting $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ for X in the above inequality, we have

$$(1-\mu)I + \mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le \mathrm{K}(h,2)\big((1-\mu)I + \mu\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^{-1}\big)^{-1}.$$

Multiplying both sides of the above inequality by $A^{\frac{1}{2}}$, we can deduce the required inequality (3.1). □

We prove an operator inequality obtained by Krnić *et al.* in [11] by a different method. But our method is more transparent and simpler than the one given in [11].

Proposition 3.2 *Let $A, B \in \mathcal{B}^{++}(H)$ and $\mu \in [0,1]$. Then*

$$A\nabla_\mu B \le A!_\mu B + 2R(A\nabla B - A!B), \tag{3.2}$$

where $R = \max\{1-\mu, \mu\}$. Equality holds if and only if $A = B$.

Proof By the inequality (2.7), for $x > 0$ and $\mu \in [0,1]$, we have

$$(1-\mu) + \mu x^{-1} \le \big((1-\mu) + \mu x\big)^{-1} + 2R\left[\frac{1+x^{-1}}{2} - \left(\frac{1+x}{2}\right)^{-1}\right].$$

For a positive invertible operator T and $\mu \in [0,1]$, it follows that

$$(1-\mu)I + \mu T^{-1} \le \big((1-\mu)I + \mu T\big)^{-1} + 2R\left[\frac{I+T^{-1}}{2} - \left(\frac{I+T}{2}\right)^{-1}\right].$$

Putting $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in the above inequality and multiplying both sides by $A^{\frac{1}{2}}$, we deduce the inequality (3.2). □

Note that, by (1.4) and (3.2), we have

$$\begin{aligned} 0 &< A!_\mu B \\ &\le A!_\mu B + 2r(A\nabla B - A!B) \\ &\le A\nabla_\mu B \\ &\le A!_\mu B + 2R(A\nabla B - A!B). \end{aligned}$$

Based on Theorem 2.1, we have the following.

Theorem 3.1 *Let $A, B \in \mathcal{B}^{++}(H)$ and positive real numbers m, M satisfy $0 < mI \le A, B \le MI$. Then for $\mu \in [0,1]$,*

$$A\nabla_\mu B - 2r(A\nabla B - A!B) \le \mathrm{K}(\sqrt{h}, 2)^{R'}\mathrm{K}(h,2)^R A!_\mu B, \tag{3.3}$$

where $h = \frac{M}{m}$, $r = \min\{\mu, 1-\mu\}$, $R = \max\{\mu, 1-\mu\}$, and $R' = \max\{2r, 1-2r\}$. Equality holds if and only if $A = B$ and $m = M$.

Proof By the inequality (2.8), we have

$$\begin{aligned} &(1-\mu) + \mu x - 2r\left[\frac{1+x}{2} - \left(\frac{1}{2} + \frac{1}{2}x^{-1}\right)^{-1}\right] \\ &\quad \le \mathrm{K}(\sqrt{x}, 2)^{R'}\mathrm{K}(x,2)^R\bigl((1-\mu) + \mu x^{-1}\bigr)^{-1} \end{aligned}$$

for any $x > 0$, and hence

$$\begin{aligned} &(1-\mu)I + \mu X - 2r\left[\frac{I+X}{2} - \left(\frac{1}{2}I + \frac{1}{2}X^{-1}\right)^{-1}\right] \\ &\quad \le \mathrm{K}(\sqrt{h}, 2)^{R'}\mathrm{K}(h,2)^R\bigl((1-\mu)I + \mu X^{-1}\bigr)^{-1} \end{aligned}$$

for the positive operator X such that $0 < \frac{1}{h}I \le X \le hI$.

By a similar process to Proposition 3.1, we can deduce the required inequality (3.3). □

Note that (3.3) is a reverse of (1.4):

$$0 < A!_\mu B \le A\nabla_\mu B - 2r(A\nabla B - A!B) \le \mathrm{K}(\sqrt{h}, 2)^{R'}\mathrm{K}(h,2)^R A!_\mu B.$$

Now, we exhibit the operator inequalities based on Theorem 2.2 and Theorem 2.3.

Theorem 3.2 *Let $A, B \in \mathcal{B}^{++}(H)$ and $\mu \in [0,1]$. The positive real numbers m, M satisfy $0 < mI \le A, B \le MI$ and $h = \frac{M}{m}$.*

(I) *If* $0 \le \mu \le \frac{1}{2}$, *then*

$$A\nabla_\mu B - 2\mu(A\nabla B - A!B) \le \mathrm{K}(\sqrt{h},2)A!_{2\mu}(A\#B). \tag{3.4}$$

Equality holds if and only if $A = B$ *and* $m = M$.

(II) *If* $\frac{1}{2} < \mu \le 1$, *then*

$$A\nabla_\mu B - 2(1-\mu)(A\nabla B - A!B) \le \mathrm{K}(\sqrt{h},2)B!_{2(1-\mu)}(A\#B). \tag{3.5}$$

Equality holds if and only if $A = B$ *and* $m = M$.

Proof If $0 \le \mu \le \frac{1}{2}$, by (I) of Theorem 2.2, we have

$$\begin{aligned}&(1-\mu) + \mu a - 2\mu\left[\frac{1+a}{2} - \left(\frac{1}{2} + \frac{1}{2}a^{-1}\right)^{-1}\right]\\ &\quad\le \mathrm{K}(\sqrt{a},2)\bigl[(1-2\mu) + 2\mu a^{-\frac{1}{2}}\bigr]^{-1}\end{aligned}$$

for any $a > 0$, and hence

$$\begin{aligned}&(1-\mu)I + \mu X - 2\mu\left[\frac{I+X}{2} - \left(\frac{1}{2}I + \frac{1}{2}X^{-1}\right)^{-1}\right]\\ &\quad\le \mathrm{K}(\sqrt{h},2)\bigl[(1-2\mu)I + 2\mu X^{-\frac{1}{2}}\bigr]^{-1}\end{aligned}$$

for the positive invertible operator X such that $0 < \frac{1}{h}I \le X \le hI$.

Substituting $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ for X in the above inequality, we have

$$\begin{aligned}&(1-\mu)I + \mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\\ &\quad - 2\mu\left[\frac{I + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2} - \left(\frac{1}{2}I + \frac{1}{2}\bigl(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\bigr)^{-1}\right)^{-1}\right]\\ &\quad\le \mathrm{K}(\sqrt{h},2)\bigl[(1-2\mu)I + 2\mu\bigl(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\bigr)^{-\frac{1}{2}}\bigr]^{-1}.\end{aligned} \tag{3.6}$$

Multiplying both sides of (3.6) by $A^{\frac{1}{2}}$, we can deduce the required inequality (3.4).

Likewise, if $\frac{1}{2} < \mu \le 1$, by (II) of Theorem 2.2, we have

$$\begin{aligned}&(1-\mu)b + \mu - 2(1-\mu)\left[\frac{1+b}{2} - \left(\frac{1}{2} + \frac{1}{2}b^{-1}\right)^{-1}\right]\\ &\quad\le \mathrm{K}(\sqrt{b},2)\bigl[(2\mu-1)b^{-\frac{1}{2}} + (2-2\mu)\bigr]^{-1}\end{aligned}$$

for any $b > 0$, and hence

$$\begin{aligned}&(1-\mu)Y + \mu I - 2(1-\mu)\left[\frac{I+Y}{2} - \left(\frac{1}{2}I + \frac{1}{2}Y^{-1}\right)^{-1}\right]\\ &\quad\le \mathrm{K}(\sqrt{h},2)\bigl[(2\mu-1)Y^{-\frac{1}{2}} + (2-2\mu)I\bigr]^{-1}\end{aligned}$$

for the positive invertible operator Y such that $0 < \frac{1}{h}I \le Y \le hI$.

Substituting $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ for Y in the above inequality, we have

$$\begin{aligned} &\mu I + (1-\mu)B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \\ &\quad - 2(1-\mu)\left[\frac{I + B^{-\frac{1}{2}}AB^{-\frac{1}{2}}}{2} - \left(\frac{1}{2}I + \frac{1}{2}I\bigl(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\bigr)^{-1}\right)^{-1}\right] \\ &\quad \leq \mathrm{K}(\sqrt{h},2)\bigl[(2\mu-1)\bigl(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\bigr)^{-\frac{1}{2}} + (2-2\mu)I\bigr]^{-1}. \end{aligned} \tag{3.7}$$

Multiplying both sides of (3.7) by $B^{\frac{1}{2}}$, we can deduce the required inequality (3.5). □

Note that, by (1.4), (3.4), and (3.5), we have

$$\begin{aligned} 0 &< A!_{\mu}B \\ &\leq A\nabla_{\mu}B - 2r(A\nabla B - A!B) \\ &\leq \begin{cases} \mathrm{K}(\sqrt{h},2)A!_{2\mu}(A\#B), & 0 \leq \mu \leq \frac{1}{2}, \\ \mathrm{K}(\sqrt{h},2)B!_{2(1-\mu)}(A\#B), & \frac{1}{2} < \mu \leq 1. \end{cases} \end{aligned}$$

Theorem 3.3 *Let $A, B \in \mathcal{B}^{++}(H)$ and $\mu \in [0,1]$.*

(I) *If $0 \leq \mu \leq \frac{1}{2}$, then*

$$A\nabla_{\mu}B - 2\mu(A\nabla B - A!B) \leq A!_{2\mu}(A\#B) + 2R'\bigl(A\nabla(A\#B) - A!(A\#B)\bigr). \tag{3.8}$$

Equality holds if and only if $A = B$.

(II) *If $\frac{1}{2} < \mu \leq 1$, then*

$$A\nabla_{\mu}B - 2(1-\mu)(A\nabla B - A!B) \leq B!_{2-2\mu}(A\#B) + 2R'\bigl(B\nabla(A\#B) - B!(A\#B)\bigr). \tag{3.9}$$

Equality holds if and only if $A = B$.

Proof If $0 \leq \mu \leq \frac{1}{2}$, by (I) of Theorem 2.3, we have

$$\begin{aligned} &(1-\mu) + \mu b - 2\mu\left[\frac{1+b}{2} - \left(\frac{1}{2} + \frac{1}{2}b^{-1}\right)^{-1}\right] - \bigl[(1-2\mu) + 2\mu b^{-\frac{1}{2}}\bigr]^{-1} \\ &\quad \leq 2\max\{1-2\mu, 2\mu\}\left[\frac{1}{2} + \frac{1}{2}b^{\frac{1}{2}} - \left(\frac{1}{2} + \frac{1}{2}b^{-\frac{1}{2}}\right)^{-1}\right] \end{aligned}$$

for any $b > 0$, and hence

$$\begin{aligned} &(1-\mu)I + \mu X - 2\mu\left[\frac{I+X}{2} - \left(\frac{1}{2}I + \frac{1}{2}X^{-1}\right)^{-1}\right] - \bigl[(1-2\mu)I + 2\mu X^{-\frac{1}{2}}\bigr]^{-1} \\ &\quad \leq 2\max\{1-2\mu, 2\mu\}\left[\frac{1}{2}I + \frac{1}{2}X^{\frac{1}{2}} - \left(\frac{1}{2}I + \frac{1}{2}X^{-\frac{1}{2}}\right)^{-1}\right] \end{aligned}$$

for the positive invertible operator X.

Substituting $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ for X in the above inequality and then multiplying both sides by $A^{\frac{1}{2}}$, we can deduce the required inequality (3.8).

Likewise, if $\frac{1}{2} < \mu \le 1$, by (II) of Theorem 2.3, we have

$$\begin{aligned}&(1-\mu)a+\mu-2(1-\mu)\left[\frac{a+1}{2}-\left(\frac{1}{2}a^{-1}+\frac{1}{2}\right)^{-1}\right]\\&\quad\le\left[(2\mu-1)+(2-2\mu)a^{-\frac{1}{2}}\right]^{-1}+2\max\{2-2\mu,2\mu-1\}\left[\frac{1}{2}+\frac{1}{2}a^{\frac{1}{2}}-\left(\frac{1}{2}+\frac{1}{2}a^{-\frac{1}{2}}\right)^{-1}\right]\end{aligned}$$

for any $a > 0$, and hence

$$\begin{aligned}&(1-\mu)Y+\mu I-2(1-\mu)\left[\frac{Y+I}{2}-\left(\frac{1}{2}Y^{-1}+\frac{1}{2}I\right)^{-1}\right]\\&\quad\le\left[(2\mu-1)I+(2-2\mu)Y^{-\frac{1}{2}}\right]^{-1}\\&\qquad+2\max\{2-2\mu,2\mu-1\}\left[\frac{1}{2}I+\frac{1}{2}Y^{\frac{1}{2}}-\left(\frac{1}{2}I+\frac{1}{2}Y^{-\frac{1}{2}}\right)^{-1}\right]\end{aligned}$$

for the positive invertible operator Y.

Substituting $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ for Y in the above inequality and then multiplying both sides by $B^{\frac{1}{2}}$, we can deduce the required inequality (3.9). □

Note that, by (1.4), (3.8), and (3.9), we have

$$\begin{aligned}&0 < A!_\mu B\\&\quad\le A\nabla_\mu B-2r(A\nabla B-A!B)\\&\quad\le\begin{cases}A!_{2\mu}(A\#B)+2R'(A\nabla(A\#B)-A!(A\#B)), & 0\le\mu\le\frac{1}{2},\\ B!_{2-2\mu}(A\#B)+2R'(B\nabla(A\#B)-B!(A\#B)), & \frac{1}{2}<\mu\le 1.\end{cases}\end{aligned}$$

Remark 3.1 These inequalities proved in Theorem 3.1-3.3 are all the reverse forms of the inequality (1.4). It is easy to see that the right-hand side of these inequalities can not be compared with each other, but they are indeed new versions of reverse ratio arithmetic-harmonic mean inequality.

4 The mixed mean inequalities

In this section, we obtain refinements of the inequalities (1.5) and (1.6) and deduce some mixed weighted arithmetic-geometric and geometric-harmonic means inequalities.

First, we need the following lemma.

Lemma 4.1 (Hermite-Hadamard's inequality [12]) *If $f:\mathbb{I}\to\mathbb{R}$ is a convex function on the interval $\mathbb{I}\subset\mathbb{R}$, then for any $x,y\in\mathbb{I}$ with $x\neq y$, we have*

$$f\left(\frac{x+y}{2}\right)\le\frac{1}{y-x}\int_x^y f(t)\,dt\le\frac{f(x)+f(y)}{2}. \tag{4.1}$$

Theorem 4.1 *Let $A,B\in\mathcal{B}^{++}(H)$. Then*

$$A\#(A\nabla B)\ge\frac{1}{3}\left[4A\nabla(A\#B)-(A\#B)\left(A\nabla(A\#B)\right)^{-1}A\right]\ge A\nabla(A\#B), \tag{4.2}$$

$$A\#(A!B)\le 3\left[4\left(A!(A\#B)\right)^{-1}-(A\#B)^{-1}\left(A!(A\#B)\right)A^{-1}\right]^{-1}\le A!(A\#B). \tag{4.3}$$

Proof Applying the Hermite-Hadamard's inequalities (4.1) to the convex function $f(t) = -t^{\frac{1}{2}}$, $t > 0$, we have

$$\left(\frac{x+y}{2}\right)^{1/2} \geq \frac{1}{3}\left[4\left(\frac{x^{1/2}+y^{1/2}}{2}\right) - (xy)^{1/2}\left(\frac{x^{1/2}+y^{1/2}}{2}\right)^{-1}\right] \geq \frac{(x^{1/2}+y^{1/2})}{2},$$

and hence

$$\left(\frac{I+C}{2}\right)^{1/2} \geq \frac{1}{3}\left[4\left(\frac{I+C^{1/2}}{2}\right) - (C)^{1/2}\left(\frac{I+C^{1/2}}{2}\right)^{-1}\right] \geq \frac{(I+C^{1/2})}{2}$$

for a positive invertible operator C.

Putting $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in the above inequality and multiplying both sides by $A^{\frac{1}{2}}$, we can obtain (4.2).

Substituting A^{-1} for A and B^{-1} for B and then taking the inverse in (4.2), we get (4.3). □

Note that the inequalities (4.2) and (4.3) are the refinements of (1.5) and (1.6), respectively.

If f is convex on a segment $[a,b]$ of a linear space, one can easily observe that (4.1) is equivalent to the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f\big((1-t)a+tb\big)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{4.4}$$

A natural generalization of the classical Hermite-Hadamard inequality to Hermitian matrices could be the double inequality

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f\big((1-t)A+tB\big)\,dt \leq \frac{f(A)+f(B)}{2}.$$

However, Moslehian [13] pointed out that this was not true. We will show that the following result is valid.

Theorem 4.2 *Let $A, B \in \mathcal{B}^{++}(H)$. Then*

$$A\#(A\nabla B) \geq \int_0^1 A\#(A\nabla_t B)\,dt \geq A\nabla(A\#B). \tag{4.5}$$

Proof Taking $f(t) = -t^{\frac{1}{2}}$, $t > 0$ in (4.4), then

$$\left(\frac{a+b}{2}\right)^{1/2} \geq \int_0^1 \big((1-t)a+tb\big)^{1/2}\,dt \geq \frac{(a^{1/2}+b^{1/2})}{2},$$

and by a method resembling Theorem 4.1, we obtain (4.5). □

In the next theorems, mixed weighted arithmetic-geometric and geometric-harmonic mean inequalities are established. First, we show the definition of operator convex (see [10], p.113).

Definition 4.1 A continuous function $f : \mathbb{I} \to \mathbb{R}$ on the interval $\mathbb{I} \subset \mathbb{R}$ is said to be operator convex if for every pair of self-adjoint operators X, Y on a Hilbert space H with spectrum in $\mathbb{I}$ and each $\nu \in [0,1]$,

$$f\bigl((1-\nu)X + \nu Y\bigr) \le (1-\nu)f(X) + \nu f(Y). \tag{4.6}$$

A function f is called operator concave if the function $-f$ is operator convex. The function $f(X) = X^s$ is operator convex on a self-adjoint operator space for $-1 \le s \le 0$ or $1 \le s \le 2$ and is operator concave for $0 \le s \le 1$.

For an operator convex function $f : \mathbb{I} \to \mathbb{R}$ on the interval $\mathbb{I} \subset \mathbb{R}$, we have the following property (see [14], p.71): For each $\nu \notin [0,1]$, self-adjoint operators X, Y and $(1-\nu)X + \nu Y$ with spectra in $\mathbb{I}$,

$$f\bigl((1-\nu)X + \nu Y\bigr) \ge (1-\nu)f(X) + \nu f(Y). \tag{4.7}$$

In the next theorems, we still use the notations $A\nabla_\nu B$, $\nu \notin [0,1]$, and $A\#_s B$, $s \in [-1,0] \cup [1,2]$.

Theorem 4.3 *Let* $A, B \in \mathcal{B}^{++}(H)$, *and* $\nu \in [0,1]$. *If* $s \in [-1,0] \cup [1,2]$, *then*

$$A\#_s(A\nabla_\nu B) \le A\nabla_\nu(A\#_s B), \tag{4.8}$$

$$A\#_s(A!_\nu B) \ge A!_\nu(A\#_s B). \tag{4.9}$$

Proof By the inequality (4.6), for every pair of self-adjoint operators X, Y and the operator convex function $f(x) = x^s$ $(x > 0)$, $s \in [-1,0] \cup [1,2]$, we have

$$\bigl[(1-\nu)X + \nu Y\bigr]^s \le (1-\nu)X^s + \nu Y^s.$$

Putting $X = I$ and $Y = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in the above inequality and multiplying both sides by $A^{\frac{1}{2}}$, we can get (4.8).

Substituting A^{-1} for A and B^{-1} for B in (4.8) and then taking the inverse of both sides, we get (4.9). □

Theorem 4.4 *Let* $A, B \in \mathcal{B}^{++}(H)$ *and* $(1-\nu)A + \nu B \in \mathcal{B}^{++}(H)$ *for* $\nu \notin [0,1]$.

(I) *If* $s \in [0,1]$, *then*

$$A\#_s(A\nabla_\nu B) \le A\nabla_\nu(A\#_s B),$$

$$A\#_s(A!_\nu B) \ge A!_\nu(A\#_s B).$$

(II) *If* $s \in [-1,0] \cup [1,2]$, *then*

$$A\#_s(A\nabla_\nu B) \ge A\nabla_\nu(A\#_s B),$$

$$A\#_s(A!_\nu B) \le A!_\nu(A\#_s B).$$

Proof By the inequality (4.7) and using the same ideas as in the proof of Theorem 4.2, we can deduce this theorem. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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